On The Four-Loop QCD Form Factors

Robert M. Schabinger

with Andreas von Manteuffel and Erik Panzer, based on: Phys. Lett. **B744** (2015) 101, JHEP **1502** (2015) 120, Phys. Rev. **D93** (2016) 125014, Phys. Rev. **D95** (2017) 034030, Phys. Rev. **D99** (2019) 105010, Phys. Rev. **D99** (2019) 094014, JHEP **1905** (2019) 073, and work in progress

Michigan State University

Outline

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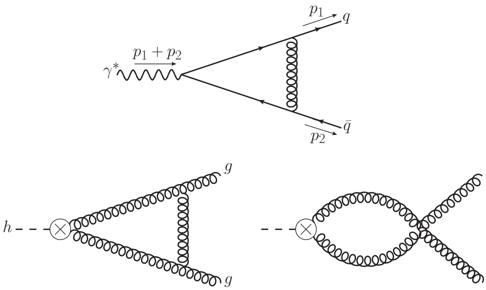

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$$q^2 \frac{\partial}{\partial q^2} \ln\left(\mathcal{F}\left(q^2/\mu^2, \alpha_s, \epsilon\right)\right) = 1/2\mathcal{K}(\alpha_s) + 1/2\mathcal{G}\left(q^2/\mu^2, \alpha_s, \epsilon\right)$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right)\mathcal{G}\left(q^2/\mu^2, \alpha_s, \epsilon\right) = \Gamma(\alpha_s)$$

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At L loops, Γ_L characterizes the leading IR divergences which cannot be understood as exponentiated lower-loop contributions.

$\Longrightarrow$ Γ_4 is the last unknown ingredient needed for N^3LL resummation!

A Dipole Formula For Gauge Theory IR Divergences?

S. Catani, Phys. Lett. **B427** (1998) 161; S. Mert Aybat *et. al.*, Phys. Rev. **D74** (2006) 074004

T. Becher and M. Neubert, JHEP **0906** (2009) 081; E. Gardi and L. Magnea, JHEP **0903** (2009) 079

The IR divergences of the simplest non-Abelian gauge theory, planar $SU(N_c)$ $\mathcal{N}=4$ super Yang-Mills, are believed to be of the form:

$$\mathcal{A}_1^{\mathcal{N}=4}(p_1,\ldots,p_n) = \exp\left\{-\frac{1}{2}\sum_{L=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^L \mu_\epsilon^{2L\epsilon}\int_0^{\mu_\epsilon^2} \mathrm{d}\mu^2\left(\mu^2\right)^{-1-L\epsilon}\right.$$
$$\left.\sum_{\substack{\mathrm{i,j}=1\\ \mathrm{i}<\mathrm{j}}}^{n}\left(\Gamma_{1;L}^{\mathcal{N}=4}\ln\left(\frac{\mu^2}{-s_{ij}}\right)+\mathcal{G}_{1;L}^{\mathcal{N}=4}\right)\frac{\mathbf{T}_\mathrm{i}\cdot\mathbf{T}_\mathrm{j}}{N_c}\right\}\sum_{L=0}^{\infty}\mathbf{H}_{1;L}^{\mathcal{N}=4}(\epsilon;p_1,\ldots,p_n)$$

At four points, this structure has been realized explicitly at strong coupling (L. F. Alday and J. Maldacena, JHEP **0706** (2007) 064). In a nutshell, the dipole conjecture is the suggestion that, with minor modifications, the above structure could hold for more general gauge theories like QCD.

When Something Sounds Too Good To Be True...

Although some three-loop evidence was collected by Dixon (Phys. Rev. **D79** (2009) 091501) for the n_f terms, the dipole conjecture fails for QCD at three loops.

S. Caron-Huot, JHEP **1505** (2015) 093; Ø. Almelid *et. al.*, Phys. Rev. Lett. **117** (2016) 172002;

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In light of the above, one might expect Casimir scaling:

$$\Gamma_L^g \stackrel{?}{=} C_A/C_F \Gamma_L^q$$

to break down as well for some L.

R. Boels *et. al.*, Phys. Rev. Lett. **119** (2017) 201601; JHEP **1801** (2018) 153

S.-O. Moch *et. al.*, JHEP **1710** (2017) 041; Phys. Lett. **B782** (2018) 627

R. N. Lee *et. al.*, JHEP **1902** (2019) 172; J. M. Henn *et. al.*, Phys. Rev. Lett. **122** (2019) 201602

Catani *et. al.*, Eur. Phys. J. **C79** (2019) 685;
T. Becher and M. Neubert, arXiv:1908.11379; G. Falcioni *et. al.* arXiv:1909.00697

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$\Longrightarrow$ To go beyond low-precision numerics, new multi-loop methods!

Calculational Method

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- Do numerator algebra on $\sim$ 50,000 diagrams with `QGraf` + `FORM`

  P. Nogueira, J. Comput. Phys. **105** (1993) 279

  J. Kuipers *et. al.*, Comput. Phys. Commun. **184** (2013) 1453

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  E. Panzer, Comput. Phys. Commun. **188** (2015) 148

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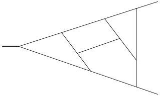
  E. Panzer, Comput. Phys. Commun. **188** (2015) 148

Even for the cusp anomalous dimensions, many new developments!

Complete Results For The Planar Master Integrals

A. von Manteuffel and RMS, JHEP **1905** (2019) 073

Complete weight 8 results obtained for all 99 master integrals, *e.g.*

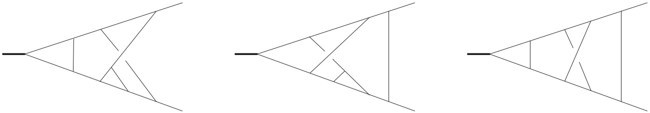


$$= \frac{1}{\epsilon^7}\left(-\frac{1}{288}\right) + \frac{1}{\epsilon^6}\left(\frac{13}{576}\right) + \frac{1}{\epsilon^5}\left(-\frac{19}{144}\zeta_2 - \frac{101}{576}\right) + \cdots - \frac{44097}{64}\zeta_7$$
$$- \frac{577}{6}\zeta_5\zeta_2 - \frac{1993}{16}\zeta_3\zeta_2^2 + \frac{9101}{72}\zeta_3^2 + \frac{240637}{1344}\zeta_2^3 - \frac{24965}{32}\zeta_5 - \frac{61}{144}\zeta_3\zeta_2 + \frac{99523}{480}\zeta_2^2$$
$$- \frac{88745}{144}\zeta_3 + \frac{13319}{72}\zeta_2 + \frac{382375}{144} + \epsilon\left(\frac{4931}{30}\zeta_{5,3} - \frac{60385}{24}\zeta_5\zeta_3 + \frac{1321}{36}\zeta_3^2\zeta_2\right.$$
$$- \frac{702196781}{1008000}\zeta_2^4 + \frac{21687}{8}\zeta_7 + \frac{9989}{24}\zeta_5\zeta_2 + \frac{339821}{1440}\zeta_3\zeta_2^2 - \frac{40519}{288}\zeta_3^2 - \frac{3681977}{5040}\zeta_2^3$$
$$\left. + \frac{155087}{48}\zeta_5 - \frac{1457}{3}\zeta_3\zeta_2 - \frac{179053}{288}\zeta_2^2 + \frac{381811}{144}\zeta_3 - \frac{10777}{24}\zeta_2 - \frac{2005247}{144}\right) + \mathcal{O}\left(\epsilon^2\right)$$

The $\mathcal{O}\left(N_{q\gamma}N_f\right)$ Quark Form Factor Calculation

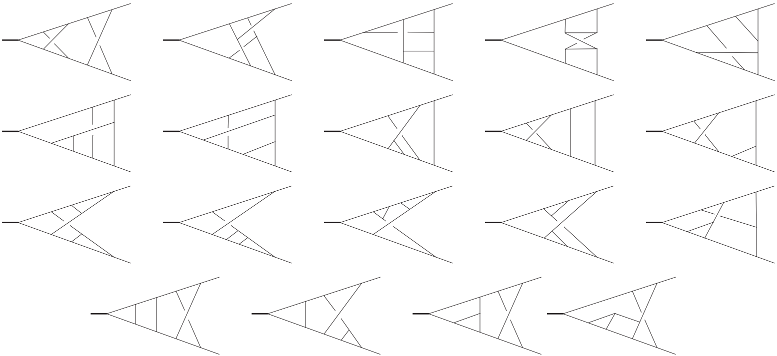
$$N_{q\gamma} = \frac{1}{e_q}\sum_{q'} e_{q'}$$

- 923 topologies to be reduced with up to 5 numerator insertions.
- The 3 non-planar top-level integral topologies are all reducible.
- 1 color structure and 45 master integrals (42 topologies).
- 42 master integrals for $\mathcal{O}(\epsilon^0)$ in the finite integral method.



The $\mathcal{O}\left(N_f^2\right)$ Gluon Form Factor Calculation

- 1781 topologies to be reduced with up to 6 numerator insertions.
- 4 of 19 non-planar top-level integral topologies are irreducible.
- 4 color structures and 163 master integrals (151 topologies).
- 130 master integrals for $\mathcal{O}(\epsilon^0)$ in the finite integral approach.



The $\mathcal{O}(N_{q\gamma}N_f)$ Bare Four-Loop Quark Form Factor

In the $\overline{\mathrm{MS}}$ scheme, we find

$$\mathcal{F}_4^q(\epsilon)\Big|_{\frac{d_F^{abc}d_F^{abc}}{N_F}} = \frac{1}{\epsilon}\left(\frac{1280}{3}\zeta_5 + \frac{32}{5}\zeta_2^2 - \frac{224}{3}\zeta_3 - 160\zeta_2 - 64\right) + 2304\zeta_3^2 + \frac{545792}{945}\zeta_2^3 - \frac{8512}{9}\zeta_5 - \frac{1888}{3}\zeta_2\zeta_3 + \frac{12448}{45}\zeta_2^2 - \frac{3520}{9}\zeta_3 - \frac{16928}{9}\zeta_2 - \frac{11296}{9} + \mathcal{O}(\epsilon)$$

in R_ξ gauge, keeping terms of $\mathcal{O}(1-\xi)$, where

$$T_F = 1/2 \qquad \text{and} \qquad \frac{d_F^{abc}d_F^{abc}}{N_F} = \frac{(N_c^2-1)(N_c^2-4)}{16N_c^2}$$

The $\mathcal{O}(N_f^2)$ Bare Four-Loop Gluon Form Factor

$$\mathcal{F}_4^g(\epsilon)\Big|_{C_A^2} = \frac{1}{\epsilon^6}\left(\frac{41}{162}\right) + \frac{1}{\epsilon^5}\left(\frac{113}{486}\right) + \frac{1}{\epsilon^4}\left(-\frac{128}{81}\zeta_2 - \frac{151}{27}\right) + \frac{1}{\epsilon^3}\left(-\frac{5167}{243}\zeta_3 + \frac{799}{243}\zeta_2 - \frac{1085641}{34992}\right) + \frac{1}{\epsilon^2}\left(-\frac{7087}{405}\zeta_2^2 - \frac{35848}{729}\zeta_3 + \frac{425}{9}\zeta_2 - \frac{1144163}{209952}\right) + \frac{1}{\epsilon}\left(-\frac{73282}{405}\zeta_5 + \frac{26396}{243}\zeta_2\zeta_3 - \frac{37894}{1215}\zeta_2^2 + \frac{28705}{324}\zeta_3 + \frac{2504629}{17496}\zeta_2 + \frac{541388327}{419904}\right) + \frac{717266}{729}\zeta_3^2 + \frac{58858}{945}\zeta_2^3 - \frac{62671}{1215}\zeta_5 - \frac{87329}{1458}\zeta_2\zeta_3 + \frac{380701}{3240}\zeta_2^2 + \frac{27036815}{52488}\zeta_3 - \frac{53253361}{104976}\zeta_2 + \frac{110016540845}{7558272} + \mathcal{O}(\epsilon)$$

$$\mathcal{F}_4^g(\epsilon)\Big|_{C_F^2} = \frac{1}{\epsilon^2}\left(\frac{1}{2}\right) + \frac{1}{\epsilon}\left(320\zeta_5 - 236\zeta_3 - \frac{74}{3}\right) + \frac{2432}{3}\zeta_3^2 + \frac{32512}{135}\zeta_2^3 + 3360\zeta_5 + \frac{128}{3}\zeta_3\zeta_2 - \frac{484}{3}\zeta_2^2 - \frac{36260}{9}\zeta_3 + \frac{439}{9}\zeta_2 - \frac{80281}{216} + \mathcal{O}(\epsilon)$$

The $\mathcal{O}(N_f^2)$ Bare Four-Loop Gluon Form Factor

$$\mathcal{F}_4^g(\epsilon)\Big|_{C_A C_F} = \frac{1}{\epsilon^4}\left(-\frac{7}{3}\right) + \frac{1}{\epsilon^3}\left(26\zeta_3 - \frac{1115}{36}\right)$$

$$+ \frac{1}{\epsilon^2}\left(\frac{806}{45}\zeta_2^2 + \frac{580}{9}\zeta_3 + \frac{148}{9}\zeta_2 - \frac{12749}{108}\right) + \frac{1}{\epsilon}\left(\frac{442}{9}\zeta_5 - \frac{1132}{9}\zeta_2\zeta_3 + \frac{1784}{45}\zeta_2^2\right.$$

$$\left. - \frac{5797}{27}\zeta_3 + \frac{1475}{18}\zeta_2 + \frac{642623}{1296}\right) - \frac{4354}{3}\zeta_3^2 - \frac{143524}{945}\zeta_2^3 - \frac{47948}{27}\zeta_5 - \frac{3688}{9}\zeta_2\zeta_3$$

$$- \frac{13820}{81}\zeta_2^2 - \frac{538853}{162}\zeta_3 - \frac{27629}{81}\zeta_2 + \frac{3697777}{324} + \mathcal{O}(\epsilon)$$

$$\mathcal{F}_4^g(\epsilon)\Big|_{\frac{d_F^{abcd} d_F^{abcd}}{N_A}} = \frac{1}{\epsilon}\left(128\zeta_3 - \frac{176}{3}\right) + 512\zeta_3^2 - 960\zeta_5 + \frac{384}{5}\zeta_2^2 + 1520\zeta_3 - \frac{9008}{9}$$

$$+ \mathcal{O}(\epsilon), \qquad \text{where} \qquad \frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{(N_c^2 - 1)(N_c^4 - 6N_c^2 + 18)}{96 N_c^2}$$

$\mathcal{O}(N_f^2)$ Casimir Scaling Works As Expected

$$\Gamma_4^q\Big|_{N_f^2} = C_A C_F\left[-\frac{224}{15}\zeta_2^2+\frac{2240}{27}\zeta_3-\frac{608}{81}\zeta_2+\frac{923}{81}\right] + C_F^2\left[\frac{64}{5}\zeta_2^2-\frac{640}{9}\zeta_3+\frac{2392}{81}\right]$$

$$\Gamma_4^q\Big|_{N_{q\gamma}N_f} = 0$$

$$\Gamma_4^g\Big|_{N_f^2} = C_A^2\left[-\frac{224}{15}\zeta_2^2+\frac{2240}{27}\zeta_3-\frac{608}{81}\zeta_2+\frac{923}{81}\right] + C_A C_F\left[\frac{64}{5}\zeta_2^2-\frac{640}{9}\zeta_3+\frac{2392}{81}\right]$$

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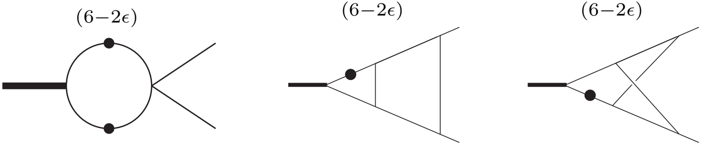
- We could numerically check the expansion coefficients of the other masters to about part per mille precision using `FIESTA 4`.

  A. V. Smirnov, Comput. Phys. Commun. **204** (2016) 189

The Finite Integral Method

Andreas von Manteuffel, Erik Panzer, and RMS,
JHEP **1502** (2015) 120, Phys. Rev. **D93** (2016) 125014

Very simple in principle, "just" choose a finite basis, *e.g.*



However...

A L-loop finite integral basis requires reducing integrals with L additional dots (*i.e.* propagators of higher multiplicity)

O. Tarasov, Phys. Rev. **D54** (1996) 6479

Syzygy-Based IBPs Without Numerators

J. Gluza *et. al.*, Phys. Rev. **D83** (2011) 045012; RMS, JHEP **1201** (2012) 077;

R. N. Lee, Moriond QCD 2014 proceedings; T. Bitoun *et. al.*, Lett. Math. Phys. **109** (2019) 497

Lee-Pomeransky representation:

$$I(\nu_1,\ldots,\nu_N) = \mathcal{N}\left[\prod_{i=1}^{N}\int_0^\infty \mathrm{d}x_i x_i^{\nu_i-1}\right] G^{-d/2} \qquad \text{with } G = \mathcal{U}+\mathcal{F}$$

linear relations from annihilators, of arbitrary order in principle:

$$\left[c_0 + \sum_{i=1}^{N} c_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{N} c_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \ldots\right] G^{-d/2} = 0$$

determine $c_0(x_1,\ldots,x_N),\ldots$ via syzygy equations:

$$c_0\left[-\frac{2}{d}G^2\right] + \sum_{i=1}^{N} c_i\left[G\frac{\partial G}{\partial x_i}\right] + \sum_{i,j=1}^{N} c_{ij}\left[G\frac{\partial^2 G}{\partial x_i \partial x_j} + \left(-\frac{d}{2}-1\right)\frac{\partial G}{\partial x_i}\frac{\partial G}{\partial x_j}\right] + \cdots = 0$$

Minimizing The # Of Integral Expansion Coefficients

- It makes a HUGE difference whether you insert zeta values for the master integrals in a random basis or a "good" finite one.
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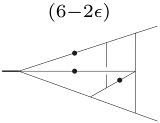

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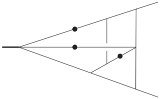

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In the conventional integral basis selected by `Reduze 2`, one would require the weight 9 ϵ expansion coefficients to recover weight 6 information at the level of the gluon form factor!

Finite + Uniform Weight Integrals

RMS, Phys. Rev. **D99** (2019) 105010; A. von Manteuffel and RMS, JHEP **1905** (2019) 073

It seems that the best you can do is to take a basis which is both finite AND uniform weight, such that non-factorizable topologies have as high a maximal weight at leading order in ϵ as possible.

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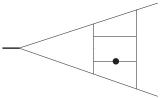
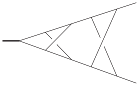


$$= \frac{35}{2}\zeta_7 - \frac{15}{4}\zeta_5\zeta_2 + \frac{18}{5}\zeta_3\zeta_2^2 - \frac{17}{4}\zeta_3^2 - \frac{131}{140}\zeta_2^3 - \frac{75}{4}\zeta_5 + \frac{1}{2}\zeta_3\zeta_2 + \frac{1}{10}\zeta_2^2 + 3\zeta_3 + \epsilon\left(-\frac{383}{20}\zeta_{5,3} - \frac{29}{2}\zeta_5\zeta_3 - \frac{9}{2}\zeta_3^2\zeta_2 + \frac{83581}{5250}\zeta_2^4 + \frac{2677}{16}\zeta_7 + 43\zeta_5\zeta_2 + \frac{261}{20}\zeta_3\zeta_2^2 - 35\zeta_3^2 - \frac{1943}{105}\zeta_2^3 - 322\zeta_5 - 6\zeta_3\zeta_2 + \frac{18}{5}\zeta_2^2 + 60\zeta_3\right) + \mathcal{O}\left(\epsilon^2\right)$$

In practice, compatibility with uniform weight is all that matters.

Outlook

Many obvious directions for future research!



$$= \frac{1}{\epsilon^8}\left(\frac{1}{72}\right) + \cdots - \frac{86152}{15}\zeta_{5,3} + \frac{47869}{12}\zeta_3\zeta_5 + \frac{31999}{36}\zeta_2\zeta_3^2$$
$$+ \frac{33270103}{21000}\zeta_2^4 + \frac{134135}{2}\zeta_7 - 8110\zeta_2\zeta_5 + \frac{23202}{5}\zeta_2^2\zeta_3 + \frac{948274}{315}\zeta_2^3 + \frac{8717}{3}\zeta_3^2$$
$$- \frac{43085}{2}\zeta_5 + \frac{5230}{3}\zeta_2\zeta_3 + \frac{117067}{30}\zeta_2^2 - \frac{76915}{6}\zeta_3 + \frac{28828}{3}\zeta_2 - \frac{183475}{24} + \mathcal{O}(\epsilon)$$

- The full quark and gluon cusp anomalous dimensions.
- Further improvements to our no-numerator syzygy generator.
- Improved understanding of finite + uniform weight bases.
- Full results for the four-loop form factors.