Heavy-quark form factors

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Outline

1. Introduction
2. Calculation
3. Results
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Considering form factors at three-loop order for the process

\[ X \to Q + \overline{Q} \]

coupling through one of the vertices

\[ \{1, \gamma_5, \gamma^\mu, \gamma^\mu \gamma_5\} \]

here:

- only non-singlet contributions, i.e. the heavy-quark pair couples directly to the external current.
- at least one heavy-quark loop
Motivation

- heavy quark production
  - continuum production $e^+ e^- \rightarrow t\bar{t}$
- particle decays
  - $H \rightarrow b\bar{b}$
  - $Z \rightarrow b\bar{b}$
  - $A \rightarrow t\bar{t}$
- technology development
Example: Vector case

\[ \bar{\psi} \Gamma_\nu^\mu \psi = -i \bar{\psi} \left( \gamma^\mu F_{V,1} + \frac{i}{2m} \sigma^{\mu\nu} q_\nu F_{V,2} \right) \psi \]
History / Previous works

- two loop
  - [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi '05]
  - [Gluza, Mitov, Moch, Riemann '09]
  - [Ablinger, Behring, Blümlein, Falcioni, De Freitas, PM, Rana, Schneider '18]

- three loop
  - light-fermionic contributions (HPLs)
    - [Lee, Smirnov, Smirnov, Steinhäuser'18]
    - [Ablinger, Blümlein, PM, Rana, Schneider'18]
  - color-planar contributions (HPLs + cyclotomic HPLs)
    - [Henn, Smirnov, Smirnov, Steinhäuser '17]
    - [Ablinger, Blümlein, PM, Rana, Schneider'18]
  - **NEW** heavy-fermionic contributions

- general infrared and high-energy structure
  - [Ahmed, Henn, Steinhäuser '17]
  - [Blümlein, PM, Rana '18]
  - [Penin et al '17]
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For the calculation of the form factors use the well-established multi-loop toolbox

✓ **QGRAF** for the generation of the diagrams

✓ use **projectors** to obtain scalar integrals

✓ **FORM** for the algebra

✓ use **integration-by-parts identities** [Chetyrkin,Tkachov]

  to reduce to an integral basis using **Crusher** [Seidel,PM]

  14 families, 104 master integral

??? calculate the required master integrals

✓ put everything together and renormalize

✓ final result still IR divergent – compare with predictions
Calculation / Strategy

Calculation of master integrals problematic since the heavy-fermionic and non-planar contributions contain structures beyond harmonic polylogarithms
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Calculation of master integrals problematic since the heavy-fermionic and non-planar contributions contain structures beyond harmonic polylogarithms.
Strategy: Sum simpler then the individual parts!
- turn everything into recurrences by considering the expansion around $q^2 = 0$
- try to derive a recurrence for the whole form factor and find a analytic solution for that

[Blümlein, Schneider '17]
choose a more appropriate variable

\[ \frac{q^2}{m^2} = -\frac{(1 - x)^2}{x} \]

\[ q^2 \rightarrow \pm \infty \equiv x \rightarrow 0 \pm \]
\[ q^2 \rightarrow 0 \equiv x \rightarrow 1 \]

around \( q^2 = 0 \), i.e. \( x = 1 \) the non-singlet form factors can be expanded in a simple power series

\[ \mathcal{F} = \sum_{n=0} C_n \left( \frac{q^2}{m^2} \right)^n \iff \mathcal{F} = \sum_{n=0} D_n (1 - x)^n = \sum_{n=0} D_n y^n \]
Start from the coupled system of diff. eqn. for the master integrals.

Insert the power series ansatz

$$\mathcal{M}_i = \sum_{j=0}^{\infty} M_j^{(i)} y^j$$

And obtain recurrences for the coefficients $M_j^{(i)}$.

Calculate 2,000 - 8,000 terms in the expansion for the master integrals.

Use these to obtain 2,000 - 8,000 terms in the expansions of the full form factors.

As initial condition we need the values at $x = 1$, i.e. on-shell propagators.

[Melnikov,v.Ritbergen]
the final expansion for the form factors has the form

\[ F = 1(\ldots) + \zeta_2(\ldots) + \zeta_3(\ldots) + \ln(2)(\ldots) + \text{Li}_4\left(\frac{1}{2}\right)(\ldots) + \cdots \]

where (\ldots) denote power series in \(y\) with \text{rational} coefficients.

this representation is unique

can we do better?

- Guess a recurrence
- and try to solve it using \text{Sigma} \cite{Schneider'07}

if recurrence can be solved, i.e. first-order factorizing, one obtains

(generalized) harmonic sums, which can be resummed using \text{HarmonicSums} \cite{Ablinger'13}
Example

- start with the sequence for $C_i$ in $\sum C_i y^i$

\[-2, 0, -\frac{1}{6}, -\frac{1}{6}, -\frac{3}{20}, -\frac{2}{15}, -\frac{5}{42}, -\frac{3}{28}, -\frac{7}{72}, -\frac{4}{45}, -\frac{9}{110}, -\frac{5}{66}, -\frac{11}{156}, \]

\[-\frac{6}{91}, -\frac{13}{210}, -\frac{7}{120}, -\frac{15}{272}, -\frac{8}{153}, -\frac{17}{342}, -\frac{9}{190}, -\frac{19}{420}, \ldots\]
Example

- start with the sequence for $C_i$ in $\sum C_i y^i$

  \[\begin{align*}
  -2, 0, -\frac{1}{6}, -\frac{1}{6}, -\frac{3}{20}, -\frac{2}{15}, -\frac{5}{42}, -\frac{3}{28}, -\frac{7}{72}, -\frac{4}{45}, -\frac{9}{110}, -\frac{5}{66}, -\frac{11}{156}, \\
  -\frac{6}{91}, -\frac{13}{210}, -\frac{7}{120}, -\frac{15}{272}, -\frac{8}{153}, -\frac{17}{342}, -\frac{9}{190}, -\frac{19}{420}, \cdots
  \end{align*}\]

- guess recurrence

  \[n^2 C_n - (n - 1)(n + 2) C_{n+1} = 0\]
Example

- start with the sequence for $C_i$ in $\sum C_i y^i$
  
  $$-2, 0, -\frac{1}{6}, -\frac{1}{6}, -3, -\frac{2}{15}, -\frac{5}{42}, -\frac{3}{28}, -\frac{7}{72}, -\frac{4}{45}, -\frac{9}{110}, -\frac{5}{66}, -\frac{11}{156}, \ldots$$

- guess recurrence
  $$n^2 C_n - (n - 1)(n + 2) C_{n+1} = 0$$

- solve the recurrence
  $$C_n = -\frac{n-1}{n(n+1)}$$
Example

- start with the sequence for \( C_i \) in \( \sum C_i y^i \)
  
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- guess recurrence
  \[n^2 C_n - (n - 1)(n + 2) C_{n+1} = 0\]

- solve the recurrence
  \[C_n = -\frac{n - 1}{n(n+1)}\]

- sum it
  \[-2 - \sum_{n=1}^{\infty} \frac{n - 1}{n(n+1)} y^n = -\frac{(y-2) \log(1-y)}{y} y \rightarrow 1-x \quad \frac{(1+x) \log(x)}{1-x}\]

  more details, see: talk by C. Schneider, Thursday morning
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We could find analytic results for all terms but for $n_h$ $n_h \zeta_2$ $n_h \zeta_3$

<table>
<thead>
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<th></th>
<th>degree</th>
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<tr>
<td>$n_h \zeta_2$</td>
<td>230</td>
<td>22</td>
<td>6</td>
</tr>
</tbody>
</table>

For leading color we could also solve the term $\propto N_c^2 n_h \zeta_2$
Results – Scalar form factor

\[ F_S = -\frac{1}{\varepsilon^3} \frac{1}{2(1+x)^2} \left\{ n_h^2 \left[ -\frac{64}{27}(1+x)^2 + \frac{64(1+x)(1+x^2)}{27(1-x)} H_0 \right] 
+ n_h \left[ \frac{4}{27}(997 + 1418x + 997x^2) - \frac{32H_0 P_8^{(5)}}{27(1-x^2)} \right] 
- n_l \left[ \frac{32}{9}(1+x)^2 - \frac{64(1+x)(1+x^2)}{27(1-x)} H_0 \right] + \frac{256(1+x^2)^2}{27(1-x)^2} H_0^2 \right\} \]
Results – Scalar form factor cont’d

$$\frac{1}{\varepsilon^2} \frac{1}{2(1 + x)^2} \left\{ n_h^2 \left[ -\frac{832}{81} (1 + x)^2 - \frac{256x(1 + x)H_0}{27(1 - x)} - \frac{128(1 + x)(1 + x^2)}{27(1 - x)} H_{-1}H_0 \right] \\
+ \frac{32(1 + x)(1 + x^2)}{27(1 - x)} H_0^2 + \frac{128(1 + x)(1 + x^2)}{27(1 - x)} H_{0, -1} - \frac{64(1 + x)(1 + x^2)}{27(1 - x)} \zeta_2 \right\} \\
+ n_h \left[ \frac{16}{27} (897 + 1786x + 897x^2) + n_l \left[ -\frac{64}{3} (1 + x)^2 + \frac{64(1 + x)(5 - 24x + 5x^2)}{81(1 - x)} H_0 \right] \\
- \frac{256(1 + x)(1 + x^2)}{27(1 - x)} H_{-1}H_0 + \frac{64(1 + x)(1 + x^2)}{27(1 - x)} H_0^2 + \frac{256(1 + x)(1 + x^2)}{27(1 - x)} H_{0, -1} \zeta_2 \right\} \\
+ \left( \frac{128H_{-1}P_7^{(5)}}{27(1 - x^2)} - \frac{16P_{13}^{(5)}}{27(1 - x^2)} \right) H_0 + \left( \frac{64P_{26}^{(5)}}{27(1 - x)^2(1 + x)} \right) \\
- \frac{1024(1 + x^2)^2}{27(1 - x)^2} H_{-1} H_{0}^2 - \frac{128(-2 + x^2)(1 + x^2)}{27(1 - x)^2} H_{0}^3 - \frac{128(1 + x)(1 + x^2)}{3(1 - x)} H_0 H_1 \zeta_2 \right\} \\
+ \left( \frac{128(1 + x)(1 + x^2)}{3(1 - x)} - \frac{128(1 + x^2)^2}{3(1 - x)^2} H_0 \right) H_{0, 1} - \left( \frac{128P_7^{(5)}}{27(1 - x^2)} - \frac{2176(1 + x^2)^2}{27(1 - x)^2} H_0 \right) \\
\times H_{0, -1} - \frac{256(1 + x^2)^2}{3(1 - x)^2} H_{0, 0, 1} - \frac{256(1 + x^2)^2}{3(1 - x)^2} H_{0, 0, -1} + \left( \frac{64P_{5}^{(5)}}{27(1 - x^2)} \right) \\
- \frac{64(1 + x^2)(-1 + 35x^2)}{27(1 - x)^2} H_0 \right) \zeta_2 - \frac{64(1 + x^2)^2}{3(1 - x)^2} \zeta_3 \right\} \}$$
Results – Scalar form factor – unsolved recurrences

\[ F_S = \ldots + n_h F_{S,1}^{(0)}(x) + n_h \zeta_2 F_{S,2}^{(0)}(x) + n_h \zeta_3 F_{S,3}^{(0)}(x) \]

\[
F_{S,1}^{(0)}(x) = -\frac{874750}{243} - \frac{731018}{729} y^2 - \frac{731018}{729} y^3 - \frac{316061833}{437400} y^4 - \frac{96756433}{218700} y^5 + \ldots + O(y^{2001})
\]

\[
F_{S,2}^{(0)}(x) = \frac{343864}{81} + \frac{2421832}{3645} y^2 + \frac{2421832}{3645} y^3 + \frac{16041283}{36450} y^4 + \frac{3932123}{18225} y^5 + \ldots + O(y^{2001})
\]

\[
F_{S,3}^{(0)}(x) = \frac{62968}{27} - \frac{22516}{81} y^2 - \frac{22516}{81} y^3 - \frac{21262303}{97200} y^4 - \frac{7752703}{48600} y^5 + \ldots + O(y^{2001}).
\]
Results – High-Energy region

Expansion about $x = 1$, log singularity for $x \rightarrow 0$

shown: 20, 50, 100, 200, 500 terms in the expansion
Results – High-Energy region

Ratios with respect to the result obtained with 2,000 terms

shown: 20, 50, 100, 200, 500 terms in the expansion
Expansion about $z = q^2/m^2 = 0$, square-root singularity for $z \to 4$

shown: 20, 50, 100, 200, 500 terms in the expansion
Calculated the heavy-fermionic corrections to the heavy-quark form factors in an expansion about $q^2 = 0$
Many parts can be resummed and are available analytically
For some parts only recurrences and thus deep expansions exist
Results for pole terms agree with predictions
ToDo: full color
ToDo: singlet contributions
ToDo: find a solution for the non-first-order factorizing recurrences