RADCOR 2019, Avignon, September 12, 2019

A refined machinery to calculate large moments from coupled systems of linear differential equations

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Recurrence solving

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(N), \dots, a_\delta(N)$: polynomials in N
$h(N)$: expression in indefinite nested sums
defined over hypergeometric products.

$$a_0(N)F(N) + \dots + a_\delta(N)F(N+\delta) = h(N);$$

together with initial values $F(0), \dots, F(\delta - 1) \in \mathbb{K}$

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DECIDE constructively if $F(N)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

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DECIDE constructively if $F(N)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$S_{2,1}(N) = \sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j} \qquad \text{(harmonic sums)}$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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DECIDE constructively if $F(N)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^{n} \frac{2^k}{k} \sum_{i=1}^{k} \frac{2^{-i}}{i} \sum_{j=1}^{i} \frac{S_1(j)}{j}$$ (generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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DECIDE constructively if $F(N)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^{N} \frac{1}{(1+2k)^2} \sum_{j=1}^{k} \frac{1}{j^2} \sum_{i=1}^{j} \frac{1}{1+2i}$$ (cyclotomic harmonic sums)

J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{j=1}^{N} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \qquad \text{(binomial sums)}$$

J. Ablinger, J. Blümlein, C. G. Raab and C. Schneider, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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DECIDE constructively if $F(N)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2 \binom{2k}{k}}$$

(generalized binomial sums)

see talks of J. Ablinger and K. Schönwald

Sigma.m is based on difference ring/field theory

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23. S.A. Abramov, M. Bronstein, M. Petkovšek, CS, in preparation

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

Ablinger's MultiIntegrate.m $\big\downarrow$ (9 hours)

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

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$$a_0(\varepsilon, N) F(\varepsilon, N) + a_1(\varepsilon, N) F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N) F(\varepsilon, N+5) = 0$$

recurrence solver $\downarrow$

$F(\varepsilon, N) =$ expression in terms of special functions

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$a_0(\varepsilon,N),\dots,a_\delta(\varepsilon,N)$: polynomials in ε, N
$h_l(N), h_{l+1}(N),\dots,h_\lambda(N)$:
expressions in indefinite nested sums
defined over hypergeometric products.

$$a_0(\varepsilon,N)F(\varepsilon,N)+\dots+a_\delta(\varepsilon,N)F(\varepsilon,N+\delta)$$
$$=h_l(N)\varepsilon^l+h_{l+1}(N)\varepsilon^{l+1}+\dots h_\lambda(N)\varepsilon^r+O(\varepsilon^{r+1});$$

together with ε-expansions of $F(0),\dots,F(\delta-1)$ up to a certain order.

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$$a_0(\varepsilon, N)F(\varepsilon, N) + \dots + a_\delta(\varepsilon, N)F(\varepsilon, N+\delta)$$
$$= h_l(N)\varepsilon^l + h_{l+1}(N)\varepsilon^{l+1} + \dots h_\lambda(N)\varepsilon^r + O(\varepsilon^{r+1});$$

together with ε-expansions of $F(0), \dots, F(\delta-1)$ up to a certain order.

DECIDE constructively if the coefficients $F_i(N)$ of

$$F(N) = F_l(N)\,\varepsilon^l + F_{l+1}(N)\,\varepsilon^{l+1} + \dots + F_\lambda(N)\,\varepsilon^r + O(\varepsilon^{r+1})$$

can be given in terms of indefinite nested sums defined over hypergeometric products.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F(N)\Big] \\
+&a_1(\varepsilon, N)\Big[F(N+1)\Big] \\
+& \\
&\vdots \\
+&a_\delta(\varepsilon, N)\Big[F(N+\delta)\Big]
\end{aligned}
$$

$$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F(N+1)\Big]\\
+&\\
&\vdots\\
+&a_\delta(\varepsilon,N)\Big[F(N+\delta)\Big]\\
&\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon, N)\Big[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big] \\
+ & a_1(\varepsilon, N)\Big[F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big] \\
+ & \\
\vdots & \\
+ & a_\delta(\varepsilon, N)\Big[F(N+\delta)\Big] \\
& = h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots
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&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
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+&\\
&\vdots\\
+&a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
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given (in terms of indefinite nested sums and products)

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&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
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&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_\delta(0,N)F_0(N+\delta)=h_0(N)}
$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_\delta(0,N)F_0(N+\delta)=h_0(N)}
$$

REC solver: Given the initial values $F_0(1),F_0(2),\dots,F_0(\delta)$, **decide** if $F_0(N)$ can be written in terms of indefinite nested sums and products.

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

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\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_\delta(0,N)F_0(N+\delta)=h_0(N)}
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\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
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+&\\
&\vdots\\
+&a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_\delta(0,N)F_0(N+\delta)=h_0(N)}$$

$$\begin{aligned}
&\quad a_0(\varepsilon,N)\Big[F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_1(N{+}1)\varepsilon+F_2(N{+}1)\varepsilon^2+\dots\Big]\\
&+\\
&\;\vdots\\
&+a_\delta(\varepsilon,N)\Big[F_1(N{+}\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0'(N)+h_1'(N)\varepsilon+h_2'(N)\varepsilon^2+\dots
\end{aligned}$$

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big]\\
&+a_1(\varepsilon, N)\Big[F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon, N)\Big[F_1(N+\delta)\varepsilon + F_2(N+\delta)\varepsilon^2 + \dots\Big]\\
&\qquad\qquad = \underbrace{h'_0(N)}_{\mathbf{=0}} + h'_1(N)\varepsilon + h'_2(N)\varepsilon^2 + \dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_1(N)+F_2(N)\varepsilon+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_1(N+1)+F_2(N+1)\varepsilon+\dots\Big]\\
+&\\
\vdots&\\
+&a_\delta(\varepsilon,N)\Big[F_1(N+\delta)+F_2(N+\delta)\varepsilon+\dots\Big]=h_1'(N)+h_2'(N)\varepsilon+\dots
\end{aligned}
$$

Repeat to get $F_1(N), F_2(N), \dots$

Remark: Works the same for Laurent series.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+N+1}(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^N$$

↓ (package `MultiIntegrate.m`)

$$a_0(\varepsilon,N)F(N)+a_1(\varepsilon,N)F(N+1)+\cdots+a_5(\varepsilon,N)F(N+5)=0$$

$$F(2)=\frac{20}{27\varepsilon^3}-\frac{40}{27\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{1393}{486}+\frac{5\zeta_2}{18}\right)+\dots$$

$$\vdots$$

$$F(6)=\frac{22}{147\varepsilon^3}-\frac{535}{2058\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{630043}{1234800}+\frac{11\zeta_2}{196}\right)+\dots$$

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↓ (summation package `Sigma.m`)

$$\boxed{F(N)=F_{-3}(N)\,\varepsilon^{-3}+F_{-2}(N)\,\varepsilon^{-2}+F_{-1}(N)\,\varepsilon^{-1}+\dots}$$

We get

$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$

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$$F_{-2}(N) = -\frac{4(-1)^N\left(3N^3+18N^2+31N+18\right)}{3(N+1)^3(N+2)^2} - \frac{4\left(6N^3+32N^2+51N+26\right)}{3(N+1)^3(N+2)^2}$$

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$$\begin{aligned}F_{-1}(N) = (-1)^N \Bigg(& \frac{2\left(9N^5+81N^4+295N^3+533N^2+500N+204\right)}{3(N+1)^4(N+2)^3} + \frac{\zeta_2}{(N+1)(N+2)} \\ & + \frac{2\left(18N^5+150N^4+490N^3+755N^2+536N+132\right)}{3(N+1)^4(N+2)^3} + \frac{(2N+3)\zeta_2}{(N+1)^2(N+2)} \\ & + \left(-\frac{4}{(N+1)^2(N+2)} + \frac{4(-1)^N}{(N+1)(N+2)}\right) S_2 \\ & + \left(\frac{4(-1)^N}{3(N+1)(N+2)} - \frac{4(N+9)}{3(N+1)^2(N+2)}\right) S_{-2}\end{aligned}$$

Solving coupled systems

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \ldots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \ldots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \ldots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \ldots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \ldots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

given

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$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$d_0(x)\hat{I}_1(x) + d_1(x)D_x\hat{I}_1(x) + \dots + d_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$d_0(x)\hat{I}_1(x) + d_1(x)D_x\hat{I}_1(x) + \dots + d_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \mathsf{LinCom}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \mathsf{LinCom}(..., D^i\hat{R}_i(x), \dots)$$

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$\swarrow$

DE-solver
(see, e.g., [arXiv:1810.12261])

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DE-solver
(see, e.g., [arXiv:1810.12261])

REC-solver

The DE-REC approach

DE system

$$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

The DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$ $\hat{I}_k(x) = \mathsf{expr}_k(D_x^i \hat{I}_1(x)), k > 1$

The DE-REC approach

DE system

$$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system

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$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

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$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

↓ coeff. comparison w.r.t. x^N ↓

linear recurrence
$$\sum_i a_i(N) I_1(N+i) = r(N)$$

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linear recurrence

$$\sum_i a_i(N) I_1(N+i) = r(N)$$

← `Sigma's` REC-solver

closed form solutions of $I_1(N)$ in the class of nested sums and products – if this is possible

The DE-REC approach (SolveCoupledSystem package)

DE system
$$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system
$$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$$
$$\mathbf{\hat{I}_k(x)} = \text{expr}_{\mathbf{k}}(\mathbf{D_x^i \hat{I}_1(x)}), \mathbf{k} > \mathbf{1}$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

extract coefficient
SumProduction package

coeff. comparison w.r.t. x^N

linear recurrence
$$\sum_i a_i(N) I_1(N+i) = r(N)$$

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closed form solutions of
$I_1(N), \mathbf{I_2(N)}, \ldots, \mathbf{I_n(N)}$
in the class of nested sums and products
– if this is possible

J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

General strategy:

$$\Big\downarrow \text{IBP methods}$$

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$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \ldots$$

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Concrete calculations:

- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, 2014. arXiv:1406.4654 [hep-ph].
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, 2014. arXiv:1405.4259 [hep-ph].
- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, 2014. arXiv:1402.0359 [hep-ph].
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), 2015. arXiv:1508.01449 [hep-ph].
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, 2015. arXiv:1504.08217 [hep-ph].
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, 2015. arXiv:1409.1135.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$. Physical Review D 94(11), 2016. arXiv:1609.06255 [hep-ph].
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, 2016. arXiv:1509.08324 [hep-ph].
- J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit, 2018. arXiv:1804.07313 [hep-ph].
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D, 2018. arXiv:1712.09889.

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$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) \underbrace{+\varepsilon^{0} F_{0}(N) + \cdots + \varepsilon^{6} F_{6}(N)}_{\text{expensive calculation}}$$

$$\Big\downarrow \text{plug into } \hat{D}(x) = \sum_{N=0}^{\infty} D(N) x^N$$

$$D(N) = \varepsilon^{-3} D_{-3}(N) + \varepsilon^{-2} D_{-2}(N) + \varepsilon^{-1} D_{-1}(N) + \varepsilon^{0} D_{0}(N) + \ldots$$

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$$D(N) = \underbrace{\varepsilon^{-3} D_{-3}(N) + \varepsilon^{-2} D_{-2}(N) + \varepsilon^{-1} D_{-1}(N)}_{\text{often nice}} + \underbrace{\varepsilon^{0} D_0(N)}_{\text{partially nice}} + \ldots$$

Computing large moments and guessing recurrences

The DE-REC approach (SolveCoupledSystem package)

DE system

$$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

$$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$$

$$\mathbf{\hat{I}_k(x)} = \text{expr}_{\mathbf{k}}(\mathbf{D_x^i \hat{I}_1(x)}), \mathbf{k > 1}$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

extract coefficient

SumProduction package

coeff. comparison w.r.t. x^N

linear recurrence

$$\sum_i a_i(N) I_1(N+i) = r(N)$$

`Sigma's` REC-solver

closed form solutions of $I_1(N), \mathbf{I_2(N)}, \ldots, \mathbf{I_n(N)}$ in the class of nested sums and products – if this is possible

J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

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→ OreSys package (S. Gerhold) — uncoupling algorithm →

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← `Sigma`'s REC-solver ←

closed form solutions of $I_1(N)$

in the class of nested sums and products

– if th

Given $r(0), r(1), \ldots, r(8000)$,
compute $I_1(0), I_1(1), \ldots, I_1(8000)$

J. Blümlein, CS, Physics Letters B 771, pp. 31-36., 2017; arXiv:1701.04614 [hep-ph].

General strategy:

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$N = 0, 1, \dots, 8000$

$$\Big\downarrow \text{ plug into } \hat{D}(x) = \sum_{N=0}^{\infty} D(N) x^N$$

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$N = 0, 1, \dots, 8000$

coupled systems for $D(N)$

`SolveCoupledSystem.m` ↓

large no. of moments

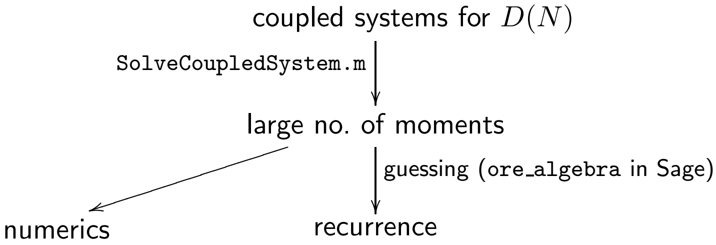
coupled systems for $D(N)$

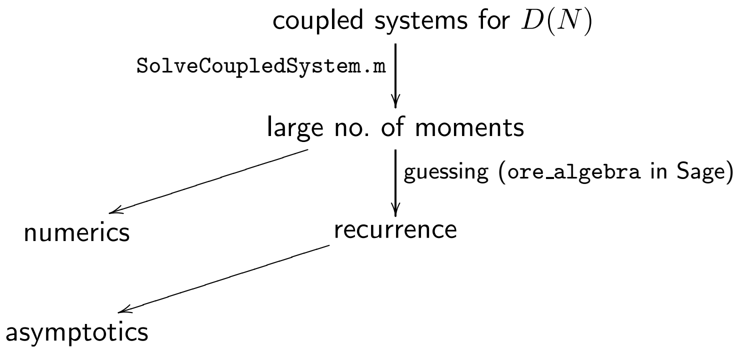
`SolveCoupledSystem.m` ↓

large no. of moments

↙

numerics


coupled systems for $D(N)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence


coupled systems for $D(N)$
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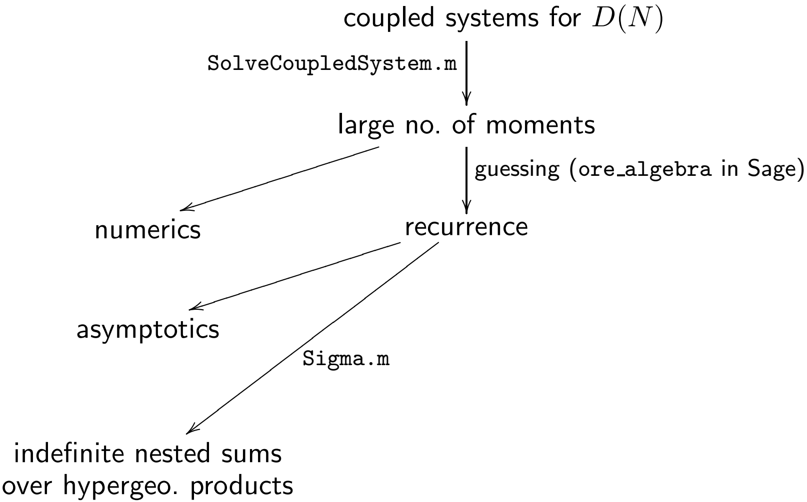
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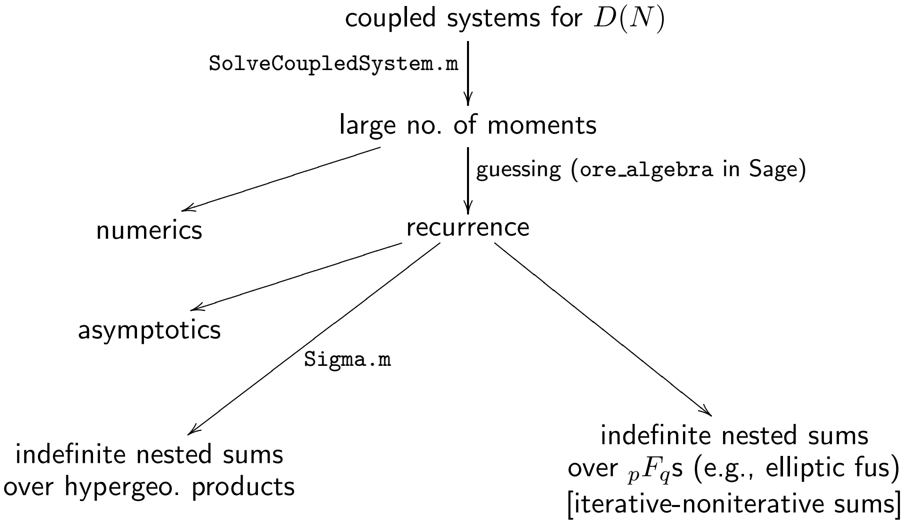
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asymptotics
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over hypergeo. products
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]

coupled systems for $D(N)$

↓ `SolveCoupledSystem.m`

large no. of moments

↓ guessing (`ore_algebra` in Sage)

recurrence

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all N solution

Concrete calculations of large moments:

- The three-loop splitting functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{N_F})}$ [Nucl.Phys.B./2017]
  1. computed ~ 2400 moments
  2. guessed all recurrences
  3. solved all recurrences in terms of harmonic sums.

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  $\rightarrow$ Johannes Blümlein's talk (this afternoon)

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- The heavy fermion contributions to the massive three loop form factors

  $\rightarrow$ Peter Marquard's talk (yesterday)

  Requires a refined large moment machinery

Refinements of the large moment method

Basic approach

$$\Big\downarrow \text{uncoupling}$$

$$d_0(x,\varepsilon)\hat{I}_1(x,\varepsilon) + d_1(x,\varepsilon)D_x\hat{I}_1(x,\varepsilon) + \dots + d_\lambda(x,\varepsilon)D_x^\lambda\hat{I}_1(x,\varepsilon) = r(x,\varepsilon)$$

$$\Big\downarrow$$

$$a_0(N,\varepsilon)I_1(\varepsilon,N)+a_1(x,\varepsilon)I_1(\varepsilon,N+1)+\dots+a_\delta(\varepsilon,N)I_1(\varepsilon,N+\delta) = h(\varepsilon,N)$$

$$\Big\downarrow \delta \text{ initial values}$$

$$I_1(\varepsilon,N) = F_{-3}(N)\,\varepsilon^{-3} + F_{-2}(N)\,\varepsilon^{-2} + F_{-1}(N)\,\varepsilon^{-1} + \dots$$

for $N = 0, 1, \dots, 8000$

Basic approach

$$\Big\downarrow \text{uncoupling}$$

$$d_0(x,\varepsilon)\hat{I}_1(x,\varepsilon) + d_1(x,\varepsilon)D_x\hat{I}_1(x,\varepsilon) + \dots + d_\lambda(x,\varepsilon)D_x^\lambda\hat{I}_1(x,\varepsilon) = r(x,\varepsilon)$$

$$\Big\downarrow$$

$$a_0(N,\varepsilon)I_1(\varepsilon,N)+a_1(x,\varepsilon)I_1(\varepsilon,N+1)+\dots+a_\delta(\varepsilon,N)I_1(\varepsilon,N+\delta) = h(\varepsilon,N)$$

$$\boxed{\delta \leq \lambda + \max_{0\leq i\leq\lambda} \deg_x(d_i)}$$

$$\Big\downarrow \delta \text{ initial values}$$

$$I_1(\varepsilon,N) = F_{-3}(N)\,\varepsilon^{-3} + F_{-2}(N)\,\varepsilon^{-2} + F_{-1}(N)\,\varepsilon^{-1} + \dots$$

for $N = 0, 1, \dots, 8000$

Naive improvement

Compute

$$g(x,\varepsilon) = \operatorname*{gcd_x}_{0\le i\le\lambda} d_i(x,\varepsilon)$$

$$d_0(x,\varepsilon)\hat{I}_1(x,\varepsilon) + d_1(x,\varepsilon)D_x\hat{I}_1(x,\varepsilon) + \cdots + d_\lambda(x,\varepsilon)D_x^\lambda\hat{I}_1(x,\varepsilon) = r(x,\varepsilon)$$

$$\downarrow$$

$$a_0(N,\varepsilon)I_1(\varepsilon,N)+a_1(x,\varepsilon)I_1(\varepsilon,N+1)+\cdots+a_\delta(\varepsilon,N)I_1(\varepsilon,N+\delta) = h(\varepsilon,N)$$

$$\boxed{\delta \le \lambda + \max_{0\le i\le\lambda} \deg_x(d_i)}$$

$$\downarrow \; \delta \text{ initial values}$$

$$I_1(\varepsilon,N) = F_{-3}(N)\,\varepsilon^{-3} + F_{-2}(N)\,\varepsilon^{-2} + F_{-1}(N)\,\varepsilon^{-1} + \ldots$$

for $N = 0, 1, \ldots, 8000$

Naive improvement

Compute

$$g(x,\varepsilon) = \gcd_x_{0\le i\le\lambda} d_i(x,\varepsilon)$$

$$\frac{d_0(x,\varepsilon)}{g(x,\varepsilon)}\hat{I}_1(x,\varepsilon) + \frac{d_1(x,\varepsilon)}{g(x,\varepsilon)}D_x\hat{I}_1(x,\varepsilon) + \cdots + \frac{d_\lambda(x,\varepsilon)}{g(x,\varepsilon)}D_x^\lambda\hat{I}_1(x,\varepsilon) = \frac{r(x,\varepsilon)}{g(x,\varepsilon)}$$

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Major improvement

Compute

$$g(x) = \gcd_{x \atop 0\le i\le\lambda} d_i(x,0)$$

$$\frac{d_0(x,\varepsilon)}{g(x)}\hat{I}_1(x,\varepsilon) + \frac{d_1(x,\varepsilon)}{g(x)}D_x\hat{I}_1(x,\varepsilon) + \cdots + \frac{d_\lambda(x,\varepsilon)}{g(x)}D_x^\lambda\hat{I}_1(x,\varepsilon) = \frac{r(x,\varepsilon)}{g(x)}$$

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Ex: $(\lambda = 4)$

$$\delta = 17 \to \delta = 4$$

Major improvement

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Iterate to get $F_{-2}(N), F_{-1}(N), \ldots$

Ex: $(\lambda = 4)$

$$\delta = 17 \to \delta = 4$$

Strategy 1: uncouple with ε:

$$D_x \begin{pmatrix} \hat{I}_1(x,\varepsilon) \\ \dots \\ \hat{I}_\lambda(x,\varepsilon) \end{pmatrix} = A(x,\varepsilon) \begin{pmatrix} \hat{I}_1(x,\varepsilon) \\ \dots \\ \hat{I}_\lambda(x,\varepsilon) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x,\varepsilon) \\ \dots \\ \hat{R}_\lambda(x,\varepsilon) \end{pmatrix}$$

↓ uncoupling is too hard for $\lambda \geq 5$

$$d_0(x,\varepsilon)\hat{I}_1(x,\varepsilon) + d_1(x,\varepsilon)D_x\hat{I}_1(x,\varepsilon) + \cdots + d_\lambda(x,\varepsilon)D_x^\lambda\hat{I}_1(x,\varepsilon) = \hat{r}(x,\varepsilon)$$

↓ compute large moments (see above)

$$I_1(\varepsilon,N) = F_{-3}(N)\,\varepsilon^{-3} + F_{-2}(N)\,\varepsilon^{-2} + F_{-1}(N)\,\varepsilon^{-1} + \dots$$

Strategy 2: uncouple without ε:

$$D_x \begin{pmatrix} \hat{F}_{-3}(x) \\ \dots \\ \hat{F}_{-3}(x) \end{pmatrix} = A(x, 0) \begin{pmatrix} \hat{F}_{-3}(x) \\ \dots \\ \hat{F}_{-3}(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_{-3,1}(x) \\ \dots \\ \hat{R}_{-3,\lambda}(x) \end{pmatrix}$$

$\downarrow$ uncoupling

$$d_0(x)\hat{F}_{-3}(x) + d_1(x)D_x\hat{F}_{-3}(x) + \dots + d_\lambda(x)D_x^\lambda\hat{F}_{-3}(x) = \hat{r}(x)$$

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Heavy Fermion Contributions to the Massive Three Loop Form Factors

$\rightarrow$ Peter Marquard's talk [to appear in Nucl. Phys. B]

- 41 systems

number of systems	order	
16	$\lambda = 1$	
15	$\lambda = 2$	
2	$\lambda = 3$	Strategy 1
3	$\lambda = 4$	
3	$\lambda = 5$	
1	$\lambda = 6$	Strategy 2
1	$\lambda = 7$	

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    $\rightarrow$ recurrences for ζ_3 contributions of ε^0 ($\leq$ order 29)
    but not for the constant-free contributions
  - 6000 moments: 20 days (242 total CPU days)
    $\rightarrow$ no further recurrences
  - 8000 moments: 43 days (597 total CPU days)
    $\rightarrow$ recurrences for constant-free contributions ($\leq$ order 54)

General strategy:

$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{D}(x) = \text{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

$\downarrow$ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$

$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) \underbrace{+\varepsilon^0 F_0(N) + \cdots + \varepsilon^6 F_6(N)}_{\text{only numbers}}$$

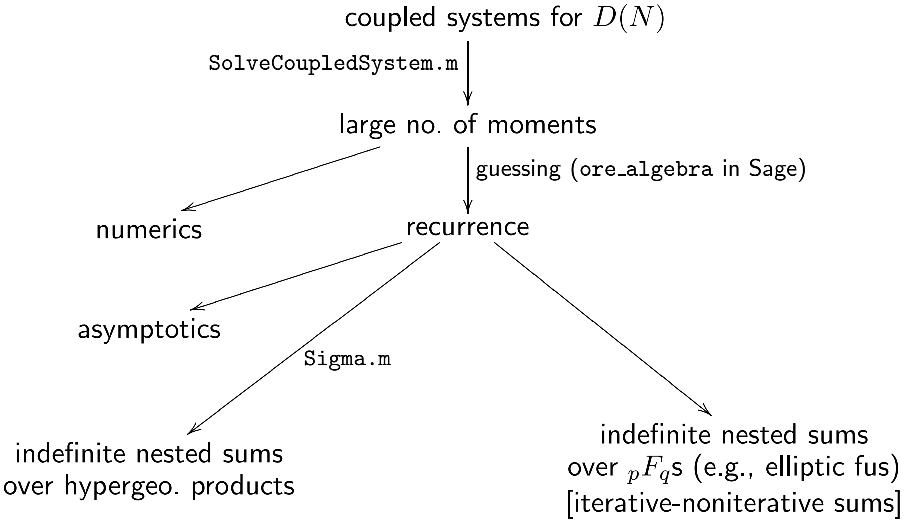
$\downarrow$ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N) x^N$

$$D(N) = \underbrace{\varepsilon^{-3} D_{-3}(N) + \varepsilon^{-2} D_{-2}(N) + \varepsilon^{-1} D_{-1}(N)}_{\text{nice}} + \underbrace{\varepsilon^0 D_0(N)}_{\text{partially nice}} + \ldots$$

all N solution

Conclusion

1. We presented the Large Moment Methods and its flexibility


coupled systems for $D(N)$
SolveCoupledSystem.m
large no. of moments
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]

Conclusion

1. We presented the Large Moment Methods and its flexibility
2. Suitable for challenging problems

Concrete calculations of large moments:

- The three-loop splitting functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{N_F})}$ [Nucl.Phys.B./2017]
  1. computed ~ 2400 moments
  2. guessed all recurrences
  3. solved all recurrences in terms of harmonic sums.

- The massive Wilson coefficient A_{Qg}:
  1. computed 2000 moments
  2. guessed and solved some recurrences.

- The second calculation of the polarized three–loop splitting functions in the M–scheme (Moch/Vermaseren/Vogt 2014)
  1. computed 6000 moments
  2. guessed all recurrences and solved them

  $\rightarrow$ Johannes Blümlein's talk (this afternoon)

- The heavy fermion contributions to the massive three loop form factors

  $\rightarrow$ Peter Marquard's talk (yesterday)

  Requires a refined large moment machinery

Conclusion

1. We presented the Large Moment Methods and its flexibility
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3. Refined versions (Strategies 1 and 2)

Conclusion

1. We presented the Large Moment Methods and its flexibility
2. Suitable for challenging problems
3. Refined versions (Strategies 1 and 2)
4. Relies on advanced technologies

Used Packages to solve DE systems

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= << **SumProduction.m**

SumProduction by Carsten Schneider © RISC-Linz

In[5]:= << **OreSys.m**

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[6]:= << **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

Simple integrals: symbolic summation, Matad (M. Steinhauser), literature...
Guessing recurrences: OreAlgebra (M. Kauers)