A new formulation of loop-tree duality at higher loops

Zoltán Szőr

Johannes Gutenberg-Universität Mainz

with Robert Runkel, Juan Pablo Vesga and Stefan Weinzierl

based on
[arXiv:1902.02135, arXiv:1906.02218]

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Motivation

Precision measurements require accurate theoretical predictions!

The goal

- Automated higher loop amplitude calculations

The dream

- Cancel infrared-divergences on the integrand level

Loop-tree duality: relates loop graphs to trees

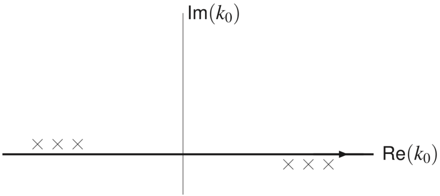
Part I: loop-tree duality for graphs

Part II: from graphs to amplitude-like objects

Loop-tree duality

Perform energy integration using Cauchy's residue theorem.

$$f(\Gamma) = \frac{P_\Gamma}{\prod_j (k_j^2 - m_j^2 + i\delta)}$$



$$I = \int \frac{\mathrm{d}^D k}{(2\pi)^D} f(\Gamma) = -i \int \frac{\mathrm{d}^{D-1} k}{(2\pi)^D} \sum_i (-1)^{\mathrm{sgn}(k_{i,0})} \mathrm{res}(f, k_{i,0}^{(\pm)})$$

Taking residues $\longleftrightarrow$ cutting edges to obtain trees

$$(-1)^{\mathrm{sgn}(k_{i,0})} \mathrm{res}(f, k_{i,0}^{(\pm)}) = \frac{P_\Gamma}{\prod_{j \neq i} (k_j^2 - m_j^2 + i\delta(E_j - E_i))}$$

[Catani, Gleisberg, Krauss, Rodrigo, Winter '08]

Loop-tree duality @ one-loop

- Modified $i\delta$ prescription: dual propagators
  [Catani, Gleisberg, Krauss, Rodrigo, Winter '08]
- Contour deformation
  [Gong, Nagy, Soper '08]
- UV counterterms
  [Becker, Reuschle, Weinzierl '10]
- Dual cancellations
  [Buchta, Chachamis, Draggiotis, Malamos, Rodrigo '14]
- Cancellation of IR divergences at the integrand level
  (see William's talk!)
  [Sbrolini, Driencourt-Mangin, Hernandez-Pinto, Rodrigo '16]

Works for amplitudes as well!

Loop-tree duality @ many-loops

- Modified $i\delta$ prescription: linear or non-linear dependence
  [Bierenbaum, Catani, Draggiotis, Rodrigo '10]
  [Runkel, Z.Sz., Vesga, Weinzierl '19]
- Presence of higher poles
  [Bierenbaum, Buchta, Draggiotis, Malamos, Rodrigo '12]
  [Baumeister, Mediger, Pečovnik, Weinzierl '19]
- Combinatorial factors
  [Runkel, Z.Sz., Vesga, Weinzierl '19]

How to go beyond graphs?

Loop-tree duality à la Mainz

Take the l-fold residue

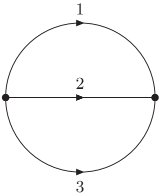
$$\int \prod_{i=1}^{l} \frac{\mathrm{d}^D k_i}{(2\pi)^D} f(\Gamma) = (-i)^l \sum_{\sigma \in \mathcal{C}_\Gamma} \int_{+/-} \prod_{i=1}^{l} \frac{\mathrm{d}^{D-1} k_{\sigma_i}}{(2\pi)^{D-1}} S_{\sigma\alpha} (-1)^{n_\sigma^{(\alpha)}} \mathrm{res}(f, E_\sigma^{(\alpha)})$$

- Set of cutted edges $\sigma = (\sigma_1, \ldots, \sigma_l) \in \mathcal{C}_\Gamma$
- Set of energy signs $\alpha = (\alpha_1, \ldots, \alpha_l) \in \{-1, 1\}^l$
- Implicit summation in $\int_{+/-}$ for all α
- Combinatorial factor $S_{\sigma\alpha}$

Taking the l-fold residue $\leftrightarrow$ cutting l edges to obtain trees
(if all propagators are to power one)

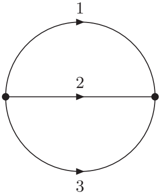
$$(-1)^{n_\sigma^{(\alpha)}} \mathrm{res}(f, E_\sigma^{(\alpha)}) = \prod_{i=1}^{l} \frac{1}{2\sqrt{\vec{k}_{\sigma_i}^2 + m_{\sigma_i}^2}} \frac{P_\Gamma}{\prod_{j \notin \sigma} (k_j^2 - m_j^2 + i\delta s_j(\sigma))}$$

Symmetry might lead astray



$$
\begin{aligned}
&= \frac{1}{4}\mathrm{Cut}(1^+,2^+) + \frac{1}{4}\mathrm{Cut}(1^+,2^-) + \frac{1}{4}\mathrm{Cut}(1^-,2^+) + \frac{1}{4}\mathrm{Cut}(1^-,2^-) \\
&+ \frac{1}{4}\mathrm{Cut}(1^+,3^+) + \frac{1}{4}\mathrm{Cut}(1^+,3^-) + \frac{1}{4}\mathrm{Cut}(1^-,3^+) + \frac{1}{4}\mathrm{Cut}(1^-,3^-) \\
&+ \frac{1}{2}\mathrm{Cut}(2^+,3^+) \qquad\qquad + \frac{1}{2}\mathrm{Cut}(2^-,3^-)
\end{aligned}
$$

The actual symmetric form



$$
\begin{aligned}
=&\frac{1}{3}\mathrm{Cut}(1^+,2^+)+\frac{1}{6}\mathrm{Cut}(1^+,2^-)+\frac{1}{6}\mathrm{Cut}(1^-,2^+)+\frac{1}{3}\mathrm{Cut}(1^-,2^-)\\
+&\frac{1}{3}\mathrm{Cut}(1^+,3^+)+\frac{1}{6}\mathrm{Cut}(1^+,3^-)+\frac{1}{6}\mathrm{Cut}(1^-,3^+)+\frac{1}{3}\mathrm{Cut}(1^-,3^-)\\
+&\frac{1}{3}\mathrm{Cut}(2^+,3^+)+\frac{1}{6}\mathrm{Cut}(2^+,3^-)+\frac{1}{6}\mathrm{Cut}(2^-,3^+)+\frac{1}{3}\mathrm{Cut}(2^-,3^-)
\end{aligned}
$$

The correct procedure

The representation in terms of cuts is not unique!

$$I = \int \sum_{\sigma \in \mathcal{C}_\Gamma} \sum_{\pi \in S_l} \sum_{\alpha \in \{-1,1\}^l} C^{\tilde{\sigma}\tilde{\pi}\tilde{\alpha}}_{\sigma\pi\alpha} \mathrm{Cut}(\sigma, \alpha)$$

We sum over:

- cutted edges σ
- energy signs α
- order of picking up the residues π

But we have free choices for:

- labeling the loop momenta $\tilde{\sigma}$
- order of integration $\tilde{\pi}$
- closing the contours below or above $\tilde{\alpha}$

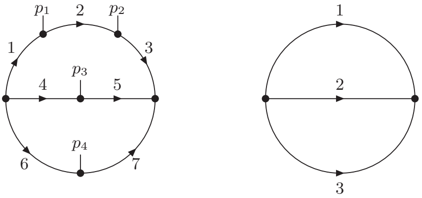
[Capatti, Hirschi, Kermanschah, Ruijl '19]

Combinatorial factors

Average over $\tilde{\sigma}$, $\tilde{\pi}$, $\tilde{\alpha}$ and sum over π.

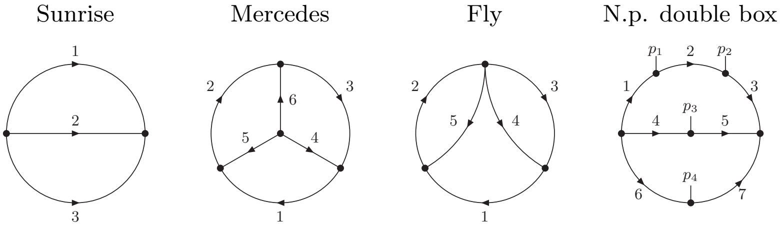
$$S_{\sigma\alpha} = \frac{(-1)^{l+n_\sigma^{(\alpha)}}}{2^l l! |\mathcal{C}_{\Gamma^{\text{chain}}}|} \sum_{\pi \in S_l} \sum_{\tilde{\sigma} \in \mathcal{C}_\Gamma} \sum_{\tilde{\pi} \in S_l} \sum_{\tilde{\alpha} \in \{-1,1\}^l} \frac{C_{\sigma\pi\alpha}^{\tilde{\sigma}\tilde{\pi}\tilde{\alpha}}}{N^{\text{chain}}(\sigma)}$$

$S_{\sigma\alpha}$ can be computed once for all for a given chaingraph.



Cut	$(1^+, 2^+)$	$(1^+, 2^-)$
$S_{\sigma\alpha}$	$1/3$	$1/6$

Integrals in $D = 1$



	Sunrise	Mercedes
I_{LTD}	$\frac{1}{398684} \approx 2.50825 \cdot 10^{-6}$	$\frac{3264791}{253676278437997615200} \approx 1.28699 \cdot 10^{-14}$
I_{MC}	$(2.5083 \pm 0.0001) \cdot 10^{-6}$	$(1.28699 \pm 0.00008) \cdot 10^{-14}$

	Fly	N.p. double box
I_{LTD}	$\frac{19}{653441364576} \approx 2.90768 \cdot 10^{-11}$	$9.50190 \cdot 10^{-19}$
I_{MC}	$(2.9077 \pm 0.0002) \cdot 10^{-11}$	$(9.504 \pm 0.005) \cdot 10^{-19}$

Part I: loop-tree duality for graphs

Part II: from graphs to amplitude-like objects

Loop-tree duality for amplitudes

Consider the n-leg, l-loop renormalised scattering amplitude in ϕ^3 theory

$$\mathcal{A}_{n,l} = \mathcal{A}_{n,l}^{\text{bare}} + \mathcal{A}_{n,l}^{\text{CT}} = \sum_{\Gamma \in \mathcal{U}_{l,n}^{n.s.}} \frac{1}{S_\Gamma} \int \prod_{i=1}^{l} \frac{\mathrm{d}^D k_i}{(2\pi)^D} f(\Gamma)$$

Apply loop-tree duality for each graph...

$$\mathcal{A}_{n,l} = (-i)^l \sum_{\Gamma \in \mathcal{U}_{l,n}^{n.s.}} \frac{1}{S_\Gamma} \sum_{\sigma \in \mathcal{C}_\Gamma} \int_{+/-} \prod_{i=1}^{l} \frac{\mathrm{d}^{D-1} k_{\sigma_i}}{(2\pi)^{D-1}} S_{\sigma\alpha} (-1)^{n_\sigma^{(\alpha)}} \operatorname{res}(f, E_\sigma^{(\alpha)})$$

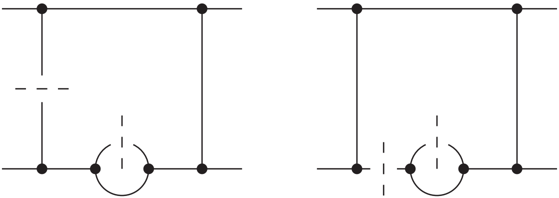
Relabel loop momenta and average over all possibilities

$$\mathcal{A}_{n,l} = \frac{(-i)^l}{l!} \int_{+/-} \prod_{i=1}^{l} \frac{\mathrm{d}^{D-1} k_i}{(2\pi)^{D-1}} \sum_{\Gamma \in \mathcal{U}_{l,n}^{n.s.}} \sum_{\sigma \in \mathcal{C}_\Gamma} \sum_{S_l} \frac{1}{S_\Gamma} S_{\sigma\alpha} (-1)^{n_\sigma^{(\alpha)}} \operatorname{res}(f, E_\sigma^{(\alpha)})$$

We have to address:

- propagators with higher powers
- symmetry factors

Higher poles and UV counterterms



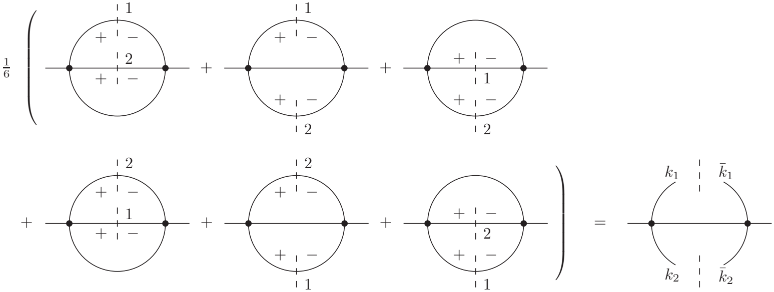
One can construct UV counterterms in integral representation that

- match the UV singularity structure locally
- are proper counterterms when integrated
- cancel higher pole cuts, if field and mass renormalisation performed in the on-shell scheme

Process independent!

[Baumeister, Mediger, Pečovnik, Weinzierl '19]

Symmetry factors



$$\sum_{\Gamma \in \mathcal{U}_{l,n}^{l-marked}} \frac{1}{S_{\pi_{\text{forget}}(\Gamma)}} f(\pi_{\text{forget history}}(\Gamma)) = \sum_{\Gamma \in \mathcal{U}_{0,n+2l}^{l-sewed}} f(\iota(\Gamma))$$

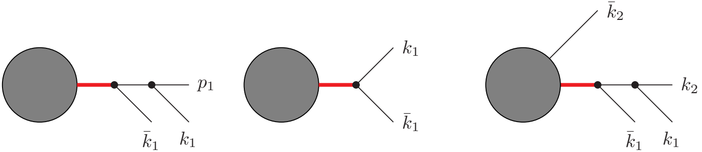
$$\int_{+/-} \sum_{\Gamma \in \mathcal{U}_{l,n}^{n.s.}} \sum_{\sigma \in \mathcal{C}_\Gamma} \sum_{S_l} = \sum_{\mathcal{U}_{l,n}^{l-marked,n.s.}} \longrightarrow \sum_{\mathcal{U}_{0,n+2l}^{l-sewed,n.s.}}$$

The regularised forward limit

Sewing: $\lim_{\bar{k}_1 \to -k_1} \longrightarrow$ forward limit

$$\mathcal{A}_{n,l} = \frac{(-i)^l}{l!} \int \prod_{i=1}^{l} \frac{\mathrm{d}^{D-1} k_i}{(2\pi)^{D-1} 2\sqrt{\vec{k}_i^2 + m_i^2}} \sum_{\Gamma \in \mathcal{U}_{0,n+2l}^{l-sewed,n.s.}} S_{\sigma\alpha} f(\Gamma)$$

Singular graphs



Regularised forward limit

$$R_f \widetilde{\mathcal{A}}_{0,n+2l} = \lim_{\bar{k}_1 \to -k_1} \cdots \lim_{\bar{k}_l \to -k_l} \sum_{\Gamma \in \mathcal{U}_{0,n+2l}^{l-sewed,n.s.}} S_{\sigma\alpha} f(\Gamma)$$

Integrands of loop amplitudes

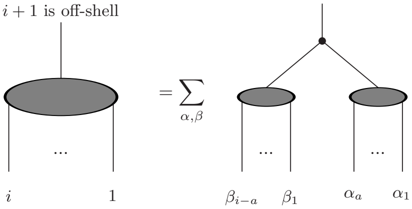
Renormalized loop amplitudes $\longleftrightarrow$ phase space integrals of tree-like objects

$$\mathcal{A}_{n,l} = \frac{(-i)^l}{l!} \int \prod_{i=1}^{l} \frac{\mathrm{d}^{D-1} k_i}{(2\pi)^{D-1} 2\sqrt{\vec{k}_i^2 + m_i^2}} R_f \widetilde{\mathcal{A}}_{0,n+2l}$$

Recall: $R_f \widetilde{\mathcal{A}}_{0,n+2l}$ contains 'cutted' UV counterterms

Tree amplitudes can be computed via off-shell recurrence relations!
True for tree-like objects up to three-loops with slight modifications!

Recurrence relations



$$\mathcal{A}_n(p_1,\ldots,p_n) = (p_1+\cdots+p_{n-1})^2 \mathcal{J}(p_1,\ldots,p_{n-1})\Big|_{p_1+\cdots+p_{n-1}\to -p_n}$$

Compare two sets

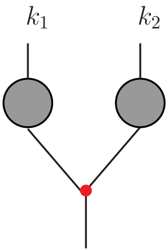
- Generate loop graphs with **QGRAF**
- Performs cuts to obtain trees
- Search for identical graphs to cancel symmetry factors

- Generate tree graphs via recurrence relations
- Keep only non-singular graphs

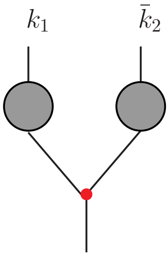
Agreement up to $n+2l=9\,, l\leq 3$ with $\mathcal{O}(10^5)$ graphs!

Combinatorial factors in recurrence relations

Up to three-loops combinatorial factors can be incorporated into recurrence relations



$= \frac{1}{\sqrt{3}}$



$= \frac{1}{\sqrt{6}}$

Recall

Cut	$(1^+, 2^+)$	$(1^+, 2^-)$
$S_{\sigma\alpha}$	$1/3$	$1/6$

Extension to gauge theories

Scalar theory is OK, but what about gauge theories?

Simple extension of rules

- Sum over all particle flavours for sewed external legs
- Sewing: summing over polarizations, unphysical included
- Include (-1) for each fermion and ghost sewing
- Include tadpoles graphs for theories with non-vanishing vacuum

Summary

We have seen that

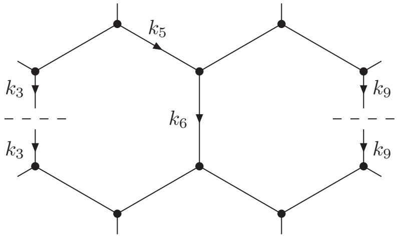
- Loop graphs $\longrightarrow$ weighted sum of trees
- Loop amplitudes $\longrightarrow$ phase space integral of tree-like object
- Compute tree-like object via recurrence relations up to three-loop

We are closer to the goal, but still dreaming the dream!
(the Sun is coming up though)

Thank you for your attention!

Modified $i\delta$ prescription

$$s_j(\sigma) = E_j \sum_{a \in \{j\} \cup \pi} \frac{1}{E_a}$$



Obtain the set π as follows

- Cut the edges $\{e_{\sigma_1}, \ldots, e_{\sigma_l}\}$.
- In addition cut the edge $e_j \notin \{e_{\sigma_1}, \ldots, e_{\sigma_l}\}$ and obtain two trees, T_1 and T_2.
- Pick T_1 and orient the external momenta such that all momenta are outgoing.
- The set π is given by the external legs of T_1 which come from cuts. One index may appear twice.
- $s_j(\sigma)$ is invariant under $T_1 \leftrightarrow T_2$
- For example $s_5(\sigma) = \frac{E_3 + E_5}{E_3}$