Simple differential equations for Feynman integrals associated to elliptic curves

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Part I: **Differential equations**

Part II: **One elliptic curve, one variable**

Part III: **One elliptic curve, several variables**

Standard tools

- Integration-by-parts identities

  Tkachov ’81, Chetyrkin ’81

- the method of differential equations

  Kotikov ’90, Remiddi ’97, Gehrmann and Remiddi ’99

- Laporta algorithm and computer implementations

  Laporta ’01,

  `REDUZE` von Manteuffel, Studerus ’12,

  `FIRE` Smirnov ’15,

  `KIRA` Maierhöfer, Usovitsch, Uwer ’ 17

Notation

$N_F = N_{\mathrm{Fibre}}$: Number of master integrals,
master integrals denoted by $I = (I_1, ..., I_{N_F})$.

$N_B = N_{\mathrm{Base}}$: Number of kinematic variables,
kinematic variables denoted by $x = (x_1, ..., x_B)$.

$N_L = N_{\mathrm{Letters}}$: Number of letters,
differential one-forms denoted by $\omega = (\omega_1, ..., \omega_L)$.

Differential equations

System of differential equations

$$dI + AI = 0,$$

where $A(\varepsilon, x)$ is a matrix-valued one-form

$$A = \sum_{i=1}^{N_B} A_i dx_i.$$

The matrix-valued one-form A satisfies the integrability condition

$$dA + A \wedge A = 0 \qquad \text{(flat Gauß-Manin connection)}.$$

Computation of Feynman integrals reduced to solving differential equations!

Simple differential equations

The system of differential equations is particular simple, if A is of the form

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k,$$

where

- C_k is a $N_F \times N_F$-matrix, whose entries are (rational or integer) numbers,

- the only dependence on ε is given by the explicit prefactor,

- the differential one-forms ω_k have only simple poles.

Henn '13

Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \to M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \quad = \quad \gamma^* \omega_j.$$

The iterated integral is defined by (Chen '77)

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) \quad = \quad \int\limits_0^{\lambda} d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Computation of Feynman integrals reduced to transforming the system of differential equations to a simple form!

Multiple polylogarithms

If all ω_k's are of the form

$$\omega_k = d \ln p_k(x) \,,$$

where the p_k's are polynomials in the variables x, then (after factorisation of univariate polynomials)

$$f_j = \frac{d\lambda}{\lambda - z_j}$$

and all iterated integrals are multiple polylogarithms:

$$G(z_1, \ldots, z_k; \lambda) = \int_0^{\lambda} \frac{d\lambda_1}{\lambda_1 - z_1} \int_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} \ldots \int_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k - z_k}$$

Transformations

- Change the basis of the master integrals

$$I' = UI,$$

where $U(\varepsilon, x)$ is a $N_F \times N_F$-matrix. The new connection matrix is

$$A' = UAU^{-1} + UdU^{-1}.$$

- Perform a coordinate transformation on the base manifold:

$$x_i' = f_i(x), \qquad 1 \leq i \leq N_B.$$

The connection transforms as

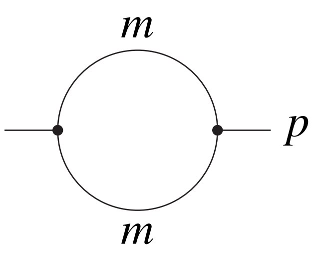
$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x_j'} dx_j'.$$

Change of coordinates

A change of variables is already required for the one-loop two-loop function, where one encounters ($x = p^2/m^2$)

$$\frac{dx}{\sqrt{-x(4-x)}}.$$

Here, a change of variables in the base manifold



$$x = -\frac{(1-x')^2}{x'}$$

will rationalise the square root and transform

$$\frac{dx}{\sqrt{-x(4-x)}} = \frac{dx'}{x'}$$

Transformations in the case of multiple polylogarithm

- Change the basis of the master integrals

$$I' = UI$$

Systematic algorithms if U is rational in the kinematic variables:

Henn '13; Gehrmann, von Manteuffel, Tancredi, Weihs '14; Argeri et al. '14; Lee '14; Meyer '16; Prausa '17; Gituliar, Magerya '17; Lee, Pomeransky '17;

- Perform a coordinate transformation on the base manifold:

$$x_i' = f_i(x)$$

Algorithms to rationalise square roots:

Becchetti, Bonciani, '17, Besier, van Straten, S.W., '18

Simple differential equations

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles.}$$

This form can be reached for many Feynman integrals evaluating to multiple polylogarithms.

Remark: Two-loop electroweak-QCD corrections to Drell-Yan

Heller, von Manteuffel, Schabinger, '19

Simple differential equations beyond multiple polylogarithms

Can the system of differential equations be brought into the form

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles}$$

for Feynman integrals not evaluating to multiple polylogarithms?

Some explicit examples:

Integral	ε-form	simple poles	comments
equal mass sunrise	yes	yes	$N_B = 1$, 1 elliptic curve
unequal mass sunrise	yes	yes	$N_B = 3$, 1 elliptic curve
topbox	yes	?	$N_B = 2$, 3 elliptic curves

Part II

One elliptic curve, one variable

(The equal mass sunrise integral)

$$x = \frac{p^2}{m^2}$$

The elliptic curve

How to get the elliptic curve?

- From the Feynman graph polynomial:

$$-x_1 x_2 x_3 x + (x_1 + x_2 + x_3)(x_1 x_2 + x_2 x_3 + x_3 x_1) = 0$$

- From the maximal cut:

$$v^2 - (u-x)(u-x+4)\left(u^2+2u+1-4x\right) = 0$$

Baikov '96; Lee '10; Kosower, Larsen, '11; Caron-Huot, Larsen, '12; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

The periods ψ_1, ψ_2 of the elliptic curve are solutions of the homogeneous differential equation.

Adams, Bogner, S.W., '13; Primo, Tancredi, '16

Variables

Recall

$$x = \frac{p^2}{m^2}.$$

Set

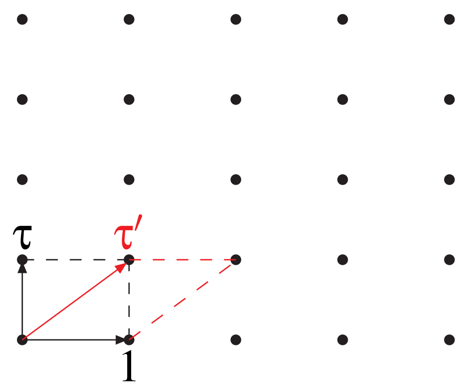
$$\tau = \frac{\psi_2}{\psi_1}, \qquad q = e^{2i\pi\tau}.$$

Change variable from x to τ (or q).

Bloch, Vanhove, '13

Bases of lattices

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).
Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$.



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

The ε-form of the differential equation for the sunrise

It is not possible to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.

However by factoring off the (non-algebraic) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form:

$$I_1 = 4\varepsilon^2 S_{110}\left(2-2\varepsilon,x\right), \quad I_2 = -\varepsilon^2 \frac{\pi}{\psi_1} S_{111}\left(2-2\varepsilon,x\right), \quad I_3 = \frac{1}{\varepsilon}\frac{1}{2\pi i}\frac{d}{d\tau} I_2 + \frac{1}{24}\left(3x^2-10x-9\right)\frac{\psi_1^2}{\pi^2} I_2.$$

If in addition one makes a (non-algebraic) change of variables from x to τ, one obtains

$$\frac{d}{d\tau}\vec{I} \quad = \quad \varepsilon\, A(\tau)\, \vec{I},$$

where $A(\tau)$ is an ε-independent 3×3-matrix whose entries are modular forms.

The ε-form of the differential equation for the sunrise

The matrix $A(\tau)$ is given by

$$A(\tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -f_2(\tau) & 1 \\ \frac{1}{4} f_3(\tau) & f_4(\tau) & -f_2(\tau) \end{pmatrix},$$

where f_2, f_3 and f_4 are modular forms of $\Gamma_1(6)$ of modular weight 2, 3 and 4, respectively.

I_1, I_2 and I_3 are expressed as iterated integrals of modular forms to all orders in ε.

Adams, S.W., ’17, ’18

Simple poles at $\tau = i\infty$

A modular form $f_k(\tau)$ is by definition holomorphic at the cusp and has a q-expansion

$$f_k(\tau) = a_0 + a_1 q + a_2 q^2 + \ldots, \qquad q = \exp(2\pi i \tau)$$

The transformation $q = \exp(2\pi i \tau)$ transforms the point $\tau = i\infty$ to $q = 0$ and we have

$$2\pi i f_k(\tau) d\tau = \frac{dq}{q} \left(a_0 + a_1 q + a_2 q^2 + \ldots \right).$$

Thus a modular form non-vanishing at the cusp $\tau = i\infty$ has a simple pole at $q = 0$.

Part III

One elliptic curve, several variables

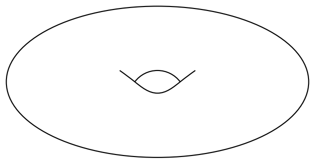
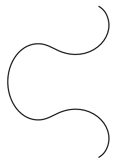
(The unequal mass sunrise integral)

$$x = \frac{p^2}{m_3^2}, \qquad y_1 = \frac{m_1^2}{m_3^2}, \qquad y_2 = \frac{m_2^2}{m_3^2}$$

Moduli spaces

$\mathcal{M}_{g,n}$: Space of isomorphism classes of smooth (complex, algebraic) curves of genus g with n marked points.

Recall:

real surface  $\Leftrightarrow$  complex curve

$$\dim \mathcal{M}_{g,n} \quad = \quad 3g + n - 3.$$

Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n-3$.

Sphere has a unique shape

Use Möbius transformation to fix $z_{n-2}=1, \quad z_{n-1}=\infty, \quad z_n=0$

Coordinates are $(z_1, \ldots, z_{n-3})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.

One coordinate describes the shape of the torus

Use translation to fix $z_n = 0$

Coordinates are $(\tau, z_1, \ldots, z_{n-1})$

In particular:

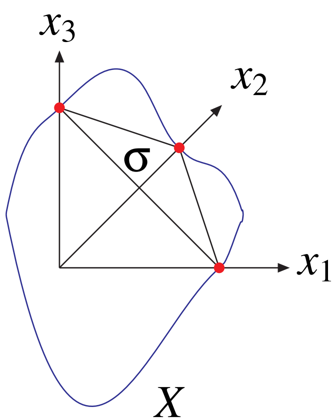
$\dim \mathcal{M}_{1,1} = 1$	with coordinate τ,	(equal mass sunrise)
$\dim \mathcal{M}_{1,3} = 3$	with coordinates τ, z_1, z_2,	(unequal mass sunrise).

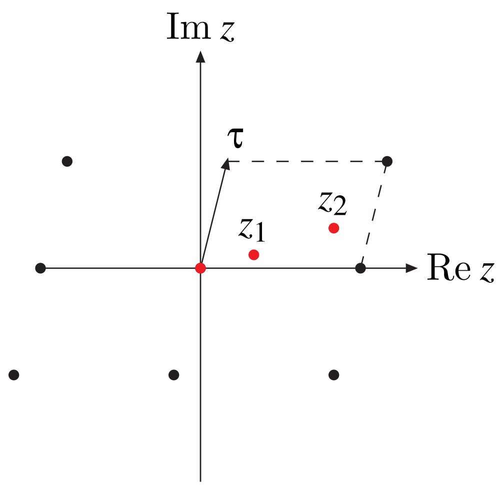
How to find z_1 and z_2?

In the Feynman parameter representation there are two objects of interest:

- the domain of integration σ,
- the zero set X of the second graph polynomial.

X and σ intersect at three points:





The differential equation for the unequal mass sunrise integral

There 8 master integrals. After a redefinition of the basis of master integrals and a change of coordiantes from $(x, y_1, y_2) = (p^2/m_3^2, m_1^2/m_3^2, m_2^2/m_3^2)$ to (τ, z_1, z_2) one finds

$$A \quad = \quad \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles},$$

where ω_k is either

$$(2\pi)^{2-k} f_k(\tau) \frac{d\tau}{2\pi i},$$

where $f_k(\tau)$ is a modular form, or of the form

$$\omega_k(z_i, \tau) \quad = \quad (2\pi)^{2-k} \left[g^{(k-1)}(z_i, \tau) \, dz_i + (k-1) \, g^{(k)}(z_i, \tau) \frac{d\tau}{2\pi i} \right],$$

where $g^{(k)}(z, \tau)$ are functions appearing in the expansion of the Kronecker function.

Bogner, Müller-Stach, S.W., ’19

The Kronecker function

$$F(z,\alpha,\tau) \quad = \quad \pi\theta_1'(0,q)\frac{\theta_1(\pi(z+\alpha),q)}{\theta_1(\pi z,q)\,\theta_1(\pi\alpha,q)} = \frac{1}{\alpha}\sum_{k=0}^{\infty} g^{(k)}(z,\tau)\alpha^k, \qquad q = e^{i\pi\tau}$$

Properties of $g^{(k)}(z,\tau)$:

- only simple poles as a function of z
- quasi-periodic as a function of z: Periodic by 1, quasi-periodic by τ.
- almost modular: Nice modular transformation properties only spoiled by divergent Eisenstein series $E_1(z,\tau)$.

Brown, Levin, '11,

Broedel, Duhr, Dulat, Penante, Tancredi, '18

Conclusions

Computation of Feynman integrals is trivial, as soon as the system of differential equations is transformed to

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles.}$$

This form can be reached for
- many Feynman integrals evaluating to multiple polylogarithms
- a few non-trivial elliptic examples

Open question: Any Feynman integral can be obtained from a system of differential equations of this form.

A constructive proof would gives us an algorithm to compute any Feynman integral.