Wilson-line geometries and the relation between IR singularities of form factors and the large-x limit of DGLAP splitting functions

Calum Milloy

with Giulio Falcioni and Einan Gardi arXiv:1909.00697

RADCOR 2019

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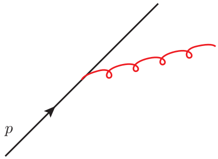

- Represents classical trajectory of particle
- Massive particles $n^2 \neq 0$, massless particles $n^2 = 0$
- Colour representation of A depends on the rep. of particle
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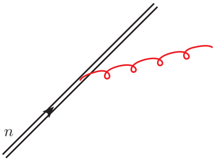

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- gauge invariant
- eikonal
- Casimir scaling $C_A \leftrightarrow C_F$

Wilson-line geometries

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$$\text{RG eq.}\quad \frac{d}{d\log\mu}\log W_i \;=\; \gamma_{\text{cusp}}\,\log(\ldots)\;+\;\Gamma_i$$

[Korchemskaya, Korchemsky '92]

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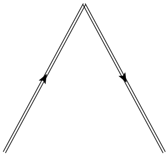

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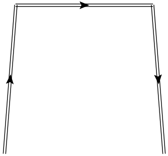

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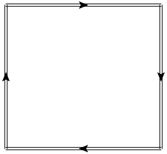


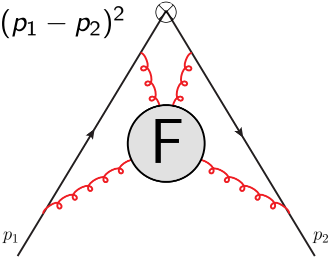
[Korchemskaya, Korchemsky '92]

Wilson-line geometries and the relation between IR singularities of form factors and the large-x limit of DGLAP splitting functions

IR singularities of form factors

Consider form factor of two massless partons

Depends on $Q^2 = (p_1 - p_2)^2$



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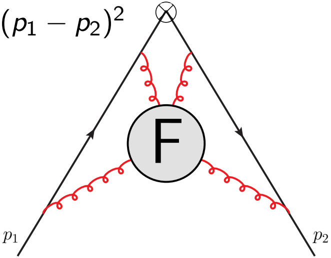

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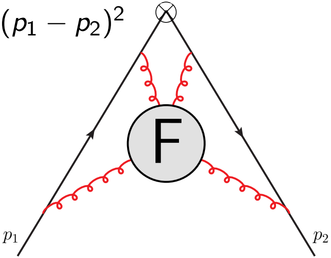

infrared divergences

$$\frac{1}{\epsilon^2} : \gamma^i_{\text{cusp}}$$ cusp anomalous dimension

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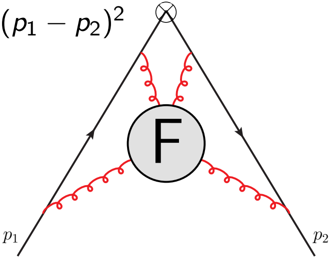

infrared divergences

$\frac{1}{\epsilon} : \gamma_G^i$ "collinear AD"
collinear singularities depend
on the species of partons

Wilson-line geometries and the relation between IR singularities of form factors and the large-x limit of DGLAP splitting functions

Large-x limit of DGLAP splitting functions

- Parton distribution functions $\frac{d}{d \log \mu} f_{ik} = \sum_j P_{ij} f_{jk}$
- Diagonal splitting functions, P_{qq}, P_{gg}
- At large-x behave as

$$P_{ii} = \frac{\gamma^i_{\text{cusp}}}{(1-x)_+} + B^i_\delta \delta(1-x) + \mathcal{O}((1-x)^0)$$

[Korchemsky '89]

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"B-delta" [Korchemsky '89]

Wilson-line geometries and the relation between IR singularities of form factors and the large-x limit of DGLAP splitting functions

The relation

- form factor: $F = \frac{\gamma^i_{\text{cusp}}}{\epsilon^2} + \frac{\gamma^i_G}{\epsilon} + \mathcal{O}(\epsilon^0)$
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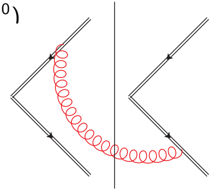
  [Sterman '87][Catani, Trentadue '89][Laenen, Magnea '06][Becher, Neubert, Xu '08]
- (generalised)-Casimir scaling $\Gamma^g_{\text{DY}}/C_A = \Gamma^q_{\text{DY}}/C_F$

  [Ravindran, Smith, van Neerven '05][Moch, Vermaseren, Vogt '05]



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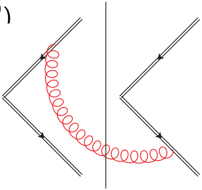

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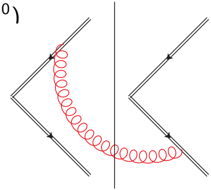

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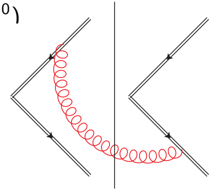

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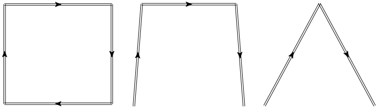
- Yes! Building block of finite and infinite segments emerges
- $\Gamma_{\square} = 4(\Gamma_{\sqcap} - \Gamma_{\wedge})$



Why care about $\gamma_G - 2B_\delta$?

$\Gamma_\wedge$ is physical [Fadin, Kotsky, Fiore '95][Blumlein, Ravindran, van Neerven '98][Erdoğan, Sterman '15]

$\Gamma_\wedge$ appears in the Regge trajectory to two loops

$$\alpha(t,\epsilon)^{(2)} = \left(\frac{\alpha_s}{\pi}\right)^2 \frac{1}{4}\left(-\frac{C_A \hat{b}_0}{\epsilon^2} + \frac{\gamma_{\text{cusp}}^{g\,(2)}}{\epsilon} + 2\Gamma_\wedge^{g\,(2)} + C_A \hat{b}_0 \zeta_2 \right)$$

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The virtuality of B_δ

By assuming B_δ is purely virtual $\gamma_G - 2B_\delta \stackrel{?}{=} -\Gamma_\wedge$ [Dixon, Magnea, Sterman '08]

By explicit calculation B_δ includes real corrections [Falcioni, Gardi, CM '19]

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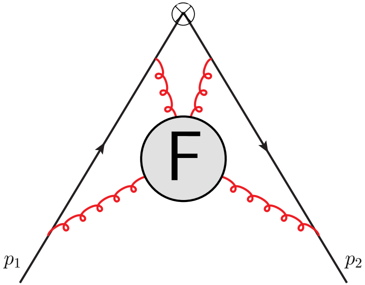
Testing universality in factorisation

We shall derive $\gamma_G - 2B_\delta$ by factorising both a form factor and a PDF in analogous ways

Infrared factorisation of form factor

[Collins '89]



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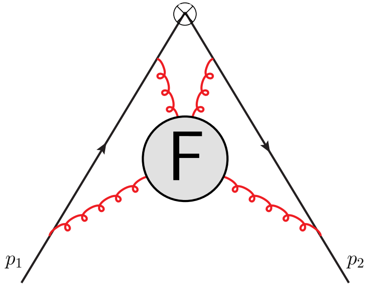
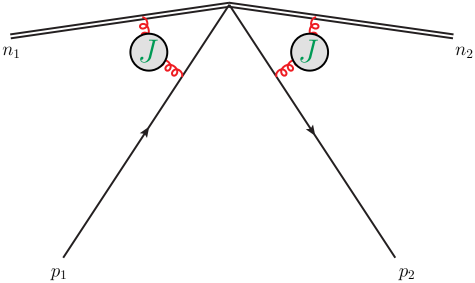
Separate the divergences: collinear $u(p_i)J_i = \langle 0|\, T\,[W_{n_i}(\infty,0)\psi(0)]\,|p_i\rangle$



F



J_1 J_2

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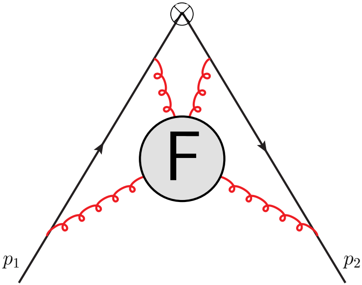
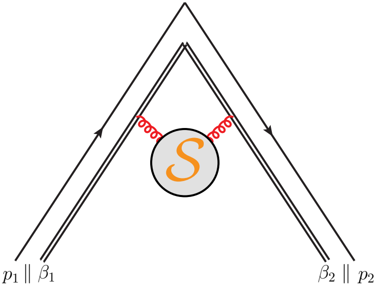
Separate the divergences: collinear soft $\mathcal{S} = \langle 0 | \, T\, [W_{\beta_1}(\infty, 0) W_{\beta_2}(0,\infty)]\, |0\rangle$
$= W_{\wedge}$





F

$\mathcal{S}\,\frac{J_1}{}\,\frac{J_2}{}$

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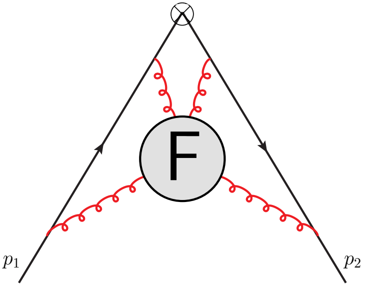
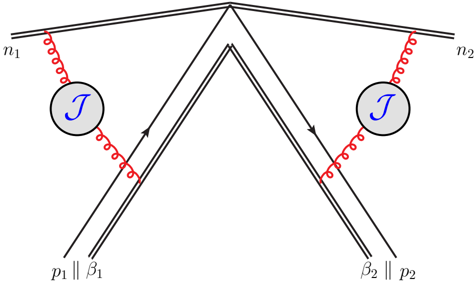
Divide by eikonal jets for double counting of soft-collinear region





F

$$\mathcal{S} \frac{J_1}{\mathcal{J}_1} \frac{J_2}{\mathcal{J}_2}$$

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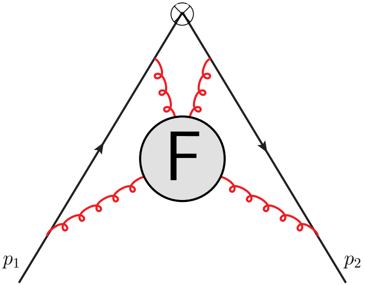

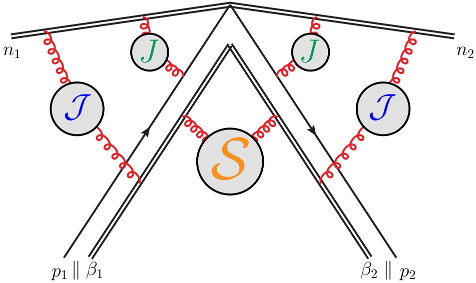

$$\mathrm{F} = H\,\mathcal{S}\,\frac{J_1}{\mathcal{J}_1}\frac{J_2}{\mathcal{J}_2}$$

Isolating hard-collinear singularities

Poles of form factor

$\mathcal{S}$ and $\mathcal{J}$ are scaleless $\implies$ only poles $\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right)$

$$F\big|_{\text{pole}} = \frac{J_1\big|_{\text{pole}}}{\mathcal{J}_1}\frac{J_2\big|_{\text{pole}}}{\mathcal{J}_2}\mathcal{S}$$

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Anomalous dimensions in the infrared

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Anomalous dimensions in the infrared

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$$\gamma_G =$$

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Anomalous dimensions in the infrared

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$$\gamma_G = 2\gamma_{J/\mathcal{J}}$$

hard-collinear

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Anomalous dimensions in the infrared

form factor

soft

$$\gamma_G = 2\gamma_{J/\mathcal{J}} - \Gamma_\wedge$$

hard-collinear

Parton distribution functions

Light-cone PDF [Collins, Soper '82]

$$f_{qq} = \frac{1}{2}\int \frac{dy}{2\pi} e^{-iyxp\cdot u} \langle q|\bar{\psi}_q(yu)\gamma\cdot u W_u(y,0)\psi_q(0)|q\rangle$$

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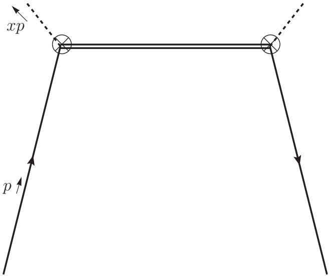

cross-section level

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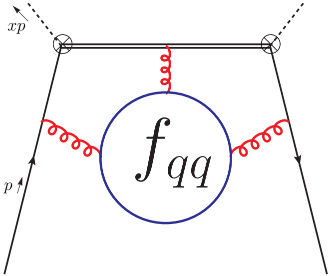
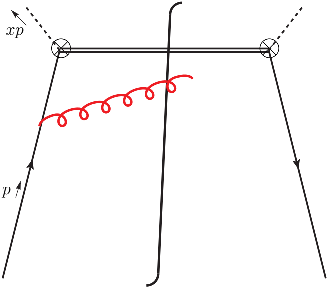
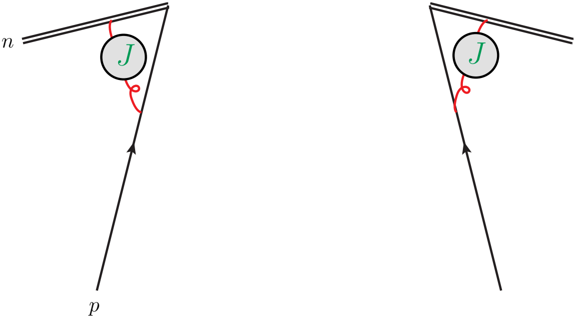


PDF factorisation at large x

- As $x \to 1$ the external parton loses less and less momentum
- Soft gluon radiation dominates
- Implies factorisation [Korchemsky '89] [Berger '02]

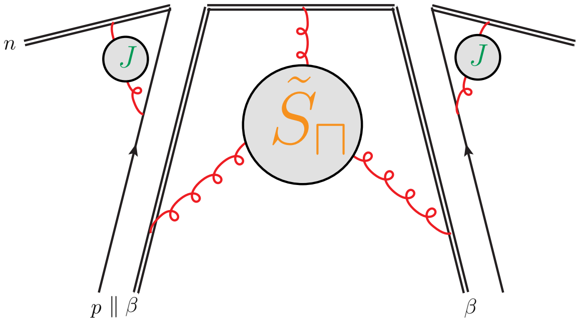


PDF factorisation at large x



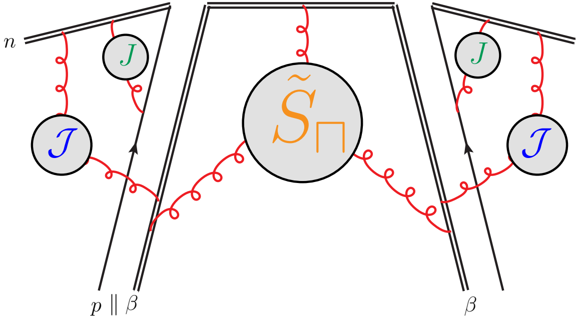
$J_L J_R$

PDF factorisation at large x



$$\tilde{S}_{\sqcap}(N) = \int_0^1 x^{N-1} \int \frac{dy}{2\pi} e^{iy(1-x)p\cdot u} W_{\sqcap}$$

PDF factorisation at large x



$$f(N) = H \frac{J_L J_R}{\mathcal{J}_L \mathcal{J}_R} \tilde{S}_\sqcap(N) + \mathcal{O}\left(\frac{\log N}{N}\right)$$

PDF factorisation at large x

$$f(N) = H \frac{J_L J_R}{\mathcal{J}_L \mathcal{J}_R} \tilde{S}_{\sqcap}(N) + \mathcal{O}\left(\frac{\log N}{N}\right)$$

- Bare PDFs are scaleless $\implies$ only poles
- In a minimal subtraction scheme, renormalised PDFs are also pure poles
- $\mathcal{J}_i$ and $\tilde{S}_{\sqcap}(N)$ are pure poles
- We must have $H J_L J_R \to J_L|_{\text{pole}} J_R|_{\text{pole}}$

Large-N factorisation

Renormalised PDF

$$f^{\text{ren}}(N) = \frac{J_L|_{\text{pole}}}{\mathcal{J}_L}\frac{J_R|_{\text{pole}}}{\mathcal{J}_R}\tilde{S}_{\sqcap}(N) + \mathcal{O}\left(\frac{\log N}{N}\right)$$

The same hard-collinear behaviour as the form factor

Large-N factorisation

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RG equation

$$2P(N)$$

Large-N factorisation

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RG equation

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$$- 2\gamma_{\text{cusp}} \log N + 2B_\delta$$

[Korchemsky '89]

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RG equation

$$-2\gamma_{\text{cusp}} \log N + 2B_\delta = 2\gamma_{J/\mathcal{J}} - 2\gamma_{\text{cusp}} \log N - \Gamma_{\sqcap}$$

Large-N factorisation

Renormalised PDF

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RG equation

$$2B_\delta = 2\gamma_{J/\mathcal{J}} - \Gamma_{\sqcap}$$

splitting function

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hard-collinear

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RG equation

$$2B_\delta = 2\gamma_{J/\mathcal{J}} - \Gamma_{\sqcap}$$

splitting function

soft, includes real contributions

hard-collinear

Our relation

Form factor

$$\gamma_G = 2\gamma_{J/\mathcal{J}} - \Gamma_\wedge$$

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Universality of hard-collinear behaviour leads to

$$\boxed{\gamma_G - 2B_\delta = \Gamma_{\sqcap} - \Gamma_{\wedge}}$$

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Check at two loops

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Extract $\gamma_{J/\mathcal{J}}$

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Check at two loops

Extract $\gamma_{J/\mathcal{J}}$ then find $\Gamma_\sqcap$

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Check at two loops

Extract $\gamma_{J/\mathcal{J}}$ then find $\Gamma_{\sqcap}$

$$\Gamma_{\sqcap}^{(2)} = \frac{C_i}{2}\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 4\zeta_3\right]\right)$$

Our relation

Form factor

$$\gamma_G = 2\gamma_{J/\mathcal{J}} - \Gamma_\wedge$$

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Compute $\Gamma_\sqcap$ explicitly

$W_\sqcap$

- Non-Abelian exponentiation $W = \exp(\sum_{\text{webs}} w)$
- For non-lightlike infinite lines $W_\sqcap$ is known to two loops

  [Korchemsky, Marchesini '93]

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$W_{\sqcap}$

- Non-Abelian exponentiation $W = \exp(\sum_{\text{webs}} w)$
- For non-lightlike infinite lines $W_{\sqcap}$ is known to two loops [Korchemsky, Marchesini '93]
- For lightlike infinite lines we encounter scaleless integrals
- Carefully disentangle UV/IR divergences
- Only recently understood how to do for $W_{\wedge}$ [Erdoğan, Sterman '14]
- Extend to include a geometry with a finite segment $W_{\sqcap}$

Calculating $\Gamma_{\sqcap}$

Coordinate space

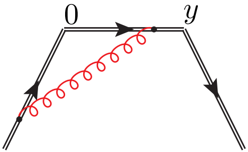
μ  $\nu = -\frac{1}{4\pi^{2-\epsilon}} \frac{\Gamma(1-\epsilon)}{\left[-x^2 + i0\right]^{1-\epsilon}}\, g_{\mu\nu}$

Calculating $\Gamma_{\sqcap}$

Coordinate space

$$\mu \;\text{(gluon propagator)}\; \nu = -\frac{1}{4\pi^{2-\epsilon}} \frac{\Gamma(1-\epsilon)}{\left[-x^2+i0\right]^{1-\epsilon}}\, g_{\mu\nu}$$

One loop



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$$\mu \quad \nu = -\frac{1}{4\pi^{2-\epsilon}} \frac{\Gamma(1-\epsilon)}{\left[-x^2 + i0\right]^{1-\epsilon}} \, g_{\mu\nu}$$

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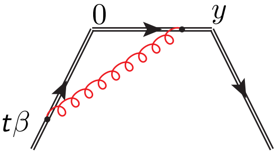

Calculating $\Gamma_{\sqcap}$

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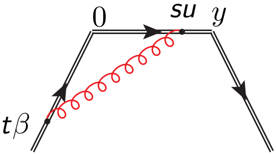
One loop



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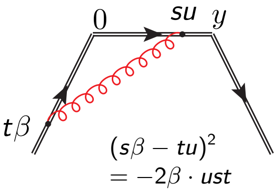


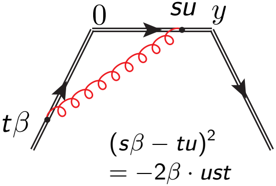
Calculating $\Gamma_{\sqcap}$

Coordinate space

μ  $\nu = -\frac{1}{4\pi^{2-\epsilon}} \frac{\Gamma(1-\epsilon)}{\left[-x^2 + i0\right]^{1-\epsilon}} \, g_{\mu\nu}$

One loop

$$d_1 = \frac{\alpha_s}{\pi} (\mu^2 \pi)^{\epsilon} u \cdot \beta C_i \Gamma(1-\epsilon) \int_0^{\infty} dt \int_0^{y} ds (-2\beta \cdot ust + i0)^{-1+\epsilon}$$



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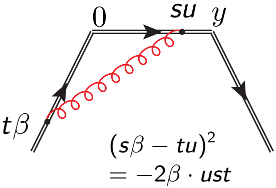

Want PDFs to vanish for $x > 1$

$\rho = i(\beta\cdot uy - i0)$ no singularities in lower-half y-plane

Change variables $t = -i\sqrt{2}\lambda,\ s = -i\sqrt{2}\dfrac{\sigma}{u\cdot\beta}$

Calculating $\Gamma_{\sqcap}$

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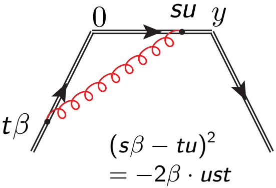

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Calculating $\Gamma_\sqcap$

Coordinate space

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One loop

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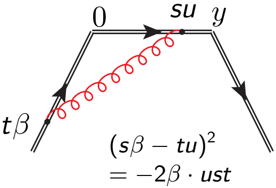

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$$\log W_\sqcap^{\text{bare}} = -C_i \int_0^\infty \frac{d\lambda}{\lambda} \int_0^{\rho/\sqrt{2}} \frac{d\sigma}{\sigma} \frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi} e^{-\epsilon\gamma_E}\Gamma(1-\epsilon)$$

One loop cont.

Renormalisation of lightlike semi-infinte lines [Erdoğan, Sterman '14]

$$\log W_{\sqcap}^{\text{bare}} = -C_i \int_0^\infty \frac{d\lambda}{\lambda} \int_0^{\rho/\sqrt{2}} \frac{d\sigma}{\sigma} \frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi} e^{-\epsilon\gamma_E}\Gamma(1-\epsilon)$$

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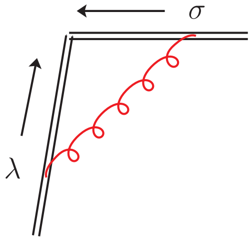
cusp



One loop cont.

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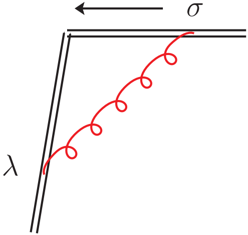
line



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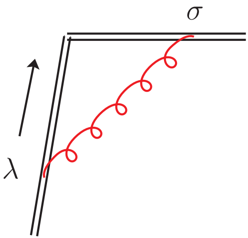

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One loop cont.

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One loop cont.

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One loop cont.

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$$= \frac{\alpha_s(\mu^2)}{\pi} \frac{C_i}{\epsilon} \log\left(\frac{\rho\mu}{\sqrt{2}}\right)$$

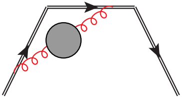
$$\gamma_{\text{cusp}} = \frac{\alpha_s}{\pi} C_i$$

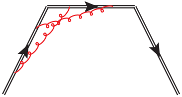
double UV pole but single IR pole!

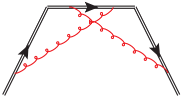
$$\Gamma_{\sqcap} = 0$$

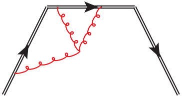
Calculating $\Gamma_{\sqcap}$ at two loops

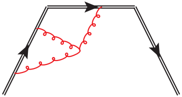
- Work in the exponent
  $\implies$ only require maximally non-Abelian diagrams

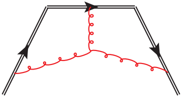
$d^{(2)}_{\mathrm{SE}} =$ 

$d^{(2)}_{X_2} =$ 

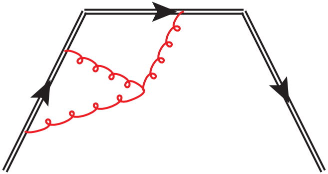
$d^{(2)}_{X_3} =$ 

$d^{(2)}_{Y_s} =$ 

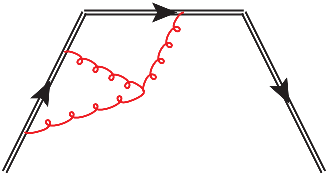
$d^{(2)}_{Y_L} =$ 

$d^{(2)}_{3s} =$ 

$d_{Y_L}^{(2)}$

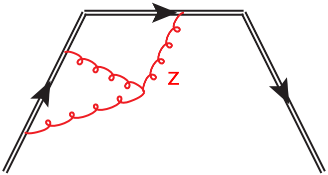


$d_{Y_L}^{(2)}$



$$d_{Y_L}^{(2)} = K_Y \int d^d z \int_{-\infty}^{0} ds_1 \int_{s_1}^{0} ds_2 \int_0^y dt_3 \left[\beta \cdot \frac{\partial}{\partial s_2 \beta} - \beta \cdot \frac{\partial}{\partial s_1 \beta}\right]$$
$$\times \left[-(s_1\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(s_2\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(ut_3 - z)^2 + i0\right]^{-1+\epsilon}$$

$d^{(2)}_{Y_L}$



$$d^{(2)}_{Y_L} = K_Y \int d^d z \int_{-\infty}^{0} ds_1 \int_{s_1}^{0} ds_2 \int_0^y dt_3 \left[\beta \cdot \frac{\partial}{\partial s_2 \beta} - \beta \cdot \frac{\partial}{\partial s_1 \beta}\right]$$
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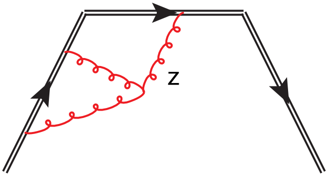
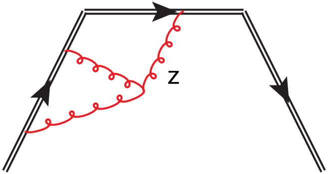
$d^{(2)}_{Y_L}$



in coordinate space
the three gluon vertex is given by
derivatives of positions

$$d^{(2)}_{Y_L} = K_Y \int d^d z \int_{-\infty}^{0} ds_1 \int_{s_1}^{0} ds_2 \int_0^y dt_3 \left[\beta \cdot \frac{\partial}{\partial s_2 \beta} - \beta \cdot \frac{\partial}{\partial s_1 \beta}\right]$$
$$\times \left[-(s_1\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(s_2\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(ut_3 - z)^2 + i0\right]^{-1+\epsilon}$$

$d_{Y_L}^{(2)}$

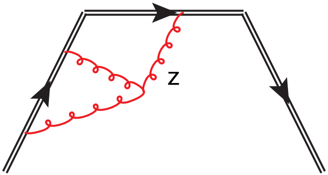


- compute by integration-by-parts

$$d_{Y_L}^{(2)} = K_Y \int d^d z \int_{-\infty}^{0} ds_1 \int_{s_1}^{0} ds_2 \int_0^y dt_3 \left[\beta \cdot \frac{\partial}{\partial s_2 \beta} - \beta \cdot \frac{\partial}{\partial s_1 \beta}\right]$$
$$\times \left[-(s_1\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(s_2\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(ut_3 - z)^2 + i0\right]^{-1+\epsilon}$$

$$d_E^{(2)}(v, u\cdot\beta) = K_Y \int d^d z \int_{-\infty}^{0} dt \int_0^y ds$$
$$\left[-(\beta t - z)^2\right]^{\epsilon-1} \left[-(us - z)^2\right]^{\epsilon-1} \left[-(v - z)^2\right]^{\epsilon-1}$$

$d_{Y_L}^{(2)}$



- compute by integration-by-parts
- have to be careful about endpoint terms at ∞ [Erdoğan, Sterman '14]

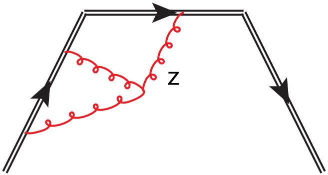
$$d_{Y_L}^{(2)} = K_Y \int d^d z \int_{-\infty}^{0} ds_1 \int_{s_1}^{0} ds_2 \int_0^y dt_3 \left[\beta \cdot \frac{\partial}{\partial s_2 \beta} - \beta \cdot \frac{\partial}{\partial s_1 \beta}\right]$$

$$\times \left[-(s_1\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(s_2\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(ut_3 - z)^2 + i0\right]^{-1+\epsilon}$$

$$d_E^{(2)}(\infty, u \cdot \beta) = K_Y \int d^d z \int_{-\infty}^{0} dt \int_0^y ds$$

$$\left[-(\beta t - z)^2\right]^{\epsilon-1} \left[-(us - z)^2\right]^{\epsilon-1} \left[-(\infty - z)^2\right]^{\epsilon-1}$$

$d_{Y_L}^{(2)}$



- compute by integration-by-parts
- have to be careful about endpoint terms at ∞ [Erdoğan, Sterman '14]
  - vanishes! different from finite length

$$d_{Y_L}^{(2)} = K_Y \int d^d z \int_{-\infty}^{0} ds_1 \int_{s_1}^{0} ds_2 \int_0^y dt_3 \left[\beta \cdot \frac{\partial}{\partial s_2 \beta} - \beta \cdot \frac{\partial}{\partial s_1 \beta}\right]$$

$$\times \left[-(s_1\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(s_2\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(ut_3 - z)^2 + i0\right]^{-1+\epsilon}$$

$$d_E^{(2)}(\infty, u\cdot\beta) = K_Y \int d^d z \int_{-\infty}^{0} dt \int_0^y ds$$

$$\left[-(\beta t - z)^2\right]^{\epsilon-1} \left[-(us - z)^2\right]^{\epsilon-1} \left[-(\infty - z)^2\right]^{\epsilon-1} = 0$$

Two loops together

$$\log W_{\sqcap}^{\text{bare}} = C_i \int_0^\infty \frac{d\lambda}{\lambda} \int_0^{\frac{\rho}{\sqrt{2}}} \frac{d\sigma}{\sigma} \Bigg\{ \frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi} e^{-\epsilon\gamma_E} \Gamma(1-\epsilon) \left[-1 + \frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi} \frac{11C_A - 4T_f n_f}{12\epsilon} \right]$$

$$+ \left(\frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi} e^{-\epsilon\gamma_E} \right)^2 \left[\frac{C_A}{4} \left(\frac{3(-4+3\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2(3-8\epsilon+4\epsilon^2)} - 4\pi\Gamma(-2\epsilon)\cot\left(\frac{\pi\epsilon}{2}\right)\right)\right.$$

$$\left. - T_f n_f \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{3-8\epsilon+4\epsilon^2} \right] \Bigg\}$$

Two loops together

$$\log W_{\sqcap}^{\text{bare}} = -\int_0^\infty \frac{d\lambda}{\lambda}\int_0^{\frac{\rho}{\sqrt{2}}} \frac{d\sigma}{\sigma}\Bigg\{ \left(\frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi}\right)\left[1+\frac{\epsilon^2}{2}\zeta_2+\mathcal{O}(\epsilon^3)\right] + \left(\frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi}\right)^2\left[\gamma_{\text{cusp}}^{(2)}+\epsilon\left(\Gamma_{\sqcap}^{(2)}+\frac{3\hat{b}_0\zeta_2}{2}\right)+\mathcal{O}(\epsilon^2)\right]\Bigg\}.$$

$$\Gamma_{\sqcap}^{(2)} = \frac{C_i}{2}\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27}-4\zeta_3\right]\right)$$

Two loops together

$$\log W_{\sqcap}^{\text{bare}} = -\int_0^{\infty} \frac{d\lambda}{\lambda} \int_0^{\frac{\rho}{\sqrt{2}}} \frac{d\sigma}{\sigma} \left\{ \left(\frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi} \right) \left[1 + \frac{\epsilon^2}{2}\zeta_2 + \mathcal{O}(\epsilon^3) \right] + \left(\frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi} \right)^2 \left[\gamma_{\text{cusp}}^{(2)} + \epsilon \left(\Gamma_{\sqcap}^{(2)} + \frac{3\hat{b}_0\zeta_2}{2} \right) + \mathcal{O}(\epsilon^2) \right] \right\}.$$

$$\Gamma_{\sqcap}^{(2)} = \frac{C_i}{2} \left(-2\hat{b}_0\zeta_2 - \frac{56}{27} T_f n_f + C_A \left[\frac{202}{27} - 4\zeta_3 \right] \right)$$

Renormalisation

$$\log W_{\sqcap} = \alpha_s(\mu^2)\frac{1}{\epsilon} \log\left(\frac{\rho\mu}{\sqrt{2}} \right) + \alpha_s(\mu^2)^2 \left\{ -\frac{\hat{b}_0}{2\epsilon^2} \log\left(\frac{\rho\mu}{\sqrt{2}} \right) + \frac{1}{\epsilon} \left(\frac{1}{4} \Gamma_{\sqcap}^{(2)} + \frac{1}{2} \gamma_{\text{cusp}}^{(2)} \log\left(\frac{\rho\mu}{\sqrt{2}} \right) \right) \right\}$$

Two loops together

$$\log W_{\sqcap}^{\text{bare}} = -\int_0^{\infty} \frac{d\lambda}{\lambda} \int_0^{\frac{\rho}{\sqrt{2}}} \frac{d\sigma}{\sigma} \Bigg\{ \left(\frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi} \right) \left[1 + \frac{\epsilon^2}{2}\zeta_2 + \mathcal{O}(\epsilon^3) \right] + \left(\frac{\alpha_s\left(\frac{1}{\lambda\sigma}\right)}{\pi} \right)^2 \left[\gamma_{\text{cusp}}^{(2)} + \epsilon \left(\Gamma_{\sqcap}^{(2)} + \frac{3\hat{b}_0\zeta_2}{2} \right) + \mathcal{O}(\epsilon^2) \right] \Bigg\}.$$

$$\Gamma_{\sqcap}^{(2)} = \frac{C_i}{2} \left(-2\hat{b}_0\zeta_2 - \frac{56}{27} T_f n_f + C_A \left[\frac{202}{27} - 4\zeta_3 \right] \right)$$

Renormalisation

$$\log W_{\sqcap} = -\frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \left\{ 2\gamma_{\text{cusp}} \log\left(\frac{\rho\mu}{\sqrt{2}} \right) + \Gamma_{\sqcap} \right\}$$

- Again we see differing UV/IR behaviour
- $\mu \to \xi$ in log would give double UV/IR

Wilson-line geometry

Divergences localised

- Only when all vertices approach cusps or lines

Wilson-line geometry

Divergences localised

- Only when all vertices approach cusps or lines

Confirmation at two loops

Wilson-line geometry

Divergences localised

- Only when all vertices approach cusps or lines

Confirmation at two loops

- $\Gamma_{\wedge}^{(2)} = \frac{C_i}{4}\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 1\zeta_3\right]\right)$ [Erdoğan, Sterman '14]

Wilson-line geometry

Divergences localised

- Only when all vertices approach cusps or lines

Confirmation at two loops

- $\Gamma^{(2)}_{\wedge} = \frac{C_i}{4}\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 1\zeta_3\right]\right)$ [Erdoğan, Sterman '14]
- $\Gamma^{(2)}_{\sqcap} = \frac{C_i}{2}\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 4\zeta_3\right]\right)$

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- $\Gamma^{(2)}_{\wedge} = \frac{C_i}{4}\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 1\zeta_3\right]\right)$ [Erdoğan, Sterman '14]
- $\Gamma^{(2)}_{\sqcap} = \frac{C_i}{2}\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 4\zeta_3\right]\right)$
- $\Gamma^{(2)}_{\square} = C_i\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 7\zeta_3\right]\right)$ [Korchemskaya, Korchemsky '92]

Wilson-line geometry

Divergences localised

- Only when all vertices approach cusps or lines

Confirmation at two loops

- $\Gamma_{\wedge}^{(2)} = \frac{C_i}{4}\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 1\zeta_3\right]\right)$ [Erdoğan, Sterman '14]
- $\Gamma_{\sqcap}^{(2)} = \frac{C_i}{2}\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 4\zeta_3\right]\right)$
- $\Gamma_{\square}^{(2)} = C_i\left(-2\hat{b}_0\zeta_2 - \frac{56}{27}T_f n_f + C_A\left[\frac{202}{27} - 7\zeta_3\right]\right)$ [Korchemskaya, Korchemsky '92]
- Differ only in endpoint contributions i.e. finite vs. infinite length
- Blind to global geometry
- $\Gamma_{\sqcap} - \Gamma_{\wedge} = \Gamma_{\square}/4 \implies \boxed{\gamma_G - 2B_\delta = \Gamma_{\square}/4}$

Conclusions

Summary

- We factorised a form factor and PDF
- $\boxed{\gamma_G - 2B_\delta \; = \Gamma_\sqcap - \Gamma_\wedge \; = \Gamma_\square/4}$
- Relation was checked by explicit two-loop calculation of $\Gamma_\sqcap$

Outlook

- Building block of finite and infinite lines, more geometries
- $\Gamma_\square = 2\Gamma_{\text{DY}}$ three-loop check using Γ_{DY} [Li, von Manteuffel, Schabinger, Zhu '15]
- $W_\square = 2W_{\text{DY}}$?
- Connection to Regge trajectory

Extra slides

Constraint equations for the functions

subleading AD

cusp AD

eikonal

$$\log \mathcal{J}_i = -\frac{1}{4}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\left(\Gamma_{\mathcal{J}}\left(\alpha_s(\lambda^2,\epsilon)\right) + \gamma_{\rm cusp}(\alpha_s(\lambda^2,\epsilon))\log\frac{2(\beta_i\cdot n_i)^2\mu^2}{n_i^2\lambda^2}\right)$$

$$\log \mathcal{S} = -\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\left(\Gamma_{\wedge}(\alpha(\lambda^2,\epsilon)) + \gamma_{\rm cusp}(\alpha_s(\lambda^2,\epsilon))\log\left(\frac{\beta_1\cdot\beta_2\mu^2}{\lambda^2}\right)\right)$$

$$\log J_i\big|_{\rm pole} = \frac{1}{4}\int_0^{p_n^2} \frac{d\lambda^2}{\lambda^2}\Big[-\gamma_{\rm cusp}(\alpha_s(\lambda^2,\epsilon))\log\left(\frac{p_n^2}{\lambda^2}\right) + \Gamma_{\wedge}(\alpha_s(\lambda^2,\epsilon)) - \Gamma_{\mathcal{J}}(\alpha_s(\lambda^2,\epsilon)) + \gamma_G(\alpha_s(\lambda^2,\epsilon))\Big]$$

form factor

ρ

$\rho = i(\beta \cdot uy - i0)$

$$g(x) = e^{iy(1-x)} f(i(\beta \cdot uy - i0))$$

- If $x > 1$ then as $y \to -i\infty$, $g(x) \to 0$
- Since $\rho = 0$ only in upper-half plane

Momentum-space d_{Y_L}

$$d_{Y_L}^{(2)} = K_Y \int d^d z \int_{-\infty}^{0} ds_1 \int_{s_1}^{0} ds_2 \int_0^y dt_3 \left[\beta \cdot \frac{\partial}{\partial s_2 \beta} - \beta \cdot \frac{\partial}{\partial s_1 \beta}\right] \times \left[-(s_1\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(s_2\beta - z)^2 + i0\right]^{-1+\epsilon} \left[-(ut_3 - z)^2 + i0\right]^{-1+\epsilon}$$

convert to propagators momentum space

$$\sim \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2d}} \int_{-\infty}^{0} ds_1 \int_{s_1}^{0} ds_2 \int_0^y dt_3 \left[\beta \cdot \frac{\partial}{\partial s_2 \beta} - \beta \cdot \frac{\partial}{\partial s_1 \beta}\right] \frac{e^{-ik_1\cdot\beta s_1} e^{-ik_2\cdot\beta s_2} e^{i(k_1+k_2)\cdot ut_3}}{k_1^2 k_2^2 (k_1+k_2)^2}$$

take derivatives and integrate over s_2 and s_1

$$\sim \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2d}} \int_0^y dt_3\, e^{i(k_1+k_2)\cdot ut_3} \frac{(-i)^3}{k_1^2 k_2^2 (k_1+k_2)^2} \times \left\{\frac{1}{-i(k_1\cdot\beta + i0)} - \frac{2}{-i\left[(k_1+k_2)\cdot\beta + i0\right]}\right\}.$$

reconvert proagators from momentum space back to coordinate space

$$d_{Y_L}^{(2)} = K_Y \int d^d z \int_0^y dt_3 \int_{-\infty}^{0} ds_1 \left[-(z-ut_3)^2 + i0\right]^{\epsilon-1} \left\{\left[-(z-\beta s_1)^2 + i0\right]^{\epsilon-1} \left[-z^2 + i0\right]^{\epsilon-1} - 2\left[-(z-\beta s_1)^2 + i0\right]^{2\epsilon-2}\right\}.$$

same as integration-by-parts procedure