Master Integrals for the two-loop non-planar QCD corrections to top-quark pair production in the quark-annihilation channel

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Outline

- Introduction
- Differential equation method
- Canonical basis approach
- Master Integrals computation
- Conclusions and outlooks

Introduction

Top-Antitop production

- Top-Antitop production is one of the major QCD backgrounds at the LHC.
- The top quark is the heaviest particle in the SM. It has a very short lifetime, about $10^{-25}s$, which implies that it does not hadronize $\Rightarrow$ it is possible to study top quark properties as a single quark.
- Top-Antitop production total and differential cross sections are known numerically to NNLO in QCD. [Barnreuter, Czakon, Fiedler, Heymes, Mitov (2012-2016)]

An analytic calculation of top-antitop production QCD two-loop corrections is still important:

- It provides an independent check of the numerical results.
- It could be a faster and cheaper (in terms of CPU usage) way to compute two-loop QCD corrections to the process.

Top-Antitop production

- Production channels:
  - $q\bar{q} \to t\bar{t}$: quarks-annihilation channel $\to$ subdominant at LHC energies.
  - $gg \to t\bar{t}$: gluon-fusion channel $\to$ dominant at LHC enegies.
- Interference of the two-loop amplitude in the quark-annihilation channel with the tree-level amplitude:

$$\begin{aligned} A_{2L}^{q\bar{q}} &= (N^2-1)\left[N^2 A^{q\bar{q}} + B^{q\bar{q}} + \frac{C^{q\bar{q}}}{N^2} + N N_l D_l^{q\bar{q}} + \frac{N_l}{N} E_l^{q\bar{q}} + N N_h D_h^{q\bar{q}} \right. \\ &\left. + \frac{N_h}{N} E_h^{q\bar{q}} + N_l^2 F_l^{q\bar{q}} + N_l N_h F_{lh}^{q\bar{q}} + N_h^2 F_h^{q\bar{q}} \right] \end{aligned}$$

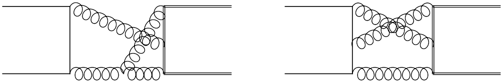
N color number, N_l number of light-fermion, N_h number of heavy-fermion.

$q\bar{q}$ channel status

- All the 10 color coefficients are known numerically at NNLO. [Czakon '08]
- The two-loops infrared poles are known analytically. [Ferroglia, Neubert, Pecjak, L.L. Yang '09]
- The color coefficients $A^{q\bar{q}}, D_l^{q\bar{q}}, E_l^{q\bar{q}}, D_h^{q\bar{q}}, E_h^{q\bar{q}}, F_l^{q\bar{q}}, F_{lh}^{q\bar{q}}, F_h^{q\bar{q}}$ have been computed analytically in terms of GPLs. [Bonciani, Ferroglia, Gehrmann, Maitre, Studerus '08]
- For the color coefficients $B^{q\bar{q}}$ and $C^{q\bar{q}}$ an analytic expression is not yet available $\Rightarrow$ Analytic solution to certain Feynman integrals families is still missing, while others have already been computed [Di Vita, Laporta, Mastrolia, Primo, Schubert '18] [von Manteuffel, Studerus '13].

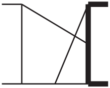
Two-Loop QCD corrections

- Example of Feynman Diagrams needed for the colour coefficients $B^{q\bar{q}}$ and $C^{q\bar{q}}$:



Thin lines massless quarks, thick lines top quarks, curly lines gluons.

- Topologies:



(A) (B)

- MIs of topology B have been recently computed [Di Vita, Laporta, Mastrolia, Primo, Schubert '18]
- MIs of topology A are new. They have been computed by [MB, Bonciani, Casconi, Ferroglia, Lavacca, von Manteuffel '19] and by [Di Vita, Gehrmann, Laporta, Mastrolia, Primo, Schubert '19].

Scattering Amplitudes

Feynman Diagrams + Feynman Rules $\Rightarrow$ Scattering amplitudes can be written formally as $\mathcal{A} = c \times I$

- c is obtained by the contraction of all the tensor and color structures compatible with the process.
- I are scalar integrals associated to the loop structure of the amplitude.

Tipically large number of integrals to be solved $\Rightarrow$ We need an efficient computation method

Identities by-parts relations (IBPs)

A generic scalar integrals I can be written as:

$$I(a_1, \cdots, a_n) = \int \prod_{j=1}^{l} \mathcal{D}^d k_j \prod_{m=1}^{n} \frac{1}{D_m^{a_m}}$$

- D_m are linearly independent functions of scalar products of the kind $p_i \cdot p_j$,$p_i \cdot k_j$ and $k_i \cdot k_j$, where p_i are external momenta, and $a_1, \cdots, a_n$ are integer numbers.
- IBPs [Chetyrkin, Tkachov '81]:

$$\int \prod_{j=1}^{l} \mathcal{D}^d k_j \frac{\partial}{\partial k_i^{\mu}} \left(v_\mu \prod_{m=1}^{n} \frac{1}{D_m^{a_m}} \right) = 0$$

  relations among scalar integrals.

IBPs $\Rightarrow$ All the scalar integrals involved in an higher-order computation can be written as a finite linear combination of Master Integrals.

We need to compute a lower number of integrals!

Differential equation method

Differential Equation Method

The MIs $\vec{f}(\vec{x}, \epsilon)$ satisfy a system of linear first order coupled PDE with respect to the kinematical invariants $\vec{x}$

$$d\vec{f}(\vec{x}, \epsilon) = dA(\vec{x}, \epsilon)\vec{f}(\vec{x}, \epsilon)$$

[Remiddi '97]

- d is the total differential with respect to the kinematical invariants.
- $dA(\vec{x}, \epsilon)$ is the one-form matrix associated to the PDE system.

Canonical Basis

In some cases exists a particular integrals basis, the Canonical Basis [Henn '13], in which the system of PDEs takes the form

$$d\tilde{f}(\vec{x},\epsilon) = \epsilon d\tilde{A}(\vec{x})\tilde{f}(\vec{x},\epsilon)$$

with $d\tilde{A}(\vec{x})$ logarithmic differential one-form

$$\tilde{A} = \sum_k A^{(k)} \log l_k(\vec{x})$$

A_k rational numbers matrices, $l_k(\vec{x})$ algebraic functions of kinematical invariants.

Canonical Basis approach

Canonical Basis Approach

- The solution of system of differential in canonical basis is given in terms of Chen Iterated Integrals [Chen '77]:

$$\vec{f}(\vec{x},\epsilon) = \mathbb{P}\left(\epsilon \int_\gamma d\tilde{A}(\vec{x})\right) \vec{f}(\vec{x}_0,\epsilon)$$

  γ is an integration path in kinematic invariants space, $\vec{f}(\vec{x}_0,\epsilon)$ is the boundary conditions vector.

- Solution suitable for ϵ-expansion:

$$\vec{f}(x) = \vec{f}^{(0)}(\vec{x}_0) + \sum_{k=1}^{\infty} \epsilon^k \sum_{j=1}^{k} \int_0^1 dt_1 \frac{\partial \tilde{A}(t_1)}{\partial t_1} \int_0^{t_1} dt_2 \frac{\partial \tilde{A}(t_2)}{\partial t_2} \dots \int_0^{t_{j-1}} dt_j \frac{\partial \tilde{A}(t_j)}{\partial t_j} \vec{f}^{(k-j)}(\vec{x}_0)\,.$$

- Solution is given order-by-order in ϵ in terms of Goncharov Multiple Polylogarithms (GPLs) [Goncharov '95, Remiddi, Vermaseren '05].
- Algorithmic integration procedure $\Rightarrow$ Efficient method to compute MIs.
- Numerical routines for GPLs such as GiNaC [Vollinga, Weinzierl '05] $\Rightarrow$ Phenomenology application.

The explicit integration order-by-order in ϵ of the Chen Iterated integral solution is closely related to the class of functions to which belong $I_k(x)$.

- Rational functions: the solution is directly expressible in terms of GPLs.
- Algebraic functions: the solution is not directly expressible in terms of GPLs $\Rightarrow$ we have to linearize all the roots that appear in the PDE system.

Polylogarithms Representation

If the Alphabet of the differential equation system $I_k(\vec{x})$ has a rational dependence on the kinematical invariants, then the matrix $d\tilde{A}(\vec{x})$ has the form

$$d\tilde{A}(\vec{x}) \simeq \sum_i \sum_k \frac{A_{i,k} dx_i}{x_i - l_{i,k}}$$

$A_{i,k}$ are real numbers and $l_{i,k}$ are algebraic functions of all the kinematical invariants except for x_i.
This implies that the master integrals can be expressed in terms of Goncharov multiple polylogarithms (GPLs):

$$G(\vec{\alpha}; z) = G(\alpha_1, \cdots, \alpha_n; z) = \int_0^z \frac{dt}{t - \alpha_1} G(\alpha_2, \cdots, \alpha_n; t) \, ,$$

$$G(\alpha_1; z) = \int_0^z \frac{dt}{t - \alpha_1} \quad \alpha_1 \neq 0 \, , \qquad G(\vec{0}_n; z) = \frac{\log^n z}{n!} \, .$$

$\vec{\alpha} = \{\alpha_1, \cdots, \alpha_n\}$ with $c_i \in \mathbb{C}$ is called weight vector, the dimension of $\vec{\alpha}$ is the weight of the GPL.

Semi-algorithmic Approach

It is possible to find the Canonical Basis for the system of differential equations by following the semi-algorithmic approach described in [Gehrmann, von Manteuffel, Tancredi, Weihs '14]:

- Step 1: we choose the powers a_i of the scalar products in order to get a differential equation:

$$df_i(x) = \left(H_{0,ij}(x) + \epsilon H_{1,ij}(x)\right) f_j(x) + \Omega_{ij}(x,\epsilon) g_j(x)$$

the first term of the equation is the homogeneous part, while the non-homogeneous part has the structure:

$$\Omega_{ij}(x,\epsilon) = \omega_{0,ij}(x) + \epsilon\omega_{1,ij}(x) + \sum_a \frac{\omega_{a,ij}(x)}{\epsilon + p_a}$$

with $p_a \in \mathbb{R}$

Semi-algorithmic Approach

- Step 2: We eliminate the term $H_{0,ij}(x)$ rotating the homogeneous part of the system by the transformation $\tilde{f}_i(\vec{x},\epsilon) = h_{0,ia}(\vec{x})\vec{f}_a(\vec{x},\epsilon)$
- $h_{0,ij}(x)$ satisfies the differential equation:

$$dh_{0,ij}(x) = -h_{0,ia}(x)H_{0,aj}(x)$$

- In the new basis the system of differential equations is:

$$d\tilde{f}_i(x) = \epsilon\tilde{H}_{1,ij}(x)\tilde{f}_j(x) + \tilde{\Omega}_{ij}(x,\epsilon)g_j(x)$$

Semi-algorithmic Approach

- Step 3: We put in canonical form the non-homogeneous part shifting the $\tilde{f}_i$ as:

$$\tilde{f}_i(x) \to \tilde{f}_i(x) + \left(\tilde{\omega}_{0,ij}(x) + \sum_a \frac{\tilde{\omega}_{a,ij}(x)}{\epsilon + p_a} \right) g_j(x)$$

- In order to remove the first and third term in $\tilde{\Omega}_{ij}$, $\tilde{\omega}_{0,ij}$ and $\tilde{\omega}_{a,ij}$ have to satisfy the differential equations:

$$d\tilde{\omega}_{0,ij} - \tilde{H}_{1,ib}\tilde{\omega}_{a,bj} + \tilde{\omega}_{a,ib}G_{a,bj} + \omega_{0,ij} = 0\,,$$

$$d\tilde{\omega}_{a,ij} + p_a\tilde{H}_{ib}\tilde{\omega}_{a,bj} - p_a\tilde{\omega}_{a,ib}G_{1,bj} + \omega_{a,ij} = 0$$

where $G_{1,ij}$ is the matrix relative to the system of differential equations for the subtopologies g_j:

$$dg_i(x) = \epsilon G_{1,ij}(x)g_j(x)$$

Master Integrals Computation

Computation settings

- Kinematics:
  incoming momenta: $p_1^2 = p_2^2 = 0$; outcoming momenta: $p_3^2 = p_4^2 = m_t^2$.
  Mandelstam variables: $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$,
  $s + t + u = 2m_t^2$
- Scalar products:

$$
\begin{aligned}
D_i \quad = \quad & \{-k_1^2,\, -k_2^2,\, -(p_1+k_1)^2\,,\, -(p_1+k_1+k_2)^2\,,\, -(k_1+p_1+p_2)^2\,,\, -(k_2+p_1+p_2)^2\,, \\
& -(k_1+k_2+p_1+p_2)^2\,,\, m_t^2-(k_1+k_2+p_3)^2\,,\, m_t^2-(k_2+p_3)^2\}\,.
\end{aligned}
$$

- Topology A:

$$
\int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \, \frac{D_4^{-a_4}\, D_6^{-a_6}}{D_1^{a_1}\, D_2^{a_2}\, D_3^{a_3}\, D_5^{a_5}\, D_7^{a_7}\, D_8^{a_8}\, D_9^{a_9}}\,,
$$

- Topology B:

$$
\int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \, \frac{D_5^{-b_5}\, D_6^{-b_6}}{D_1^{b_1}\, D_2^{b_2}\, D_3^{b_3}\, D_4^{b_4}\, D_7^{b_7}\, D_8^{b_8}\, D_9^{b_9}}\,.
$$

$$
\mathcal{D}^d k_i = \frac{d^d k_i}{i\pi^{\frac{d}{2}}} e^{\epsilon\gamma_E} \left(\frac{m_t^2}{\mu^2}\right)^{\epsilon}
$$

a_i and b_i are integer numbers with $a_4, a_6, b_5, b_6 \leqslant 0$.

Roots Linearization

The differential equations system depends on the following two roots of the kinematical invariants:

$$\sqrt{s(s-4m_t^2)}, \quad \sqrt{m_t^2 s(m_t^2-t)(s+t-m_t^2)},$$

It is possible to linearize them by means of the change of variables:

$$\begin{aligned} s &= -m_t^2 \frac{(1-w)^2}{w}, \\ t &= -m_t^2 \frac{1-w+w^2-z^2}{z^2-w}. \end{aligned}$$

- Non-physical region: $s, t < 0 \Rightarrow 0 < w < 1, \ \sqrt{w} < z < \sqrt{1-w+w^2}$.
- Physical region:
  $s > 4m_t^2, \ t_{min} < t < t_{max}, \ u < 0 \Rightarrow -1 < w < 0, \ 0 < -w < z < 1.$

$$\begin{aligned} t_{min} &= m_t^2 - \frac{s}{2} - \frac{1}{2}\sqrt{s(s-4m_t^2)}, \\ t_{max} &= m_t^2 - \frac{s}{2} + \frac{1}{2}\sqrt{s(s-4m_t^2)}. \end{aligned}$$

Integration

Canonical Basis for the system of differential equations + No roots of kinematical invariants $\Rightarrow$ the solution can be integrated order-by-order in the ϵ-expasion in terms of GPLs.

- Alphabet: the matrix $\tilde{A}(\vec{x}) = \sum_k \tilde{A}^{(k)} \log(l_k)$ which defines the differential equations is characterized by the alphabet

$$\{l_k\} = \Big\{w, w-1, w+1, z, z-1, z+1, w-z, w+z, w-z^2, w^2-w+1-z^2, \\ w^2 - z^2(w^2-w+1), w^2-3w+z^2+1\Big\}.$$

- w-argument weights:

$$\Big\{0, 1, -1, z, -z, z^2, \frac{1-\sqrt{4z^2-3}}{2}, \frac{1+\sqrt{4z^2-3}}{2}, \frac{z(z-\sqrt{4-3z^2})}{2(z^2-1)}, \\ \frac{z(z+\sqrt{4-3z^2})}{2(z^2-1)}, \frac{3-\sqrt{5-4z^2}}{2}, \frac{3+\sqrt{5-4z^2}}{2}\Big\},$$

- z-argument weights:

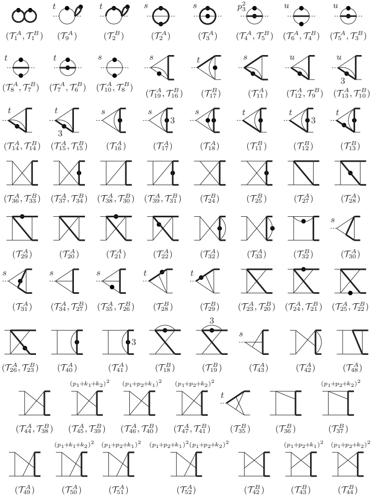
$$\{0, -1, 1, -i, i\}\,.$$

Numerical checks

The solution for the master integrals has been succesfully tested numerically both on the physical and non-physical regions.

- Numerical checks are performed evaluating numerically pre-canonical master integrals by means of public softwares SecDec [Borowka, Heinrich, Jones, Kerner, Schlenk, Zirke '15] and Fiesta [Smirnov, Tentyukov '14].
- Numerical checks for six and seven denominators integrals are made writing canonical MIs in terms of quasi-finite integrals [Panzer '14], [von Manteuffel, Panzer, Schabinger '15]. The numerical evaluation of the latter is more precise and allows for a better comparison with the analytic solution.
- All MIs are numerically cross-checked against the analytic solution obtained by Di Vita, Gehrmann, Laporta, Mastrolia, Primo, Schubert '19.

Top-Antitop production



- IBPs: We solved IBPs using FIRE [Smirnov] and Reduze2 [von Manteuffel, Studerus] we find 52 master integrals to be solved for topology A, and 44 for top B.
- Canonical Basis: We find the canonical basis for the system of differential equations.
- Roots Linearization: We were able to linearize all the roots of the kinematical invariants.
- All the master integrals are written in terms of GPLs.
- Numerical check of the solutions, both in the physical and non-physical regions.

Conclusions and outlooks

Amplitude

The computation of the MIs presented in this talk is the final ingredient needed to obtain an analytic expression for the two-loop corrections to the amplitude in the quark-annihilation channel for the $t\bar{t}$ production.

- Full square amplitude for the color coefficients $B^{q\bar{q}}$ and $C^{q\bar{q}}$ written in terms of canonical MIs;
- Optimization of the analytic expression: rewrite the whole result in terms of Li_n;
- UV Renormalization;
- Numerical check of the IR poles;
- Numerical check against numerical calculations [Czakon '08].

Conclusions

- We evaluated analytically the last unknown MIs needed for the two-loop corrections to the amplitude in the quark-annihilation channel for the $t\bar{t}$ production process.
- The analytic expression is obtained by means of the Differential equations method.
- We found the Canonical basis for the differential equations system, we linearized the square roots of kinematical invariants, and we express the solution in terms of GPLs up to weight 4.
- The solution is been tested numerically against SecDec and Fiesta, and it is been cross-checked with Di Vita, Gehrmann, Laporta, Mastrolia, Primo, Schubert '19.