# Threshold resummation at next-to-leading power

Robert Szafron

Technische Universität München

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# Outline

- Hard function
- Kinematic corrections
- Soft function
- Fixed order expansion
- Higgs threshold production

$$\frac{d\sigma}{dz} = \sum_{n=0}^{\infty} \alpha_s^n \left[ c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left( c_{nm} \left[ \frac{\ln^m(1-z)}{1-z} \right]_+ + d_{nm} \ln^m(1-z) \right) + \dots \right]$$

- Leading power ○
- Next-to-leading power ○
- Leading Log: $m = 2n - 1 \longrightarrow \alpha_s \ln(1-z) + \alpha_s^2 \ln^3(1-z) + \dots$

# The Drell-Yan process - the leading power threshold factorization

$$A(p_A)B(p_B) \to \gamma^*(Q) + X$$

$$\frac{d\sigma_{\mathrm{DY}}}{dQ^2} = \frac{4\pi\alpha_{\mathrm{em}}^2}{3N_c Q^4} \sum_{a,b} \int_0^1 dx_a dx_b\, f_{a/A}(x_a) f_{b/B}(x_b)\, \hat{\sigma}_{ab}^{\mathrm{LP}}(z)$$

[G. P. Korchemsky, G. Marchesini, 1993]

[T. Becher, M. Neubert, G. Xu, 0710.0680; S. Moch, A. Vogt, hep-ph/0508265]

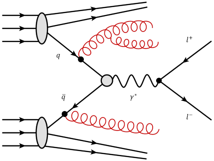
Factorization of partonic cross-section

$$\hat{\sigma}^{\,\mathrm{LP}}(z) = |C(Q^2)|^2\; Q\; S_{\mathrm{DY}}(Q(1-z))$$

$z = Q^2/\hat{s}$ partonic threshold $z \to 1$

Leading Power Soft function

$$S_{\mathrm{DY}}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega/2}\, \frac{1}{N_c}\, \mathrm{Tr}\, \langle 0|\bar{\mathbf{T}}(Y_+^\dagger(x^0)Y_-(x^0))\, \mathbf{T}(Y_-^\dagger(0)Y_+(0))|0\rangle$$



# Next-to-leading power

SCET offers systematic and intuitive approach to power expansion.

Sources of power suppression:

- Subleading operators →*see talk by Martin* [M. Beneke, M. Garny, R. S. and J. Wang, 1712.04416; M. Beneke, M. Garny, R. S. and J. Wang, 1808.04742 ]
- Time-ordered products of subleading Lagrangian (see also [M. Beneke, M. Garny, R. S. and J. Wang, 1907.05463 ] ); which are factorized into
    - collinear functions → *see talk by Sebastian*
    - generalized soft function – this talk
- Phase space expansion – kinematic corrections –this talk

We work in position space SCET [M. Beneke and T. Feldmann, hep-ph/0211358]

$$\mathcal{L}_i^{(0)} = \bar{\xi}_i \left[ i n_{i-} D + i \not{D}_{\perp i} \frac{1}{i n_{i+} D} i \not{D}_{\perp i} \right] \frac{\not{n}_{i+}}{2} \xi_i$$

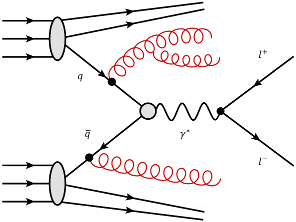
$$\mathcal{L} = \sum_i \left[ \mathcal{L}_i^{(0)} + \mathcal{L}_i^{(1)} + \ldots \right]$$

For example

$$\mathcal{L}_{i\xi q}^{(1)} = \overline{q}_s(x_{i-})\, W_i^\dagger(x)\, i\not{D}_{\perp i}\, \xi_i(x) + \text{h.c.}$$

# DY cross-section at NLP

[M. Beneke, A. Broggio, M. Garny, S. Jaśkiewicz, R. S., L. Vernazza, J. Wang, 1809.10631]



$$A(p_A)B(p_B) \to \gamma^*(Q) + X$$

$z = \frac{Q^2}{\hat{s}}$ threshold $z \to 1$

$$\Omega \sim Q(1-z) \ll Q$$

Factorization theorem valid at *LL accuracy*

$$\begin{aligned}\hat{\sigma}(z) \quad = \quad & H(\hat{s}) \times Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x} \\ & \times \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J^{(O)}_{2\xi}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}\end{aligned}$$

Scales:

- hard $\mu_h \sim Q$
- collinear $\mu_c \sim \sqrt{Q\Omega}$ (no LL collinear contribution at NLP)
- soft $\mu_s \sim \Omega$

# Factorization of time-ordered products at NLP

We separate the Lagrangian insertions into collinear and soft parts

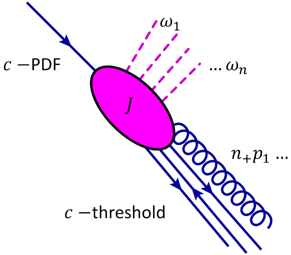
$$\mathcal{L}^{(n)}(z) = \mathcal{L}_c^{(n)}(z) \otimes \mathcal{L}_s^{(n)}(z_-)$$

- Soft fields are multipole expanded – convolution variable is one-dimensional
- We perform Fourier transform for each $z_-$
- We gather all the collinear structures that correspond to a given soft structure, use equation of motion for the soft building blocks
- Order $\lambda^2$ amplitude is $\mathcal{A} \sim \int d^4z \left\langle X_s \left| \mathbf{T}\left\{\chi_c(tn_+), \mathcal{L}^{(2)}(z)\right\}\right| q_{\text{PDF}} \right\rangle$
- States factorize i.e. $\langle X_s| = \langle X_s|_s \, \langle 0|_c$
- $\mathcal{A} \sim \int dz_- \left\langle X_s \left| \mathcal{L}_s^{(2)}(z_-) \right| 0 \right\rangle_s \left\langle 0 \left| \mathbf{T}\left\{\chi_c, \mathcal{L}_c^{(2)}\right\}\right| q_{\text{PDF}} \right\rangle_c$

This gives an NLP collinear function

$$i\left(\int d^4z \, e^{i\,\omega\,\frac{n_+z}{2}}\right) \times \mathbf{T}\left[\chi_c(tn_+) \times \mathcal{L}_c^{(n)}(z)\right]$$

$$= J\,(t;\omega)\,\chi_c^{\text{PDF}}(tn_+)$$



Collinear function is a non-local object

*See talk by Sebastian tomorrow*

# Hard Function

$$\begin{aligned}
\hat{\sigma}(z) &= \boxed{H(\hat{s})} \\
&\times \quad Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x} \\
&\times \quad \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J^{(O)}_{2\xi}(x_a n_+ p_A;\omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}
\end{aligned}$$

# Hard function

When considering power corrections we have to be careful about kinematic factors.
Consider the hard function

$$\bar{\psi}\gamma_\mu\psi(0) = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t})\, J_\mu^{A0}(t,\bar{t}), \qquad H(\hat{s},\mu_h) = |C^{A0}(-\hat{s})|^2$$

We can obtain power corrections from expansion of $\hat{s}$ around $Q^2$

$$H(\hat{s}) = H(Q^2) + Q^2(1-z)H'(Q^2) + \ldots$$

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$$H(\hat{s}) = H(Q^2) + Q^2(1-z)H'(Q^2) + \ldots$$

This correction modifies the LP factorization

$$\hat{\sigma}(z) = H(Q^2)\, Q S_{\mathrm{DY}}(Q(1-z)) + Q^2(1-z)H'(Q^2)\, Q S_{\mathrm{DY}}(Q(1-z))$$

with

$$H(\hat{s}) = 1 + \mathcal{O}\left(\alpha_s \ln^2\left(\frac{\mu}{\mu_h}\right)\right) \quad \text{and} \quad S_{\mathrm{DY}}(\Omega) = \delta(\Omega) + \mathcal{O}\left(\alpha_s\right)$$

This contributions starts at $\alpha_s^2 \ln^2\left(\frac{\mu}{\mu_h}\right)$ so at the LL accuracy it is enough to replace $H(\hat{s})$ by $H(Q^2)$

# Kinematic corrections

$$\begin{aligned}\hat{\sigma}(z) &= H(\hat{s}) \\ &\times \boxed{Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x}} \\ &\times \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J_{2\xi}^{(O)}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}\end{aligned}$$

# Kinematic corrections I

At LP we only need the soft function at $x = x_0$ but for now consider the soft function for generic $x$

$$\widetilde{S}_0(x) = \frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\bar{\mathbf{T}}(Y_+^\dagger(x)Y_-(x))\,\mathbf{T}(Y_-^\dagger(0)Y_+(0))|0\rangle$$

Use partonic center-of-mass frame $x_a\vec{p}_A + x_b\vec{p}_B = 0$
Momentum $\vec{p}_{X_s}$ of the soft hadronic final state is balanced by the lepton-pair $\vec{q} + \vec{p}_{X_s} = 0$

$$\vec{q} \sim \lambda^2, \quad q^0 = \sqrt{\hat{s}} + \mathcal{O}(\lambda^2)$$

Energy of the soft radiation

$$[x_1 p_1 + x_2 p_2 - q]^0 = p^0_{X_s} = \sqrt{\hat{s}} - \sqrt{Q^2 + \vec{q}^{\,2}} = \frac{\Omega_*}{2} - \frac{\vec{q}^{\,2}}{2Q} + \mathcal{O}\left(\lambda^6\right)$$

with

$$\Omega_* = 2Q\frac{1-\sqrt{z}}{\sqrt{z}} = Q(1-z) + \frac{3}{4}Q(1-z)^2 + \mathcal{O}\left(\lambda^6\right)$$

# Kinematic corrections II

Expansion of the kinematic factors leads to

$$Q \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x}\, \widetilde{S}_0(x)$$

$$\to \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega_*/2} \left(1 + \frac{ix^0 \partial^2_{\vec{x}}}{2Q} + \mathcal{O}\left(\lambda^4\right)\right) \widetilde{S}_0(x^0,\vec{x})_{|\vec{x}=0}$$

$$\to S_{\mathrm{DY}}(Q(1-z)) + \frac{1}{Q} S_{K1}(Q(1-z)) + \frac{1}{Q} S_{K2}(Q(1-z)) + \mathcal{O}(\lambda^4)$$

NLP kinematic soft functions

$$\begin{aligned} S_{K1}(\Omega) &= \frac{\partial}{\partial\Omega} \partial^2_{\vec{x}} S_0\left(\Omega,\vec{x}\right)_{|\vec{x}=0} \\ S_{K2}(\Omega) &= \frac{3}{4}\,\Omega^2 \frac{\partial}{\partial\Omega} S_0(\Omega,\vec{x})_{|\vec{x}=0} \end{aligned}$$

# Kinematic corrections III

It is more convenient to introduce

$$\Delta_{ab}(z) = \frac{\hat{\sigma}_{ab}(z)}{z}$$

$\Delta^{\rm LP}_{ab}(z) = \hat{\sigma}^{\rm LP}_{ab}(z)$ but $\Delta^{\rm NLP}_{ab}(z)$ receives additional NLP correction

$$(1-z) \times \hat{\sigma}_{\rm LP}(z)$$

which leads to

$$S_{K3}(\Omega) = \Omega\, S_0(\Omega, \vec{x})_{|\vec{x}=0}$$

Factorization theorem for $\Delta(z) = \Delta_{q\bar{q}}(z)$:

$$\begin{aligned} \Delta(z) &= H(Q^2) \\ &\times Q\left\{ S_{\rm DY}(Q(1-z)) + \sum_{i=1}^{3} \frac{1}{Q} S_{Ki}(Q(1-z)) \right. \\ &\left. +2 \cdot \frac{1}{2} \int d\omega\, J^{(O)}_{2\xi}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x, \omega) + \bar{c}\text{-term} \right\} \end{aligned}$$

No further expansion in $\lambda$ is needed!

# Example: expansion of the soft function RGE I

In position space, renormalization of the LP soft function is multiplicative

$$\frac{d}{d\ln\mu}\,\widetilde{S}_0(x) = \left[2\Gamma_{\mathrm{cusp}}L - 2\gamma_W\right]\widetilde{S}_0(x)$$

$$L \equiv \ln\left(-\frac{1}{4}n_-x n_+x \mu^2 e^{2\gamma_E}\right)$$

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Expansion of the soft function, $x = (x^0, 0, 0, z)$

$$\widetilde{S}_0(x) = \widetilde{S}_0(x_0) + \ldots + \frac{1}{2}\vec{\partial}_z^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0}z^2 + \ldots$$

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Expansion of the log generates inhomogeneous term

$$L = L_0 - \frac{z^2}{(x^0)^2} + \mathcal{O}\left(\frac{z^4}{(x^0)^4}\right)$$

$$L_0 \equiv \ln\left(-\frac{1}{4}(x^0)^2\mu^2 e^{2\gamma_E}\right)$$

# Example: expansion of the soft function RGE II

Coefficient of $z^2$ gives

$$\frac{d}{d\ln\mu}\,\frac{1}{2}\vec{\partial}_z^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0} = \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\,\frac{1}{2}\vec{\partial}_z^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0} - \frac{2}{(x^0)^2}\widetilde{S}_0(x_0)$$

Define soft functions

$$\begin{aligned}\widetilde{S}_3(x_0) &= \frac{ix_0}{2}\vec{\partial}_z^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0}\\ \widetilde{S}_{x_0}(x_0) &= \frac{-2i}{x^0-i\varepsilon}\widetilde{S}(x_0)\end{aligned}$$

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Soft functions mix

$$\begin{aligned}\frac{d}{d\ln\mu}\widetilde{S}_3(x_0) &= \left[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\right]\widetilde{S}_3(x_0) + \widetilde{S}_{x_0}(x_0)\\ \frac{d}{d\ln\mu}\widetilde{S}_{x_0}(x_0) &= \left[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\right]\widetilde{S}_{x_0}(x_0)\end{aligned}$$

Note: $\widetilde{S}_3(x_0) = \mathcal{O}(\alpha_s L_0)$ and $\widetilde{S}_{x_0}(x_0) = 1 + \mathcal{O}\left(\alpha_s L_0^2\right)$

$\widetilde{S}_{x_0}(x_0)$ corresponds to $\theta$-soft function [I. Moult, I. Stewart, G. Vita, H. Xing Zhu,1804.04665 ]

# RGE for kinematic soft functions

Proceeding like in the example we obtain

$$\begin{aligned}\frac{d}{d\ln\mu}\,\vec{S}(x^0) &= \Big[2\Gamma_{\text{cusp}}L_0 - 2\gamma_W\Big]\,\mathbf{1}\,\vec{S}(x) \\ &+ \Gamma_{\text{cusp}}\begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -6 & +3 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}\vec{S}(x^0)\end{aligned}$$

with $\vec{S}(x^0) = \left(\widetilde{S}_{K1}, \widetilde{S}_{K2}, \widetilde{S}_{K3}, \widetilde{S}_{x_0}\right)^T$

$$\frac{d}{d\ln\mu}\,\widetilde{S}_{K1+K2+K3}(x^0) = \Big[2\Gamma_{\text{cusp}}L_0 - 2\gamma_W\Big]\,\widetilde{S}_{K1+K2+K3}(x^0) - 6\,\Gamma_{\text{cusp}}\,\widetilde{S}_{K3}(x^0)$$

Note: $\widetilde{S}_{K1+K2+K3}(x^0) = \mathcal{O}(\alpha_s)$

*No LL kinematic corrections to all orders!*

# Soft function

$$\begin{aligned}\hat{\sigma}(z) &= H(\hat{s}) \\ &\times \; Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\, \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x} \\ &\times \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J_{2\xi}^{(O)}(x_a n_+ p_A;\omega)\, \boxed{\widetilde{S}_{2\xi}(x,\omega)} + \bar{c}\text{-term} \right\}\end{aligned}$$

# NLP time-ordered products

Soft operator in position space

$$\widetilde{\mathcal{S}}_{2\xi}\left(x, z_{-}\right) = \bar{\mathbf{T}}\left[Y_{+}^{\dagger}(x) Y_{-}(x)\right] \mathbf{T}\left[Y_{-}^{\dagger}(0) Y_{+}(0) \frac{i\partial_{\perp}^{\nu}}{in_{-}\partial} \mathcal{B}_{\perp\nu}^{+}(z_{-})\right]$$

with decoupled soft fields

$$\mathcal{B}_{\pm}^{\mu} = Y_{\pm}^{\dagger}\left[iD_{s}^{\mu} Y_{\pm}\right]$$

Lagrangian is already multipole expanded → soft fields depend only on $z_{-}$

$$\mathcal{L}_{2\xi}^{(2)} = \frac{1}{2} \bar{\chi}_{c} z_{\perp}^{\mu} z_{\perp}^{\nu}\left[i\partial_{\nu} in_{-}\partial \mathcal{B}_{\mu}^{+}\right] \frac{\not{n}_{+}}{2} \chi_{c}$$

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In the factorization theorem we need only vacuum matrix element

$$S_{2\xi}(\Omega,\omega) = \int \frac{dx^0}{4\pi}\int\frac{d(n_+z)}{4\pi}\,e^{ix^0\Omega/2-i\omega(n_+z)/2}\,\frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\widetilde{\mathcal{S}}_{2\xi}(x^0,z_-)|0\rangle$$

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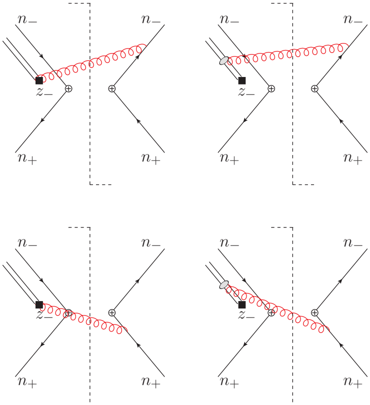

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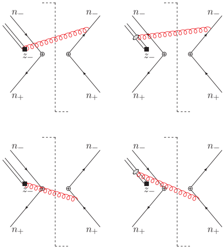


$$S_{2\xi}(\Omega,\omega) = \frac{\alpha_s C_F}{2\pi}\left\{\theta(\Omega)\delta(\omega)\left(-\frac{1}{\epsilon}+\ln\frac{\Omega^2}{\mu^2}\right)+\left[\frac{1}{\omega}\right]_+\theta(\omega)\theta(\Omega-\omega)\right\}$$

# Soft function renormalization

We assume that renormalization in the momentum space is a convolution in $\Omega$ and $\omega$

$$\begin{aligned} S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} \quad = \quad & \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} \\ & + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}} \end{aligned}$$

Renormalization through mixing with the same $S_{x_0}$ as in the case of kinematic corrections

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

How to determine $Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega')$ at one loop?

# Soft operator

Let us consider an operator rather than its matrix element

$$\begin{aligned}\mathcal{S}_{2\xi}\left(\Omega,\omega\right) &= \int\frac{dx^0}{4\pi}\int\frac{d\left(n_+z\right)}{4\pi}e^{i\left(x^0\Omega-n_+z\omega\right)/2}\,\overline{\mathbf{T}}\left[Y_+^\dagger\left(x_0\right)Y_-\left(x_0\right)\right]\\ &\quad\times\mathbf{T}\left[Y_-^\dagger\left(0\right)Y_+\left(0\right)\frac{i\partial_{\perp\mu}}{in_-\partial}\mathcal{B}_+^\mu\left(z_-\right)\right]\end{aligned}$$

Generalize renormalization equation to

$$\left[\mathcal{S}_A\left(\Omega,\omega_i\right)\right]_{\text{ren}} = \sum_B\int d\Omega' d\omega_j'\,\mathcal{Z}_{AB}\left(\Omega,\omega_i;\Omega',\omega_j'\right)\left[\mathcal{S}_B\left(\Omega',\omega_j'\right)\right]_{\text{bare}}$$

$$Z_{2\xi\,2\xi} = \frac{1}{N_c}\sum_{a,c}\left(\mathcal{Z}_{2\xi\,2\xi}\right)_{aa,cc}$$

For the leading $1/\epsilon^2$ pole we find that

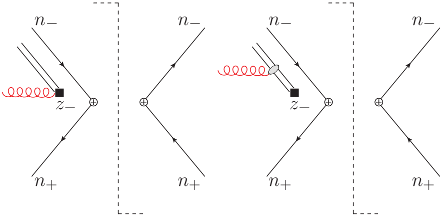
$$\left(\mathcal{Z}_{2\xi\,2\xi}\right)_{ab,cd}\equiv\delta_{ac}\delta_{bd}\mathcal{Z}_{2\xi\,2\xi}+\mathcal{O}(\epsilon^{-1})$$

hence

$$Z_{2\xi\,2\xi}=\mathcal{Z}_{2\xi\,2\xi}+\mathcal{O}(\epsilon^{-1})$$

# Soft matrix elements

Problem of finding Z-factor reduced to operator renormalization

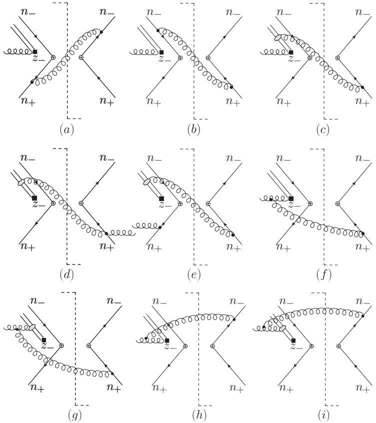


Tree level matrix element is not zero

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}} = g_s T^A \left(\frac{p_\perp \cdot \epsilon^*_\perp}{n_-p} - \frac{p_\perp^2 n_-\epsilon^*}{(n_-p)^2}\right)\delta(\Omega)\delta(\omega - n_-p).$$

Dependence on the external momentum allows to determine full dependence on $\omega'$

# One loop “real” diagrams

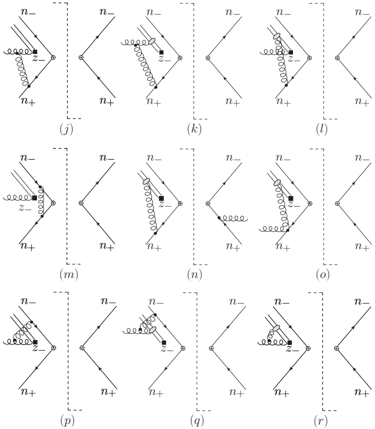


$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{a)}_{\text{1-loop}} = \left[\frac{\alpha_s}{2\pi}\frac{C_F}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{b)}_{\text{1-loop}} = \left[\frac{\alpha_s}{2\pi}\frac{C_F}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{c)}_{\text{1-loop}} = \left[-\frac{\alpha_s}{4\pi}\frac{C_A}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

# One loop "virtual" diagrams



$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{j)+k)}_{\text{1-loop}} =$$

$$\left[\frac{\alpha_s}{4\pi}\,\frac{C_A}{\epsilon^2}+\mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

# Diagonal part of the anomalous dimension

We find the sum of virtual and real contribution to give a result exactly equal to the corresponding cusp anomalous dimension of the leading power soft function

$$Z^{(1)}_{2\xi\,2\xi}\left(\Omega,\omega;\Omega',\omega'\right)=-\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon^2}\delta\left(\Omega-\Omega'\right)\delta\left(\omega-\omega'\right)$$

$$\Gamma_{2\xi\,2\xi}\left(\Omega,\omega;\Omega',\omega'\right)=4\frac{\alpha_s C_F}{\pi}\ln\frac{\mu}{\mu_s}\delta\left(\Omega-\Omega'\right)\delta\left(\omega-\omega'\right)$$

- $C_A$ part cancels!
- leading pole is diagonal in color indices
- result is proportional to the tree level but the dependence on $\Omega'$ must be extrapolated from the LP result

# LL soft function RGE

We checked our result by explicit two-loop computation of the soft function. Both methods lead to the same AD matrix → non-trivial check of

- the choice of $S_{x_0}$
- the correctness of our procedure to extract leading poles
- the relation between soft operator and soft function renormalization

At the LL we have

$$\frac{d}{d\ln\mu}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega) \end{pmatrix} = \frac{\alpha_s}{\pi}\begin{pmatrix} 4C_F\ln\frac{\mu}{\mu_s} & -C_F\delta(\omega) \\ 0 & 4C_F\ln\frac{\mu}{\mu_s} \end{pmatrix}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega) \end{pmatrix}$$

with a solution

$$\begin{aligned} S_{2\xi}^{\mathrm{LL}}(\Omega,\omega,\mu) &= \frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\exp\left[-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\theta(\Omega)\delta(\omega) \\ &= C_F\frac{\alpha_s}{\pi}\ln\frac{\mu_s}{\mu}\exp\left[-2C_F\frac{\alpha_s}{\pi}\ln^2\frac{\mu_s}{\mu}\right]\theta(\Omega)\delta(\omega) \end{aligned}$$

# LL resummation

The resummed collinear function does not contribute to the LL result, we only need tree level result

$$J^{\mu\rho}_{2\xi;\alpha\beta,abde}(n_+p,n_+p';\omega)=-\frac{g_\perp^{\mu\rho}}{n_+p}\delta(n_+p-n_+p')\delta_{\alpha\beta}\delta_{ad}\delta_{eb}\;+\mathcal{O}\left(\alpha_s\ln\left(\frac{\mu}{\mu_c}\right)\right)$$

The resummed cross-section is

$$\begin{aligned}\Delta^{\mathrm{LL}}(z) &= \Delta^{\mathrm{LL}}_{\mathrm{LP}}(z)\\ &-\exp\left[4S^{\mathrm{LL}}(\mu_h,\mu)-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\times\frac{8C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\,\theta(1-z)\end{aligned}$$

where at LL accuracy

$$S^{\mathrm{LL}}(\mu_1,\mu_2)=-\frac{\alpha_sC_F}{2\pi}\ln^2\frac{\mu_2}{\mu_1}\quad\text{and}\quad\frac{1}{\beta_0}\ln\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)}=\frac{\alpha_s}{2\pi}\ln\frac{\mu_2}{\mu_1}$$

# Fixed order expanded result

- R. Hamberg, W. L. van Neerven and T. Matsuura, 1991
- D. de Florian, J. Mazzitelli, S. Moch and A. Vogt, 2014

$$
\begin{aligned}
\Delta^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu) \quad = \quad & -\theta(1-z)\Bigg\{4C_F\frac{\alpha_s}{\pi}\Big[\ln(1-z)-L_\mu\Big] \\
& +8C_F^2\left(\frac{\alpha_s}{\pi}\right)^2\Big[\ln^3(1-z)-3L_\mu\ln^2(1-z)+2L_\mu^2\ln(1-z)\Big] \\
& +8C_F^3\left(\frac{\alpha_s}{\pi}\right)^3\Big[\ln^5(1-z)-5L_\mu\ln^4(1-z)+8L_\mu^2\ln^3(1-z)-4L_\mu^3\ln^2(1-z)\Big] \\
& +\frac{16}{3}C_F^4\left(\frac{\alpha_s}{\pi}\right)^4\Big[\ln^7(1-z)-7L_\mu\ln^6(1-z)+18L_\mu^2\ln^5(1-z)-20L_\mu^3\ln^4(1-z) \\
& \qquad\qquad +8L_\mu^4\ln^3(1-z)\Big] \\
& +\frac{8}{3}C_F^5\left(\frac{\alpha_s}{\pi}\right)^5\Big[\ln^9(1-z)-9L_\mu\ln^8(1-z)+32L_\mu^2\ln^7(1-z)-56L_\mu^3\ln^6(1-z) \\
& \qquad\qquad +48L_\mu^4\ln^5(1-z)-16L_\mu^5\ln^4(1-z)\Big]\Bigg\}+\mathcal{O}(\alpha_s^6\times(\log)^{11})\,,
\end{aligned}
$$

$L_\mu = \ln(\mu/Q)$.

# Higgs threshold production I

$$A(p_A)B(p_B) \to H(q) + X(p_X)$$

Threshold variable

$$z \equiv \frac{m_H^2}{\hat{s}}$$

$$\mathcal{L}_{\text{eff}} = \frac{\alpha_s(\mu)}{3\pi} C_t(m_t,\mu)\ \frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A\ \ln\left(1 + \frac{H}{\nu}\right)$$

# Higgs threshold production I

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LP current

$$F^A_{\mu\nu} F^{\mu\nu}_A \to 2 g^{\perp}_{\mu\nu} n_- \partial \mathcal{A}^{\nu A}_{\bar{c}\perp} n_+ \partial \mathcal{A}^{\mu A}_{c\perp}$$

The derivation of the factorization is similar like in the DY case, with Wilson lines in the adjoint representation

$$Y_\pm(x) \to \mathcal{Y}^{AB}_\pm = \mathcal{P} \exp\left\{ g_s \int_{-\infty}^{0} ds f^{ABC} n_\mp A^C_s(x + s n_\mp) \right\}$$

# Higgs threshold production II

Important differences with respect to the Drell-Yan case are:

- Derivatives in the current produce additional factor of $\hat{s}$ compared to DY case

$$S_{K3}(\Omega) = 2\,\Omega\,S_0(\Omega, \vec{x})_{|\vec{x}=0}$$

Kinematic corrections do not cancel for the Higgs production!

# Higgs threshold production II

Important differences with respect to the Drell-Yan case are:

- Derivatives in the current produce additional factor of $\hat{s}$ compared to DY case

  $$S_{K3}(\Omega) = 2\,\Omega\,S_0(\Omega,\vec{x})_{|\vec{x}=0}$$

  Kinematic corrections do not cancel for the Higgs production!
- After using EOM for soft fields, the tree-level collinear function takes form

  $$J^{ABC}_{2\xi\ \mu\rho}(n_+p, n_+p';\omega) = -2iT_R f^{ABC} g^{\perp}_{\mu\rho}\left[2 - 2n_+p'\frac{\partial}{\partial n_+p}\right]\delta(n_+p - n_+p')$$

  The derivative term does not contribute to the DY LL but it contributes to the Higgs case due a factor $\hat{s}$ in the current

# NLP Resummation for Higgs threshold production

$$\begin{aligned} S^{\mathrm{LL}}_{2\xi}(\Omega,\omega,\mu) &= \frac{2C_A}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\exp\left[-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\theta(\Omega)\delta(\omega) \\ S^{\mathrm{LL}}_{\mathrm{K}}(\Omega,\omega,\mu) &= \frac{8C_A}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\exp\left[-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\theta(\Omega)\delta(\omega) \end{aligned}$$

Convoluting $S_{2\xi}$ with tree-level $J_{2\xi}$ and adding kinematic correction we obtain the LL resummed cross-section

$$\begin{aligned} \Delta^{\mathrm{LL}}(z) &= \Delta^{\mathrm{LL}}_{\mathrm{LP}}(z) \\ &\quad -\exp\left[4S^{\mathrm{LL}}(\mu_h,\mu)-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\times\frac{8C_A}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\,\theta(1-z) \end{aligned}$$

where at LL accuracy

$$S^{\mathrm{LL}}(\mu_1,\mu_2) = -\frac{\alpha_s C_A}{2\pi}\ln^2\frac{\mu_2}{\mu_1}$$

The result has the same form as Drell-Yan with $C_F \leftrightarrow C_A$

# Summary and Conclusions

- We achieved LL factorization of NLP time-order products into "radiative jet functions" – see also talk by Sebastian
- NLP LL threshold resummation for Drell-Yan and Higgs production is completed, simple relation $C_F \leftrightarrow C_A$ holds to all orders
- Renormalization of new, generalized soft functions is understood at LL accuracy but we must better understand its renormalization properties
- Work in progress: Extension to the quark gluon channel
- Subleading power factorization using SCET has many other applications e.g.
    - QED corrections in flavor physics [M. Beneke, C. Bobeth, R.S, 1908.07011]
    - Thrust resummation in $H \to gg$ [I. Moult, I. Stewart, G. Vita, H. Xing Zhu, 1804.04665]
    - N-jettines subtraction [M. Ebert, I. Moult, I. Stewart, F. Tackmann, G. Vita, H. Xing Zhu, 1807.10764]

# Auxiliary slide: Hard function running

Well known RGE for two-jet operator

$$\frac{d}{d\ln\mu}H(Q^2,\mu)=\left(2\Gamma_{\rm cusp}\ln\frac{Q^2}{\mu^2}+2\gamma\right)H(Q^2,\mu)$$

$$\Gamma_{\rm cusp}=\frac{\alpha_s}{\pi}C_F+\mathcal{O}(\alpha_s^2),\qquad \gamma=-\frac{3}{2}\frac{\alpha_s}{\pi}C_F+\mathcal{O}(\alpha_s^2),$$

The general solution RGE reads

$$H(Q^2,\mu)=\exp\left[4S(\mu_h,\mu)-2a_\gamma(\mu_h,\mu)\right]\left(\frac{Q^2}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu)}H(Q^2,\mu_h)$$

where

$$S(\nu,\mu)=-\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)}d\alpha\,\frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)}\int\limits_{\alpha_s(\nu)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')},$$

$$a_\Gamma(\nu,\mu)=-\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)}d\alpha\,\frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)},\qquad a_\gamma(\nu,\mu)=-\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)}d\alpha\,\frac{\gamma(\alpha)}{\beta(\alpha)}$$

# Auxiliary slide: Soft function in position space

At the one-loop order in dimensional regularization with $d = 4 - 2\epsilon$, the bare soft function must have a simple dependence

$$\tilde{S}_{0,\text{bare}}(x) = 1 + \frac{\alpha_s}{\pi}\left(-n_- x n_+ x \mu^2\right)^{\epsilon} f\left(\epsilon, \frac{x^2}{n_+ x n_- x}\right)$$

Explicit evaluation gives

$$\begin{aligned}
\widetilde{S}_{0,\text{bare}}(x) &= 1 + \frac{\alpha_s C_F}{\pi}\,\frac{\Gamma(1-\epsilon)}{\epsilon^2} e^{-\epsilon\gamma_E} \\
&\quad \times \left(-\frac{1}{4} n_- x n_+ x \mu^2 e^{2\gamma_E}\right)^{\epsilon} \left(\frac{x^2}{n_- x n_+ x}\right)^{1+\epsilon} {}_2F_1\left(1, 1, 1-\epsilon; 1 - \frac{x^2}{n_- x n_+ x}\right) \\
&= 1 + \frac{\alpha_s C_F}{\pi}\left(\frac{1}{\epsilon^2} + \frac{L}{\epsilon} + \frac{L^2}{2} + \frac{\pi^2}{12} + \text{Li}_2\left(1 - \frac{x^2}{n_- x n_+ x}\right) + \mathcal{O}(\epsilon)\right)
\end{aligned}$$

where we defined

$$L \equiv \ln\left(-\frac{1}{4} n_- x n_+ x \mu^2 e^{2\gamma_E}\right).$$

# Auxiliary slide: Kinematic soft functions at $\mathcal{O}(\alpha_s)$

Expanding the kinematic factors in the factorization formula we obtain further corrections related to the LP soft function

$$\begin{aligned} S_{K1}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(\ \frac{1}{\epsilon}+2\ln\frac{\mu}{\Omega}-2\right)\theta(\Omega) \\ S_{K2}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(\ \frac{3}{\epsilon}+6\ln\frac{\mu}{\Omega}+6\right)\theta(\Omega) \\ S_{K3}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(-\frac{4}{\epsilon}-8\ln\frac{\mu}{\Omega}\right)\theta(\Omega) \end{aligned}$$

$$\sum_{i=1}^{3} S_{Ki}(\Omega) = 2\,\frac{\alpha_s C_F}{\pi}\,\theta(\Omega)$$

*At $\mathcal{O}(\alpha_s)$ no LL kinematic corrections!*

# Auxiliary slide: Soft function renormalization

We assume that renormalization in the momentum space is a convolution in $\Omega$ and $\omega$

$$\begin{aligned} S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} \quad &= \quad \int d\Omega' \int d\omega' \, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} \\ &\quad + \int d\Omega' \, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}} \end{aligned}$$

Renormalization through mixing

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

# Auxiliary slide: Soft function renormalization

We assume that renormalization in the momentum space is a convolution in $\Omega$ and $\omega$

$$\begin{aligned} S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} &= \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} \\ &\quad + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}} \end{aligned}$$

Aside:

Is the convolution assumption too strong?

- Dependence of Z on $\Omega'$ cannot be uniquely determined - at LP we determine it from the known properties of Wilson loop renormalization in position space – multiplicative renormalization in position space
- Dependence on $\omega'$ can by determined under additional assumptions

# Auxiliary slide: Soft function renormalization

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$$\begin{aligned} S_{2\xi}(\Omega,\omega)_{|\text{ren}} \quad &= \quad \int d\Omega' \int d\omega' \, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega') \, S_{2\xi}(\Omega',\omega')_{|\text{bare}} \\ &\quad + \int d\Omega' \, Z_{2\xi,x_0}(\Omega,\omega;\Omega') \, S_{x_0}(\Omega')_{|\text{bare}} \end{aligned}$$

Renormalization through mixing

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

How to determine $\mathcal{O}\left(\alpha_s\right)$ of the diagonal Z-factor?

# Auxiliary slide: Alternative approach without operator renormalization

Renormalization condition for the two-loop soft function $S_{2\xi}^{(2)}$

$$\begin{aligned} S_{2\xi}^{(2)} + Z_{2\xi\,x_0}^{(1)} S_{x_0}^{(1)} + Z_{2\xi\,x_0}^{(2)} S_{x_0}^{(0)} + Z_{2\xi\,2\xi}^{(1)} S_{2\xi}^{(1)} &= \text{finite} \\ S_{x_0}^{(1)} + Z_{x_0\,x_0}^{(1)} S_{x_0}^{(0)} &= \text{finite} \\ S_{2\xi}^{(1)} + Z_{2\xi\,x_0}^{(1)} S_{x_0}^{(0)} &= \text{finite} \end{aligned}$$

Following structure

$$\Gamma = \alpha_s(\mu) \begin{pmatrix} \Gamma_{AA} \ln \frac{\mu}{\mu_s} + \gamma_{AA} & \gamma_{AB} \\ \gamma_{BA} & \Gamma_{BB} \ln \frac{\mu}{\mu_s} + \gamma_{BB} \end{pmatrix}$$

implies

$$Z_{AB}^{(2)} = \frac{1}{4} Z_{AB}^{(1)} \left( Z_{AA}^{(1)} + 3 Z_{BB}^{(1)} \right) + \mathcal{O}\left(\frac{1}{\epsilon^2}\right) \quad A \neq B.$$

$$S_{2\xi}^{(2)} - \frac{1}{4} Z_{2\xi\,x_0}^{(1)} \left( 3 Z_{2\xi\,2\xi}^{(1)} + Z_{x_0\,x_0}^{(1)} \right) S_{x_0}^{(0)} = \mathcal{O}\left(\frac{1}{\epsilon^2}\right)$$

Two loop result agrees with one-loop operator renormalization

# Auxiliary slide: Fixed order check

For arbitrary $\mu$ we then find

$$\Delta^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu) \quad = \quad \exp\left[4S^{\mathrm{LL}}(\mu_h,\mu) - 4S^{\mathrm{LL}}(\mu_s,\mu)\right] \times \frac{-8C_F}{\beta_0} \ln \frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\, \theta(1-z)$$

Note $\Delta^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu_c)$ has the same form → *no LL in collinear function!*

$$S^{\mathrm{LL}}(\mu_1,\mu_2) = -\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{\mu_2}{\mu_1} \quad \text{and} \quad \frac{1}{\beta_0} \ln \frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)} = \frac{\alpha_s}{2\pi} \ln \frac{\mu_2}{\mu_1}$$

Our result

$$\Delta^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu) = \frac{\hat{\sigma}^{\mathrm{LL}}_{\mathrm{NLP}}(z,\mu)}{z} \quad = \quad \exp\left[2\frac{\alpha_s C_F}{\pi} \ln^2 \frac{\mu}{\mu_s} - 2\frac{\alpha_s C_F}{\pi} \ln^2 \frac{\mu}{\mu_h}\right] \times(-4)\frac{\alpha_s C_F}{\pi} \ln \frac{\mu_s}{\mu}\, \theta(1-z)$$

agrees with

- R. Hamberg, W. L. van Neerven and T. Matsuura, 1991, full fixed order NNLO computation
- D. de Florian, J. Mazzitelli, S. Moch and A. Vogt, 2014 approximate results for $\mu = \mu_h$ up to $N^4LO$

# Auxiliary slide: RGE for kinematic soft functions – Higgs case

Proceeding like in the example we obtain

$$\begin{aligned}\frac{d}{d\ln\mu}\,\vec{S}(x^0) &= \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\,\mathbf{1}\,\vec{S}(x) \\ &+ \Gamma_{\rm cusp}\begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -6 & +3 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}\vec{S}(x^0)\end{aligned}$$

with $\vec{S}(x^0) = \left(\widetilde{S}_{K1}, \widetilde{S}_{K2}, \widetilde{S}_{K3}, \widetilde{S}_{x_0}\right)^T$

$$\begin{aligned}\frac{d}{d\ln\mu}\,\widetilde{S}_{K1+K2+K3}(x^0) =& \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\,\widetilde{S}_{K1+K2+K3}(x^0) \\ &-4\Gamma_{\rm cusp}\widetilde{S}_{x_0}(x^0) - 6\,\Gamma_{\rm cusp}\,\widetilde{S}_{K3}(x^0)\,,\end{aligned}$$

Note: $\widetilde{S}_{K1+K2+K3}(x^0) = \mathcal{O}\left(\alpha_s \ln\frac{\mu}{\mu_s}\right)$