The Gravitational Potential of Two Point Masses at Five Loops

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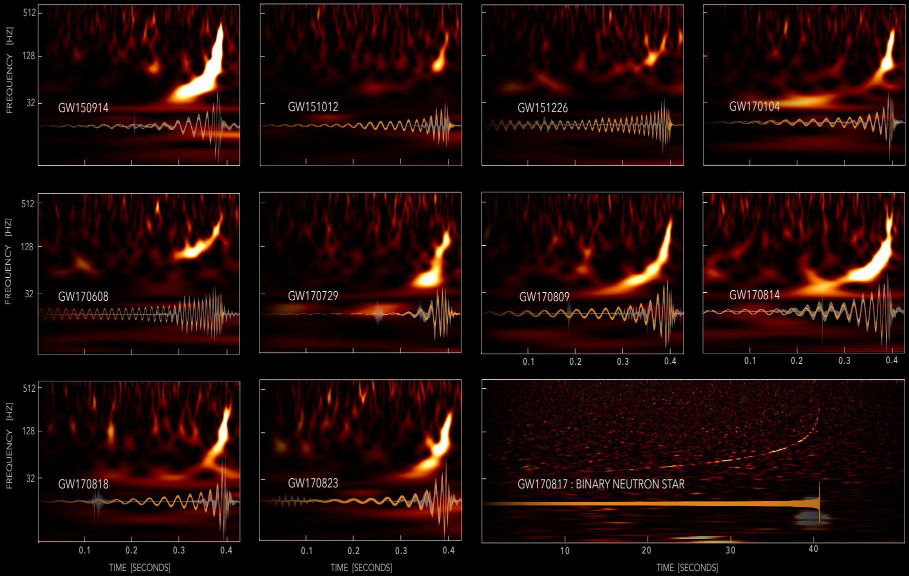
Avignon, 11 September 2019

J. Blümlein, A. Maier, P. Marquard, arXiv:1902.11180

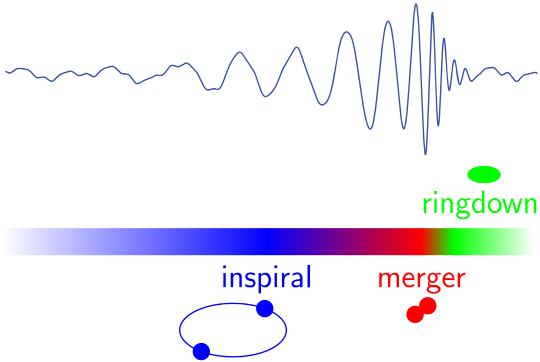
Gravitational-Wave Transient Catalog-1



LIGO-VIRGO DATA: HTTPS://DOI.ORG/10.7935/82H3-HH23 WAVELET (UNMODELED) EINSTEIN'S THEORY S. GHONGE, K. JANI | GEORGIA TECH

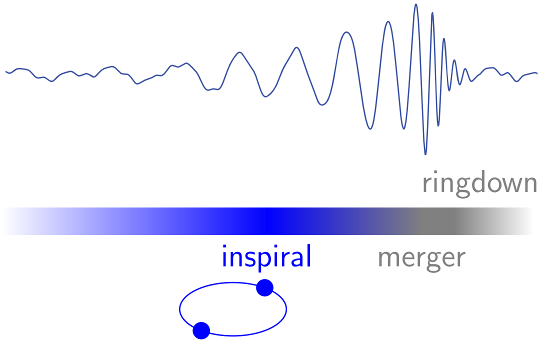
Gravitational waves

[LIGO Scientific Collaboration and Virgo Collaboration 2016]



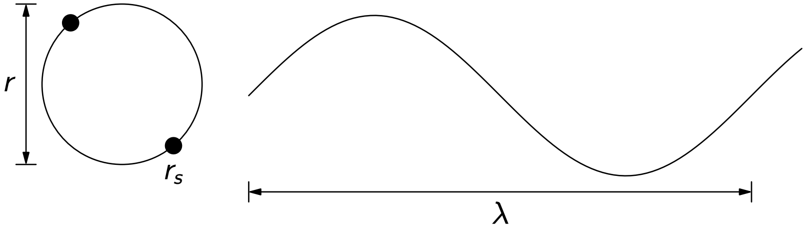
Gravitational waves

[LIGO Scientific Collaboration and Virgo Collaboration 2016]



Compact binary systems

Power counting

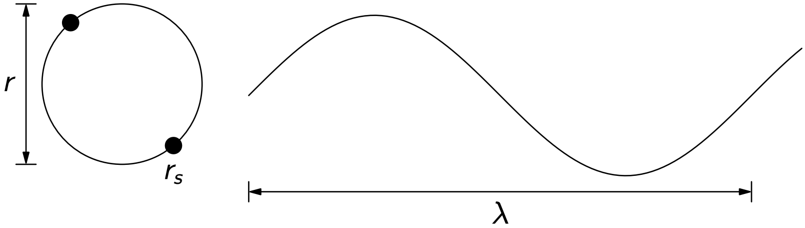


- Masses comparable: $m \equiv m_1 \sim m_2$
  Generalisation to different masses straightforward
- Nonrelativistic system: $v \ll 1$
- Virial theorem: $mv^2 \sim \frac{Gm^2}{r}$

> *Post-Newtonian (PN) expansion*:
> Combined expansion in $v \sim \sqrt{Gm/r} \ll 1$

Post-Newtonian expansion

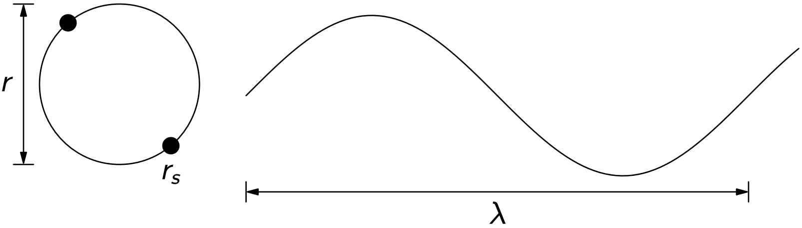
Scales



- $\omega \approx \frac{2v}{r} \Rightarrow$ $\boxed{\lambda \sim \frac{r}{v}}$
- $r_s = 2GM \Rightarrow$ $\boxed{r_s \sim r v^2}$

Post-Newtonian expansion

Scales at LIGO/VIRGO



- $10\,\text{km} \lesssim \lambda \lesssim 10\,000\,\text{km}$
- $10\,\text{km} \lesssim r_s \lesssim 100\,\text{km}$
- Inspiral: $0.1 \lesssim v \lesssim 0.5$, $100\,\text{km} \lesssim r \lesssim 1000\,\text{km}$

	black holes	neutron stars
masses	$\sim 10\text{–}50\ m_\odot$	$\sim 1 m_\odot$
radiated energy	$\sim 1\text{–}5\ m_\odot$	$\geq 0.04\ m_\odot$
redshift	$\sim 0.1\text{–}0.5$	~ 0.01

[LIGO Scientific Collaboration and the Virgo Collaboration, O1/O2 Catalog, 2018]

Post-Newtonian expansion

Theory status

Conservative dynamics, no spin: complete results up to 4PN (v^8)

- ADM Hamiltonian formalism [Damour, Jaranowski, Schäfer 2016]
- Fokker Lagrangian in harmonic coordinates

  [Bernard, Blanchet, Bohé, Faye, Marchant, Marsat 2017]
- Non-relativistic effective field theory

  [Foffa, Mastrolia, Porto, Rothstein, Sturani, Sturm 2017–2019]

Method in this talk:
non-relativistic effective field theory [Goldberger, Rothstein 2004]

General Relativity

- Here: point-like objects
  - No spin
  - No finite-size effects (neutron stars: 5PN, black holes: 6PN)
- Harmonic gauge fixing: $\partial_\mu(\sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}) = 0$
  $g = \det(g^{\mu\nu})$
- Dimensional regularisation: $d = 3 - 2\epsilon$

$$S_{\mathrm{GR}} = S_{\mathrm{EH}} + S_{\mathrm{GF}} + S_{pp}$$

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int d^{d+1}x\,\sqrt{-g}R$$

$$S_{\mathrm{GF}} = -\frac{1}{32\pi G}\int d^{d+1}x\,\sqrt{-g}\,\Gamma_\mu\Gamma^\mu$$

$$S_{\mathrm{pp}} = -\sum_i m_i\int d\tau_i = -\sum_i m_i\int dt\,\sqrt{-g_{\mu\nu}\frac{\partial x_i^\mu}{\partial t}\frac{\partial x_i^\nu}{\partial t}}$$

$$R = g^{\mu\nu}R_{\mu\nu}$$

$$\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta}$$

Non-relativistic effective theory

[Goldberger, Rothstein 2004]

Similar to non-relativistic QCD

[Caswell, Lepage 1985; Pineda, Soto 1997; Luke, Manohar, Rothstein 2000; . . .]

Full theory: General relativity		Effective theory: NRGR
$S_{\text{GR}} = S_{\text{EH}} + S_{\text{GF}} + S_{pp}$	$\longrightarrow$	$S_{\text{NRGR}} = \int dt \frac{1}{2} m_i v_i^2 + \frac{G m_1 m_2}{r} + \dots$
potential gravitons: $k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{1}{r}$		classical potentials
radiation gravitons: $k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{v}{r}$		radiation gravitons

Potential matching

Expansion of action

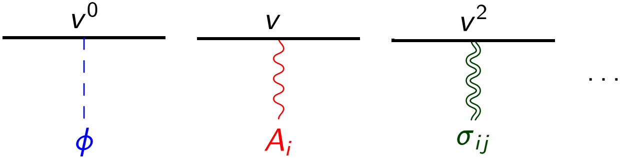
Expand S_{GR} in $v \sim \sqrt{Gm/r} \ll 1$, e.g.

$$S_{\mathrm{pp}} = -\sum_i m_i \int dt \sqrt{-g_{\mu\nu} \frac{\partial x_i^\mu}{\partial t} \frac{\partial x_i^\nu}{\partial t}} = -\sum_i m_i \int dt \sqrt{-g_{00}} + \mathcal{O}(v_i)$$

Coupling to spatial components of metric suppressed

Temporal Kaluza-Klein decomposition [Kol, Smolkin 2010]

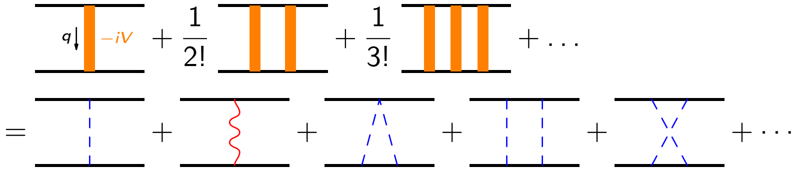
$$g^{\mu\nu} = e^{2\phi} \begin{pmatrix} -1 & A_j \\ A_i & e^{-2\frac{d-1}{d-2}\phi}(\delta_{ij} + \sigma_{ij}) - A_i A_j \end{pmatrix}$$



Potential matching

Diagrammatic expansion

Equate amplitude in effective and full theory:

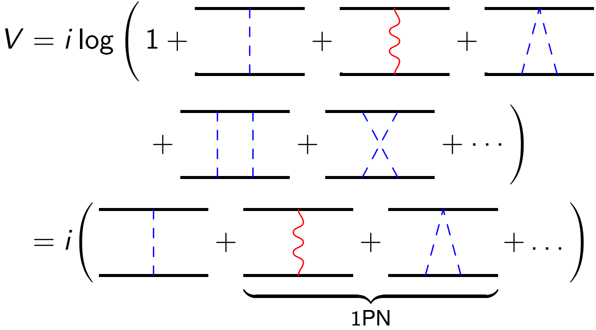


All momenta potential, $p_0 \sim \frac{v}{r} \ll p_i \sim \frac{1}{r}$
$\hookrightarrow$ expand propagators:

$$\frac{1}{\vec{p}^{\,2} - p_0^2} = \frac{1}{\vec{p}^{\,2}} + \frac{p_0^2}{\vec{p}^{\,4}} + \mathcal{O}(v^4)$$

Potential matching

Diagrammatic expansion



Potential matching

Known results

Confirmation of previous results:

- 1PN: [Goldberger, Rothstein 2004]
- 2PN: [Gilmore, Ross 2008]
- 3PN: [Foffa, Sturani 2011; Blümlein, Maier, Marquard 20XX]
- 4PN:
  - “static” contribution $v = 0$:

    [Foffa, Mastrolia, Sturani, Sturm 2016; Damour, Jaranowski 2017]
  - $v \neq 0$: [Foffa, Sturani 2019; Foffa, Porto, Rothstein, Sturani 2019]

New:

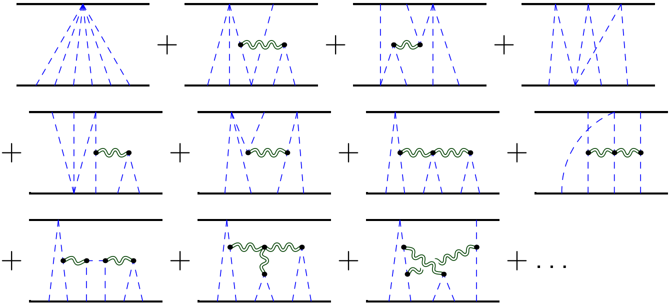
- **5PN static contribution:**

  [Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 27 Feb 2019; Blümlein, Maier, Marquard 28 Feb 2019]

Potential matching

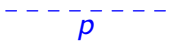
Static 5PN calculation

$$-iV^{S}_{\mathrm{5PN}} =$$

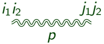


Potential matching

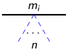
Feynman rules



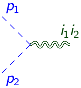
$$= -\frac{i}{2c_d\vec{p}^{\,2}}$$



$$= -\frac{i}{2\vec{p}^{\,2}}\left(\delta_{i_1j_1}\delta_{i_2j_2}+\delta_{i_1j_2}\delta_{i_2j_1}+(2-c_d)\delta_{i_1i_2}\delta_{j_1j_2}\right)$$

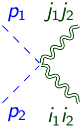


$$= -i\frac{m_i}{m_{\rm Pl}^n}$$



$$= i\frac{c_d}{2m_{\rm Pl}}(V^{i_1i_2}_{\phi\phi\sigma}+V^{i_2i_1}_{\phi\phi\sigma})$$

$$V^{i_1i_2}_{\phi\phi\sigma} = \vec{p}_1\cdot\vec{p}_2\delta^{i_1i_2}-2p_1^{i_1}p_2^{i_2}$$

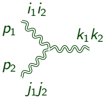


$$= i\frac{c_d}{16m_{\rm Pl}^2}(V^{i_1i_2,j_1j_2}_{\phi\phi\sigma\sigma}+V^{i_2i_1,j_1j_2}_{\phi\phi\sigma\sigma}+V^{i_1i_2,j_2j_1}_{\phi\phi\sigma\sigma}+V^{i_2i_1,j_2j_1}_{\phi\phi\sigma\sigma})$$

$$V^{i_1i_2,j_1j_2}_{\phi\phi\sigma\sigma} = \vec{p}_1\cdot\vec{p}_2(\delta^{i_1i_2}\delta^{j_1j_2}-2\delta^{i_1j_1}\delta^{i_2j_2})-2(p_1^{i_1}p_2^{i_2}\delta^{j_1j_2}+p_1^{j_1}p_2^{j_2}\delta^{i_1i_2})+8\delta^{i_1j_1}p_1^{i_2}p_2^{j_2}$$

Potential matching

Feynman rules



$$= \frac{i}{32 m_{\rm Pl}} \left(\tilde{V}^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} + \tilde{V}^{i_2 i_1, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} \right)$$

$$\tilde{V}^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} = V^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_2 j_1, k_1 k_2}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_1 j_2, k_2 k_1}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_2 j_1, k_2 k_1}_{\sigma\sigma\sigma}$$

$$\begin{aligned}
V^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} = {} & (\vec{p}_1^{\,2} + \vec{p}_1 \cdot \vec{p}_2 + \vec{p}_2^{\,2}) \Big(- \delta^{j_1 j_2} \left(2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2} \right) \\
& \quad + 2 \left[\delta^{i_1 j_1} \left(4\delta^{i_2 k_1} \delta^{j_2 k_2} - \delta^{i_2 j_2} \delta^{k_1 k_2} \right) - \delta^{i_1 i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} \right] \Big) \\
& + 2 \Big\{ 4 \left(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2} \right) \delta^{i_1 j_1} \delta^{j_2 k_1} \\
& \quad + 2 \left[\left(p_1^{i_1} + p_2^{i_1} \right) p_2^{i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} - p_1^{k_1} p_2^{k_2} \delta^{i_1 j_1} \delta^{i_2 j_2} \right] \\
& \quad + \delta^{j_1 j_2} \left[p_1^{k_1} p_2^{k_2} \delta^{i_1 i_2} + 2 \left(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2} \right) \delta^{i_1 k_1} - \left(p_1^{i_1} + p_2^{i_1} \right) p_2^{i_2} \delta^{k_1 k_2} \right] \\
& \quad + p_2^{j_2} \Big(4 p_1^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} + p_1^{j_1} \left(2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2} \right) \\
& \qquad + 2 \left[\delta^{i_1 j_1} \left(p_1^{i_2} \delta^{k_1 k_2} - 2 p_1^{k_2} \delta^{i_2 k_1} \right) - p_1^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} \right] \Big) \\
& \quad + p_1^{j_2} \Big(p_1^{j_1} \left(2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2} \right) - 4 p_2^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} \\
& \qquad + 2 \left[p_2^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} + \delta^{i_1 j_1} \left(2 p_2^{k_2} \delta^{i_2 k_1} - p_2^{i_2} \delta^{k_1 k_2} \right) \right] \Big) \Big\}
\end{aligned}$$

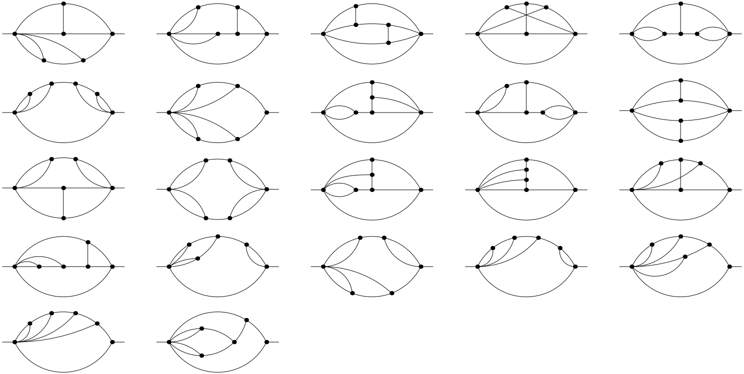
$$c_d = 2 \frac{d-1}{d-2}, \qquad m_{\rm Pl} = \sqrt{32\pi G}$$

Potential matching

Diagram families

Massless propagators:

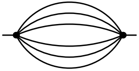
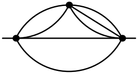
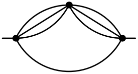
$$P_f(q) = \int \frac{d^d l_1}{\pi^{d/2}} \cdots \frac{d^d l_5}{\pi^{d/2}} \frac{\mathcal{N}(q, l_1, \ldots, l_5)}{\vec{p}_1^{\,2a_1} \cdots \vec{p}_{10}^{\,2a_{10}}}$$

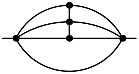


Potential matching

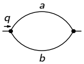
Master integrals

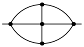
Reduction to master integrals (`crusher`): [Chetyrkin, Tkachov 1981; Laporta 2000]

$V^S_{5\text{PN}} = c_0$  $+ c_1$  $+ c_2$ 

$+ c_3$  $+ \mathcal{O}(\epsilon)$

c_j: Laurent series in $\epsilon = \frac{3-d}{2}$, polynomials in m_1, m_2, r^{-1}, G^{-1}

Master integrals factorise into 

and known 

[Lee, Mingulov 2015]

Potential matching

Result

$$V_{\mathrm{N}}^{S} = -\frac{G}{r} m_1 m_2$$

$$V_{1\mathrm{PN}}^{S} = \frac{G^2}{2r^2} m_1 m_2 (m_1 + m_2)$$

$$V_{2\mathrm{PN}}^{S} = -\frac{G^3}{r^3} m_1 m_2 \left[\frac{1}{2}(m_1^2 + m_2^2) + 3 m_1 m_2 \right]$$

$$V_{3\mathrm{PN}}^{S} = \frac{G^4}{r^4} m_1 m_2 \left[\frac{3}{8} \left(m_1^3 + m_2^3\right) + 6 m_1 m_2 (m_1 + m_2) \right]$$

$$V_{4\mathrm{PN}}^{S} = -\frac{G^5}{r^5} m_1 m_2 \left[\frac{3}{8} \left(m_1^4 + m_2^4\right) + \frac{31}{3} m_1 m_2 \left(m_1^2 + m_2^2\right) + \frac{141}{4} m_1^2 m_2^2 \right]$$

$$V_{5\mathrm{PN}}^{S} = \frac{G^6}{r^6} m_1 m_2 \left[\frac{5}{16}(m_1^5 + m_2^5) + \frac{91}{6} m_1 m_2 (m_1^3 + m_2^3) + \frac{653}{6} m_1^2 m_2^2 (m_1 + m_2) \right]$$

Potential matching

Velocity corrections

Full corrections include velocities and *higher time derivatives*:

$$\mathcal{L}_{2PN} = +\frac{G^3}{r^3} m_1 m_2 \left[\frac{1}{2}(m_1^2 + m_2^2) + 3 m_1 m_2\right]$$
$$+ G m_1 m_2 r \left[\frac{15}{8}\vec{a}_1\vec{a}_2 - \frac{1}{8}(\vec{a}_1\vec{r})(\vec{a}_2\vec{r})\right]$$
$$+ (\text{terms depending on } \vec{v}_1, \vec{v}_2)$$

Can be eliminated using

- *Total time derivatives* $\delta\mathcal{L} \propto \frac{d}{dt}F(\vec{r}, \vec{v}_1, \vec{v}_2)$
- *Equations of motion* $\delta\mathcal{L} \propto \left(\vec{a}_1 + \frac{G m_2}{r^3}\vec{r}\right)\left(\vec{a}_2 - \frac{G m_1}{r^3}\vec{r}\right)$

$$\mathcal{L}_{2PN} = -\frac{G^3}{r^3} m_1 m_2 \left[\frac{1}{4}(m_1^2 + m_2^2) + \frac{5}{4} m_1 m_2\right]$$
$$+ (\text{terms depending on } \vec{v}_1, \vec{v}_2)$$

Conclusions

- Inspiral phase of compact binary systems described well by *Post-Newtonian (PN) expansion* $v \sim \sqrt{Gm/r} \ll 1$
- Effective field theories and calculational methods from particle physics very effective for high PN orders
- Static gravitational potential now known at five loops (5PN)

Backup

Potential matching

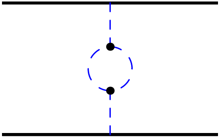
Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)



$\sim \frac{\hbar}{mvr}$

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Potential matching

Diagram selection

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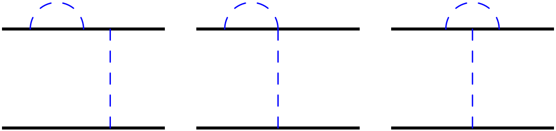

Absorbed into renormalisation of sources

- No source-reducible diagrams [Fischler 1977]

Potential matching

Diagram selection

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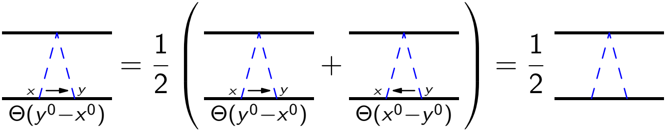
  Initially *time-ordered* diagrams:



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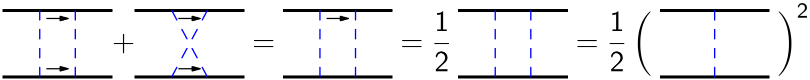

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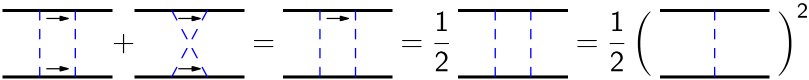
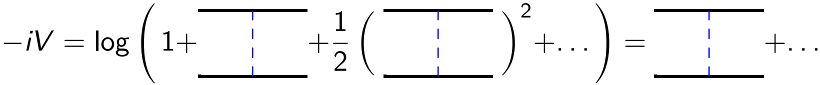
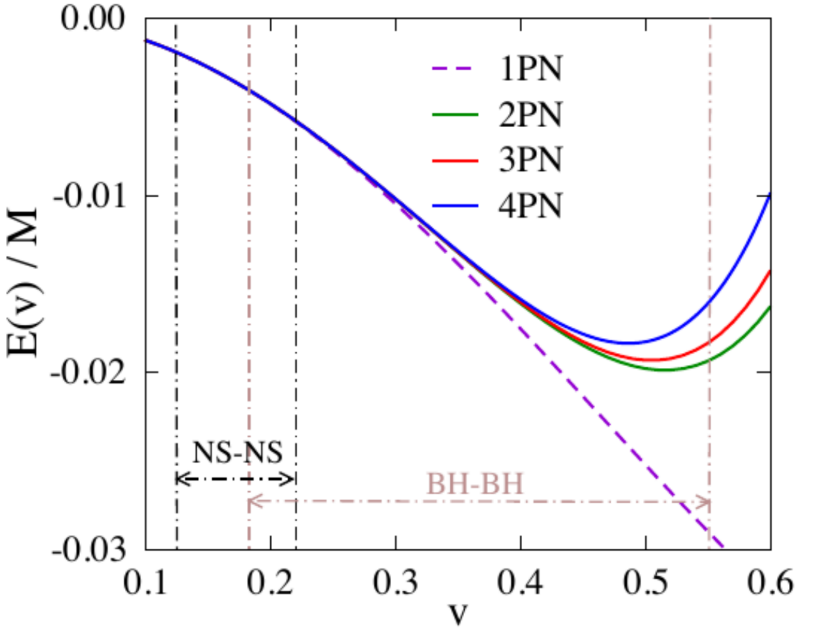




Potential matching

Number of diagrams

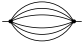
	QGRAF	source irred	no source loops	no tadpoles	sym
N	1	1	1	1	1
1PN	2	2	2	2	1
2PN	19	19	19	15	5
3PN	360	276	258	122	8
4PN	10081	5407	4685	1815	50
5PN	332020	128080	101570	27582	154

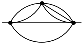


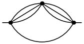
[Buonanno, QCD Meets Gravity IV, 2018]

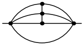
Potential matching

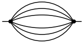
Results for master integrals

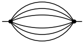
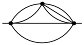
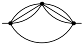
 $= e^{5\epsilon\gamma_E} \dfrac{\Gamma\left(6-\frac{5d}{2}\right)\Gamma^6\left(-1+\frac{d}{2}\right)}{\Gamma(-6+3d)}$

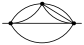
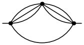
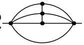
 $= e^{5\epsilon\gamma_E} \dfrac{\Gamma\left(7-\frac{5d}{2}\right)\Gamma\left(3-d\right)\Gamma\left(2-\frac{d}{2}\right)\Gamma^7\left(-1+\frac{d}{2}\right)\Gamma(5-2d)}{\Gamma\left(5-\frac{3}{2}d\right)\Gamma(-2+d)\Gamma\left(-3+\frac{3}{2}d\right)\Gamma\left(-7+3d\right)}$

 $= e^{5\epsilon\gamma_E} \dfrac{\Gamma\left(7-\frac{5d}{2}\right)\Gamma^2(3-d)\Gamma^7\left(-1+\frac{d}{2}\right)\Gamma\left(-6+\frac{5d}{2}\right)}{\Gamma\left(6-2d\right)\Gamma^2\left(-3+\frac{3d}{2}\right)\Gamma\left(-7+3d\right)}$

 $= 6\pi^{7/2}\left[\dfrac{2}{\epsilon} - 4 - 4\ln(2) - \left(48 + 8\ln(2) - 4\ln^2(2) - 105\zeta_2\right)\epsilon + \mathcal{O}(\epsilon^2)\right]$

$V^S_{5PN} \overset{\epsilon=0}{=} \dfrac{G^6}{r^6}(m_1 m_2)\pi^{-7/2}\left\{\dfrac{15}{32}(m_1^5+m_2^5)\right[$  $\left]_{\epsilon^0} + \dfrac{91}{4} m_1 m_2 (m_1^3+m_2^3)\right[$  $\left]_{\epsilon^0}\right.$

$+ m_1^2 m_2^2 (m_1+m_2)\left(\left[\dfrac{293}{4}\right.\right.$  $- \dfrac{45}{16}$  $+ \dfrac{45}{32}$  $\Big]_{\epsilon^0}$

$+ \Big[\dfrac{519}{16}$  $- \dfrac{627}{32}$  $+ 2$  $\left.\left.\Big]_{\epsilon^{-1}}\right)\right\}$