The Gravitational Potential of Two Point Masses at Five Loops

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Avignon, 11 September 2019

Gravitational waves

[LIGO Scientific Collaboration and Virgo Collaboration 2016]
Gravitational waves

[LIGO Scientific Collaboration and Virgo Collaboration 2016]
Compact binary systems

Power counting

- Masses comparable: \( m \equiv m_1 \sim m_2 \)
  Generalisation to different masses straightforward
- Nonrelativistic system: \( v \ll 1 \)
- Virial theorem: \( m v^2 \sim \frac{Gm^2}{r} \)

Post-Newtonian (PN) expansion:
Combined expansion in \( v \sim \sqrt{Gm/r} \ll 1 \)
Post-Newtonian expansion

Scales

- $\omega \approx \frac{2v}{r} \Rightarrow \lambda \sim \frac{r}{v}$
- $r_s = 2GM \Rightarrow r_s \sim rv^2$
Post-Newtonian expansion
Scales at LIGO/VIRGO

- \(10 \text{ km} \lesssim \lambda \lesssim 10000 \text{ km}\)
- \(10 \text{ km} \lesssim r_s \lesssim 100 \text{ km}\)
- Inspiral: \(0.1 \lesssim \nu \lesssim 0.5\), \(100 \text{ km} \lesssim r \lesssim 1000 \text{ km}\)

<table>
<thead>
<tr>
<th></th>
<th>black holes</th>
<th>neutron stars</th>
</tr>
</thead>
<tbody>
<tr>
<td>masses</td>
<td>(\sim 10-50 , m_{\odot})</td>
<td>(\sim 1 , m_{\odot})</td>
</tr>
<tr>
<td>radiated energy</td>
<td>(\sim 1-5 , m_{\odot})</td>
<td>(\geq 0.04 , m_{\odot})</td>
</tr>
<tr>
<td>redshift</td>
<td>(\sim 0.1-0.5)</td>
<td>(\sim 0.01)</td>
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[LIGO Scientific Collaboration and the Virgo Collaboration, O1/O2 Catalog, 2018]
Post-Newtonian expansion

Theory status

Conservative dynamics, no spin: complete results up to 4PN ($\nu^8$)

- **ADM Hamiltonian formalism** [Damour, Jaranowski, Schäfer 2016]
- **Fokker Lagrangian in harmonic coordinates**
  [Bernard, Blanchet, Bohé, Faye, Marchant, Marsat 2017]
- **Non-relativistic effective field theory**
  [Foffa, Mastrolia, Porto, Rothstein, Sturani, Sturm 2017–2019]

Method in this talk:

non-relativistic effective field theory  [Goldberger, Rothstein 2004]
General Relativity

- Here: point-like objects
  - No spin
  - No finite-size effects (neutron stars: 5PN, black holes: 6PN)
- Harmonic gauge fixing: \( \partial_\mu (\sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}) = 0 \)
  \[ g = \text{det}(g^{\mu\nu}) \]
- Dimensional regularisation: \( d = 3 - 2\epsilon \)

\[
S_{GR} = S_{EH} + S_{GF} + S_{pp}
\]
\[
S_{EH} = \frac{1}{16\pi G} \int d^{d+1}x \; \sqrt{-g} R
\]
\[
S_{GF} = - \frac{1}{32\pi G} \int d^{d+1}x \; \sqrt{-g} \; \Gamma_\mu \Gamma^\mu
\]
\[
S_{pp} = - \sum_i m_i \int d\tau_i = - \sum_i m_i \int dt \; \sqrt{-g_{\mu\nu}} \frac{\partial x^\mu_i}{\partial t} \frac{\partial x^\nu_i}{\partial t}
\]
\[
R = g^{\mu\nu} R_{\mu\nu}
\]
\[
\Gamma^\mu = g^{\alpha\beta} \Gamma^\mu_{\alpha\beta}
\]
Non-relativistic effective theory

[Goldberger, Rothstein 2004]

Similar to non-relativistic QCD


Full theory:  
General relativity

\[ S_{GR} = S_{EH} + S_{GF} + S_{pp} \]

Effective theory:  
NRGR

\[ S_{NRGR} = \int dt \left( \frac{1}{2} m_i \dot{v}_i^2 + \frac{G m_1 m_2}{r} + \ldots \right) \]

potential gravitons:  
\[ k_0 \sim \frac{v}{r}, \quad \vec{k} \sim \frac{1}{r} \]

classical potentials

radiation gravitons:  
\[ k_0 \sim \frac{v}{r}, \quad \vec{k} \sim \frac{v}{r} \]

radiation gravitons
Potential matching

Expansion of action

Expand $S_{\text{GR}}$ in $v \sim \sqrt{Gm/r} \ll 1$, e.g.

$$S_{pp} = - \sum_i m_i \int dt \sqrt{-g_{\mu\nu} \frac{\partial x^\mu_i}{\partial t} \frac{\partial x^\nu_i}{\partial t}} = - \sum_i m_i \int dt \sqrt{-g_{00}} + \mathcal{O}(v_i)$$

Coupling to spatial components of metric suppressed

Temporal Kaluza-Klein decomposition [Kol, Smolkin 2010]

$$g^{\mu\nu} = e^{2\phi} \begin{pmatrix}
-1 & A_j \\
A_i & e^{-2 \frac{d-1}{d-2} \phi} (\delta_{ij} + \sigma_{ij}) - A_i A_j
\end{pmatrix}$$

\begin{array}{c|c|c|c}
\phi & A_i & \sigma_{ij} & \cdots \\
\hline
\nu^0 & \nu & \nu^2 & \cdots \\
\end{array}
Equate amplitude in effective and full theory:

\[ q \downarrow -iV + \frac{1}{2!} + \frac{1}{3!} + \cdots \]

\[ = + + + + \cdots \]

All momenta potential, \( p_0 \sim \frac{v}{r} \ll p_i \sim \frac{1}{r} \)

\[ \leftrightarrow \text{expand propagators:} \]

\[ \frac{1}{\tilde{p}^2 - p_0^2} = \frac{1}{\tilde{p}^2} + \frac{p_0^2}{\tilde{p}^4} + \mathcal{O}(v^4) \]
Potential matching

Diagrammatic expansion

\[ V = i \log \left( 1 + \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array} + \begin{array}{c}
\text{Diagram 3}
\end{array} + \cdots \right) \]

\[ = i \left( \begin{array}{c}
\text{Diagram 4} \\
\text{Diagram 5}
\end{array} + \begin{array}{c}
\text{Diagram 6}
\end{array} + \cdots \right) \]

1PN
Confirmation of previous results:

- **1PN:** [Goldberger, Rothstein 2004]
- **2PN:** [Gilmore, Ross 2008]
- **3PN:** [Foffa, Sturani 2011; Blümlein, Maier, Marquard 20XX]
- **4PN:**
  - “static” contribution $\nu = 0$:
    [Foffa, Mastrolia, Sturani, Sturm 2016; Damour, Jaranowski 2017]
  - $\nu \neq 0$: [Foffa, Sturani 2019; Foffa, Porto, Rothstein, Sturani 2019]

New:

- **5PN static contribution:**
  [Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 27 Feb 2019; Blümlein, Maier, Marquard 28 Feb 2019 ]
Potential matching

Static 5PN calculation

$$-iV_{5PN}^S = \ldots$$
Potential matching

Feynman rules

\[ p = -\frac{i}{2c_d \vec{p}^2} \]

\[ \frac{i_{i_1i_2}j_{j_1j_2}}{p} = -\frac{i}{2\vec{p}^2} (\delta_{i_1j_1} \delta_{i_2j_2} + \delta_{i_1j_2} \delta_{i_2j_1} + (2 - c_d)\delta_{i_1i_2} \delta_{j_1j_2}) \]

\[ m_i = -i \frac{m_i}{m^m_{Pl}} \]

\[ V^{i_1i_2}_{\phi \phi \sigma} = \vec{p}_1 \cdot \vec{p}_2 \delta^{i_1i_2} - 2p_1^i p_2^i \]

\[ = \frac{c_d}{2m_{Pl}} (V^{i_1i_2}_{\phi \phi \sigma} + V^{i_2i_1}_{\phi \phi \sigma}) \]

\[ V^{i_1i_2,j_{j_1j_2}}_{\phi \phi \sigma} = \vec{p}_1 \cdot \vec{p}_2 (\delta^{i_1i_2} \delta^{j_1j_2} - 2\delta^{i_1j_1} \delta^{i_2j_2}) - 2(p_1^i p_2^j \delta^{i_1j_2} + p_1^i p_2^j \delta^{i_2j_1}) + 8\delta^{i_1j_1} p_1^i p_2^j \]

\[ V^{i_1i_2,j_{j_1j_2}}_{\phi \phi \sigma} = \vec{p}_1 \cdot \vec{p}_2 (\delta^{i_1i_2} \delta^{j_1j_2} - 2\delta^{i_1j_1} \delta^{i_2j_2}) - 2(p_1^i p_2^j \delta^{i_1j_2} + p_1^i p_2^j \delta^{i_2j_1}) + 8\delta^{i_1j_1} p_1^i p_2^j \]
Potential matching

Feynman rules

\[ \frac{i}{32 m_P} (\tilde{V}_{i_1 i_2 j_1 j_2, k_1 k_2} + V_{i_1 i_2 j_1 j_2, k_1 k_2}) \]

\[ \tilde{V}_{i_1 i_2 j_1 j_2, k_1 k_2} = V_{i_1 i_2 j_1 j_2, k_1 k_2} + V_{i_1 i_2 j_1 j_2, k_1 k_2} + V_{i_1 i_2 j_1 j_2, k_1 k_1} + V_{i_1 i_2 j_1 j_2, k_2 k_1} \]

\[ V_{i_1 i_2 j_1 j_2, k_1 k_2} = (p_1^2 + \bar{p}_1 \cdot \bar{p}_2 + p_2^2) \left( -\delta_{i_1 j_2} \left( 2\delta_{i_1 k_1} \delta_{i_2 k_2} - \delta_{i_1 i_2} \delta_{k_1 k_2} \right) \right) \]

\[ + 2 \left[ \delta_{i_1 j_1} \left( 4\delta_{i_2 k_1} \delta_{j_2 k_2} - \delta_{i_2 j_2} \delta_{k_1 k_2} \right) - \delta_{i_1 i_2} \delta_{j_1 k_1} \delta_{j_2 k_2} \right] \]

\[ + 2 \left\{ 4 \left( p_1^{k_1} p_2^{i_2} - p_1^{i_2} p_2^{k_1} \right) \delta_{i_1 j_1} \delta_{i_2 j_1} \right\} \]

\[ + 2 \left[ \left( p_1^{i_1} + p_2^{i_1} \right) p_2^{i_2} \delta_{i_1 k_1} \delta_{i_2 k_2} - p_1^{k_1} p_2^{k_2} \delta_{i_1 j_1} \delta_{i_2 j_2} \right] \]

\[ + \delta_{i_1 j_2} \left[ p_1^{k_1} p_2^{k_2} \delta_{i_1 i_2} + 2 \left( p_1^{i_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2} \right) \delta_{i_1 k_1} - \left( p_1^{i_1} + p_2^{i_2} \right) p_2^{i_2} \delta_{k_1 k_2} \right] \]

\[ + p_2^{j_2} \left( 4 p_1^{i_2} \delta_{i_1 k_1} \delta_{j_1 k_2} + p_1^{j_1} \left( 2\delta_{i_1 k_1} \delta_{i_2 k_2} - \delta_{i_1 i_2} \delta_{k_1 k_2} \right) \right) \]

\[ + 2 \left[ \delta_{i_1 j_1} \left( p_1^{i_2} \delta_{k_1 k_2} - 2 p_1^{k_2} \delta_{i_2 k_1} \right) - p_1^{k_2} \delta_{i_1 i_2} \delta_{j_1 k_1} \right] \]

\[ + p_2^{j_1} \left( p_1^{j_1} \left( 2\delta_{i_1 k_1} \delta_{i_2 k_2} - \delta_{i_1 i_2} \delta_{k_1 k_2} \right) - 4 p_2^{j_1} \delta_{i_1 k_1} \delta_{j_1 k_2} \right) \]

\[ + 2 \left[ p_2^{k_1} \delta_{j_1 i_2} \delta_{j_1 k_1} + \delta_{i_1 j_1} \left( 2 p_2^{k_2} \delta_{i_2 k_1} - p_2^{i_2} \delta_{k_1 k_2} \right) \right] \]
Potential matching

Diagram families

Massless propagators:

\[ P_f(q) = \int \frac{d^d l_1}{\pi^{d/2}} \cdots \frac{d^d l_5}{\pi^{d/2}} \frac{N(q, l_1, \ldots, l_5)}{\bar{p}_1^{2a_1} \cdots \bar{p}_{10}^{2a_{10}}} \]
Potential matching

Master integrals

Reduction to master integrals (crusher): [Chetyrkin, Tkachov 1981; Laporta 2000]

\[ \mathcal{V}_{5PN}^S = c_0 + c_1 + c_2 + c_3 + \mathcal{O}(\epsilon) \]

\( c_j \): Laurent series in \( \epsilon = \frac{3-d}{2} \), polynomials in \( m_1, m_2, r^{-1}, G^{-1} \)

Master integrals factorise into \( \begin{array}{c} a \\ \quad b \end{array} \) and known \( \begin{array}{c} a \\ b \end{array} \)

[Lee, Mingulov 2015]
Potential matching

Result

\[
V_N^S = -\frac{G}{r} m_1 m_2
\]
\[
V_{1PN}^S = \frac{G^2}{2r^2} m_1 m_2 (m_1 + m_2)
\]
\[
V_{2PN}^S = -\frac{G^3}{r^3} m_1 m_2 \left[ \frac{1}{2} (m_1^2 + m_2^2) + 3m_1 m_2 \right]
\]
\[
V_{3PN}^S = \frac{G^4}{r^4} m_1 m_2 \left[ \frac{3}{8} (m_1^3 + m_2^3) + 6m_1 m_2 (m_1 + m_2) \right]
\]
\[
V_{4PN}^S = -\frac{G^5}{r^5} m_1 m_2 \left[ \frac{3}{8} (m_1^4 + m_2^4) + \frac{31}{3} m_1 m_2 (m_1^2 + m_2^2) + \frac{141}{4} m_1^2 m_2^2 \right]
\]
\[
V_{5PN}^S = \frac{G^6}{r^6} m_1 m_2 \left[ \frac{5}{16} (m_1^5 + m_2^5) + \frac{91}{6} m_1 m_2 (m_1^3 + m_2^3) + \frac{653}{6} m_1^2 m_2^2 (m_1 + m_2) \right]
\]
Potential matching

Velocity corrections

Full corrections include velocities and *higher time derivatives*:

\[
\mathcal{L}_{2\text{PN}} = + \frac{G^3}{r^3} m_1 m_2 \left[ \frac{1}{2} (m_1^2 + m_2^2) + 3m_1 m_2 \right] \\
+ G m_1 m_2 r \left[ \frac{15}{8} \ddot{a}_1 \ddot{a}_2 - \frac{1}{8} (\dddot{a}_1 \dot{r})(\dddot{a}_2 \dot{r}) \right] \\
+ (\text{terms depending on } \vec{v}_1, \vec{v}_2)
\]

Can be eliminated using

- *Total time derivatives* \( \delta \mathcal{L} \propto \frac{d}{dt} F(\vec{r}, \vec{v}_1, \vec{v}_2) \)

- *Equations of motion* \( \delta \mathcal{L} \propto (\dddot{a}_1 + \frac{G m_2}{r^3} \dot{r}) \left( \dddot{a}_2 - \frac{G m_1}{r^3} \dot{r} \right) \)

\[
\mathcal{L}_{2\text{PN}} = - \frac{G^3}{r^3} m_1 m_2 \left[ \frac{1}{4} (m_1^2 + m_2^2) + \frac{5}{4} m_1 m_2 \right] \\
+ (\text{terms depending on } \vec{v}_1, \vec{v}_2)
\]
Conclusions

- Inspiral phase of compact binary systems described well by \( \text{Post-Newtonian (PN) expansion} \quad v \sim \sqrt{G m/r} \ll 1 \)
- Effective field theories and calculational methods from particle physics very effective for high PN orders
- Static gravitational potential now known at five loops (5PN)
Backup
Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]
Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)

\[ \sim \frac{\hbar}{mvr} \]

- No single-source corrections
- No source-reducible diagrams [Fischler 1977]
Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections

Absorbed into renormalisation of sources

- No source-reducible diagrams [Fischler 1977]
Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams \[\text{[Fischler 1977]}\]

Initially *time-ordered* diagrams:

\[
\Theta(y_0 - x_0) = \frac{1}{2} \left( \Theta(y_0 - x_0) + \Theta(x_0 - y_0) \right) = \frac{1}{2} \]

Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially time-ordered diagrams:

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{diagram1} \\
\includegraphics[width=0.3\textwidth]{diagram2}
\end{array}
\end{align*}
\]

\[= \frac{1}{2} \begin{array}{c}
\includegraphics[width=0.3\textwidth]{diagram3}
\end{array} = \frac{1}{2} \left( \begin{array}{c}
\includegraphics[width=0.3\textwidth]{diagram4}
\end{array} \right)^2 \]
Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially *time-ordered* diagrams:

\[
\begin{align*}
\phantom{\frac{1}{2}} & \quad \frac{1}{2} \\
\frac{1}{2} & \quad \frac{1}{2} (\text{Diagram})^2
\end{align*}
\]

\[-iV = \log \left( 1 + \frac{1}{2} \left( (\text{Diagram})^2 + \ldots \right) \right) = \text{Diagram} + \ldots\]
## Potential matching
### Number of diagrams

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<th>source irred</th>
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<th>no tadpoles</th>
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<td>27582</td>
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[Buonanno, QCD Meets Gravity IV, 2018]
Potential matching

Results for master integrals

\[
\begin{align*}
\left(\begin{array}{c}
\end{array}\right) &= e^{5\epsilon \gamma_E} \frac{\Gamma \left(6 - \frac{5d}{2}\right) \Gamma^6 \left(-1 + \frac{d}{2}\right)}{\Gamma \left(-6 + 3d\right)} \\
\left(\begin{array}{c}
\end{array}\right) &= e^{5\epsilon \gamma_E} \frac{\Gamma \left(7 - \frac{5d}{2}\right) \Gamma \left(3 - d\right) \Gamma \left(2 - \frac{d}{2}\right) \Gamma^7 \left(-1 + \frac{d}{2}\right) \Gamma \left(5 - 2d\right)}{\Gamma \left(5 - \frac{3}{2}d\right) \Gamma \left(-2 + d\right) \Gamma \left(-3 + \frac{3}{2}d\right) \Gamma \left(-7 + 3d\right)} \\
\left(\begin{array}{c}
\end{array}\right) &= e^{5\epsilon \gamma_E} \frac{\Gamma \left(7 - \frac{5d}{2}\right) \Gamma^2 \left(3 - d\right) \Gamma^7 \left(-1 + \frac{d}{2}\right) \Gamma \left(-6 + \frac{5d}{2}\right)}{\Gamma \left(6 - 2d\right) \Gamma^2 \left(-3 + \frac{3d}{2}\right) \Gamma \left(-7 + 3d\right)} \\
\left(\begin{array}{c}
\end{array}\right) &= 6\pi^{7/2} \left[ \frac{2}{\epsilon} - 4 - 4 \ln(2) - \left(48 + 8 \ln(2) - 4 \ln^2(2) - 105\zeta_2\right) \epsilon + \mathcal{O}(\epsilon^2) \right]
\end{align*}
\]

\[
V_{5\text{PN}}^{S, \epsilon=0} = \frac{G^6}{r^6} (m_1 m_2) \pi^{-7/2} \left\{ \frac{15}{32} (m_1^5 + m_2^5) \left[ \begin{array}{c}
\end{array}\right]_{\epsilon^0} + \frac{91}{4} m_1 m_2 (m_1^3 + m_2^3) \left[ \begin{array}{c}
\end{array}\right]_{\epsilon^0}
\right.
\]

\[
+ m_1^2 m_2^2 (m_1 + m_2) \left( \left[ \frac{293}{4} - \frac{45}{16} \right]_{\epsilon^0} + \frac{45}{32} \right)
\]

\[
\left. + \left[ \frac{519}{16} - \frac{627}{32} + 2 \right] \right]_{\epsilon^{-1}} \right\}
\]