From sum-integrals to continuum integrals and back

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based on recent work with
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and earlier work with
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Motivation

- Finite-temperature field theory
  - ▷ fairly mature subject; textbooks [Kapusta 89; LeBellac 00; Kapusta/Gale 06; Laine/Vuorinen 17]
  - ▷ relevant in cosmology (mostly weak int; QCD as background)
    early univ, equilibration, $T_{max} = ?$
    DM searches, relic densities
  - ▷ relevant in HIC (mainly QCD)
    fireball lifetime ~ 10 fm/c; $T_{max} \sim 10^2$ MeV
    particle yields, jet quenching, plasma hydro

- equilibrium thermodynamics: imaginary time formalism, $t \to i\tau$
  - ▷ (grand) canonical ensemble, $Z(T,\mu) = \mathrm{Tr}[e^{-(\hat{H}-\mu\hat{N})/T}]$
  - ▷ path int quant, fields periodic: $Z = \int \mathcal{D}\phi\, e^{-\int_0^{1/T}\int d^dx\, \mathcal{L}_E}$ $\Leftarrow$ $d = 3 - 2\varepsilon$
  - ▷ Fourier trafo discrete; mom-space measure $T\sum_{n\in\mathbb{Z}}\int \frac{d^dp}{(2\pi)^d} \equiv \SumInt_P$
  - ▷ bosonic prop $\sim [(2n\pi T)^2 + \vec{p}^{\,2} + m^2]^{-1}$
  - ▷ Dirac prop $\sim [i\gamma_0((2n+1)\pi T + i\mu) + i\vec{\gamma}\vec{p} + m]^{-1}$

- upshot: integrals $\to$ sum-integrals

Motivation

- clean sub-problem: vacuum-type sum-integrals
  - ▷ relevance: free energy $f = -T \ln Z$ of a thermal system
  - ▷ EoS, expansion rate, etc.
  - ▷ in many settings, QCD effects dominant [Linde/IR problem tamed by EFT's]

- even cleaner: massless bosonic (think gluons) vacuum-type sum-integrals
  - ▷ state-of-the-art: 1-, 2-loop OK; 3-loop; isolated 4loop cases.

Motivation

- first example: LO / 1-loop bosonic tadpole
- recall $T=0$ case: $J_\nu(m) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{1}{[p^2+m^2]^\nu} = [m^2]^{d/2-\nu} \times \frac{\Gamma(\nu-d/2)}{(4\pi)^{d/2}\,\Gamma(\nu)}$
- at $T \neq 0$ therefore [writing $P^2 = P_0^2 + \vec{p}^{\,2}$ with $P_0 = 2n\pi T$, and d-dim vector $\vec{p}$]

$$I_\nu^\eta(d) \equiv \sum\!\!\!\!\!\!\int_P \frac{(P_0)^\eta}{[P^2]^\nu} = \delta_\eta J_\nu(0) + [1+(-1)^\eta]\,T \sum_{n=1}^{\infty} (2n\pi T)^\eta J_\nu(2n\pi T)$$

$$= 0 + \frac{[1+(-1)^\eta]\,T\,\zeta(2\nu-\eta-d)}{(2\pi T)^{2\nu-\eta-d}} \frac{\Gamma(\nu-\frac{d}{2})}{(4\pi)^{d/2}\,\Gamma(\nu)}$$

  - ▷ note that 'thermal part' has the form $\zeta(n_{\text{even}}-d)$

- massless sum-integral ⇔ massive (T=0) integral
- relevance: free E, selfE's, Debye screening masses, etc.
  - ▷ example: blackbody radiation / Stefan-Boltzmann law at LO

    $f_{QED} = -\frac{\pi^2 T^4}{90}[2 + 4\,\frac{7}{8} N_f]$ [↔ expansion rate of univ at $T \sim$ MeV]

    $f_{QCD} = -\frac{\pi^2 T^4}{90}[2(N_c^2-1) + 4N_c\,\frac{7}{8} N_f]$

Motivation

- next step: NLO / 2-loop
  - ▷ a number of worked-out examples in the literature
  - ▷ general observation: factorization $\sum\!\!\!\!\!\!\int_{PQ}(\cdots) \sim [\sum\!\!\!\!\!\!\int_{P}(\cdots)] \times [\sum\!\!\!\!\!\!\int_{Q}(\cdots)]$
  - ▷ confirmed by (thermal adaptation) of IBP
  - ▷ $\Rightarrow$ Q: is this a theorem? [A: YES (for bos, $m = \mu = 0$)]
- at higher orders (or with $\frac{1}{\varepsilon}$ from IBP pre-factors) need higher ε-terms of 2-loop sum-ints
  - ▷ generic analytic results (in d) would be useful
- goal: devise a constructive proof of 2-loop factorization

Setup

- recall from 1-loop: massless sum-integral $\Leftrightarrow$ massive (T=0) integral
- define massive 2-loop vacuum integral in d dimensions [we are interested in $d = 3 - 2\varepsilon$]

$$B^{\nu_1,\nu_2,\nu_3}_{m_1,m_2,m_3} \equiv \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[m_1^2+p^2]^{\nu_1}\,[m_2^2+q^2]^{\nu_2}\,[m_3^2+(p-q)^2]^{\nu_3}}$$

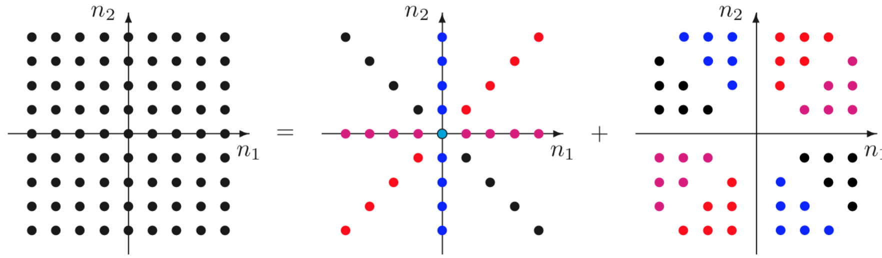
- define massless bosonic 2-loop vacuum sum-integral [$\nu \equiv \nu_1+\nu_2+\nu_3$ and $\eta \equiv \eta_1+\eta_2+\eta_3$]

$$L^{\eta_1,\eta_2,\eta_3}_{\nu_1,\nu_2,\nu_3} \equiv \sum\!\!\!\!\!\!\int_{PQ} \frac{(P_0)^{\eta_1}\,(Q_0)^{\eta_2}\,(P_0-Q_0)^{\eta_3}}{[P^2]^{\nu_1}\,[Q^2]^{\nu_2}\,[(P-Q)^2]^{\nu_3}}$$

$$= \frac{T^2}{(2\pi T)^{2\nu-\eta-2d}} \sum_{n_1,n_2\in\mathbb{Z}} n_1^{\eta_1}\,n_2^{\eta_2}\,(n_1-n_2)^{\eta_3}\,B^{\nu_1,\nu_2,\nu_3}_{n_1,n_2,n_1-n_2}$$

- remaining task: do double sum over known analytic result for B [Davydychev/Tausk 1992]
  - ▷ known result is in terms of Appell's hypergeometric function F_4
  - ▷ not practical: four infinite sums
- can do (much) better: 'masses' are linearly related $\Rightarrow$ finite sums
  - ▷ examine B from scratch, at special kinematic point

Setup



- sort out the cases where masses of B can vanish
  - ▷ decompose double-sum into sectors where 'masses' are always positive
  - ▷ take into account that B depends on m_i^2, use the integral's symmetry

$$
\begin{aligned}
L^{\eta_1,\eta_2,\eta_3}_{\nu_1,\nu_2,\nu_3} &= \frac{T^2\,[1+(-1)^\eta]}{(2\pi T)^{2\nu-\eta-2d}} \Big\{ \frac{1}{2}\,\delta_{\eta_1}\delta_{\eta_2}\delta_{\eta_3}\, B^{\nu_1\nu_2\nu_3}_{0,0,0} \\
&+ \zeta(2\nu-\eta-2d)\Big[(-)^{\eta_3}\delta_{\eta_1} B^{\nu_1,\nu_2,\nu_3}_{0,1,1} + \delta_{\eta_2} B^{\nu_2,\nu_1,\nu_3}_{0,1,1} + \delta_{\eta_3} B^{\nu_3,\nu_1,\nu_2}_{0,1,1} + 2^{\eta_3}(-)^{\eta_2} B^{\nu_1,\nu_2,\nu_3}_{1,1,2}\Big] \\
&+ \bar{H}^{\nu_3,\nu_2,\nu_1}_{\eta_3,\eta_2,\eta_1} + (-)^{\eta_3}\bar{H}^{\nu_3,\nu_1,\nu_2}_{\eta_3,\eta_1,\eta_2} + (-)^{\eta_2} H^{\nu_2,\nu_1,\nu_3}_{\eta_2,\eta_1,\eta_3} + (-)^{\eta_2} H^{\nu_1,\nu_2,\nu_3}_{\eta_1,\eta_2,\eta_3} \Big\}
\end{aligned}
$$

- 1st line: no-scale case of $B \rightarrow$ massless tadpole, $= 0$ in dim reg
- 2nd line: single-scale cases of B, double sum trivial $\rightarrow \zeta$ [explicit results not needed here]
- 3rd line: two types of double sums, each over B with $\boxed{m_1 + m_2 = m_3}$

$$
\begin{aligned}
H^{\nu_a,\nu_b,\nu_c}_{\eta_a,\eta_b,\eta_c} &\equiv \sum_{n_1>n_2>0} n_1^{\eta_a} n_2^{\eta_b} (n_1+n_2)^{\eta_c}\, B^{\nu_a,\nu_b,\nu_c}_{n_1,n_2,n_1+n_2} \\
\bar{H}^{\nu_a,\nu_b,\nu_c}_{\eta_a,\eta_b,\eta_c} &\equiv \sum_{n_1>n_2>0} (n_1-n_2)^{\eta_a} n_2^{\eta_b} n_1^{\eta_c}\, B^{\nu_a,\nu_b,\nu_c}_{n_1-n_2,n_2,n_1}
\end{aligned}
$$

Continuum integral B

- recall: need 2-loop massive vacuum integral $B^{\nu_1,\nu_2,\nu_3}_{m_1,m_2,m_3}$ at $\boxed{m_3 = m_1 + m_2}$ (all $m_i > 0$)

- IBP gives a recurrence that allows to shrink one line [Tarasov 1997]

$$2uB^{\nu_1\nu_2\nu_3} = \left\{\frac{\mathbf{1}^-}{m_1}\left[\frac{c+\nu_2}{m_2} - \frac{c+\nu_3}{m_3}\right] + \frac{\mathbf{2}^-}{m_2}\left[\frac{c+\nu_1}{m_1} - \frac{c+\nu_3}{m_3}\right] + \frac{\mathbf{3}^-}{m_3}\left[\frac{c-\nu_1}{m_1} + \frac{c-\nu_2}{m_2}\right]\right\} B^{\nu_1\nu_2\nu_3}$$

[$u \equiv d+3-2\nu$ and $c \equiv d+2-\nu$ as well as $\nu = \nu_1+\nu_2+\nu_3$]

- can one solve this explicitly?
  - ▷ trivial boundary cond: $B^{000} = 0$, $B^{\nu_1 00} = 0$, $B^{\nu_1\nu_2 0} = J_{\nu_1}(m_1)\, J_{\nu_2}(m_2)$ etc.

- experimental math: look at some low (index-) weight examples [$x_{ij} \equiv m_i/m_j$ are mass ratios]

$$B^{111} = \frac{(d-2)}{2(d-3)}\left\{-\frac{B^{011}}{m_2 m_3} - \frac{B^{101}}{m_1 m_3} + \frac{B^{110}}{m_1 m_2}\right\}$$

$$B^{211} = \frac{(d-2)}{4(d-5)}\left\{\frac{B^{011}}{m_2^2 m_3^2} + \left[(d-4)\frac{m_3}{m_1} - 1\right]\frac{B^{101}}{m_1^2 m_3^2} - \left[(d-4)\frac{m_2}{m_1} + 1\right]\frac{B^{110}}{m_1^2 m_2^2}\right\}$$

$$\ldots$$

Continuum integral B

- observe lots of structure $\Rightarrow$ boldly conjecture the full result

$$B^{\nu_1\nu_2\nu_3} \overset{?!}{=} B^{110}_{110} \sum_{j=1-\nu_1}^{\nu_2-1} (-1)^\nu \, c^{(\nu)}_{\nu_1,\nu_2;j} \, m_1^{d-\nu+j} \, m_2^{d-\nu-j} + (231) + (312)$$

  - ▷ coefficients $c^{(\nu)}_{\nu_a,\nu_b;j}$ are rational functions in d
  - ▷ symmetries $c^{(\nu)}_{\nu_a,\nu_b;j} = c^{(\nu)}_{\nu_b,\nu_a;-j}$ (with special case $c^{(\nu)}_{\nu_a,\nu_b;0} = c^{(\nu)}_{\nu_b,\nu_a;0}$)
  - ▷ conjecture confirmed via recurrence to weight 18

- conjecture proven via induction over weight ν [details in forthcoming paper]
  - ▷ relying on the IBP recurrence
  - ▷ lots of rearrangements of sums; add cleverly constructed zero
  - ▷ proof is constructive: gives fast algorithm to recursively construct c's

- at higher ν, c's contain huge numerator polynomials; plus lots of structure not shown here
- obtained some interesting new analytic results, e.g. for B^{aac} and perms, such as

$$B^{aac} = \sum_{k=0}^{a-1} \mathrm{rat}^{ac}_k(d) \left\{ \frac{B^{110}}{(m_1m_2)^{2a+c-2}} \frac{(m_1+m_2)^{2k}}{(m_1m_2)^k} + \sum_{j=0}^{c+k-1} \mathrm{rat}^{ac}_{kj}(d) \left[\frac{B^{101}}{(m_1m_3)^{2a+c-2}} \left(\frac{m_1}{m_3}\right)^{j-k} + (1 \leftrightarrow 2) \right] \right\}$$

  - ▷ needed B^{11c} as derived directly from corresponding limit of F_4 representation
  - ▷ coeffs rat known analytically

Continuum integral B

- to fix $\boldsymbol{c}$'s in practice, yet another recurrence is most useful (= fast)
- mixing a number of IBP and dimensional relations [extracted from Tarasov 1997]

$$
\begin{aligned}
(d-2)(d+3-2\nu)B^{\nu_1,\nu_2,\nu_3}(d) &= \lambda(\mathbb{1}^-,\mathbb{2}^-,\mathbb{3}^-)\,\mathbb{d}^-\,B^{\nu_1,\nu_2,\nu_3}(d) \\
\lambda(a,b,c) &= a^2+b^2+c^2-2(ab+bc+ca) \\
\mathbb{d}^-\,B^{\nu_1,\nu_2,\nu_3}(d) &= \frac{1}{16\pi^2}\,B^{\nu_1,\nu_2,\nu_3}(d-2)
\end{aligned}
$$

  - ▷ reduces the weight $\boldsymbol{\nu}$ by two in each step; use until one $\nu_i \longrightarrow 0$ or -1
  - ▷ lift the neg. index via $\mathbb{3}^-\,B^{\nu_1,\nu_2,0} = \{2m_1m_2 + \mathbb{1}^- + \mathbb{2}^-\}\,B^{\nu_1,\nu_2,0}$ and perms
  - ▷ recurrence does not contain explicit mass-factors
  - ▷ know the boundary integrals for arbitrary dimension $\boldsymbol{d}$

- there is much more to be discovered..
- important: IBP rel asserts that B is polynomial in masses; allows to tackle sums

Back to sum-integrals

- reminder to self: wanted to evaluate two-loop sum-integral L

$$
\begin{aligned}
L^{\eta_1,\eta_2,\eta_3}_{\nu_1,\nu_2,\nu_3} &= \frac{T^2\,[1+(-1)^\eta]}{(2\pi T)^{2\nu-\eta-2d}}\Big\{\frac{1}{2}\,\delta_{\eta_1}\delta_{\eta_2}\delta_{\eta_3}\,B^{\nu_1\nu_2\nu_3}_{0,0,0} \\
&+ \zeta(2\nu-\eta-2d)\Big[(-)^{\eta_3}\delta_{\eta_1}B^{\nu_1,\nu_2,\nu_3}_{0,1,1} + \delta_{\eta_2}B^{\nu_2,\nu_1,\nu_3}_{0,1,1} + \delta_{\eta_3}B^{\nu_3,\nu_1,\nu_2}_{0,1,1} + 2^{\eta_3}(-)^{\eta_2}B^{\nu_1,\nu_2,\nu_3}_{1,1,2}\Big] \\
&+ \bar{H}^{\nu_3,\nu_2,\nu_1}_{\eta_3,\eta_2,\eta_1} + (-)^{\eta_3}\bar{H}^{\nu_3,\nu_1,\nu_2}_{\eta_3,\eta_1,\eta_2} + (-)^{\eta_2}H^{\nu_2,\nu_1,\nu_3}_{\eta_2,\eta_1,\eta_3} + (-)^{\eta_2}H^{\nu_1,\nu_2,\nu_3}_{\eta_1,\eta_2,\eta_3}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
H^{\nu_a,\nu_b,\nu_c}_{\eta_a,\eta_b,\eta_c} &\equiv \sum_{n_1>n_2>0} n_1^{\eta_a}n_2^{\eta_b}(n_1+n_2)^{\eta_c}\,B^{\nu_a,\nu_b,\nu_c}_{n_1,n_2,n_1+n_2} \\
\bar{H}^{\nu_a,\nu_b,\nu_c}_{\eta_a,\eta_b,\eta_c} &\equiv \sum_{n_1>n_2>0} (n_1-n_2)^{\eta_a}n_2^{\eta_b}n_1^{\eta_c}\,B^{\nu_a,\nu_b,\nu_c}_{n_1-n_2,n_2,n_1}
\end{aligned}
$$

  - ▷ need to to perform the remaining (Matsubara) double sums $H\,,\bar{H}$
  - ▷ having the 'conjecture' at hand, the mass structure of B is explicit
  - ▷ can work out the sums without specifying the coefficient functions $c(d)$

- we will now show how the sums combine to
  - (a) evaluate to single and double zeta values only
  - (b) cancel all $\zeta(i,j)$ in the sum of all four terms of 3rd line of L-decomposition
  - (c) cancel all remaining single $\zeta(i)$ in 2nd line of L-decomposition
  - (d) leave us with products $\zeta(i)\,\zeta(j)$ containing only $\zeta(n_{\text{even}}-d)$

- all of this allows us to write a compact final result, containing products of 1-loop tadpoles

proof of statement (a)

- show that double sums evaluate to MZV only
- use the 'conjecture' for H
  - ▷ re-express the 'unwanted mass' in the prefactor in terms of the two others (introduces an additional binomial sum)
  - ▷ shift summation indices and use c-symmetries [def $\ell_i \equiv \nu - \eta_i - j - k$]

$$\frac{H^{\nu_2\nu_1\nu_3}_{\eta_2\eta_1\eta_3} + H^{\nu_1\nu_2\nu_3}_{\eta_1\eta_2\eta_3}}{B^{110}_{110}}$$

$$= \sum_{j=1-\nu_1}^{\nu_2-1} (-1)^\nu \, c^{(\nu)}_{\nu_1,\nu_2;j} \sum_{k=0}^{\eta_3} \binom{\eta_3}{k} \sum_{n_1>n_2>0} \left(n_2^{d-\ell_1} n_1^{d-2\nu+\eta+\ell_1} + n_1^{d-\ell_1} n_2^{d-2\nu+\eta+\ell_1} \right) + (231) + (312)$$

- this contains only two combinations of double sums [1st instance contains non-MZV, cancels in sum]

$$\sum_{n_1>n_2>0} \left(\frac{1}{n_1^\alpha} + \frac{1}{n_2^\alpha}\right) \frac{1}{(n_1+n_2)^\beta} = \zeta(\beta,\alpha) - \frac{1}{2^\beta}\zeta(\alpha+\beta)$$

$$\sum_{n_1>n_2>0} \left(\frac{1}{n_1^\alpha}\frac{1}{n_2^\beta} + \frac{1}{n_2^\alpha}\frac{1}{n_1^\beta}\right) = \zeta(\alpha)\zeta(\beta) - \zeta(\alpha+\beta)$$

proof of statement (b)

- show that no MZV's contribute
- massaging $\bar{H}$ as above

$$\frac{\bar{H}^{\nu_3\nu_2\nu_1}_{\eta_3\eta_2\eta_1}}{B^{110}_{110}} = \sum_{j=1-\nu_1}^{\nu_2-1} (-1)^j \, c^{(\nu)}_{\nu_1,\nu_2;j} \sum_{k=0}^{\eta_3} \binom{\eta_3}{k} (-1)^{k-\eta_3} \sum_{n_1>n_2>0} n_2^{d-2\nu+\eta+\ell_1} n_1^{d-\ell_1} + (231) + (312)$$

- double sums results in multiple zeta values [def $\ell_i = \nu - \eta_i - j - k$, $d_i = \ell_i - d$ and $e_i = 2\nu - d - \eta - \ell_1$]

$$\frac{\bar{H}^{\nu_3\nu_2\nu_1}_{\eta_3\eta_2\eta_1} + (-1)^{\eta_3} \bar{H}^{\nu_3\nu_1\nu_2}_{\eta_3\eta_1\eta_2} + (-1)^{\eta_2} [H^{\nu_2\nu_1\nu_3}_{\eta_2\eta_1\eta_3} + H^{\nu_1\nu_2\nu_3}_{\eta_1\eta_2\eta_3}]}{(-1)^\nu \, B^{110}_{110}}$$

$$= \sum_{j=1-\nu_1}^{\nu_2-1} c^{(\nu)}_{\nu_1,\nu_2;j} \sum_{k=0}^{\eta_3} \binom{\eta_3}{k} \left\{ (-)^{\ell_3} \Big[\zeta(d_1, e_1) + \zeta(e_1, d_1) \Big] + (-)^{\eta_2} \Big[\zeta(e_1)\zeta(d_1) - \zeta(e_1 + d_1) \Big] \right\} + (231) + (312)$$

- simplify via shuffle rel $\zeta(a, b) + \zeta(b, a) = \zeta(a)\zeta(b) - \zeta(a + b)$

$$\frac{\bar{H}^{\nu_3\nu_2\nu_1}_{\eta_3\eta_2\eta_1} + (-1)^{\eta_3} \bar{H}^{\nu_3\nu_1\nu_2}_{\eta_3\eta_1\eta_2} + (-1)^{\eta_2} [H^{\nu_2\nu_1\nu_3}_{\eta_2\eta_1\eta_3} + H^{\nu_1\nu_2\nu_3}_{\eta_1\eta_2\eta_3}]}{(-1)^\nu \, B^{110}_{110}}$$

$$= \sum_{j=1-\nu_1}^{\nu_2-1} c^{(\nu)}_{\nu_1,\nu_2;j} \sum_{k=0}^{\eta_3} \binom{\eta_3}{k} \left\{ \Big[(-1)^{\ell_3} + (-1)^{\eta_2} \Big] \Big[\zeta(e_1)\zeta(d_1) - \zeta(e_1 + d_1) \Big] \right\} + (231) + (312)$$

proof of statement (c)

- show that only products of single zeta values remain in L
- happy with the products of zetas (hints at 1-loop squared)
- remaining single zetas have arguments $e_i + d_i = 2\nu - 2d - \eta \Rightarrow$ pull out of sums
  - ▷ k-sums are trivial: $\sum_{k=0}^{N} \binom{N}{k} x^k = (1+x)^N$, $\sum_{k=0}^{N} \binom{N}{k} (-1)^k = \delta_N$
  - ▷ j-sum can then be seen to be nothing but the coefficients of single-scale cases of B

$$
\begin{aligned}
& \bar{H}^{\nu_3\nu_2\nu_1}_{\eta_3\eta_2\eta_1} + (-1)^{\eta_3} \bar{H}^{\nu_3\nu_1\nu_2}_{\eta_3\eta_1\eta_2} + (-1)^{\eta_2} [H^{\nu_2\nu_1\nu_3}_{\eta_2\eta_1\eta_3} + H^{\nu_1\nu_2\nu_3}_{\eta_1\eta_2\eta_3}] \Big|_{\text{single zetas}} \\
= \quad & -\zeta(2\nu - 2d - \eta)\,(-1)^{\nu}\, B^{110}_{110} \sum_{j=1-\nu_1}^{\nu_2-1} c^{(\nu)}_{\nu_1,\nu_2;j} \left[(-1)^{\nu-\eta_3-j} \delta_{\eta_3} + (-1)^{\eta_2} 2^{\eta_3} \right] + (231) + (312) \\
= \quad & -\zeta(2\nu - 2d - \eta) \left[\delta_{\eta_3} B^{\nu_3\nu_1\nu_2}_{011} + \delta_{\eta_2} B^{\nu_2\nu_1\nu_3}_{011} + (-1)^{\eta_2} \delta_{\eta_1} B^{\nu_1\nu_2\nu_3}_{011} + (-1)^{\eta_2} 2^{\eta_3} B^{\nu_1\nu_2\nu_3}_{112} \right]
\end{aligned}
$$

  - ▷ cancels exactly against the 2nd line of L-decomposition

proof of statement (d)

- show that only $\zeta(n-d)$ at even integers n contribute
- re-write the specific combinations of prefactors [using $\ell_i \equiv \nu - \eta_i - j - k$]

$$
\begin{aligned}
L^{\eta_1\eta_2\eta_3}_{\nu_1\nu_2\nu_3} \quad &= \quad \frac{T^2[1+(-1)^\eta]}{(2\pi T)^{2\nu-\eta-2d}}\, B^{110}_{110} \times \\
&\times \quad \sum_{j=1-\nu_1}^{\nu_2-1} (-1)^\nu\, c^{(\nu)}_{\nu_1,\nu_2;j} \sum_{k=0}^{\eta_3} \binom{\eta_3}{k} (-1)^{\eta_2}\, [1+(-1)^{\ell_1}]\, \zeta(\ell_1-d)\, \zeta(2\nu-\eta-\ell_1-d) \\
&\quad + (231) + (312)
\end{aligned}
$$

- normalization factor $B^{110}_{110} \equiv \frac{\Gamma^2(1-d/2)}{(4\pi)^d} \equiv [J_1(d)]^2$

Result

- convert $\zeta \to$ 1-loop sum-ints I^{η}_{ν} [use $\sigma_n \equiv \texttt{Max}[n,1]$ to take care of numerators]

$$\zeta(2n-d) = \frac{(2\pi T)^{2n-d}}{T\, J_1(d)}\, \hat{I}_n \;, \quad \text{with 1-loop sum-int} \quad \hat{I}_n \equiv \frac{\Gamma(\sigma_n)}{2\left(1-\frac{d}{2}\right)_{\sigma_n-1}}\, I^{2\sigma_n-2n}_{\sigma_n}$$

- final result [recall $\nu = \nu_1+\nu_2+\nu_3$, $\eta = \eta_1+\eta_2+\eta_3$ and $\ell_i = \nu - \eta_i - j - k$]

$$\mathrm{L}^{\eta_1\eta_2\eta_3}_{\nu_1\nu_2\nu_3} = [1+(-1)^{\eta}] \sum_{j=1-\nu_1}^{\nu_2-1} (-1)^{\nu}\, c^{(\nu)}_{\nu_1,\nu_2;j} \sum_{k=0}^{\eta_3} \binom{\eta_3}{k} (-1)^{\eta_2}\, [1+(-1)^{\ell_1}]\, \hat{I}_{\frac{\ell_1}{2}}\, \hat{I}_{\frac{2\nu-\eta-\ell_1}{2}} + (231) + (312)$$

- some examples (checks against literature, and new):

$$L^{000}_{111} = 0$$

$$L^{220}_{311} = -\frac{(d-4)(d^2-8d+19)}{4(d-7)(d-5)}\, I^0_2\, I^0_1$$

$$L^{000}_{114} = -\frac{4\, I^0_3\, I^0_3}{(d-9)(d-7)(d-4)(d-2)} - \frac{6\, I^0_2\, I^0_4}{(d-9)(d-2)}$$

$$L^{000}_{116} = -\frac{36\, I^0_4\, I^0_4}{(d-13)(d-11)(d-9)(d-6)(d-4)(d-2)} - \frac{48\, I^0_3\, I^0_5}{(d-13)(d-11)(d-4)(d-2)} - \frac{10\, I^0_2\, I^0_6}{(d-13)(d-2)}$$

Outlook

- thermal field theory: phenomenologically relevant for cosmology and HIC
  - ▷ perturbative tools (by far) not as well developed/automatized as for collider physics
  - ▷ here: first derivation of a parametric IBP solution

- massless 2-loop vacuum-type sum-integrals completely understood
  - ▷ get closed form for coeffs c?

- generalizations of 2-loop case?
  - ▷ massless fermions
  - ▷ massive particles
    (different game: not even massive 1-loop sum-int known analytically)
  - ▷ chemical potentials

- generalizations to higher loops?
  - ▷ some strikingly simple relations known via IBP
    e.g. 3-loop massless benz-type sum-integral vanishes
  - ▷ other known ε-expansions of 3-loop masters show more complicated structure
  - ▷ useful to exploit known properties of 3d massive integrals at $T = 0$?
  - ▷ few useful analytic results available

Continuum integral B: minimizing c's

- express the coefficients $c^{(\nu)}$ at $j = 0$ in terms of the $j > 0$ ones
  - ▷ compare 'conjecture' with well-known result for 1-scale case $B^{\nu_1\nu_2\nu_3}_{0,m,m}$ [e.g. Vladimirov 1980]

$$c^{(\nu)}_{\nu_a,\nu_b;0} = \beta^{\nu-\nu_a-\nu_b,\nu_a,\nu_b} - \sum_{j=1-\nu_a}^{\nu_b-1} (-1)^j \, c^{(\nu)}_{\nu_a,\nu_b;j} \, [1-\delta_j]$$

$$\beta^{\nu_1\nu_2\nu_3} \equiv \frac{\Gamma(\nu_1+\nu_2-d/2)\Gamma(\nu_1+\nu_3-d/2)\Gamma(d/2-\nu_1)\Gamma(\nu-d)}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma^2(1-d/2)\Gamma(d/2)\Gamma(\nu_1+\nu-d)}$$

  - ▷ in view of symmetries of the c's, it is now sufficient to specify them for $j > 0$