Diagrammatic Coaction of Two-Loop Feynman Integrals

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Introduction and Motivation

- The family of polylogarithms is the simplest class of iterated integral that appears in Laurent expansions of Feynman integrals. They are known to possess a coaction $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ which encodes the functional identities of the polylogarithms, as well as their analytic structure. [Goncharov 0103059; Brown 1102.1310]
- Polylogarithms can be used to express one-loop Feynman integrals, so we can compute their coaction. There is a remarkably simple diagrammatic interpretation of the result using master integrals and cut Feynman integrals. [Abreu, Britto, Duhr, Gardi 1704.07931]
- Can this be generalised to integrals beyond one-loop? What about other classes of integral such as hypergeometric functions evaluating to polylogs?

Coaction of Polylogarithms

- The family of polylogarithms is defined by

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t), \qquad G(; z) = 1$$

The weight of a polylog is the number of integrations.

- We will write the coaction in the form:

$$\Delta G(\vec{a}; z) = \sum_{\vec{b} \subseteq \vec{a}} G(\vec{b}; z) \otimes G_{\vec{b}}(\vec{a}; z)$$

Same integration contour as $G(\vec{a}; z)$, integrand modified to keep only the poles in $\vec{b}$

Poles in $\vec{b}$ replaced with residues i.e. same integrand as $G(\vec{a}; z)$, contour modified to encircle poles in $\vec{b}$

- Example: $\Delta G(a, b; z)$. We have, for instance

$$G_a(a, b; z) = \text{Res}_{u=a} \int_0^z \frac{du}{u - a} \int_0^u \frac{dv}{v - b} = \int_0^a \frac{dv}{v - b} = G(b; a)$$

Computing the other terms similarly, we get

$$\begin{aligned} &\Delta G(a, b; z) \\ =&1 \otimes G(a, b; z) + G(a; z) \otimes G(b; a) + G(b; z) \otimes [G(a; z) - G(a; b)] + G(a, b; z) \otimes 1 \end{aligned}$$

- This reproduces the conventional way of writing the coaction.

One-Loop Graphs: Basis and Cuts

- We will consider the basis of pure one-loop integrals

$$e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \prod_{i=1}^{n} \frac{1}{(k+q_i)^2 - m_i^2}, \qquad D = 2\left\lceil \frac{n}{2} \right\rceil - 2\epsilon$$

  normalised by their leading singularity, where γ_E is the Euler-Mascheroni constant. All one-loop integrals are related to this set by dimension shift identities, tensor reduction and IBP relations.
- Their cuts can be defined by parametrising the momenta

$$\begin{aligned} q_1 &= (q_{11}, \vec{0}_{D-1}) \\ q_2 &= (q_{21}, q_{22}, \vec{0}_{D-2}) \\ &\dots \\ k &= k_0(1, \beta\cos\theta_1, \beta\cos\theta_2\sin\theta_1, \dots) \end{aligned}$$

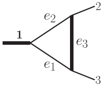
  writing the integral in the new measure

$$\begin{aligned} &\int d^D k \\ =&\alpha \int_{-\infty}^{+\infty} dk_0 k_0^{D-1} \int_0^1 d\beta \beta^{D-2} \int_0^{\pi} d\theta_1 \sin^{D-3}\theta_1 \int_0^{\pi} d\theta_2 \sin^{D-4}\theta \dots \end{aligned}$$

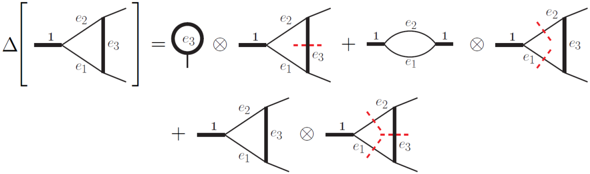
  and replacing a subset of the propagators by their residues in the variables $k_0, \beta, \cos\theta_1, \cos\theta_2, \dots$.
- We can define multiple independent cuts by choosing different integration contours.

Coaction of One-Loop Graphs - Example 1

- Consider a simple one loop graph such as the triangle



where bold lines indicate internal masses or non-null external momenta. Evaluating the Laurent expansion in ϵ and applying the coaction on polylogs order by order in ϵ, we find:



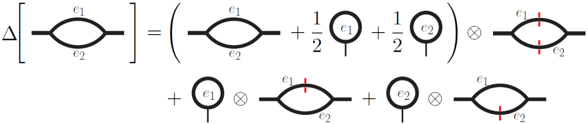
where each graph represents the corresponding element of our pure one-loop basis and dashed red lines denote cuts.

- The coaction in this case takes the form

$$\Delta J_E = \sum_{\emptyset \subsetneq X \subseteq E} J_X \otimes \mathcal{C}_X J_E$$

with terms missing due to vanishing cuts and subgraphs.

- Generally there is an extra ‘deformation’ term, for example in



- There is a general formula valid for all configurations of masses and external kinematics:

One Loop Coproduct

$$\Delta J_E = \sum_{\emptyset \subsetneq X \subseteq E} \left(J_X + a_X \sum_{e \in X} J_{X \setminus e}\right) \otimes \mathcal{C}_X J_E \qquad a_X = \begin{cases} 0 & \text{if } |X| \text{ odd} \\ \frac{1}{2} & \text{if } |X| \text{ even} \end{cases}$$

- The deformation term arises due to the singularity at infinite momentum.
- This general formula obeys various consistency checks e.g. ϵ pole structure. [Abreu, Britto, Duhr, Gardi 1704.07931]
- We can cast this result in the form

$$\Delta \int_\gamma \omega = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

similar to the coproduct of polylogs, but now the ω_i are master integrands and the γ_j are cut contours dual to each other in the sense that $\int_{\gamma_j} \omega_i = \delta_{ij} + \mathcal{O}(\epsilon)$

Coaction of Hypergeometric Functions

- We have now seen two coactions $\Delta \int_\gamma \omega = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$ with the $\{\omega_i\}$ and $\{\gamma_i\}$ obeying $\int_{\gamma_j} \omega_i = \delta_{ij} + \mathcal{O}(\epsilon)$
- This property also holds for coactions of hypergeometric functions that evaluate to polylogs e.g. ${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 du u^{a-1}(1-u)^{c-a-1}(1-uz)^{-b}$ has coaction:

${}_2F_1$ Coaction

$$\begin{aligned}
&\Delta\left[{}_2F_1\left(m+a\epsilon, n+b\epsilon; p+c\epsilon; z\right)\right] \\
=&{}_2F_1\left(1+a\epsilon, b\epsilon; 1+c\epsilon; z\right) \otimes {}_2F_1\left(m+a\epsilon, n+b\epsilon; p+c\epsilon; z\right) \\
&-\frac{b\epsilon}{1+c\epsilon} {}_2F_1\left(1+a\epsilon, 1+b\epsilon; 2+c\epsilon; z\right) \\
&\otimes \Bigg\{ z^{1-m-a\epsilon} \frac{\Gamma(1-n-b\epsilon)\Gamma(p+c\epsilon)}{\Gamma(1+m-n+(a-b)\epsilon)\Gamma(p-m+(c-a)\epsilon)} \\
&\quad {}_2F_1\left(m+a\epsilon, 1+m-p+(a-c)\epsilon; 1+m-n+(a-b)\epsilon; \frac{1}{z}\right)\Bigg\}
\end{aligned}$$

- It is sufficient to check this formula on a basis of two independent ${}_2F_1$ functions.
- Evaluating second entries at weight zero lets us project onto a basis.
- See talk by Ruth Britto on Friday for more details.

Computing Diagrammatic Coactions

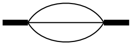
- Feynman integrals can be expressed using hypergeometric functions, so their coaction follows from the coactions of these classes of function. We can then compute the coaction without having to expand in ϵ and work with polylogs.
- We also need the coactions of some prefactors that occur:

$$\begin{aligned}\Delta(a^\epsilon) =& a^\epsilon \otimes a^\epsilon \\ \Delta e^{\gamma_E \epsilon}\Gamma(1+\epsilon) =& e^{\gamma_E \epsilon}\Gamma(1+\epsilon) \otimes e^{\gamma_E \epsilon}\Gamma(1+\epsilon)\end{aligned}$$

which follow from $\Delta\left(\log^n a\right) = \sum_{i\leq n} \begin{pmatrix} n \\ i \end{pmatrix} \log^{n-i} a \otimes \log^i a$,

and $\Gamma(1+\epsilon) = \exp\left(-\gamma_E \epsilon + \sum_{k\geq 2} \frac{(-\epsilon)^k \zeta_k}{k}\right)$.

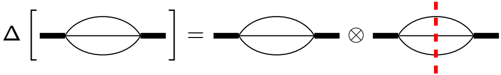
- Using the coassociativity property $\Delta(ab) = \Delta(a)\Delta(b)$ we can compute the coaction of Feynman integrals. Example:

$\Delta\Bigg[$  $\Bigg]$

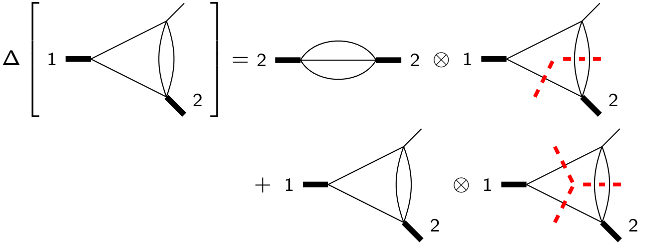
$$\begin{aligned} =& \Delta\left[e^{2\gamma_E \epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}(-p^2)^{-2\epsilon}\right] \\ =& e^{2\gamma_E \epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}(-p^2)^{-2\epsilon} \otimes e^{2\gamma_E \epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}(-p^2)^{-2\epsilon}\end{aligned}$$

Coaction of Two-Loop Graphs - Simple Cases

- Is there a diagrammatic coaction for polylogarithmic integrals at two-loops? Let's take some simple examples, starting with a sunset with no internal masses evaluated in $D = 2 - 2\epsilon$:



- How about a less trivial case, a triangle with a double edge. Choosing a pure integral with a numerator inserted:

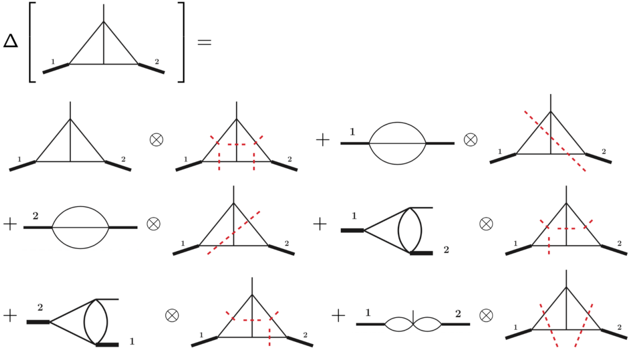


which follows from the coaction of the ${}_2F_1$

$$\Delta\, {}_2F_1(1-\epsilon, 1; 1+\epsilon; z) = 1 \otimes \frac{\epsilon}{(-1+2\epsilon)z}\, {}_2F_1\left(1-\epsilon, 1; 2-2\epsilon; 1-\frac{1}{z}\right) + {}_2F_1(1-\epsilon, 1; 1+\epsilon; z) \otimes \frac{\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)} z^{\epsilon} \left(1-\frac{1}{z}\right)^{2\epsilon}$$

Coaction of Two-Loop Graphs - Simple Cases 2

- We can proceed in the same fashion to graphs with more propagators, for instance:



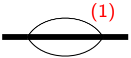
- This graph is expressed in terms of the function $_3F_2(a,b,c;d,e;z) = \frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)} \int_0^1 du u^{c-1}(1-u)^{e-c-1} {}_2F_1(a,b;d;uz)$
- So far we see a pattern similar to the one loop case without deformation terms

$$\Delta J_E = \sum_{\substack{\emptyset \subsetneq X \subseteq E \\ J_X \text{ has two loops}}} J_X \otimes \mathcal{C}_X J_E$$

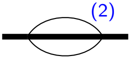
- A general two-loop graph has multiple master integrals. What happens in these cases?

Coaction of Two Loop Graphs - One Mass Sunset

- Consider the sunset graph with one internal mass and select a pure basis of integrals in $D = 2 - 2\epsilon$:

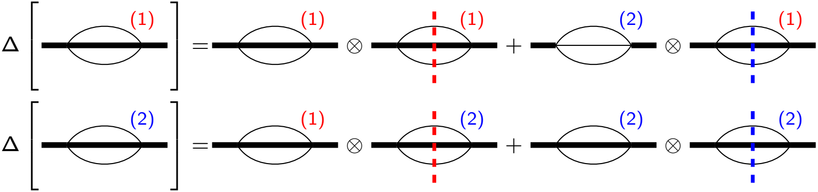


$$= e^{2\gamma_E \epsilon} \int \frac{d^D k_1}{i\pi^{D/2}} \int \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{k_1^2 k_2^2 [(k_1 + k_2 + p)^2 - m^2]}$$



$$= e^{2\gamma_E \epsilon} \int \frac{d^D k_1}{i\pi^{D/2}} \int \frac{d^D k_2}{i\pi^{D/2}} \frac{(k_1 + k_2)^2}{k_1^2 k_2^2 [(k_1 + k_2 + p)^2 - m^2]}$$

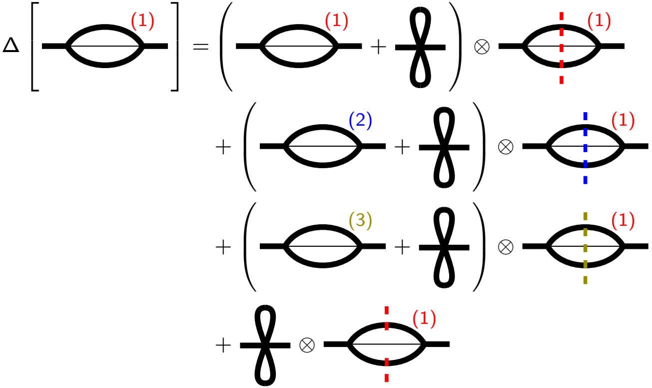
We can find cut contours dual to these master integrals and compute the coaction:

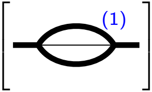


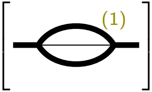
- So now the pairing between graphs and cuts in the coaction is preserved, but with a sum over a basis of integrals and cuts for each topology.
- The coaction of each master integral takes the same form, consistent with $\Delta \int_\gamma \omega = \sum \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$

Coaction of Two Loop Graphs - Two Mass Sunset

- We have not seen any deformation terms at two-loops. The first two-loop case with such a term is the two mass sunset. We can pick three pure master integrals and take their coactions:



with Δ 

and Δ 

taking similar forms.

- So there are deformation terms appearing for one loop subgraphs with an even number of propagators. Their coefficient in this case is different from the one loop case.

Conclusions and Outlook

Conclusions

- There is a well understood diagrammatic coaction for one-loop Feynman integrals.
- We can find a closed form expression for the coactions of hypergeometric functions that expand to polylogs.
- For the two-loop case, we observe the same pairing between graphs and cuts as at one-loop, but now with the potential for multiple master integrals at each topology. We are restricted to only two loop subgraphs in the coaction.

Outlook

- Does the coproduct suggest a canonical choice of basis for the master integrals and a canonical definition for the cuts that are dual to them?
- When do deformation terms occur at two-loops and what is the rule to determine their coefficients? Can we derive this using a condition $\int_{\gamma_i} \omega_j = \delta_{ij} + \mathcal{O}(\epsilon)$ like the one loop case?
- Is there a coaction on graphs that evaluate to elliptic polylogs and does it take the same diagrammatic form as we have seen for simpler cases?