Intersection Theory and Higgs physics

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September 10, 2019
Introduction

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Vector Space of Feynman Integrals and Multivariate Intersection Numbers

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INTEGRALS AND DIFFERENTIAL FORMS

In this work, we focus on integrals of the hypergeometric type.

\[
I = \int_c w(x) \, dx ,
\]
For state-of-the art two-loop scattering amplitude calculations
Feynman diagrams $\rightarrow \mathcal{O}(10000)$ Feynman integrals

Linear relations bring this down to $\mathcal{O}(100)$ *master integrals*
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Linear relations may be derived using IBP (integration by part) identities

$$\int \frac{d^d k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu} \frac{q^\mu N(k)}{D_{1}^{a_1}(k) \cdots D_{P}^{a_P}(k)} = 0$$

Systematic by Laporta’s algorithm $\Rightarrow$ Solve a huge linear system.
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Feynman diagrams $\rightarrow \mathcal{O}(10000)$ Feynman integrals

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Linear relations may be derived using IBP (integration by part) identities

$$\int \frac{d^d k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu} q^\mu N(k) \frac{D_{a_1}^{a_1}(k) \cdots D_{a_P}^{a_P}(k)}{D_{b_1}^{b_1}(k) \cdots D_{b_Q}^{b_Q}(k)} = 0$$

Systematic by Laporta’s algorithm $\Rightarrow$ Solve a huge linear system.

The linear relations are often informally referred to as IBPs as well.
The linear relations form a vector space

\[ I = \sum_{i \in \text{masters}} c_i I_i \]

Subsectors are sub-spaces.
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Subsectors are sub-spaces.

Not all vector spaces are inner product spaces

\[ \langle v \rangle = \sum_i \langle vw_j \rangle (C^{-1})_{ji} \langle v_i \rangle \quad \text{with} \quad C_{ij} = \langle v_i w_j \rangle \]

\[ = \sum_i c_i \langle v_i \rangle \]
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\[ = \sum_i c_i \langle v_i \mid \]

If only there were a way to define an inner product for Feynman integrals...
The loop-by-loop version of Baikov representation can often decrease

\[ I = \int \frac{d^d k_1}{\pi^{d/2}} \cdots \int \frac{d^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} = K \int_C d^n x \frac{B^\gamma(x)N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \]

The \( x_i \) are Baikov variables, \( B \) is the Baikov Polynomial, \( C = \{ B > 0 \} \).

\[ n = L(L+1)/2 + EL \quad \gamma = (d - E - L - 1)/2 \]

Baikov representation

\[ I = \int \frac{d^d k_1}{\pi^{d/2}} \cdots \int \frac{d^d k_L}{\pi^{d/2}} \frac{N(k)}{D^{a_1}_1(k) \cdots D^{a_P}_P(k)} = K \int C d^n x \frac{\mathcal{B}^\gamma(x) N(x)}{x^{a_1}_1 \cdots x^{a_P}_P} \]

The \( x_i \) are Baikov variables, \( \mathcal{B} \) is the Baikov Polynomial, \( C = \{ \mathcal{B} > 0 \} \).

\[ n = \frac{L(L+1)}{2} + EL \quad \gamma = \frac{(d-E-L-1)}{2} \]


The loop-by-loop version of Baikov representation can often decrease \( n \)

\[ I = \tilde{K} \int_C d^n x \left( \prod_{j=1}^{2L-1} \mathcal{B}_j^{\gamma_j}(x) \right) \frac{N(x)}{x^{a_1}_1 \cdots x^{a_P}_P} \]

Baikov representation

\[ I = \int \frac{d^d k_1}{\pi^{d/2}} \cdots \int \frac{d^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} = K \int_C d^n x \frac{B^\gamma(x)N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \]

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\[ I = \tilde{K} \int_C d^n x \left( \prod_{j=1}^{2L-1} B^\gamma_j(x) \right) \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \]


Baikov representation is suitable for *generalized unitarity cuts*

\[ \int d x \rightarrow \oint d x. \text{ Preserves linear relations.} \]

\[ I = \int_C d^n x \frac{B^\gamma(x)N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_C u \phi \]

\[ u = B^\gamma \text{ is a multivariate function in } \{x\} \]

\[ \phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} d x_1 \wedge \cdots \wedge d x_n \text{ is a form} \]
Theory

\[ I = \int_C d^n x \frac{B^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_C u \phi = \langle \phi | C \rangle \omega \]

\( u = B^\gamma \) is a multivariate function in \( \{ x \} \)

\( \phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} dx_1 \wedge \cdots \wedge dx_n \) is a form

\( \omega = d \log(u) \) is the twist

\( \langle \phi | C \rangle \omega \) is a pairing of a twisted cycle (C) and a twisted co-cycle (\( \phi \)) (equivalence classes of contours and integrands respectively)

\( \text{dim of the set of } \phi \text{s, is the number of master integrals.} \)
\[ I = \int_{\mathcal{C}} d^n x \frac{B^\gamma(x)N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u\phi = \langle \phi | \mathcal{C} \rangle_\omega \]

\[ u = B^\gamma \text{ is a multivariate function in } \{x\} \]
\[ \phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} dx_1 \wedge \cdots \wedge dx_n \text{ is a form} \]
\[ \omega = d \log(u) \text{ is the twist} \]

\[ \langle \phi | \mathcal{C} \rangle_\omega \text{ is a pairing of a twisted cycle } (\mathcal{C}) \text{ and a twisted co-cycle } (\phi) \]

(dim of the set of \(\phi\)s, is the number of master integrals.

*Lee Pomeransky criterion:*

\(\text{nr. of master integrals} = \text{nr. of solutions to } \omega = 0\)

The *intersection number* $\langle \phi | \xi \rangle$ is a pairing of a twisted co-cycle $\phi$ with a *dual* twisted co-cycle $\xi$.

Lives up to all criteria for being a scalar product.
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Lives up to all criteria for being a scalar product.

When there is one integration variable $z$ ($\phi$ and $\xi$ are one-forms)

$$\langle \phi | \xi \rangle_\omega = \sum_{p \in \mathcal{P}} \text{Res}_{z=p}(\psi_p \xi) \quad (d + \omega)\psi_p = \phi$$

$\mathcal{P}$ is the set of poles of $\omega$. 

References:

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References:


HF, F. Gasparotto, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera, *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*.
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HF, F. Gasparotto, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera, *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*. 
Summary of theory:

\[ I = \sum_{i \in \text{masters}} c_i I_i \iff \langle \phi | C \rangle = \sum_i c_i \langle \phi_i | C \rangle \]

with \( I = \int_C u \phi \). \( u \) is multivariate function, \( \phi \) is a form (rational pre-factor), \( \omega = \text{d log}(u) \).
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For one-forms:

\[ \langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p} (\psi_p \xi) \quad (d + \omega) \psi_p = \phi \]

solve with series ansatz
Example (double box)

Massless double box:

\[ I = \int d^8 x \frac{uN(x)}{x_1^{a_1} \cdots x_7^{a_1}} \rightarrow I_{7\times\text{cut}} = \int u_{7\times\text{cut}} \phi \quad u_{7\times\text{cut}} = z^{d/2-3}(z+s)^{2-d/2}(z-t)^{d-5} \]

\[ \omega = \left( \frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t} \right) dz \Rightarrow \nu = 2 \]
Example (double box)

Massless double box:

\[
I = \int d^8 x \frac{uN(x)}{x_1^a \ldots x_7^a} \quad \Rightarrow \quad I_{7\times\text{cut}} = \int u_{7\times\text{cut}} \phi \\
\omega = \left( \frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t} \right) dz \quad \Rightarrow \quad \nu = 2
\]

We want to reduce

\[
I_{1111111;1} = c_1 I_{1111111;0} + c_2 I_{1111111;1} + \text{lower}
\]

\[
\phi = z^2 \, dz, \quad \phi_1 = 1 \, dz, \quad \phi_2 = z \, dz, \quad \xi_1 = \left( \frac{1}{z} - \frac{1}{z+s} \right) \, dz, \quad \xi_2 = \left( \frac{1}{z+s} - \frac{1}{z-t} \right) \, dz,
\]

\[
c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad \text{with} \quad C_{ij} = \langle \phi_i | \xi_j \rangle
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Example (double box)

\[ c_i = \langle \phi|\xi_j \rangle (C^{-1})_{ji} \quad \text{with} \quad C_{ij} = \langle \phi_i|\xi_j \rangle \]

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We need 6 intersection numbers: \[ \{ \langle \phi|\xi_1 \rangle, \langle \phi|\xi_2 \rangle, \langle \phi_1|\xi_1 \rangle, \langle \phi_1|\xi_2 \rangle, \langle \phi_2|\xi_1 \rangle, \langle \phi_2|\xi_2 \rangle \} \]
\[ c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad \text{with} \quad C_{ij} = \langle \phi_i | \xi_j \rangle \]

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We need 6 intersection numbers: \( \{ \langle \phi | \xi_1 \rangle, \langle \phi | \xi_2 \rangle, \langle \phi_1 | \xi_1 \rangle, \langle \phi_1 | \xi_2 \rangle, \langle \phi_2 | \xi_1 \rangle, \langle \phi_2 | \xi_2 \rangle \} \)

Using \( \langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p} (\psi_p \xi) \) with \( (d + \omega) \psi_p = \phi \), we get

\[ \langle \phi | \xi_1 \rangle = \frac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)}, \]

\[ \langle \phi | \xi_2 \rangle = \frac{s(s+t)(3d-4)(3d-14)s + 2(d-6)(d-5)t)}{4(d-5)(d-4)(d-3)}, \]

\[ \langle \phi_1 | \xi_1 \rangle = -\frac{s}{d-5}, \quad \langle \phi_1 | \xi_2 \rangle = \frac{s+t}{d-5}, \]

\[ \langle \phi_2 | \xi_1 \rangle = \frac{s((3d-14)s + 2(d-5)t)}{2(d-5)(d-4)}, \quad \langle \phi_2 | \xi_2 \rangle = \frac{-(3d-14)s(s+t)}{2(d-5)(d-4)}. \]
\[ c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad \text{with} \quad C_{ij} = \langle \phi_i | \xi_j \rangle \]

\[ \phi = z^2 \, dz, \quad \phi_1 = 1 \, dz, \quad \phi_2 = z \, dz, \quad \xi_1 = \left( \frac{1}{z} - \frac{1}{z+s} \right) \, dz, \quad \xi_2 = \left( \frac{1}{z+s} - \frac{1}{z-t} \right) \, dz, \]

We need 6 intersection numbers: \( \left\{ \langle \phi | \xi_1 \rangle, \langle \phi | \xi_2 \rangle, \langle \phi_1 | \xi_1 \rangle, \langle \phi_1 | \xi_2 \rangle, \langle \phi_2 | \xi_1 \rangle, \langle \phi_2 | \xi_2 \rangle \right\} \)

Using \( \langle \phi | \xi \rangle = \sum_{p \in P} \text{Res}_{z=p}(\psi_p \xi) \) with \((d + \omega)\psi_p = \phi\), we get

\[ \langle \phi | \xi_1 \rangle = \frac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)}, \]

\[ \langle \phi | \xi_2 \rangle = \frac{s(s+t)(3d-4)(3d-14)s + 2(d-6)(d-5)t}{4(d-5)(d-4)(d-3)}, \]

\[ \langle \phi_1 | \xi_1 \rangle = -\frac{s}{d-5}, \quad \langle \phi_1 | \xi_2 \rangle = \frac{s+t}{d-5}, \]

\[ \langle \phi_2 | \xi_1 \rangle = \frac{s((3d-14)s + 2(d-5)t)}{2(d-5)(d-4)}, \quad \langle \phi_2 | \xi_2 \rangle = -\frac{(3d-14)s(s+t)}{2(d-5)(d-4)}. \]

\[ I_{1111111; -2} = c_0 I_{1111111; 0} + c_1 I_{1111111; -1} + \text{lower} \quad c_0 = \frac{(d-4)st}{2(d-3)}, \quad c_1 = \frac{2t - 3(d-4)s}{2(d-3)}, \]

in agreement with FIRE
Further cases

We did $O(30)$ examples in the paper arXiv:1901.11510
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Contributions to NLO Higgs+jet production

$H + j$ “Family A”:
Further cases

Contributions to NLO Higgs+jet production

\[ H + j \text{ “Family A”:} \]

\[
  u_{7 \times \text{cut}} = z^{d-5} (z^2 + sz + m_t^2) \frac{4-d}{2} \left( (m_H^2 - s) z^2 + 2(m_H^2 - s) stz + st \left( 4m_t^2 (m_H^2 - s - t) + st \right) \right)^{\frac{d-5}{2}}
\]

There are four master integrals.
Further cases

Contributions to NLO Higgs+jet production

\[ H + j \ “Family A”: \]

\[
u_{7x\text{cut}} = z^{d-5}(z^2 + sz + m_t^2 s) \frac{4-d}{2} \left( (m_H^2 - s)^2 z^2 + 2(m_H^2 - s) stz + st(4m_t^2 (m_H^2 - s - t) + st) \right)^{\frac{d-5}{2}}\]

There are four master integrals.

\[ I_{11111111;0} = c_1 I_{11111111;0} + c_2 I_{12111111;0} + c_3 I_{11112111;0} + c_4 I_{11111112;0} + \text{lower} \]

The intersection procedure gives cs in agreement with Kira.
Further cases

Contributions to NLO Higgs+jet production

\[ u_{7 \times \text{cut}} = z^{d-5} (z^2 + sz + m_t^2 s)^{\frac{4-d}{2}} \left( (m_H^2 - s)^2 z^2 + 2(m_H^2 - s) stz + st \left( 4m_t^2 (m_H^2 - s - t) + st \right) \right)^{\frac{d-5}{2}} \]

There are four master integrals.

\[ I_{1111111; -1} = c_1 I_{1111111; 0} + c_2 I_{1211111; 0} + c_3 I_{1111211; 0} + c_4 I_{1111112; 0} + \text{lower} \]

The intersection procedure gives \( c_s \) in agreement with Kira.

It also works for e.g. \( H+j \) fam. F

see arXiv:1907.13156 for fam. F. 
Does it only work for maximal cuts?

\[ \int_C u^\hat\varphi \, dn \, z = \sum_i c_i I_i \]

with

\[ c_i = \langle \varphi | \xi_j \rangle (C - 1)_{ji} C_{ij} \]

but now

\[ \langle \varphi | \xi \rangle \]

is a multivariate intersection number.
Does it only work for maximal cuts? NO!
Does it only work for maximal cuts? NO!

\[ I = \int_C u \hat{\phi} d^n z = \sum_i c_i I_i \quad \text{with} \quad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle \]

but now \( \langle \phi | \xi \rangle \) is a multivariate intersection number

multivariate

Does it only work for maximal cuts? NO!

\[ I = \int_C u \phi \, d^n z = \sum_i c_i I_i \quad \text{with} \quad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle \]

but now \( \langle \phi | \xi \rangle \) is a \textit{multivariate intersection number}


\[ n \langle \phi^{(n)} | \xi^{(n)} \rangle = - \sum_{p \in \mathcal{P}_n} \text{Res}_{z_n = p} \left( n - 1 \langle \phi^{(n)} | h_i^{(n-1)} \rangle \psi_i^{(n)} \right) , \]

\[ \left( \delta_{ij} \partial z_n - \hat{\Omega}^{(n)}_{ij} \right) \psi_j^{(n)} = \hat{\xi}_i^{(n)} , \]

\[ \hat{\Omega}^{(n)}_{ij} = - \left( C_{(n-1)}^{-1} \right)_{ik} n - 1 \langle e_k^{(n-1)} | (\partial z_n - \hat{\omega}_n) h_j^{(n-1)} \rangle , \]

\[ \hat{\xi}_i^{(n)} = \left( C_{(n-1)}^{-1} \right)_{ij} n - 1 \langle e_j^{(n-1)} | \xi^{(n)} \rangle , \]

\[ \left( C_{(n-1)} \right)_{ij} \equiv n - 1 \langle e_i^{(n-1)} | h_j^{(n-1)} \rangle . \]
We have done the full reduction of

\[
\nu = 3 \phi = (x_2 x_1) - \frac{1}{2} \phi_1 = (x_1 x_2 x_3 x_4) - \frac{1}{2} \phi_2 = (x_1 x_3) - \frac{1}{2} \phi_3 = (x_2 x_4)
\]

and

\[
c_i = \langle \phi | \xi_j \rangle \left( C^{-1} \right)_{ji}
\]
We have done the full reduction of

\[ u(x) = \left( (st - sx_4 - tx_3)^2 - 2tx_1(s(t+2x_3-x_2-x_4)+tx_3) \right. \]
\[ + s^2 x_2^2 + t^2 x_1^2 - 2sx_2\left(t(s-x_3)+x_4(s+2t)\right) \right)^{d-5} \]

In particular

\[ \nu = 3 \quad \phi = (x_1^2 x_2^2 x_3 x_4)^{-1} \quad \phi_1 = (x_1 x_2 x_3 x_4)^{-1} \quad \phi_2 = (x_1 x_3)^{-1} \quad \phi_3 = (x_2 x_4)^{-1} \]

\[ c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \]

with

The results are in agreement with FIRE.
\[ I = \int_C u \phi \quad \rightarrow \quad I = \sum_i c_i I_i \quad \text{with} \quad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle \]

Can find integral relations without the use of IBPs.
\[ I = \int_C u\phi \quad \rightarrow \quad I = \sum_i c_i I_i \quad \text{with} \quad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle \]

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“IBPs without IBPs”
\[ I = \int_C u\phi \quad \rightarrow \quad I = \sum_i c_i I_i \quad \text{with} \quad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle \]

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“IBPs without IBPs”

Future work:

- Full understanding of the multivariate case
Perspectives

\[ I = \int_C u\phi \quad \rightarrow \quad I = \sum_i c_i I_i \quad \text{with} \quad c_i = \langle \phi | \xi_j \rangle (C^{-1})_{ji} \quad C_{ij} = \langle \phi_i | \xi_j \rangle \]

Can find integral relations without the use of IBPs.

“IBPs without IBPs”

Future work:

- Full understanding of the multivariate case
- Classify hypergeometric functions (See Manoj’ talk)
- Clarify connection to co-action (see talks of J. Matthew and R. Britto)
- Make an optimized algorithm for sub-sectors
- Combine with rational reconstruction

Thank you for listening!

H. Frellesvig
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“IBPs without IBPs”

Future work:

- Full understanding of the multivariate case
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Hjalte Frellesvig