Intersection Theory and Higgs physics

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Introduction

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Decomposition of Feynman integrals on the maximal cut by intersection numbers

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ABSTRACT: We elaborate on the recent idea of a direct decomposition of Feynman integrals onto a basis of master integrals on maximal cuts using intersection numbers. We begin by showing an application of the method to the derivation of contiguity relations for special functions, such as the Euler beta function, the Gauss ${}_2F_1$ hypergeometric function, and the Appell F_1 function. Then, we apply the new method to decompose Feynman integrals whose maximal cuts admit 1-form integral representations, including examples that have from two to an arbitrary number of loops, and/or from zero to an arbitrary number of legs. Direct constructions of differential equations and dimensional recurrence relations for Feynman integrals are also discussed. We present two novel approaches to decomposition-by-intersections in cases where the maximal cuts admit a 2-form integral representation, with a view towards the extension of the formalism to n-form representations. The decomposition formulae computed through the use of intersection numbers are directly verified to agree with the ones obtained using integration-by-parts identities.

KEYWORDS: Scattering Amplitudes, Differential and Algebraic Geometry

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Vector Space of Feynman Integrals and Multivariate Intersection Numbers

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(Dated: July 4, 2019)

Feynman integrals obey linear relations governed by intersection numbers, which act as scalar products between vector spaces. We present a general algorithm for constructing multivariate intersection numbers relevant to Feynman integrals, and show for the first time how they can be used to solve the problem of integral reduction to a basis of master integrals by projections, and to directly derive functional equations fulfilled by the latter. We apply it to the derivation of contiguity relations for special functions admitting multi-fold integral representations, and to the decomposition of a few Feynman integrals at one- and two-loops, as first steps towards potential applications to generic multi-loop integrals.

INTRODUCTION

Scattering amplitudes encode crucial information about collision phenomena in our universe, from the smallest to the largest scales. Within the perturbative field-theoretical approach, the evaluation of multi-loop Feynman integrals is a mandatory operation for the determination of scattering amplitudes and related quantities.

Linear relations among Feynman integrals can be exploited to simplify the evaluation of scattering amplitudes: they can be used both for decomposing scattering amplitudes in terms of a basis of functions, referred to as *master integrals* (MIs), and for the evaluation of the latter. The standard procedure used to derive relations among Feynman integrals in dimensional regularization makes use of *integration-by-parts identities* (IBPs) [1], which are found by solving linear systems of equations [2] (see [3, 4] and references therein for reviews). Algebraic manipulations in these cases are very demanding, and efficient algorithms for solving large-size systems of linear equations have been recently devised, by making use of finite field arithmetic and rational functions reconstruction [5–7].

In [8], it was shown that *intersection numbers* [9] of differential forms can be employed to define (what amounts to) a *scalar product* on a *vector space* of Feynman integrals in a given family. Using this approach, projecting any multi-loop integral onto a basis of MIs is conceptually no different from decomposing a generic vector into a basis of a vector space. Within this new approach, relations among Feynman integrals can be derived avoiding the generation of intermediate, auxiliary expressions which are needed when applying Gauss elimination, as in the standard IBP-based approaches.

In the initial studies, [8, 10], this novel decomposition method was applied to the realm of special mathematical functions falling in the class of Lauricella functions, as well as to Feynman integrals on maximal cuts, *i.e.* with on-shell internal lines, mostly admitting a one-fold integral representation. Those results concerned a partial construction of Feynman integral relations, mainly limited to the determination of the coefficients of the MIs with the same number of denominators as the integral to decompose, which was achieved by means of intersection numbers for *univariate* forms.

In this paper, we make an important step further, and address the *complete* integral reduction, for the determination of *all* coefficients, including those associated to MIs corresponding to sub-graphs. In the current work, we discuss the one-loop massless four-point integral as a paradigmatic case, although the algorithm has been successfully applied to several other cases at one- and two-loop.

Generic Feynman integrals admit multi-fold integral representations. Their complete decomposition requires the evaluation of intersection numbers for multivariate rational differential forms. Intersection numbers of multivariate forms have been previously studied in [11–19]. Recently, a new recursive algorithm was introduced in [20]. In this letter, we present its refined implementation and application to Feynman integrals, which provide a major step towards large-scale applicability of our strategy for the reduction to MIs. The results of this work show potential for further applications ranging from particle physics, through condensed matter and statistical mechanics, to gravitational-wave physics, while making new connections to mathematics.

INTEGRALS AND DIFFERENTIAL FORMS

In this work, we focus on integrals of the hypergeometric type,

$$I = \int_{\mathcal{C}} u(\mathbf{z})\, \varphi(\mathbf{z}) \,, \qquad (1)$$

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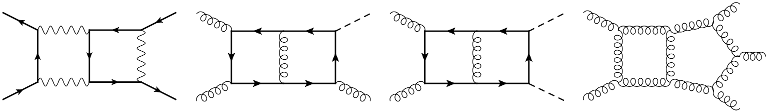
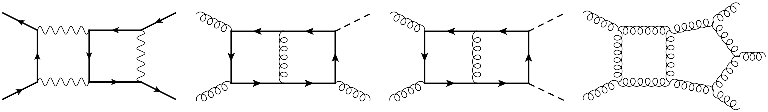


For state-of-the art two-loop scattering amplitude calculations
Feynman diagrams $\rightarrow \mathcal{O}(10000)$ Feynman integrals

Linear relations bring this down to $\mathcal{O}(100)$ *master integrals*



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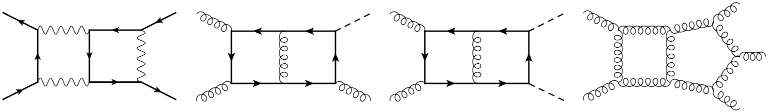
Linear relations may be derived using IBP (integration by part) identities

$$\int \frac{\mathrm{d}^d k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu} \frac{q^\mu N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} = 0$$

Systematic by Laporta's algorithm $\Rightarrow$ Solve a huge linear system.



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Systematic by Laporta's algorithm $\Rightarrow$ Solve a huge linear system.

The linear relations are often informally referred to as IBPs as well.

The linear relations form a vector space

$$I = \sum_{i \in \text{masters}} c_i I_i$$

Subsectors are sub-spaces.

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Not all vector spaces are *inner product spaces*

$$\langle v| = \sum_i \langle v w_j \rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle v_i| \qquad \text{with} \qquad \mathbf{C}_{ij} = \langle v_i w_j \rangle$$

$$= \sum_i c_i \, \langle v_i|$$

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If only there were a way to define an inner product for Feynman integrals...

Baikov representation

$$I = \int \frac{\mathrm{d}^d k_1}{\pi^{d/2}} \cdots \int \frac{\mathrm{d}^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} \ = \ K \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

The x_i are Baikov variables, $\mathcal{B}$ is the Baikov Polynomial, $\mathcal{C} = \{\mathcal{B} > 0\}$.

$$n = L(L{+}1)/2{+}EL \qquad \gamma = (d{-}E{-}L{-}1)/2$$

P. Baikov: *Nucl. Instrum. Meth.A* **389** (1997) 347–349, [hep-ph/9611449]

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The loop-by-loop version of Baikov representation can often decrease n

$$I = \tilde{K} \int_{\mathcal{C}} \mathrm{d}^{\tilde{n}} x \, \frac{\left(\prod_{j=1}^{2L-1} \mathcal{B}_j^{\gamma_j}(x)\right) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

HF and C. Papadopoulos, *JHEP* **04** (2017) 083, [arXiv:1701.07356]

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Baikov representation is suitable for *generalized unitarity cuts*

$\int \mathrm{d}x \to \oint \mathrm{d}x$. Preserve linear relations.

J. Bosma, M. Søgaard, Y. Zhang, *JHEP* **08** (2017) 051, [arXiv:1704.04255]

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$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u \phi$$

$u = \mathcal{B}^\gamma$ is a multivariate function in $\{x\}$

$\phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a form

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$\omega = \mathrm{d}\log(u)$ is *the twist*

$\langle \phi | \mathcal{C}]_\omega$ is a pairing of a *twisted cycle* ($\mathcal{C}$) and a *twisted co-cycle* (ϕ) (equivalence classes of contours and integrands respectively)

dim of the set of ϕs, is the number of master integrals.

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Lee Pomeransky criterion:
nr. of master integrals = nr. of solutions to $\omega = 0$

R. Lee and A. Pomeransky, *JHEP* **11** (2013) 165, [arXiv:1308.6676].

The *intersection number* $\langle\phi|\xi\rangle$ is a pairing of a twisted co-cycle ϕ with a *dual* twisted co-cycle ξ.

Lives up to all criteria for being a scalar product.

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When there is one integration variable z (ϕ and ξ are one-forms)

$$\langle\phi|\xi\rangle_\omega = \sum_{p\in\mathcal{P}} \mathsf{Res}_{z=p}(\psi_p\xi) \qquad\qquad (\mathrm{d}+\omega)\psi_p = \phi$$

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HF, F. Gasparotto, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera, *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*.

Summary of theory:

$$I = \sum_{i \in \text{masters}} c_i I_i \quad \Leftrightarrow \quad \langle \phi | \mathcal{C}] = \sum_i c_i \langle \phi_i | \mathcal{C}]$$

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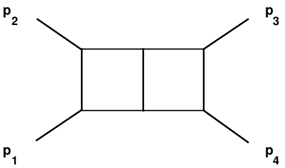
For one-forms:

$$\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p}(\psi_p \xi) \qquad (\mathrm{d} + \omega)\psi_p = \phi$$

solve with series ansatz

Example (double box)

Massless double box:



$$I = \int \mathrm{d}^8 x \, \frac{u N(x)}{x_1^{a_1} \cdots x_7^{a_1}} \quad \rightarrow \quad I_{7\times\mathsf{cut}} = \int u_{7\times\mathsf{cut}} \, \phi \qquad u_{7\times\mathsf{cut}} = z^{d/2-3}(z{+}s)^{2-d/2}(z{-}t)^{d-5}$$

$$\omega = \left(\frac{d-6}{2z} + \frac{4-d}{2(z{+}s)} + \frac{d-5}{z{-}t} \right) \mathrm{d}z \qquad \Rightarrow \qquad \nu = 2$$

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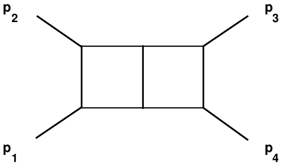

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We want to reduce

$$I_{1111111;-2} = c_1 I_{1111111;0} + c_2 I_{1111111;-1} + \text{lower}$$

$$\phi = z^2 \, \mathrm{d}z \,, \quad \phi_1 = 1 \, \mathrm{d}z \,, \quad \phi_2 = z \, \mathrm{d}z \,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z{+}s} \right) \mathrm{d}z \,, \quad \xi_2 = \left(\tfrac{1}{z{+}s} - \tfrac{1}{z{-}t} \right) \mathrm{d}z \,,$$

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We need 6 intersection numbers: $\Big\{ \langle \phi | \xi_1 \rangle, \, \langle \phi | \xi_2 \rangle, \, \langle \phi_1 | \xi_1 \rangle, \, \langle \phi_1 | \xi_2 \rangle, \, \langle \phi_2 | \xi_1 \rangle, \, \langle \phi_2 | \xi_2 \rangle \Big\}$

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Using $\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \mathsf{Res}_{z=p} (\psi_p \xi)$ with $(\mathrm{d} + \omega) \psi_p = \phi$, we get

$$\langle \phi | \xi_1 \rangle = \tfrac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi | \xi_2 \rangle = \tfrac{s(s+t)(3(d-4)(3d-14)s + 2(d-6)(d-5)t)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi_1 | \xi_1 \rangle = \tfrac{-s}{d-5} \,, \qquad \langle \phi_1 | \xi_2 \rangle = \tfrac{s+t}{d-5} \,,$$

$$\langle \phi_2 | \xi_1 \rangle = \tfrac{s((3d-14)s + 2(d-5)t)}{2(d-5)(d-4)} \,, \qquad \langle \phi_2 | \xi_2 \rangle = \tfrac{-(3d-14)s(s+t)}{2(d-5)(d-4)} \,.$$

Example (double box)

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$$\langle \phi | \xi_1 \rangle = \tfrac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi | \xi_2 \rangle = \tfrac{s(s+t)(3(d-4)(3d-14)s + 2(d-6)(d-5)t)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi_1 | \xi_1 \rangle = \tfrac{-s}{d-5} \,, \qquad \langle \phi_1 | \xi_2 \rangle = \tfrac{s+t}{d-5} \,,$$

$$\langle \phi_2 | \xi_1 \rangle = \tfrac{s((3d-14)s + 2(d-5)t)}{2(d-5)(d-4)} \,, \qquad \langle \phi_2 | \xi_2 \rangle = \tfrac{-(3d-14)s(s+t)}{2(d-5)(d-4)} \,.$$

$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower} \qquad c_0 = \frac{(d-4)st}{2(d-3)} \,, \quad c_1 = \frac{2t - 3(d-4)s}{2(d-3)} \,,$$

in agreement with FIRE

Further cases

We did $\mathcal{O}(30)$ examples in the paper arXiv:1901.11510

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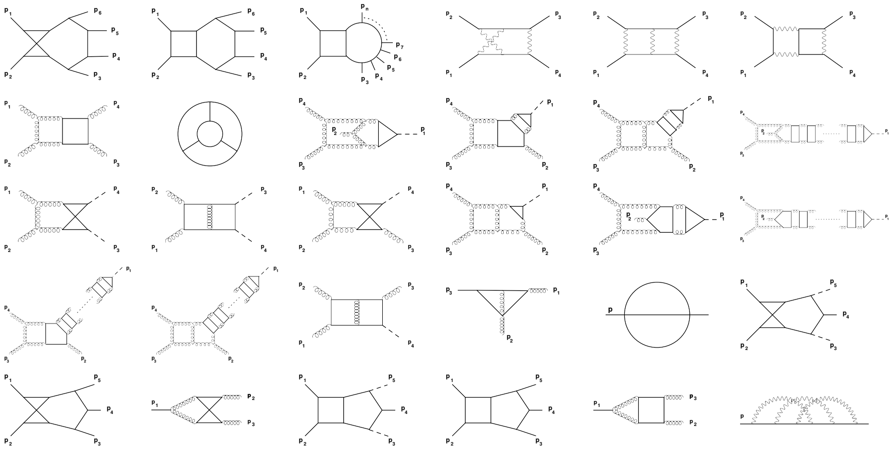
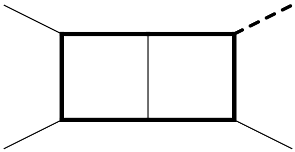


Contributions to NLO Higgs+jet production

$H + j$ "Family A":



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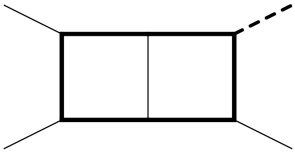

$$u_{7\times\mathsf{cut}} = z^{d-5}(z^2+sz+m_t^2 s)^{\frac{4-d}{2}}\Big((m_H^2-s)^2z^2 + 2(m_H^2-s)stz + st\big(4m_t^2(m_H^2-s-t)+st\big)\Big)^{\frac{d-5}{2}}$$

There are four master integrals.

Further cases

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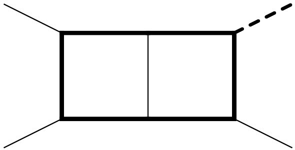

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The intersection procedure gives cs in agreement with Kira.

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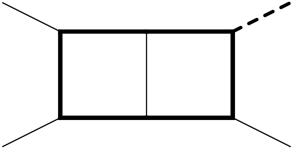

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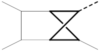
It also works for e.g. $H+j$ fam. F  $(6 \to 4)$

see arXiv:1907.13156 for fam. F.

Does it only work for maximal cuts?

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$$I = \int_{\mathcal{C}} u \hat{\phi} \, \mathrm{d}^n z = \sum_i c_i I_i \qquad \text{with} \qquad c_i = \langle \phi | \xi_j \rangle (\mathbf{C}^{-1})_{ji} \quad \mathbf{C}_{ij} = \langle \phi_i | \xi_j \rangle$$

but now $\langle \phi | \xi \rangle$ is a *multivariate intersection number*

K. Matsumoto, *Intersection numbers for logarithmic k-forms*, Osaka J. Math. **35** (1998) no. 4 873-893

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$$_{\mathbf{n}}\langle \phi^{(\mathbf{n})} | \xi^{(\mathbf{n})} \rangle = -\sum_{p \in \mathcal{P}_n} \operatorname*{Res}_{z_n = p} \Big({}_{\mathbf{n-1}}\langle \phi^{(\mathbf{n})} | h_i^{(\mathbf{n-1})} \rangle \, \psi_i^{(n)} \Big) \ ,$$

$$\Big(\delta_{ij}\, \partial_{z_n} - \hat{\boldsymbol{\Omega}}_{ij}^{(n)} \Big) \psi_j^{(n)} = \hat{\xi}_i^{(n)} \ ,$$

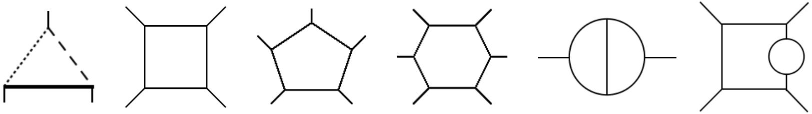
$$\hat{\boldsymbol{\Omega}}_{ij}^{(n)} = -\big(\mathbf{C}_{(\mathbf{n-1})}^{-1} \big)_{ik} \ {}_{\mathbf{n-1}}\langle e_k^{(\mathbf{n-1})} | (\partial_{z_n} - \hat{\omega}_n) h_j^{(\mathbf{n-1})} \rangle \ ,$$

$$\xi_i^{(n)} = \big(\mathbf{C}_{(\mathbf{n-1})}^{-1} \big)_{ij} \ {}_{\mathbf{n-1}}\langle e_j^{(\mathbf{n-1})} | \xi^{(\mathbf{n})} \rangle \ ,$$

$$\big(\mathbf{C}_{(\mathbf{n-1})} \big)_{ij} \equiv {}_{\mathbf{n-1}}\langle e_i^{(\mathbf{n-1})} | h_j^{(\mathbf{n-1})} \rangle \ .$$

multivariate

We have done the full reduction of



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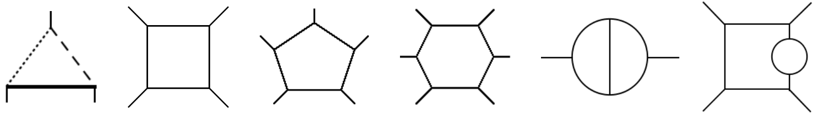
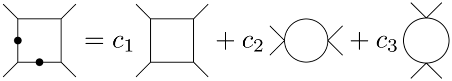


In particular



$$u(\mathbf{x}) = \Big((st - sx_4 - tx_3)^2 - 2tx_1\big(s(t + 2x_3 - x_2 - x_4) + tx_3\big) + s^2x_2^2 + t^2x_1^2 - 2sx_2\big(t(s - x_3) + x_4(s + 2t)\big)\Big)^{\frac{d-5}{2}}.$$

$$\nu = 3 \quad \phi = (x_1^2 x_2^2 x_3 x_4)^{-1} \quad \phi_1 = (x_1 x_2 x_3 x_4)^{-1} \quad \phi_2 = (x_1 x_3)^{-1} \quad \phi_3 = (x_2 x_4)^{-1}$$

$$c_i = \langle \phi | \xi_j \rangle (\mathbf{C}^{-1})_{ji} \quad \text{with} \quad \mathbf{C}_{ij} = \langle \phi_i | \xi_j \rangle$$

The results are in agreement with FIRE

Perspectives

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Thank you for listening!

Hjalte Frellesvig

Università degli Studi di Padova

Marie Curie Actions