The quark Condensate from 5-loop RG Optimized Spectral Density

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1. Introduction/Motivations

Unconventional use of perturbative expansions can give nonperturbative approximations

Versatile: relevant both at $T = 0$ or $T \neq 0$ (and/or finite density)

In this talk is considered only $T = 0$ QCD
previous context: estimate with our approach the order parameter $F_\pi(m_q = 0)/\Lambda^{\rm QCD}_{\overline{MS}}$:
$F_\pi \simeq 92.2\text{MeV} \to F_\pi(m_q = 0) \to \Lambda^{n_f=3}_{\overline{MS}} \to \alpha_S^{\overline{MS}}(\mu = m_Z)$.

N^3LO: $F_\pi^{m_q=0}/\Lambda^{n_f=3}_{\overline{MS}} \simeq 0.25 \pm .01 \to \alpha_S(m_Z) \simeq 0.1174 \pm .001 \pm .001$
(JLK, A.Neveu, PRD88 (2013))
(compares well with latest (2018) α_S lattice and world averages [PDG2018])

Here (preliminary results) on N^4LO (5-loops) upgrade of our previous $\langle \bar{q}q \rangle$ determination (JLK, A.Neveu, PRD 92 (2015))
NB heavily relies on perturbative ingredients from recently available *complete* 5-loop QCD RG functions

Chiral Symmetry Breaking (CSB) Order parameters

-Well-known facts:

1. $\langle \bar{q}q \rangle^{1/3}$, F_π,... $\sim \mathcal{O}(\Lambda_{QCD}) \simeq 330$ MeV
$\rightarrow$ large α_S at very low scale $\rightarrow$ invalidates perturbative expansion

2. F_π, $\langle \bar{q}q \rangle$,... anyway vanishing in standard perturbation:
e.g. $\langle \bar{q}q \rangle_{pert} \sim m_q^3 \sum_{n,p} \alpha_s^n \ln^p(m_q) \rightarrow 0$ for $m_q \rightarrow 0$
at any perturbative order (trivial chiral limit)

$\rightarrow$ CSB parameters are "intrinsically" NON perturbative

•Optimized perturbation (OPT): basically an (old) trick to circumvent 2.:
gives a nontrivial result for $m_q \rightarrow 0$, starting from perturbative content.

•Our more recent RG(OPT): reconciles OPT with RG invariance;
+ appears to moderate (partly circumvent) 1.

$\langle \bar{q}q \rangle$ indirect determination: Gell-Mann Oakes Renner (GMOR) relation:

$$F_\pi^2 m_\pi^2 = -(m_u + m_d)\langle \bar{q}q \rangle + \mathcal{O}(m_q^2);$$

($m_{u,d}$ from lattice or spectral sum rules).
Or $\langle \bar{q}q \rangle$ from 'first principles' BUT in simplified effective models
(Nambu-Jona-Lasinio, approximated Schwinger-Dyson Eqs.,...)
OR directly on the lattice (many recent works, e.g. Engel et al '14)

2. Optimized (or Screened) Perturbation (OPT/SPT)

Trick: add and subtract a *fake* mass m, consider $m\,\delta$ as interaction:

$$\mathcal{L}_{QCD}(m,g) \to \mathcal{L}_{QCD}(m(1-\delta), \delta\, g) \;\; \text{(NB for QCD } g \equiv 4\pi\alpha_S\text{)}$$

where $0 < \delta < 1$ interpolates between $\mathcal{L}_{free}$ and *massless* $\mathcal{L}_{int}$;
e.g. (quark) mass $m_q \to m$: auxiliary (trial) parameter

$\to$ Take any standard (renormalized) pert. series, (re)expand in δ *after:*
$m_q \to m\,(1-\delta)$; $\quad g \to \delta\, g$
finally take $\delta \to 1$ (to recover original massless theory):

•BUT m-dependence remains at any finite δ^k-order:
fixed by stationarity prescription= optimization (OPT):
$\frac{\partial}{\partial m}$(physical quantity) $= 0$ for $m = \tilde{m}_{opt}(g) \neq 0$:

•gives 2 solutions: trivial ($m = 0$), OR nontrivial *dressed* mass $\tilde{m}(g)$:
$\tilde{m} = \mathcal{O}(\mu\, e^{-1/(\beta_0 g)})$ = number $\times\, \Lambda_{QCD}$

•$T \neq 0$ similar idea: "screened perturbation" (SPT), or *resummed* "hard thermal loop (HTLpt)" (QCD) = expand around a quasi-particle mass.

Does this 'cheap trick' always work?

•Exponentially fast convergence of this procedure for $D = 1$ ϕ^4 oscillator (cancels large pert. order factorial divergences) Guida et al '95

•In Quantum Field Theories (QFT):

May be viewed as enforcing approximately at low orders the expected (all order) scale-invariance (massless limit!)

(NB *genuine* masses (e.g. $m_{quarks} \neq 0$) neglected/treated as small perturbations)

•Property: in $O(N), SU(N), \cdots$ models, at one-loop it incorporates (more easily) leading $1/N$ expansion ('cactus' or 'foam' graphs)

•At $T \neq 0$, sensibly improves the generically unstable + badly scale-dependent thermal perturbative expansions (JLK, M.B Pinto '15, '16)

•But QFT multi-loop calculations (specially $T \neq 0$) (very) difficult: $\rightarrow$ empirical convergence? not clear

•Other pb at higher order: OPT: $\partial_m(...) = 0$ has multi-solutions (some non-real!), how to choose right one without nonperturbative insight?

3. RG-compatible OPT ($\equiv$ RGOPT) (JLK, A. Neveu '2010)

Consider a *physical* quantity build from (RG invariant) perturbation
e.g. in present context $P(m,g) = m\langle\bar{q}q\rangle(m,g)$:

in addition to 'OPT' Eq.: $\frac{\partial}{\partial\, m} P^{(k)}(m,g,\delta=1)|_{m\equiv\tilde{m}} \equiv 0$,

Require (δ-modified!) $P(m,g)$ at order δ^k to satisfy (perturbative) Renormalization Group (RG) equation:

$$\mathrm{RG}\left(P^{(k)}(m,g,\delta=1)\right) = 0$$

$$\mathrm{RG} \equiv \mu\frac{d}{d\,\mu} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_m(g)\, m\frac{\partial}{\partial m}$$

$\beta(g) \equiv -2b_0 g^2 - 2b_1 g^3 + \cdots\,,\;\; \gamma_m(g) \equiv \gamma_0 g + \gamma_1 g^2 + \cdots$

$\rightarrow$ combined with OPT, RG Eq. reduces to massless form:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right] P^{(k)}(m,g,\delta=1) = 0$$

•Using both OPT AND RG completely fix $m \equiv \bar{m}$ and $g \equiv \bar{g}$.

$\rightarrow$ Parameter-free determination: $\bar{m}(g)$ = "auxiliary" RG-dressed mass but final (physical) result from $P(\bar{m},\bar{g}) \rightarrow P(\Lambda^{QCD}_{\overline{\rm MS}})$ only

OPT + RG = RGOPT main features

•Basic interpolation: why not $m \to m\,(1-\delta)^a$?
Most previous works ($T=0$ OPT, $T\neq 0$ Screened PT, HTLpt) do $a=1$
but generally $a=1$ spoils RG invariance!

•Standard OPT gives multiple $\bar{m}(g,T)$ solutions at increasing δ^k-orders

$\to$ Our approach restores RG, +requiring matching to perturbation
(i.e. Asymptotic Freedom (AF) for QCD):
$\ln\frac{\mu}{\bar{m}} \sim \frac{1}{b_0\, g} + \cdots$ for $g \to 0$, $\bar{m} \sim \Lambda_{QCD}$

$\to$ At successive orders, AF-compatible optimal $\bar{m}$ (often unique) *only* appears for a universal critical a:
$m \to m\,(1-\delta)^{\frac{\gamma_0}{b_0}}$ (in general $\frac{\gamma_0}{b_0} \neq 1$)

$\to$ RG consistency goes beyond simple "add and subtract" trick

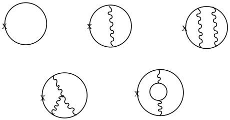
and removes any spurious solutions (incompatible with AF)

•But does not always avoid nonreal $\bar{m}$ solutions at high orders
(NB artifact of solving exactly (polynomial) RG +OPT Eqs.:
possibly cured by more perturbative approximations)

4. Perturbative QCD quark condensate $\langle \bar{q} q \rangle$

Chiral symmetry breaking order parameter:
$SU(n_f)_L \times SU(n_f)_R \to SU(n_f)_{L+R}$, n_f massless quarks. ($n_f = 2, 3$)

Perturbatively (some samples to 3-loops):



$$m \, \langle \bar{q} q \rangle (m, g)_{\overline{MS}} = 3 \frac{m^4}{2\pi^2} \left(\frac{1}{2} - \ln \frac{m}{\mu} + \alpha_s(\cdots) + \alpha_s^2(\cdots) \text{(details below)} \right)$$

NB: not finite after mass and coupling renormalization $\to$ needs extra (additive) renormalization $\to$ finite part is not separately RG-inv: related to vacuum energy anomalous dimension $\Gamma_0^{\mathcal{E}_0}$:

$$\mu \frac{d}{d\,\mu} (m \langle \bar{q} q \rangle) = -4 m^4 \Gamma_0^{\mathcal{E}_0} (\alpha_S)$$

3-loop: Chetyrkin, Kühn '94 ($\ln m$ via $\Gamma_0^{\mathcal{E}_0}$); non-log term: Chetyrkin, Maier '10
Higher (4, 5-loop) order status: details below

First attempt: direct RGOPT of $m\langle\bar{q}q\rangle$?

In principle one may apply RGOPT directly on $m\langle\bar{q}q\rangle(m,g)$:

first order (one-loop): NO non-trivial solution

Higher RGOPT orders (2- and 3-loops): right order of magnitude, but ambiguous: plagued by large, unphysical, imaginary parts
$\rightarrow$ no conclusive stability/convergence trend (appears slow at best)

Problems traced to strong sensitivity to (vacuum energy) anomalous dimensions, related to original quadratic divergences of the condensate:

with a cutoff the (dominant) one-loop quadratic divergence has correct (negative) sign (pillar of the success of Nambu-Jona-Lasinio model!)
but sign flips in dimensional regularization ($\overline{\text{MS}}$)

Yet important to keep benefits of $\overline{\text{MS}}$: high order perturbative calculations available: crucial for stability/convergence check.

$\rightarrow$ Like any other variational methods, it's sensible to start from a suitable quantity: we use the spectral density of the Dirac operator, intimately related to $\langle\bar{q}q\rangle$.

4. $\langle \bar{q}q \rangle$ and Spectral density $\rho(\lambda)$

Euclidean Dirac operator:
$i \not{D}\, u_n(x) = \lambda_n\, u_n(x)$; $\not{D} \equiv \not{\partial} + g \not{A}$;
NB $i \not{D}\, (\gamma_5 u_n(x)) = -\lambda_n\, (\gamma_5 u_n(x))$

On a lattice: $\rho(\lambda) \equiv \frac{1}{V} \langle \sum_n \delta(\lambda - \lambda_n^{[A]}) \rangle_A$

$V \to \infty$: spectrum becomes dense, and
$\langle \bar{q}q \rangle \equiv \frac{1}{V}\, \mathrm{Tr} \frac{1}{m + \not{D}} \to \langle \bar{q}q \rangle_{V \to \infty}(m) \equiv -2m \int_0^\infty d\lambda \frac{\rho(\lambda)}{\lambda^2 + m^2}$

$\rho(\lambda)$: spectral density of the (euclidean) Dirac operator.

Banks-Casher relation (1980): $\langle \bar{q}q \rangle (m \to 0) \equiv -\pi \rho(0)$
(using e.g. $\lim_{m \to 0} \frac{1}{m - i\lambda} = i\, PV(\frac{1}{\lambda}) + \pi \delta(\lambda)$)

'Washes out' large λ problems (e.g. quadratic UV divergences)

Conversely: $-\rho(\lambda) = \frac{1}{2\pi} \left(\langle \bar{q}q \rangle (i\lambda + \epsilon) - \langle \bar{q}q \rangle (i\lambda - \epsilon) \right) |_{\epsilon \to 0}$
i.e. $\rho(\lambda)$ determined by discontinuities of $\langle \bar{q}q \rangle (m)$ across imaginary axis.

Perturbative expansion: $\to \ln(m \to i\lambda)$ discontinuities
$\to$ no contributions from divergence and non-log terms (like anom. dim.)

Adapting OPT and RG Equations to spectral density

- Perturbative logarithmic discontinuities simply from

$$\ln^n\left(\frac{m}{\mu}\right) \to \frac{1}{2\mathrm{i}\pi}\left[\left(\ln\frac{|\lambda|}{\mu}+\mathrm{i}\frac{\pi}{2}\right)^n - \left(\ln\frac{|\lambda|}{\mu}-\mathrm{i}\frac{\pi}{2}\right)^n\right] \quad (1)$$

$i.e.$ $\ln\left(\frac{m}{\mu}\right) \to 1/2;\;\; \ln^2\left(\frac{m}{\mu}\right) \to \ln\frac{|\lambda|}{\mu};\;\; \ln^3\left(\frac{m}{\mu}\right) \to \frac{3}{2}\ln^2\frac{|\lambda|}{\mu} - \frac{\pi^2}{8};\cdots$

- Modified perturbation: intuitively λ plays the role of m, so:

$$\rho_{pert}(\lambda, g) \to \rho_{opt}(\lambda(1-\delta)^{\frac{4}{3}\frac{\gamma_0}{2b_0}},\, \delta g)\,;\ \text{expand in } \delta\,;\;\; \delta \to 1 \quad (2)$$

- OPT Eq.: $$\frac{\partial}{\partial\lambda}\rho_{opt}(g,\lambda) = 0 \text{ for } \lambda = \tilde{\lambda}_{opt}(g) \neq 0 \quad (3)$$

- one finds $\rho(\lambda)$ obeys RG eq.:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_m(g)\,\lambda\frac{\partial}{\partial\lambda} - \gamma_m(g)\right]\rho(g,\lambda) = 0 \quad (4)$$

$\to$ well-defined RGOPT recipe: -$\langle\bar{q}q\rangle_{pert}(m, g) \to \rho_{pert}(\lambda, g)$ from (1);
-perform (2) to order δ^k, g^k; -solve (3), (4) for optimal $\tilde{\lambda}$, $\tilde{g}$;
then $\rho(\tilde{\lambda}, \tilde{g}) \simeq \rho(0) \equiv -\langle\bar{q}q\rangle(m_q = 0)/\pi$.

5-loops condensate + spectral density: main ingredients

•4, 5-loops: $\ln^p(m)$, $p > 1$ determined from lowest orders from RG invariance

•single $\ln(m)$: determined from vacuum energy anomalous dimension:

$$\mu \frac{d}{d\,\mu}(m\langle \bar{q}q\rangle) = -4m^4 \Gamma_0^{\mathcal{E}_0}(\alpha_S)$$

crucial ingredient: 5-loops recently completed!:
Baikov, Chetyrkin PoS RADCOR2017 (2018) 025

•recently determined 5-loop RG $\beta(\alpha_S)$ and mass. dim. $\gamma_m(\alpha_s)$:
β: Baikov, Chetyrkin, Kühn 2016, Herzog, Ruijl, Ueda, Vermaseren, Vogt, 2017 Luthe, Maier, Marquard, Schröder, 2016, 2017;
γ_m: Baikov, Chetyrkin, Kühn, 2014, Luthe, Maier, Marquard, Schröder, 2016.

This completes the determination of all the $\ln m$ coefficients to 5-loops.
However...

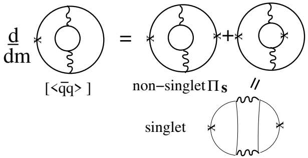
5-loops condensate + spectral density: main ingredients

... we also need the non-logarithmic 4-loop coefficient, not given by RG:
For this note the (exact) relation:

$$\frac{\partial}{\partial m}\left(\langle \bar{q}q\rangle(m)\right) = -\Pi_s(q^2=0)$$

$\Pi_s(q^2) \equiv i \int d^4x e^{iq.x}\langle 0|TJ^s(x)J^s(0)|0\rangle$: two-point scalar correlator

$\rightarrow$ relate needed 4-loop $\langle \bar{q}q\rangle(m)$ term to 4-loop Π_s (non-log term):
this was calculated by C. Sturm [2008]:
But (as is common) neglecting presumably smaller singlet contributions to $\Pi_s(0)$: needed however for the condensate:



$\rightarrow$ Accordingly our approximation (at present): neglect this unknown singlet part (only impacts the 5-loop single $\ln m$ term, details below)
NB not very difficult to calculate in principle...

Perturbative quark condensate at 5-loops

Putting all together we obtain

$$m\,\langle\bar{q}q\rangle(m,\alpha_s) = (\frac{3}{2\pi^2})\,m^4\,\Big\{\frac{1}{2} - L_m + 4(\frac{\alpha_S}{\pi})(L_m^2 - \frac{5}{6}L_m + \frac{5}{12})$$

$$+(\frac{\alpha_S}{\pi})^2\sum_{i=0}^{3} c_{3i}L_m^{3-i} + (\frac{\alpha_S}{\pi})^3\sum_{i=0}^{4} c_{4i}L_m^{4-i} + (\frac{\alpha_S}{\pi})^4\sum_{i=0}^{5} c_{5i}L_m^{5-i}\Big\}\ ,\ L_m \equiv \ln\frac{m}{\mu}$$

3-loops: $c_{30} = -18 + \frac{4}{9}\,n_f\ ,\quad c_{31} = \frac{94}{3} - \frac{10}{9}\,n_f\ ,$

$c_{32} = -29.7959 + 3.25 n_h\ ,\quad c_{33} = 16.4566 + 0.133755\,n_h$

4-loops: $c_{40} = 85.5 - 5.11111 n_f + 0.0740741 n_f^2\ ,$

$c_{41} = -224.333 + 16.7037 n_f - 0.246914 n_f^2\ ,$

$c_{42} = 342.151 - 51.1008 n_h + 0.975309 n_h^2\ ,$

$c_{43} = -375.082 + 42.6214 n_h - 0.0382074 n_h^2$

5-loops: $c_{50} = -418.95 + 42.1444 n_f - 1.38519 n_f^2 + 0.0148148 n_f^3$

$c_{51} = 1469.29 - 173.995 n_f + 6.01543 n_f^2 - 0.0617284 n_f^3$

$c_{52} = -3079.72 + 591.833 n_f - 25.9284 n_f^2 + 0.325103 n_f^3$

$c_{53} = 5102.45 - 852.446 n_h + 27.7449 n_h^2 + 0.00406919 n_h^3$

$c_{54} = (n_f - 24.5)c_{44}^{NS} - 617.146 + 309.613 n_h - 16.7381 n_h^2 - 0.0301426 n_h^3$

NB non-logarithmic 5-loop c_{55} unknown...but irrelevant (no discontinuity)

5-loop perturbative Spectral density

From Disc.[$\ln^p(m)$] thus obtain

$$-\rho^{\overline{MS}}_{pert}(\lambda,\alpha_s) = (\frac{3}{2\pi^2})\,\lambda^3\,\left\{-\frac{1}{2}+4(\frac{\alpha_S}{\pi})\left(L_\lambda-\frac{5}{12}\right)\right.$$

$$\left.+(\frac{\alpha_S}{\pi})^2\sum_{i=0}^{2}\rho_{2i}L_\lambda^{2-i}+(\frac{\alpha_S}{\pi})^3\sum_{i=0}^{3}\rho_{3i}L_\lambda^{3-i}+(\frac{\alpha_S}{\pi})^4\sum_{i=0}^{4}\rho_{4i}L_\lambda^{4-i}\right\}\,,\quad L_\lambda\equiv\ln\frac{|\lambda|}{\mu}$$

3-loops: $\rho_{20}=\frac{3}{2}c_{30},\quad \rho_{21}=c_{32},\quad \rho_{22}=\frac{1}{2}c_{32}-\frac{\pi^2}{8}c_{30}$

4-loops: $\rho_{30}=2c_{40},\ \rho_{31}=\frac{3}{2}c_{41},\ \rho_{32}=c_{42}-\frac{\pi^2}{2}c_{40},\ \rho_{33}=\frac{1}{2}c_{43}-\frac{\pi^2}{8}c_{41}$

5-loops: $\rho_{40}\ =\frac{5}{2}c_{50},\ \rho_{41}=2c_{51},\ \rho_{42}=-\frac{5}{4}\pi^2c_{50}+\frac{3}{2}c_{52},$

$$\rho_{43}=-\frac{\pi^2}{2}c_{51}+c_{53},\ \rho_{44}=\frac{1}{2}c_{54}-\frac{\pi^2}{8}c_{52}+\frac{\pi^4}{32}c_{50}$$

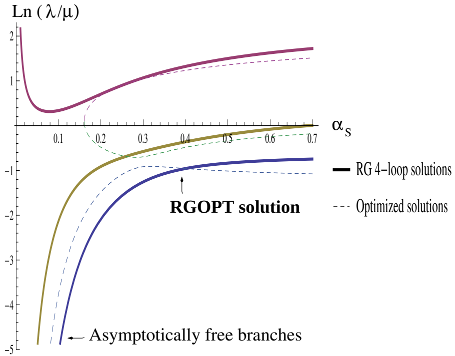
Non-logarithmic $\langle\bar{q}q\rangle$ terms do not contribute to $\rho_{pert}(\lambda)$
$\rightarrow$ complete 5-loop $\rho_{pert}(\lambda)$, except for c_{54} from c_{44}^{NS} approximation

NB $\rho_{pert}(\lambda)$ is interesting quantity irrespective of RGOPT: in particular for lattice applications (e.g. recent 'fit' to $\rho_{lattice}(\lambda)$, Nakayama et al '18)

Example: 4-loop RG and OPT solutions

Performing $\rho_{pert}(\lambda, \alpha_S) \to \rho_{opt}(\lambda(1-\delta)^{\frac{4}{3}\frac{\gamma_0}{2b_0}}, \delta\alpha_s)$; expand in δ to δ^3 (=4-loops), $\delta \to 1$: RG + OPT = order 3 polynomial Eqs.: $f_3(\alpha_S)L_\lambda^3 + f_2(\alpha_S)L_\lambda^2 + \cdots$.

Unique real solution upon requiring QCD perturbative matching (asymptotic freedom)



5. RGOPT 2,3,4-loop $\langle \bar{q}q \rangle$ results ($n_f = 2, 3$)

$n_f = 2$:

δ^k, RG order	$\ln \frac{\tilde{\lambda}}{\mu}$	$\tilde{\alpha}_S$	$\frac{-\langle \bar{q}q \rangle^{1/3}}{\bar{\Lambda}_2}(\tilde{\mu})$	$\frac{\tilde{\mu}}{\bar{\Lambda}_2}$	$\frac{-\langle \bar{q}q \rangle^{1/3}_{RGI}}{\bar{\Lambda}_2}$
δ, RG 1-loop	$-\frac{2275}{10092}$	$\frac{87\pi}{328} \simeq 0.83$	0.962	2.2	0.996
δ, RG 2-loop	-0.45	0.480	0.822	2.8	0.821
δ^2, RG 2-loop	-0.686	0.483	0.792	2.797	0.792
δ^2, RG 3-loop	-0.703	0.430	0.794	3.104	0.783
δ^3, RG 3-loop	-0.83895	0.40522	0.79252	3.308	0.77428
δ^3, RG 4-loop	-0.82164	0.39071	0.79608	3.448	0.77247

$n_f = 3$:

δ^k order	$\ln \frac{\tilde{\lambda}}{\mu}$	$\tilde{\alpha}_S$	$\frac{-\langle \bar{q}q \rangle^{1/3}}{\bar{\Lambda}_3}(\tilde{\mu})$	$\frac{\tilde{\mu}}{\bar{\Lambda}_3}$	$\frac{-\langle \bar{q}q \rangle^{1/3}_{RGI}}{\bar{\Lambda}_3}$
δ, RG 1-loop	$-\frac{283}{972}$	$\frac{27\pi}{104} \simeq 0.82$	0.965	2.35	0.987
δ, RG 2-loop	-0.56	0.474	0.799	3.06	0.789
δ^2, RG 2-loop	-0.766	0.493	0.776	2.942	0.772
δ^2, RG 3-loop	-0.788	0.444	0.780	3.273	0.766
δ^3, RG 3-loop	-0.97402	0.41367	0.76789	3.547	0.74377
δ^3, RG 4-loop	-0.96506	0.39906	0.77190	3.7094	0.74232

$$\langle \bar{q}q \rangle_{RGI} = \langle \bar{q}q \rangle(\mu)\,(2b_0\, g)^{\frac{\gamma_0}{2b_0}}\,(1 + (\frac{\gamma_1}{2b_0} - \frac{\gamma_0\, b_1}{2b_0^2})\, g + \cdots)$$

- Good stability/convergence; note: already realistic value already at 2-loops

5. RGOPT 2,3,4, 5-loop $\langle \bar{q}q \rangle (n_f = 2, 3)$ (preliminary for 5-loop!)

δ^k, RG order	$\ln \frac{\tilde{\lambda}}{\mu}$	$\tilde{\alpha}_S$	$\frac{-\langle \bar{q}q \rangle^{1/3}}{\bar{\Lambda}_2}(\tilde{\mu})$	$\frac{\tilde{\mu}}{\bar{\Lambda}_2}$	$\frac{-\langle \bar{q}q \rangle^{1/3}_{RGI}}{\bar{\Lambda}_2}$
δ, RG 2-loop	-0.45	0.480	0.822	2.8	0.821
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δ^3, RG 3-loop	-0.83895	0.40522	0.79252	3.308	0.77428
δ^3, RG 4-loop	-0.82164	0.39071	0.79608	3.448	0.77247
δ^4, RG 4-loop	-0.63250	0.47745	0.79863	2.8235	0.80556
δ^4, RG 5-loop	-0.59968	0.50310	0.79214	2.7084	0.80951

δ^k order	$\ln \frac{\tilde{\lambda}}{\mu}$	$\tilde{\alpha}_S$	$\frac{-\langle \bar{q}q \rangle^{1/3}}{\bar{\Lambda}_3}(\tilde{\mu})$	$\frac{\tilde{\mu}}{\bar{\Lambda}_3}$	$\frac{-\langle \bar{q}q \rangle^{1/3}_{RGI}}{\bar{\Lambda}_3}$
δ, RG 2-loop	-0.56	0.474	0.799	3.06	0.789
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δ^3, RG 3-loop	-0.97402	0.41367	0.76789	3.547	0.74377
δ^3, RG 4-loop	-0.96506	0.39906	0.771903	3.7094	0.74232
δ^4, RG 4-loop	-0.91690	0.40935	0.77774	3.5893	0.75229
δ^4, RG 5-loop	-0.85966	0.42196	0.77342	3.46024	0.75276

$\rightarrow n_f = 3$ appears very stable, while $n_f = 2$ changed by $\simeq 4\%$ at 5-loop:
genuine 5-loop effect? or bias of missing 4-loop singlet $c_{54}(c_{44})$??

Estimating possible impact of incomplete 4-loop singlets

• $n_f = 0$ (virtual loops) $\simeq$ "quenched" approximation: NO singlet contributions
→ 'single out' true 5-loops effects:

δ^k order	$\ln \frac{\tilde{\lambda}}{\mu}$	$\tilde{\alpha}_S$	$\frac{-\langle \bar{q}q \rangle^{1/3}}{\bar{\Lambda}_3}(\tilde{\mu})$	$\frac{\tilde{\mu}}{\bar{\Lambda}_3}$	$\frac{-\langle \bar{q}q \rangle^{1/3}_{RGI}}{\bar{\Lambda}_3}$
δ, RG 2-loop	-0.27548	0.49747	0.88139	2.49445	0.91217
δ^2, RG 2-loop	-0.54201	0.47091	0.83646	2.57953	0.85679
δ^2, RG 3-loop	-0.54434	0.41691	0.83342	2.82242	0.83522
δ^3, RG 3-loop	-0.63438	0.39314	0.84406	2.97134	0.83741
δ^3, RG 4-loop	-0.60347	0.38002	0.84653	3.06868	0.83508
δ^4, RG 4-loop	-0.54077	0.39241	0.86306	2.98846	0.85709
δ^4, RG 5-loop	-0.45296	0.42275	0.85527	2.80454	0.86069

• Alternatively: neglect (remove) singlet contributions at lower (4-loop) order:

δ^3 (4-loop) order	$\ln \frac{\tilde{\lambda}}{\mu}$	$\tilde{\alpha}_S$	$\frac{-\langle \bar{q}q \rangle^{1/3}}{\bar{\Lambda}_3}(\tilde{\mu})$	$\frac{\tilde{\mu}}{\bar{\Lambda}_3}$	$\frac{-\langle \bar{q}q \rangle^{1/3}_{RGI}}{\bar{\Lambda}_3}$
$n_f = 2$ complete	-0.82164	0.39071	0.79608	3.448	0.77247
$n_f = 2$, no singlets	-0.88242	0.38421	0.78278	3.5166	0.75721
$n_f = 3$ complete	-0.96506	0.39906	0.77190	3.7094	0.74232
$n_f = 3$, no singlets	-1.01771	0.39419	0.75982	3.7685	0.72892

Estimating possible impact of incomplete 4-loop singlets

•Finally: compare to the known 4-loop $\Pi_{v,a}$ singlet contributions:
are $\lesssim \pm 2\%$ of non-singlets (Baikov, Chetyrkin, Kühn, Rittinger '12)

$\to$ rough estimate of $c_{44}^{NS} + |c_{44}^{S}|$: take conservative $|c_{44}^{S}| \sim 10\%$ of $|c_{44}^{NS}|$:

δ^4 (5-loop)	$\frac{-\langle \bar{q}q \rangle_{RGI}^{1/3}}{\bar{\Lambda}_3}$
$n_f = 2$, $c_{44}^{S} \simeq \pm 0.1\, c_{44}^{NS}$, RG 5-loop	$0.8095^{+0.04}_{-0.03}$
$n_f = 3$, $c_{44}^{S} \simeq \pm 0.1\, c_{44}^{NS}$, RG 5-loop	$0.7528^{+0.02}_{-0.01}$

Evolution to (standard) reference scale $\mu = 2$ GeV

$$\langle \bar{q} q\rangle(\mu' = 2GeV) = \langle \bar{q} q\rangle(\tilde{\mu}) \exp[\int_{g(\tilde{\mu})}^{g(2GeV)} dg \frac{\gamma_m(g)}{\beta(g)}]$$

(equivalently extract from $\langle \bar{q} q\rangle_{RGI}$ with $\alpha_S(2\mathrm{GeV}) \simeq 0.305 \pm 0.004$)
(NB for $n_f = 3$ account for $\alpha_S(\mu \sim m_c)$ (4-loop) threshold effects)

$$\langle \bar{q} q\rangle^{1/3}_{n_f=2}(2\mathrm{GeV}) = -(0.833_{(4\text{-loop})} - 0.876_{(5\text{-loop})})\bar{\Lambda}(n_f = 2)$$

$$\langle \bar{q} q\rangle^{1/3}_{n_f=3}(2\mathrm{GeV}) = -(0.814_{(4\text{-loop})} - 0.824_{(5\text{-loop})})\bar{\Lambda}(n_f = 3)$$

•NO conclusions yet on 'intrinsical' 5-loop RGOPT uncertainty, until singlet contributions more controlled/evaluated
(NB at 5-loops the difference between using 4-loop RG and 5-loop RG is $\sim 0.5\%$ for $n_f = 2$ and less for $n_f = 3$)

Comparison with other determinations (incomplete, preliminary!)

- $n_f = 2$: using most precise $\bar{\Lambda}_2$ lattice result: $\bar{\Lambda}_2 = 331 \pm 21$ MeV (quark potential method, Karbstein et al '14):
5-loop RGOPT $\langle \bar{q}q \rangle^{1/3}_{n_f=2}(2GeV) \simeq -(290 \pm 18)$ MeV
NB compare wto our previous (4-loop) RGOPT result:
$\langle \bar{q}q \rangle^{1/3}_{n_f=2}(2GeV) \simeq -(278 \pm 18)$ MeV

-Compare with latest:
-lattice (Engel et al '14, from spectral density):
$-\langle \bar{q}q \rangle^{1/3}_{n_f=2}(\mu = 2\mathrm{GeV}) = 261 \pm 6 \pm 8$ MeV
NB combined lattice results ($n_f = 2$) (FLAG review 2019) : 266(10)
-spectral sum rules (Narison '14): $-\langle \bar{u}u \rangle^{1/3} \sim 276 \pm 7$ MeV

- $n_f = 3$: using 2018 PDG world average:
$\bar{\alpha}_S(m_Z) = 0.1181 \pm 0.0011 \to \bar{\Lambda}_3^{wa} \simeq 332 \pm 17 MeV$:
$-\langle \bar{q}q \rangle^{1/3}_{n_f=3}(2GeV, \bar{\Lambda}_3^{wa}) \simeq 274 \pm 4(rgopt) \pm 14(\bar{\Lambda}_3)\ MeV$

Compare with latest (to our knowledge) Lattice results (chiral limit, FLAG 2019 compilation): $-\langle \bar{s}s \rangle (m_q = 0) = 214 \pm 30$ to 290 ± 15...
reasons: difficult to extrapolate to $m_q \to 0$ (chiral PT less efficient for $n_f = 3$)
-e.g including $m_{u,d,s}$: $\langle \bar{s}s \rangle^{1/3}(m_q \neq 0, 2GeV) = -296(11)$ MeV (Davies et al, HPQCD 2018, from OPE+heavy-light correlators)

Summary, outlook

•Our RG-consistent version of OPT $\equiv$ *RGOPT* appears efficient to resum perturbative expansions, using only perturbative information.

•It includes 2 major differences w.r.t. most previous similar approaches:

1) OPT+ RG minimizations fix optimized $\tilde{m}$ *and* $\tilde{g} = 4\pi\tilde{\alpha}_S$
2) Requiring AF-compatible solutions uniquely fixes the basic interpolation $m \to m(1-\delta)^{\gamma_0/(2b_0)}$: discards spurious solutions and accelerates convergence.

•Application to spectral density: $\pi\rho(\lambda = 0) \equiv -\langle\bar{q}q\rangle_{m\to 0}$:

•Intrinsical RGOPT theoretical error (4,5 loop): few percents

•5-loop final determination + accuracy not conclusive yet: urgent need to evaluate missing 4-loop singlet contributions (doable)

•We find a moderate reduction of $n_f = 3$ $|\langle\bar{q}q\rangle|$ w.r.t. $n_f = 2$

•Outlook: applications to thermal/in-medium QCD (recent one see e.g. JLK, M.B Pinto, T. Restrepo, arXiv:1908.08363)

Extra Slides

pre-QCD inspiration: Gross-Neveu model

•$D = 2$ $O(2N)$ GN model shares many properties with QCD (asymptotic freedom, (discrete) chiral sym., mass gap,..)

$$\mathcal{L}_{GN} = \bar{\Psi} i \not{\partial} \Psi + \frac{g_0}{2N} (\textstyle\sum_1^N \bar{\Psi}\Psi)^2 \ (massless)$$

Standard mass-gap (massless, large N approx.):

work out $V_{eff}(\sigma \sim \langle \bar{\Psi}\Psi \rangle) \sim \frac{\sigma^2}{2g} + Tr \ln(i \not{\partial} - \sigma)$;

$\frac{\partial V_{eff}}{\partial \sigma} = 0$: $\quad \to \sigma \equiv M = \mu e^{-\frac{2\pi}{g}} \equiv \Lambda_{\overline{MS}}$

•Mass gap also known exactly for any N:

$$\frac{M_{exact}(N)}{\Lambda_{\overline{MS}}} = \frac{(4e)^{\frac{1}{2N-2}}}{\Gamma[1 - \frac{1}{2N-2}]}$$

(From $D = 2$ integrability: Bethe Ansatz) Forgacs et al '91

Massive (large N) GN model

$M(m,g) \equiv m(1+g\ln\frac{M}{\mu})^{-1}$: Resummed mass ($g/(2\pi) \to g$)
$= m(1-g\ln\frac{m}{\mu}+g^2(\ln\frac{m}{\mu}+\ln^2\frac{m}{\mu})+\cdots)$ (pert. re-expanded)

- Only fully resummed $M(m,g)$ gives right result, upon:

-identifying $\Lambda \equiv \mu e^{-1/g}$; $\to M(m,g) = \frac{m}{g\ln\frac{M}{\Lambda}} \equiv \frac{\hat{m}}{\ln\frac{M}{\Lambda}}$;
-taking *reciprocal*: $\hat{m} = M\ \ln\frac{M}{\Lambda} \to M(\hat{m}\to 0) \sim \frac{\hat{m}}{\hat{m}/\Lambda+\mathcal{O}(\hat{m}^2)} = \Lambda$

never seen in standard perturbation: $M_{pert}(m\to 0)\to 0$...

- Now (RG)OPT gives $M=\Lambda$ at *first* (and any) δ-order!
(at any order, solutions: $\ln\frac{\tilde{m}}{\mu} = -\frac{1}{\tilde{g}}$)

- At δ^2-order (2-loop), RGOPT $\sim 1-2\%$ from $M_{exact}(\text{any}N)$

- Not specific to GN: for any model, at one-loop RG approximation, RG, OPT solutions at first (and all) orders:
$\ln\frac{\tilde{m}}{\mu} = -\frac{\gamma_0}{2b_0}$; $\tilde{g} = \frac{1}{\gamma_0}$ correctly resums pure RG LL, NLL,... (as far as b_0,γ_0 dependence concerned).