Rational terms in two-loop calculations

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in collaboration with

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Motivation: Towards two-loop automation

Example of a full NNLO calculation: $gg \rightarrow t\bar{t}g$

Tree and one-loop contributions fully automated in OpenLoops 2 [arxiv:1907.13071] for all LHC processes, including real emission → strong numerical stability and CPU efficiency

Download: https://gitlab.com/openloops/OpenLoops or https://openloops.hepforge.org

Challenges for numerical two-loop calculation:

- Numerical amplitude construction → Numerator of Feynman integral in 4 dimensions

- Tensor integral reduction and computation of Master integrals  
  → computed in $d = 4 - 2\varepsilon$ dimensions

- Rational terms stemming from discrepancy between Feynman integrands with 4 and $d$ dimensional numerators ⇒ one loop: Rational terms of type $R_2$ [Ossola, Papadopoulos, Pittau]

  this talk: two-loop rational terms
Rational terms at one loop

Generic one-loop diagram (d-dimensional quantities marked with bar)

\[ \tilde{A}_1 = \int \! d\bar{q} \frac{\bar{N}(\bar{q})}{\bar{D}_0(\bar{q}) \cdots \bar{D}_{N-1}(\bar{q})} = \]

\( \text{(scalar propagators } \bar{D}_i(\bar{q}) = (\bar{q} + p_i)^2 - m_i^2 \text{ with loop momentum } \bar{q}, \text{ mass } m_i \text{ and external momentum } p_i) \)

Splitting 4 and \((d-4)\) dimensional parts of the numerator yields

\[ \tilde{A}_1 = A_1 + \Delta A_1 \quad \text{with} \quad \Delta A_1 = \tilde{A}_1 - A_1 = \int \! d\bar{q} \frac{\tilde{N}(\bar{q}) - N(q)}{\bar{D}_0(\bar{q}) \cdots \bar{D}_{N-1}(\bar{q})}, \]

where \(N(q)\) is constructed from 4-dimensional \(q, \gamma^\mu, g^{\mu\nu}\) and \(\tilde{N}(\bar{q})\) from \(d\)-dimensional \(\bar{q} = q + \bar{q}, \quad \bar{\gamma}^\mu = \gamma^\mu + \bar{\gamma}^\mu, \quad \bar{g}^{\mu\nu} = g^{\mu\nu} + \bar{g}^{\mu\nu} \)

Numerator \(\tilde{N} = \tilde{N} - N\) is of order \(\varepsilon\)

⇒ only interaction with \(\frac{1}{\varepsilon}\) UV poles contributes to finite result [Binoth, Guillet, Heinrich]

⇒ \(\Delta A_1\) is a (finite) rational term, that can be written as a local counterterm [Ossola, Papadopoulos, Pittau]
Renormalised amplitude from subtraction of divergencies (R-operation [Caswell, Kennedy])

\[ R_{\bar{A}_1} = \bar{A}_1 + \delta Z_{\bar{A}_1} T_{A_1} \quad \text{(counterterm } \delta Z_{\bar{A}_1}, \text{ tree-level structure } T_{A_1}) \]

Rational term can be written as a (local) counterterm \( \Delta A_1 = \delta Z_{\bar{A}_1} R_{A_1} \)

\[ \Rightarrow R_{\bar{A}_1} = A_1 + (\delta Z_{\bar{A}_1} + \delta Z_{R_{A_1}}) T_{A_1} =: R_{4} A_1 \]

One-loop diagram \( A_1 \) with 4-dim numerator \( \Rightarrow \) suitable for numerical implementation

Generalisation to two loops

Renormalisation procedure: Subtraction of sub-divergencies from one-loop sub-diagrams \( \bar{\gamma} \), then of local two-loop divergence:

\[ R_{\bar{A}_2} = \bar{A}_2 + \sum_{\gamma} \left( \delta Z_{\bar{\gamma}} \right) \cdot \bar{A}_2/\bar{\gamma} + \delta Z_{\bar{A}_2} T_{\bar{A}_2} \]

We show that a renormalised \( d \)-dim two-loop amplitude can be computed as

\[ R_{\bar{A}_2} = A_2 + \sum_{\gamma} \left( \delta Z_{\bar{\gamma}} + \delta Z_{R_{A_2}} \right) \cdot A_2/\gamma + \left( \delta Z_{\bar{A}_2} + \delta Z_{R_{A_2}} \right) T_{\bar{A}_2} =: R_{4} A_2. \]

with (extended) one-loop rational terms \( \delta Z_{\bar{\gamma}} \) and a two-loop rational term \( \delta Z_{R_{A_2}} \)

\( \Rightarrow \) Two and one-loop diagrams \( A_2 \) and \( A_2/\gamma \) constructed with 4-dim numerators
I. Computation of rational terms from tadpole integrals with one scale
   → Rational terms of one-loop diagrams and sub-diagrams

II. Rational terms of two-loop diagrams
   – without a global divergence
   – with a global divergence

III. Full set of QED rational terms at two loops

IV. Summary and Outlook
Rational terms from tadpole integrals: The one-loop case

\[ \bar{A}_1 = \int d\bar{q} \frac{\bar{N}(\bar{q})}{D_0(\bar{q}) \cdots D_{N-1}(\bar{q})} = R \bar{A}_1 = \bar{A}_1 + \delta Z \bar{A}_1 T A_1 \]

Define difference between diagram with \(d\)-dim and 4-dim numerator:

\[ \Delta A_1 = \bar{A}_1 - A_1 \]

Use exact decomposition of propagator denominators \( \bar{D}_i = (\bar{q} + p_i)^2 - m_i^2 \) [Chetyrkin, Misiak, Münz]

\[
\frac{1}{\bar{D}_i} = \frac{1}{\bar{q}^2 - M^2} + \frac{\Delta_i}{\bar{q}^2 - M^2} \frac{1}{D_i}
\]

with \( \Delta_i = -p_i^2 - 2\bar{q} \cdot p_i + m_i^2 - M^2 \) recursively up to a fixed order \( Z + 1 \):

\[
\frac{1}{D_i} = \sum_{n=0}^{Z} \frac{(\Delta_i)^n}{(\bar{q}^2 - M^2)^{n+1}} + \frac{(\Delta_i)^{Z+1}}{(\bar{q}^2 - M^2)^{Z+1}} \frac{1}{D_i}
\]

e.g. for \( Z = 2 \):

\[ \Rightarrow \text{Isolate divergencies + rational terms in tadpole integrals with one auxiliary scale } M^2 \]

Terms with a negative degree of divergence \( X \) (naive power counting) cancel in \( \Delta A_1 \).
One-loop example

\[ D = \text{dimension of the loop momenta, metric and } \gamma\text{-matrices in the numerator } \Rightarrow D \in \{4, d\} \]

\[ \Delta A_1 = \begin{array}{cc}
D = d & D = 4 \\
\end{array} - \begin{array}{cc}
D = d & D = 4 \\
\end{array} = \begin{array}{cc}
D = d & D = 4 \\
\end{array} - \begin{array}{cc}
D = d & D = 4 \\
\end{array} + \begin{array}{cc}
D = d & D = 4 \\
\end{array} - \begin{array}{cc}
D = d & D = 4 \\
\end{array} = \mathcal{O}(\varepsilon) \\
= \mathcal{O}(\varepsilon) \\
= \mathcal{O}(\varepsilon) \\
= \mathcal{O}(\varepsilon) \\
= \mathcal{O}(\varepsilon) \\
= \mathcal{O}(\varepsilon) \\
= \mathcal{O}(\varepsilon) \\
= \mathcal{O}(\varepsilon) \\
= \mathcal{O}(\varepsilon) \\
= \mathcal{O}(\varepsilon) \\
\]

\[ = \int d\vec{q} \frac{\tilde{N}(\vec{q}) - \mathcal{N}(q)}{(\vec{q}^2 - M^2)^3} + \mathcal{O}(\varepsilon) \]
One-loop rational terms from tadpoles

In general: \[ \Delta A_1 = \sum_{n=N}^{N+X} \int \frac{d\bar{q}}{(\bar{q}^2 - M^2)^n} (\bar{N}(\bar{q}) - N(q)) P_n(M^2, p_i^2, \bar{q} \cdot p_j) \]
polynomial \( P_n \) from decomposition of denominators

Decompose numerators:
\[ \bar{N}(\bar{q}) P_n(M^2, p_i^2, \bar{q} \cdot p_j) = \sum_{r=0}^{R} \bar{N}_{\bar{\mu}_1 \ldots \bar{\mu}_r} q^{\bar{\mu}_1} \ldots q^{\bar{\mu}_r} \]
\[ N(q) P_n(M^2, p_i^2, \bar{q} \cdot p_j) = \sum_{r=0}^{R} N_{\mu_1 \ldots \mu_r} q^{\mu_1} \ldots q^{\mu_r} \]

using \( \bar{q} \cdot p_j = q \cdot p_j \) for external 4-momenta \( p_j \). With \( A_{\mu_i} B^{\bar{\mu}_i} = A_{\mu_i} B^{\mu_i} \) for any vectors \( A, B \)

\[ \Rightarrow \Delta A_1 = \sum_{n=N}^{N+X} \sum_{r=0}^{R} (\bar{N}_{\bar{\mu}_1 \ldots \bar{\mu}_r} - N_{\mu_1 \ldots \mu_r}) \mathcal{I}_{\bar{\mu}_1 \ldots \bar{\mu}_r} \]
with \( \mathcal{I}_{\bar{\mu}_1 \ldots \bar{\mu}_r} = \int \frac{d\bar{q}}{\bar{q}^2 - M^2} \bar{q}^{\mu_1} \ldots \bar{q}^{\mu_r} \)

- Generic method to compute \( \Delta A_1 \) from tadpole integrals with one (auxiliary) scale \( M^2 \).
- Result independent of \( M^2 \) since the denominator decomposition is exact and only terms which cancel in \( \Delta A_1 \) are discarded.
- Dependence on external momenta and masses resides only in \( \bar{N}_{\bar{\mu}_1 \ldots \bar{\mu}_r} \) and \( N_{\mu_1 \ldots \mu_r} \)

\[ \Rightarrow \Delta A_1 \text{ is a rational term} \]
One-loop sub-diagrams in two-loop diagrams

One loop: All external momenta in 4 dimensions

Two loop: External momentum to a sub-diagram $\bar{\gamma}$ can be $d$-dim loop momentum $\bar{q}_1$

• Decomposition of $d$-dim denominators yields

$$
\frac{1}{(\bar{q}_1 + \bar{q}_2)^2 - m^2} = \frac{1}{\bar{q}_2^2 - M^2} + \frac{-\bar{q}_1^2 - 2\bar{q}_1 \cdot \bar{q}_2 + m^2 - M^2}{\bar{q}_2^2 - M^2} \frac{1}{(\bar{q}_1 + \bar{q}_2)^2 - m^2}
$$

$$
\Rightarrow \int \! d\bar{q}_2 \left[ \mathcal{N}(q_2, q_1) P_n(\bar{q}_1^2, \bar{q}_1, \bar{q}_2) \right] / (\bar{q}_2^2 - M^2)^n \text{ has terms } \propto q_1^2 \text{ (from $\mathcal{N}$ construction in 4-dim),} \\
\propto \bar{q}_1^2 \text{ (from denominator decomposition)}
$$

• Difference of $d$-dim and 4-dim tensor structures $\bar{t}^\alpha(\bar{q}_1) - t^\alpha(q_1)$ multiplied by one-loop UV counterterm $\delta Z \bar{\gamma}$ not negligible due to insertion into a loop diagram

Extended one-loop rational term for one-loop sub-diagram $\bar{\gamma}$ with external indices $\alpha = (\alpha_1 \alpha_2)$:

$$(\Delta \gamma)^\alpha := [\bar{\gamma}^{\bar{\alpha}} - \gamma^\alpha + (\bar{t}^{\bar{\alpha}} - t^\alpha) \delta Z \bar{\gamma}]_{\bar{\alpha} \rightarrow \alpha}$$

This has the structure $\Delta \gamma = R_2(\gamma) + \frac{\Delta R_2(\gamma)}{\varepsilon}$ with the well-known one-loop $R_2(\gamma)$.

Additional rational term $\Delta R_2(\gamma) \propto \bar{q}_1^2$ with $\bar{q}_1 = \bar{q}_1 - q_1$ requires superficial degree of divergence $\geq 2$ in sub-diagram (e.g. gauge boson self-energy).
Example: One-loop self-energy of the photon

One-loop amplitude in $d$ and 4 dimensions from denominator decomposition and tadpole method:

$$
\bar{\gamma}_{\mu\nu} = \int d\tilde{q}_2 \frac{N_{\text{sub}}^{\mu\nu}(\bar{q}_1, \bar{q}_2)}{D_0^{(2)}(\bar{q}_2) D_0^{(3)}(\bar{q}_1 + \bar{q}_2)} = \frac{i e^2}{16\pi^2} \left( \frac{-(4-2\varepsilon) q_1^2}{3\varepsilon} g^{\mu\nu} + \frac{4}{3\varepsilon} q_1^\mu q_1^\nu + C_{\text{finite}}^{\mu\nu} \right),
$$

$$
\gamma^{\mu\nu} = \int d\tilde{q}_2 \frac{N_{\text{sub}}^{\mu\nu}(q_1, q_2)}{D_0^{(2)}(\bar{q}_2) D_0^{(3)}(\bar{q}_1 + \bar{q}_2)} = \frac{i e^2}{16\pi^2} \left( \frac{-4q_1^2 - 2q_1^2}{3\varepsilon} g^{\mu\nu} + \frac{4}{3\varepsilon} q_1^\mu q_1^\nu + C_{\text{finite}}^{\mu\nu} \right),
$$

where the constant parts $C_{\text{finite}}^{\mu\nu} - C_{\text{finite}}^{\mu\nu} = \mathcal{O}(\varepsilon)$.

For the difference of the UV counterterms

$$
(\bar{t}^{\mu\nu} - t^{\mu\nu}) \delta Z_{\gamma} = \left[ (\bar{q}_1 q_1 - q_1 q_1) - (\bar{q}_1^2 g^{\mu\nu} - q_1^2 g^{\mu\nu}) \right] \frac{i e^2}{16\pi^2} \left( \frac{4}{3\varepsilon} \right) \propto \frac{q_1^2}{\varepsilon} g^{\mu\nu} \quad \text{for } \bar{\mu} \to \mu, \bar{\nu} \to \nu
$$

This yield the extended rational term

$$
(\Delta \gamma)^{\mu\nu} = \left[ (\bar{q}_1 q_1 - q_1 q_1) + (\bar{q}_1^2 g^{\mu\nu} - q_1^2 g^{\mu\nu}) \right] \delta Z_{\gamma} \left[ \bar{\mu} \to \mu, \bar{\nu} \to \nu \right] = \frac{i e^2}{16\pi^2} \left( \frac{q_1^2}{3\varepsilon} \right) \text{ known } R_2 + \frac{\tilde{q}_1^2}{3\varepsilon} \text{ new } \Delta R_2(\gamma)/\varepsilon
$$

\[ g^{\mu\nu} \]
Two-loop diagrams

Generic irreducible two-loop diagram consists of three chains and two connecting vertices $V_0$, $V_1$:

$$\bar{A}_2 = \int d\bar{q}_1 \int d\bar{q}_2 \frac{\bar{N}_1^{\bar{\alpha}_1}(\bar{q}_1) \bar{N}_2^{\bar{\alpha}_2}(\bar{q}_2) \bar{N}_3^{\bar{\alpha}_3}(\bar{q}_3)}{\prod_{i_1=0}^{N_1} \bar{D}_{i_1}^{(1)}(\bar{q}_1) \prod_{i_2=0}^{N_1} \bar{D}_{i_2}^{(2)}(\bar{q}_2) \prod_{i_3=0}^{N_1} \bar{D}_{i_3}^{(3)}(\bar{q}_3)}$$

with $\bar{q}_3 = -(\bar{q}_1 + \bar{q}_2)$

Renormalised diagram in $d$ dimensions (what we need)

Splitting in pieces with 4-dim numerators (what we can implement in OpenLoops)

$\Delta \gamma$: one-loop rational term insertion

Our task: Compute $\Delta A_2 = \bar{R} \bar{A}_2 - \bar{R}_4 A_2$ and show that it is rational.
The case with no global but one subdivergence

Use factorisation and split sub-diagram (chains 2+3) and rest (chain 1) into d-dim and 4-dim parts:

\[
\mathcal{R} \tilde{A}_2 = \int \! \! \! \int \frac{N_{1\alpha}(q_1) + \tilde{N}_{1\tilde{\alpha}}(\bar{q}_1)}{\bar{D}_0^{(1)} \cdots \bar{D}_{N_1}^{(1)}} \left[ \int \! \! \! \! \int \frac{N_{\gamma\tilde{\alpha}}(q_1, q_2)}{\bar{D}_0^{(2)} \cdots \bar{D}_{N_2}^{(2)} \bar{D}_{0}^{(3)} \cdots \bar{D}_{N_3}^{(3)}} + \tilde{t}^{\tilde{\alpha}} \delta Z_{\tilde{\gamma}} \right] \\
no divergence \quad \text{finite} \rightarrow \text{discard } \tilde{N}_{1\tilde{\alpha}}
\]

\[
= \int \! \! \! \! \int \frac{N_{1\alpha}(q_1)}{\bar{D}_0^{(1)} \cdots \bar{D}_{N_1}^{(1)}} \left[ \int \! \! \! \! \int \frac{N_{\gamma\alpha}(q_1, q_2)}{\bar{D}_0^{(2)} \cdots \bar{D}_{N_3}^{(3)}} + t^{\alpha} \delta Z_{\tilde{\gamma}} + (\Delta \gamma)^{\alpha} \right] + \mathcal{O}(\epsilon)
\]

with \((\Delta \gamma)^{\alpha} = \int \! \! \! \! \int \frac{\tilde{N}_{\gamma\tilde{\alpha}}(q_1, q_2)}{\bar{D}_0^{(2)} \cdots \bar{D}_{N_3}^{(3)}} + (\tilde{t}^{\tilde{\alpha}} - t^{\alpha}) \delta Z_{\tilde{\gamma}} = R_2(\gamma) + \frac{\Delta R_2(\gamma)}{\epsilon} \)

Diagram can be computed with 4-dim numerators:

\[
\mathcal{R} \tilde{A}_2 = A_2 + A_2/\gamma (\delta Z_{\tilde{\gamma}} + \Delta) =: \mathcal{R}_4 A_2
\]

- No two-loop rational term, i.e. \( \Delta A_2 = 0 \)
- One-loop rational terms \( R_2(\gamma) \) needed only to order \( \epsilon^0 \)
- Additional \( \bar{q}_1^2 \) term only in one-loop integrals \( \rightarrow \) need to be computed
The case with a global divergence

- Define **for each chain** \( i = 1, 2, 3 \) the **maximum degree of divergence** of the full diagram \((X \leq 0)\) and any sub-diagram constructed from chains \( i, j \neq i \) (degree of sub-divergence \( X_{ij} \)):

\[
Z_1 = \text{Max} (X, X_{12}, X_{13}), \quad Z_2 = \text{Max} (X, X_{12}, X_{23}), \quad Z_3 = \text{Max} (X, X_{23}, X_{13})
\]

- Use **exact decomposition of propagator denominators** on each chain \( i \) up to order \( Z_i + 1 \):

\[
\frac{1}{D_0^{(i)}(\bar{q}_i) \cdots D_{N_i}^{(i)}(\bar{q}_i)} = \left[ S_{Z_i}^{(i)} + F_{Z_i}^{(i)} \right] \frac{1}{D_0^{(i)}(\bar{q}_i) \cdots D_{N_i}^{(i)}(\bar{q}_i)},
\]

with \( S_{Z_i}^{(i)} \) selecting all terms contributing to a (sub-)divergence,

\[
S_{Z_i}^{(i)} \frac{1}{D_0^{(i)}(\bar{q}_i) \cdots D_{N_i}^{(i)}(\bar{q}_i)} = \sum_{n=0}^{Z_i} \frac{P_n^{(i)}(\bar{q}_i, p_j^{(i)})}{(\bar{q}_i^2 - M^2)^{N+1+n}} \quad \leftarrow \text{tadpoles with scale } M^2
\]

and \( F_{Z_i}^{(i)} \) selecting all other terms, especially those with at least one original propagator \( D_j^{(i)} \)

- In particular, terms selected by \( F_{Z_i}^{(i)} \) have **no global divergence** \( \Rightarrow \) **Apply previous case!**
The case with a global divergence

Exact decomposition ⇒ isolate divergent parts ⇒ Discard terms which cancel from calculation:

\[ \Delta A_2 = \left( S_{Z_1}^{(1)} + F_{Z_1}^{(1)} \right) \left( S_{Z_2}^{(2)} + F_{Z_2}^{(2)} \right) \left( S_{Z_3}^{(3)} + F_{Z_3}^{(3)} \right) \Delta A_2 = \Delta A_{2,\text{rat}} + \sum_{i=1}^{3} \Delta A_{2,\text{sub}}^{(i)} + \Delta A_{2,\text{fin}}, \]

with \( \Delta A_{2,\text{rat}} = S_{Z_1}^{(1)} S_{Z_2}^{(2)} S_{Z_3}^{(3)} \Delta A_2 \), \( \text{← only one-scale tadpoles} \)

\( \Delta A_{2,\text{sub}}^{(i)} = F_{Z_i}^{(1)} S_{Z_j}^{(j)} S_{Z_k}^{(k)} \Delta A_2 \), \( \text{← no global divergence, at most one} \)

\( \Delta A_{2,\text{fin}} = F_{Z_1}^{(1)} F_{Z_2}^{(2)} F_{Z_3}^{(3)} \Delta A_2 \), \( \text{← no global or sub-divergence} \) ⇒ \( O(\varepsilon) \)

⇒ \( \Delta A = \Delta A_{2,\text{rat}} \) is rational! Compute as in one-loop case from

\[
\Delta A_2 = \sum_{n_i=N_i+1}^{N_i+Z_i} \int d\bar{q}_1 \int d\bar{q}_2 \frac{\left( \mathcal{N}(\bar{q}_1, \bar{q}_2) - \mathcal{N}(q_1, q_2) \right) P_{n_1 n_2 n_3} (M^2, p_i^2, q_i \cdot p_j)}{(\bar{q}_1^2 - M^2)^{n_1} (\bar{q}_2^2 - M^2)^{n_2} (\bar{q}_3^2 - M^2)^{n_3}}
\]

\[
= \sum_{n_i=N_i+1}^{N_i+Z_i} \sum_{R} \sum_{S} \left( \mathcal{N}_{\mu_1 \ldots \mu_r \bar{\nu}_1 \ldots \bar{\nu}_s} \right) T_{n_1 n_2 n_3}^{\bar{\mu}_1 \ldots \bar{\mu}_r \bar{\nu}_1 \ldots \bar{\nu}_s}
\]

\[
\text{with } T_{n_1 n_2 n_3}^{\bar{\mu}_1 \ldots \bar{\mu}_r \bar{\nu}_1 \ldots \bar{\nu}_s} = \int d\bar{q}_1 \int d\bar{q}_2 \frac{\bar{q}_1^{\bar{\mu}_1} \ldots \bar{q}_r^{\bar{\mu}_r} \bar{q}_1^{\bar{\nu}_1} \ldots \bar{q}_s^{\bar{\nu}_s}}{(\bar{q}_1^2 - M^2)^{n_1} (\bar{q}_2^2 - M^2)^{n_2} (\bar{q}_3^2 - M^2)^{n_3}}
\]
Example: QED vertex correction

Let $D \in \{4, d\}$ be the numerator dimension. Decomposing chain 1 (with external fermion line):

\[
\Delta A_2 = \begin{pmatrix}
D = d & D = d \\
\times \delta Z_\gamma & \times \delta Z_\gamma
\end{pmatrix}
- \begin{pmatrix}
D = 4 & D = 4 \\
\times (\delta Z_\gamma + \Delta_\gamma) & \times (\delta Z_\gamma + \Delta_\gamma)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
D = d & D = d \\
\times \delta Z_\gamma & \times \delta Z_\gamma
\end{pmatrix}
- \begin{pmatrix}
D = 4 & D = 4 \\
\times (\delta Z_\gamma + \Delta_\gamma) & \times (\delta Z_\gamma + \Delta_\gamma)
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
D = d & D = d \\
\times \delta Z_\gamma & \times \delta Z_\gamma
\end{pmatrix}
- \begin{pmatrix}
D = 4 & D = 4 \\
\times (\delta Z_\gamma + \Delta_\gamma) & \times (\delta Z_\gamma + \Delta_\gamma)
\end{pmatrix}
\]

\[
= 0 + \mathcal{O}(\varepsilon) \quad \text{(case without global divergence)}
\]

\[
+ \text{more terms with original propagator denominators along outer chain}
\]

\[
= 0 + \mathcal{O}(\varepsilon) \quad \text{(case without global divergence)}
\]

black lines: original propagators $\bar{D}_j^{(i)}$; red lines: factors in denominator decomposition $\propto (\bar{q}_i^2 - M^2)$. 
Example: QED vertex correction

Decomposing chains 2,3 (photon self-energy sub-diagram):

\[ \Delta A_2 = \left( \right) - \left( \right) + \left( \right) = 0 \quad \text{(case without global divergence)} \]

Only massive tadpoles contribute to difference \( \Delta A_2 \).
Rational terms for QED in $\overline{MS}$ scheme ($\xi = 0, m = 0$)

Calculation in the GEXCOM [Chetyrkin, M.Z.] framework: QGRAF [Noguira] $\rightarrow$ Q2E+EXP
[Seidesticker, Harlander, Steinhauser] $\rightarrow$ FORM [Vermaseren] code $\rightarrow$ MATAD [Steinhauser]

$$\Delta A_{1,e} = \frac{ie^2}{16\pi^2} [-1] \gamma^\mu,$$  \hspace{1cm}  $$\Delta A_{2,e} = \frac{ie^4}{(16\pi^2)^2} \left[ \frac{19}{18\varepsilon} + \frac{247}{108} \right] \gamma^\mu$$

$$\Delta A_{1,\gamma} = \frac{ie^2}{16\pi^2} \left[ \frac{2}{3} p^2 + \frac{2}{3\varepsilon} \tilde{p}^2 \right] g^\mu\nu,$$  \hspace{1cm}  $$\Delta A_{2,\gamma} = \frac{ie^4}{(16\pi^2)^2} \left[ (p^\mu p^\nu - g^\mu\nu p^2) \left( \frac{2}{3\varepsilon} - \frac{71}{18} \right) + g^\mu\nu p^2 \left( -\frac{11}{12} \right) \right]$$

$$\Delta A_{1,ee\gamma} = \frac{ie^3}{16\pi^2} [-2] \gamma^\mu,$$  \hspace{1cm}  $$\Delta A_{2,ee\gamma} = \frac{ie^5}{(16\pi^2)^2} \left[ \frac{13}{9\varepsilon} + \frac{191}{27} \right] \gamma^\mu$$

$$\Delta A_{1,4\gamma} = \frac{ie^4}{16\pi^2} \left[ \frac{4}{3} \right] t^{\mu\nu\rho\sigma},$$  \hspace{1cm}  $$\Delta A_{2,4\gamma} = \frac{ie^6}{(16\pi^2)^2} \left[ -3 \right] t^{\mu\nu\rho\sigma}$$

with $t^{\mu\nu\rho\sigma} = g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}$

- $\Delta A_l$ are polynomial in $p$ and independent of $M^2$ $\Rightarrow$ Rational terms
- One-loop rational term for photon self-energy extended with term $\propto \tilde{p}^2 / \varepsilon$
- Full dependence on $\xi$ and electron mass $m$ in upcoming paper
Summary and Outlook

• Renormalised diagrams in $d$-dimensions can be split into objects with 4-dimensional numerators:

\[
\bar{R} \tilde{A}_2 = \left( A_2 + \sum_\gamma \left( \delta Z_\gamma + \Delta \gamma \right) \cdot A_2 / \gamma \right) + \Delta A_2
\]

with (extended) one-loop rational terms $\Delta \gamma$ and a two-loop rational term $\Delta A_2$.

⇒ Numerical implementation in automated tools, e.g. OpenLoops, possible

• We present a generic method to compute $\Delta A_l$ from tadpoles with one auxiliary scale $M^2$, which also serves as a proof that $\Delta A_l$ is rational

• Full set of QED rational terms at two-loop level

• Further tasks:
  – Compute necessary one-loop integrals involving $\tilde{q}^2$ to order $\varepsilon$
  – Implementation in OpenLoops
Backup
The one-loop procedure is directly applicable for reducible two-loop diagrams (two one-loop diagrams connected by a bridge $P$) due to the factorisation

$$R\tilde{A}_2 = R \left[ \int d\bar{q}_1 \frac{\text{Tr} \left[ \mathcal{N}^{(1)}(\bar{q}_1) \right]^{\sigma_1}}{\bar{D}_0^{(1)}(\bar{q}_1) \cdots \bar{D}_{N_1}^{(1)}(\bar{q}_1)} \right] P_{\sigma_1 \sigma_2} R \left[ \int d\bar{q}_2 \frac{\text{Tr} \left[ \bar{\mathcal{N}}^{(2)}(\bar{q}_2) \right]^{\sigma_2}}{\bar{D}_0^{(2)}(\bar{q}_2) \cdots \bar{D}_{N_2}^{(2)}(\bar{q}_2)} \right]$$

$$= R_4 \left[ \int d\bar{q}_1 \frac{\text{Tr} \left[ \mathcal{N}^{(1)}(\bar{q}_1) \right]^{\sigma_1}}{\bar{D}_0^{(1)}(\bar{q}_1) \cdots \bar{D}_{N_1}^{(1)}(\bar{q}_1)} \right] P_{\sigma_1 \sigma_2} R_4 \left[ \int d\bar{q}_2 \frac{\text{Tr} \left[ \mathcal{N}^{(2)}(\bar{q}_2) \right]^{\sigma_2}}{\bar{D}_0^{(2)}(\bar{q}_2) \cdots \bar{D}_{N_2}^{(2)}(\bar{q}_2)} \right]$$

and the fact that both terms $R \ldots$ are finite.
Example with two sub-divergences

$\gamma_1$: upper photon-fermion loop; $\gamma_2$: lower photon-fermion loop.

Contributions with one original propagator $\bar{D}_j^{(i)}$ (black) in sub-diagram $\bar{\gamma}_2$:

\[
0 = \left( \begin{array}{c}
D = d \\
D = d \\
D = d \\
D = d \\
\end{array} \right)
\bar{D}_j^{(i)}
\left( \begin{array}{c}
D = d \\
\delta Z_{\bar{\gamma}_1} \\
D = 4 \\
D = 4 \\
\end{array} \right)
- \left( \begin{array}{c}
D = 4 \\
D = 4 \\
\delta Z_{\bar{\gamma}_1} \\
D = 4 \\
\end{array} \right)
\left( \begin{array}{c}
D = d \\
D = 4 \\
D = 4 \\
\delta Z_{\bar{\gamma}_1} + \Delta \gamma_1 \\
\end{array} \right).
\]
Example with two sub-divergences

Contributions with one original propagator $\bar{D}_j^{(i)}$ (black) in sub-diagram $\bar{\gamma}_2$:

$$0 = \left(\begin{array}{c}
D = d \\
\delta Z_{\bar{\gamma}_2}
\end{array}\right) - \left(\begin{array}{c}
D = d \\
D = 4
\end{array}\right) - \left(\begin{array}{c}
D = d \\
D = 4
\end{array}\right) + \left(\begin{array}{c}
D = 4 \\
\delta Z_{\bar{\gamma}_2} + \Delta \gamma_2
\end{array}\right),$$

Contributions with one original propagator $\bar{D}_j^{(i)}$ (black) in both sub-diagrams $\bar{\gamma}_1, \bar{\gamma}_2$:

$$0 = \left(\begin{array}{c}
D = d
\end{array}\right) - \left(\begin{array}{c}
D = 4
\end{array}\right)$$