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Rational terms in two-loop calculations

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in collaboration with

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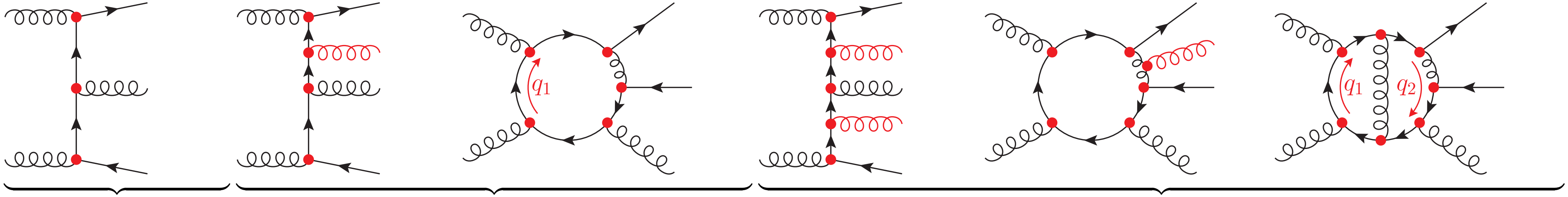
Motivation: Towards two-loop automation

Example of a full NNLO calculation: $gg \to t\bar{t}g$



Tree and one-loop contributions fully automated in **OpenLoops 2** [arxiv:1907.13071] for all LHC processes, including real emission $\rightarrow$ **strong numerical stability and CPU efficiency**

Download: `https://gitlab.com/openloops/OpenLoops` or https://openloops.hepforge.org

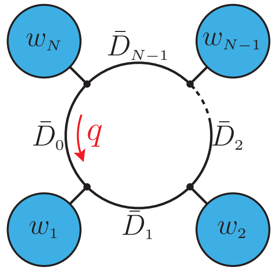
Challenges for numerical two-loop calculation:

- Numerical amplitude construction $\rightarrow$ Numerator of Feynman integral in 4 dimensions
- Tensor integral reduction and computation of Master integrals
  $\rightarrow$ computed in $d = 4 - 2\varepsilon$ dimensions
- Rational terms stemming from discrepancy between Feynman integrands with 4 and d dimensional numerators $\Rightarrow$ one loop: Rational terms of type R_2 [Ossola, Papadopoulos, Pittau]
  this talk: two-loop rational terms

Rational terms at one loop

Generic one-loop diagram (d-dimensional quantities marked with bar)

$$\bar{\mathcal{A}}_1 \;=\; \int \mathrm{d}\bar{q}\, \frac{\bar{\mathcal{N}}(\bar{q})}{\bar{D}_0(\bar{q})\cdots\bar{D}_{N-1}(\bar{q})} \;=$$



(scalar propagators $\bar{D}_i(\bar{q}) = (\bar{q}+p_i)^2 - m_i^2$ with loop momentum $\bar{q}$, mass m_i and external momentum p_i)

Splitting 4 and $(d-4)$ dimensional parts of the numerator yields

$$\bar{\mathcal{A}}_1 = \mathcal{A}_1 + \Delta\mathcal{A}_1 \qquad \text{with} \qquad \Delta\mathcal{A}_1 = \bar{\mathcal{A}}_1 - \mathcal{A}_1 = \int \mathrm{d}\bar{q}\, \frac{\bar{\mathcal{N}}(\bar{q}) - \mathcal{N}(q)}{\bar{D}_0(\bar{q})\cdots\bar{D}_{N-1}(\bar{q})},$$

where $\mathcal{N}(q)$ is constructed from 4-dimensional q, γ^μ, $g^{\mu\nu}$ and $\bar{\mathcal{N}}(\bar{q})$ from d-dimensional
$\bar{q} = q + \tilde{q}, \quad \bar{\gamma}^{\bar{\mu}} = \gamma^\mu + \tilde{\gamma}^{\tilde{\mu}}, \quad \bar{g}^{\bar{\mu}\bar{\nu}} = g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}$

Numerator $\tilde{N} = \bar{\mathcal{N}} - \mathcal{N}$ is of order ε

$\Rightarrow$ only interaction with $\frac{1}{\varepsilon}$ UV poles contributes to finite result [Binoth, Guillet, Heinrich]

$\Rightarrow$ **$\Delta\mathcal{A}_1$ is a (finite) rational term, that can be written as a local counterterm**

[Ossola, Papadopoulos, Pittau]

Renormalised amplitude from subtraction of divergencies (R-operation [Caswell, Kennedy])

$$\mathbf{R}\,\bar{\mathcal{A}}_1 \;=\; \bar{\mathcal{A}}_1 + \delta Z_{\bar{\mathcal{A}}_1} T_{\mathcal{A}_1} \qquad \text{(counterterm } \delta Z_{\bar{\mathcal{A}}_1}\text{, tree-level structure } T_{\mathcal{A}_1}\text{)}$$

Rational term can be written as a (local) counterterm $\Delta\mathcal{A}_1 = \delta Z^{R_2}_{\bar{\mathcal{A}}_1} T_{\mathcal{A}_1}$

$$\Rightarrow \boxed{\mathbf{R}\,\bar{\mathcal{A}}_1 = \mathcal{A}_1 + (\delta Z_{\bar{\mathcal{A}}_1} + \delta Z^{R_2}_{\bar{\mathcal{A}}_1}) T_{\mathcal{A}_1} =: \mathbf{R}_4\,\mathcal{A}_1}$$

One-loop diagram $\mathcal{A}_1$ with 4-dim numerator $\Rightarrow$ suitable for numerical implementation

Generalisation to two loops

Renormalisation procedure: Subtraction of sub-divergencies from one-loop sub-diagrams $\bar{\gamma}$, then of local two-loop divergence:

$$\mathbf{R}\,\bar{\mathcal{A}}_2 = \bar{\mathcal{A}}_2 + \sum_{\bar{\gamma}} \left(\delta Z_{\bar{\gamma}}\right) \cdot \bar{\mathcal{A}}_2/\bar{\gamma} + \delta Z_{\bar{\mathcal{A}}_2} T_{\bar{\mathcal{A}}_2}$$

We show that a renormalised d-dim two-loop amplitude can be computed as

$$\boxed{\mathbf{R}\,\bar{\mathcal{A}}_2 = \mathcal{A}_2 + \sum_{\bar{\gamma}} \left(\delta Z_{\bar{\gamma}} + \delta Z^{R_2}_{\bar{\gamma}}\right) \cdot \mathcal{A}_2/\gamma + \left(\delta Z_{\bar{\mathcal{A}}_2} + \delta Z^{R_2}_{\bar{\mathcal{A}}_2}\right) T_{\bar{\mathcal{A}}_2} =: \mathbf{R}_4\,\mathcal{A}_2.}$$

with (extended) one-loop rational terms $\delta Z^{R_2}_{\bar{\gamma}}$ and a two-loop rational term $\delta Z^{R_2}_{\bar{\mathcal{A}}_2}$

$\Rightarrow$ Two and one-loop diagrams $\mathcal{A}_2$ and $\mathcal{A}_2/_{\gamma}$ constructed with 4-dim numerators

Outline

I. Computation of rational terms from tadpole integrals with one scale
$\rightarrow$ Rational terms of one-loop diagrams and sub-diagrams

II. Rational terms of two-loop diagrams

– without a global divergence

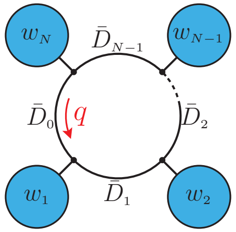
– with a global divergence

III. Full set of QED rational terms at two loops

IV. Summary and Outlook

Rational terms from tadpole integrals: The one-loop case

$$\bar{\mathcal{A}}_1 = \int \mathrm{d}\bar{q}\, \frac{\bar{\mathcal{N}}(\bar{q})}{\bar{D}_0(\bar{q})\cdots\bar{D}_{N-1}(\bar{q})} =$$



$$\mathbf{R}\,\bar{\mathcal{A}}_1 = \bar{\mathcal{A}}_1 + \delta Z_{\bar{\mathcal{A}}_1} T_{\mathcal{A}_1}$$

Define difference between diagram with d-dim and 4-dim numerator: $\Delta\mathcal{A}_1 = \bar{\mathcal{A}}_1 - \mathcal{A}_1$

Use **exact decomposition** of propagator denominators $\bar{D}_i = (\bar{q}+p_i)^2 - m_i^2$ [Chetyrkin, Misiak, Münz]

$$\frac{1}{\bar{D}_i} = \frac{1}{\bar{q}^2 - M^2} + \frac{\Delta_i}{\bar{q}^2 - M^2}\,\frac{1}{\bar{D}_i}$$

with $\Delta_i = -p_i^2 - 2\bar{q}\cdot p_i + m_i^2 - M^2$ **recursively** up to a fixed order $Z+1$:

$$\frac{1}{\bar{D}_i} = \sum_{n=0}^{Z} \frac{(\Delta_i)^n}{(\bar{q}^2 - M^2)^{n+1}} + \frac{(\Delta_i)^{Z+1}}{(\bar{q}^2 - M^2)^{Z+1}}\,\frac{1}{\bar{D}_i}$$

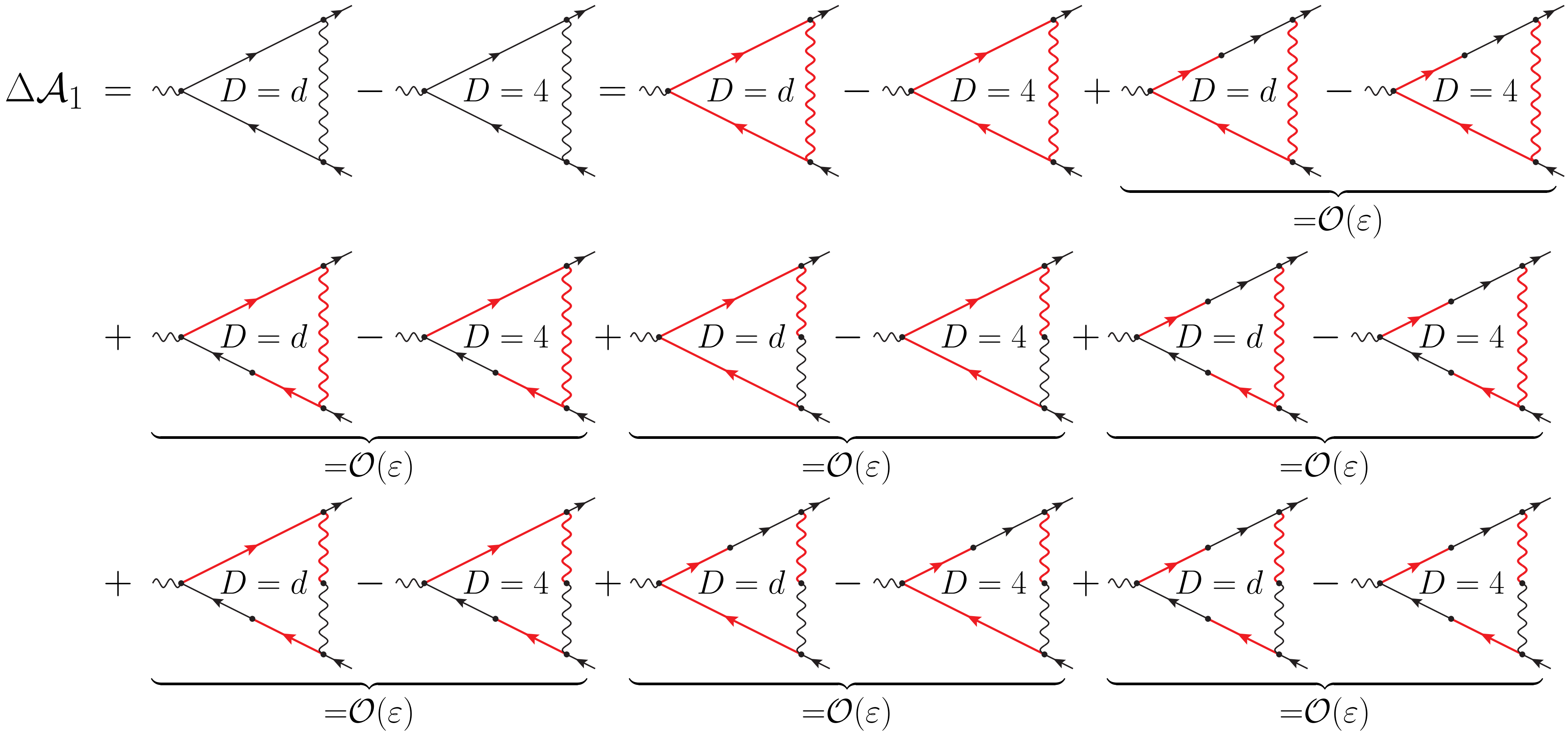
e.g. for $Z=2$: 

$\Rightarrow$ Isolate divergencies + rational terms in tadpole integrals with one auxiliary scale M^2

Terms with a negative degree of divergence X (naive power counting) cancel in $\Delta\mathcal{A}_1$.

One-loop example

$D =$ dimension of the loop momenta, metric and γ-matrices in the numerator $\Rightarrow D \in \{4, d\}$



$$= \int \mathrm{d}\bar{q}\, \frac{\bar{\mathcal{N}}(\bar{q}) - \mathcal{N}(q)}{(\bar{q}^2 - M^2)^3} + \mathcal{O}(\varepsilon)$$

One-loop rational terms from tadpoles

In general: $$\Delta\mathcal{A}_1 = \sum_{n=N}^{N+X} \int \mathrm{d}\bar{q} \, \frac{\left(\bar{\mathcal{N}}(\bar{q}) - \mathcal{N}(q)\right) \mathsf{P}_n(M^2, p_i^2, \bar{q}\cdot p_j)}{(\bar{q}^2 - M^2)^n}$$ polynomial P_n from decomposition of denominators

Decompose numerators:

$$\begin{aligned} \bar{\mathcal{N}}(\bar{q})\,\mathsf{P}_n(M^2, p_i^2, \bar{q}\cdot p_j) &= \sum_{r=0}^{R} \bar{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r} \bar{q}^{\bar{\mu}_1}\cdots\bar{q}^{\bar{\mu}_r} \\ \mathcal{N}(q)\,\mathsf{P}_n(M^2, p_i^2, \bar{q}\cdot p_j) &= \sum_{r=0}^{R} \mathcal{N}_{\mu_1\cdots\mu_r} q^{\mu_1}\cdots q^{\mu_r} \end{aligned}$$

using $\bar{q}\cdot p_j = q\cdot p_j$ for external 4-momenta p_j. With $A_{\mu_i}B^{\bar{\mu}_i} = A_{\mu_i}B^{\mu_i}$ for any vectors A, B

$$\Rightarrow \quad \Delta\mathcal{A}_1 = \sum_{n=N}^{N+X} \sum_{r=0}^{R} \left(\bar{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r} - \mathcal{N}_{\mu_1\cdots\mu_r}\right) \mathcal{I}_n^{\bar{\mu}_1\cdots\bar{\mu}_r} \qquad \text{with} \quad \mathcal{I}_n^{\bar{\mu}_1\cdots\bar{\mu}_r} = \int \mathrm{d}\bar{q} \, \frac{\bar{q}^{\bar{\mu}_1}\cdots\bar{q}^{\bar{\mu}_r}}{(\bar{q}^2 - M^2)^n}$$

- Generic method to compute $\Delta\mathcal{A}_1$ **from tadpole integrals** with one (auxiliary) scale M^2.
- Result independent of M^2 since the denominator decomposition is exact and only terms which cancel in $\Delta\mathcal{A}_1$ are discarded.
- Dependence on external momenta and masses resides only in $\bar{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r}$ and $\mathcal{N}_{\mu_1\cdots\mu_r}$
  $\Rightarrow$ $\Delta\mathcal{A}_1$ is a rational term

One-loop sub-diagrams in two-loop diagrams

One loop: All external momenta in 4 dimensions

Two loop: External momentum to a **sub-diagram** $\bar{\gamma}$ can be d-dim loop momentum $\bar{q}_1$

- Decomposition of d-dim denominators yields

$$\frac{1}{(\bar{q}_1+\bar{q}_2)^2-m^2} \;=\; \frac{1}{\bar{q}_2^2-M^2}+\frac{-\bar{q}_1^2-2\bar{q}_1\cdot\bar{q}_2+m^2-M^2}{\bar{q}_2^2-M^2}\frac{1}{(\bar{q}_1+\bar{q}_2)^2-m^2}$$

$\Rightarrow \int \mathrm{d}\bar{q}_2 \left[\mathcal{N}(q_2,q_1)\,\mathsf{P}_n(\bar{q}_1^2,\bar{q}_1\bar{q}_2)\right]/(\bar{q}_2^2-M^2)^n$ has terms $\propto q_1^2$ (from $\mathcal{N}$ construction in 4-dim),

$\propto \bar{q}_1^2$ (from denominator decomposition)

- Difference of d-dim and 4-dim tensor structures $\bar{t}^{\bar{\alpha}}(\bar{q}_1)-t^{\alpha}(q_1)$ multiplied by one-loop UV counterterm $\delta Z_{\bar{\gamma}}$ not negligible due to insertion into a loop diagram

Extended one-loop rational term for one-loop sub-diagram $\bar{\gamma}$ with external indices $\alpha=(\alpha_1\alpha_2)$:

$$(\Delta\gamma)^{\alpha} := \left[\bar{\gamma}^{\bar{\alpha}}-\gamma^{\alpha}+\left(\bar{t}^{\bar{\alpha}}-t^{\alpha}\right)\delta Z_{\bar{\gamma}}\right]_{\bar{\alpha}\to\alpha}$$

This has the structure $\boxed{\Delta\gamma = R_2(\gamma)+\frac{\Delta R_2(\gamma)}{\varepsilon}}$ with the well-known one-loop $R_2(\gamma)$.

Additional rational term $\Delta R_2(\gamma)\propto\tilde{q}_1^2$ with $\tilde{q}_1=\bar{q}_1-q_1$ requires superficial degree of divergence $\geqq 2$ in sub-diagram (e.g. gauge boson self-energy).

Example: One-loop self-energy of the photon

One-loop amplitude in d and 4 dimensions from denominator decomposition and tadpole method:

$$\bar{\gamma}^{\bar{\mu}\bar{\nu}} = \int \mathrm{d}\bar{q}_2 \frac{\bar{\mathcal{N}}^{\bar{\mu}\bar{\nu}}_{sub}(\bar{q}_1,\bar{q}_2)}{\bar{D}_0^{(2)}(\bar{q}_2)\bar{D}_0^{(3)}(\bar{q}_1+\bar{q}_2)} = \frac{ie^2}{16\pi^2}\left(\frac{-(4-2\varepsilon)\,\bar{q}_1^2}{3\varepsilon} g^{\bar{\mu}\bar{\nu}} + \frac{4}{3\varepsilon}\bar{q}_1^{\bar{\mu}}\bar{q}_1^{\bar{\nu}} + C^{\bar{\mu}\bar{\nu}}_{\text{finite}}\right),$$

$$\gamma^{\mu\nu} = \int \mathrm{d}\bar{q}_2 \frac{\mathcal{N}^{\mu\nu}_{sub}(q_1,q_2)}{\bar{D}_0^{(2)}(\bar{q}_2)\bar{D}_0^{(3)}(\bar{q}_1+\bar{q}_2)} = \frac{ie^2}{16\pi^2}\left(\frac{-4q^2-2\tilde{q}_1^2}{3\varepsilon} g^{\mu\nu} + \frac{4}{3\varepsilon}q_1^{\mu}q_1^{\nu} + C^{\mu\nu}_{\text{finite}}\right),$$

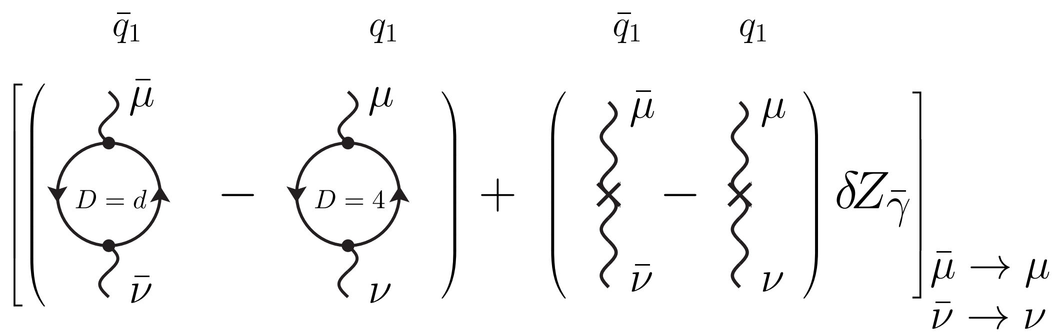
where the constant parts $C^{\bar{\mu}\bar{\nu}}_{\text{finite}} - C^{\mu\nu}_{\text{finite}} = \mathcal{O}(\varepsilon)$.

For the difference of the UV counterterms

$$\left(\bar{t}^{\bar{\mu}\bar{\nu}} - t^{\mu\nu}\right)\delta Z_{\bar{\gamma}} = \left[\left(\bar{q}_1^{\bar{\mu}}\bar{q}_1^{\bar{\nu}} - q_1^{\mu}q_1^{\nu}\right) - \left(\bar{q}_1^2 g^{\bar{\mu}\bar{\nu}} - q_1^2 g^{\mu\nu}\right)\right]\frac{ie^2}{16\pi^2}\left(\frac{-4}{3\varepsilon}\right) \propto \frac{\tilde{q}_1^2}{\varepsilon} g^{\mu\nu} \quad \text{for } \bar{\mu}\to\mu, \bar{\nu}\to\nu$$

This yield the **extended rational term**

$(\Delta\gamma)^{\mu\nu} =$

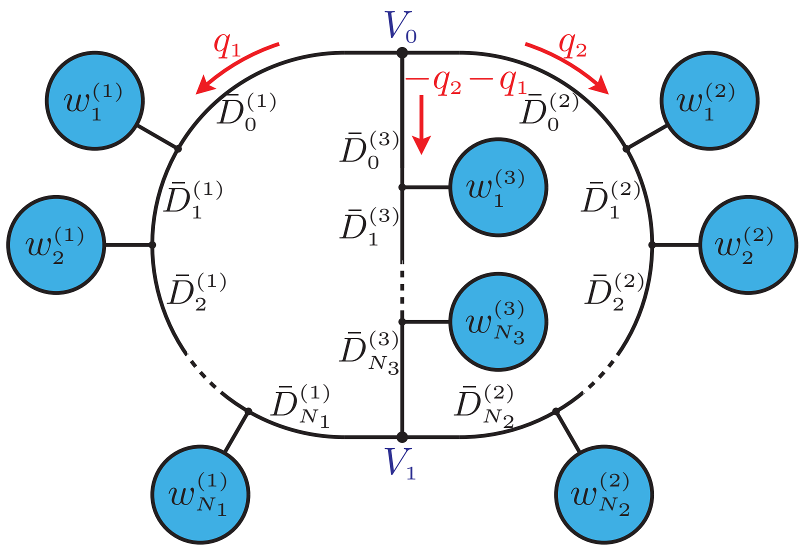


$$= \frac{ie^2}{16\pi^2}\Bigg(\underbrace{q_1^2\,\frac{2}{3}}_{\text{known } R_2} + \underbrace{\tilde{q}_1^2\,\frac{2}{3\,\varepsilon}}_{\text{new } \Delta R_2(\gamma)/\varepsilon}\Bigg) g^{\mu\nu}$$

Two-loop diagrams

Generic irreducible two-loop diagram consists of three chains and two connecting vertices V_0, V_1:

$\bar{\mathcal{A}}_2 =$



$$= \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \, \frac{\bar{\mathcal{N}}_1^{\bar{\alpha}_1}(\bar{q}_1)\, \bar{\mathcal{N}}_2^{\bar{\alpha}_2}(\bar{q}_2)\, \bar{\mathcal{N}}_3^{\bar{\alpha}_3}(\bar{q}_3)\; \Gamma^{V_0 V_1}_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3}}{\left[\prod_{i_1=0}^{N_1} \bar{D}^{(1)}_{i_1}(\bar{q}_1)\right]\left[\prod_{i_2=0}^{N_1} \bar{D}^{(2)}_{i_2}(\bar{q}_2)\right]\left[\prod_{i_3=0}^{N_1} \bar{D}^{(3)}_{i_3}(\bar{q}_3)\right]}$$

with $\bar{q}_3 = -(\bar{q}_1 + \bar{q}_2)$

Renormalised diagram in d dimensions
(what we need)

$$\mathbf{R}\,\bar{\mathcal{A}}_2 = \underbrace{\bar{\mathcal{A}}_2 + \sum_{\bar{\gamma}} \left(\delta Z_{\bar{\gamma}}\right) \cdot \bar{\mathcal{A}}_2/_{\bar{\gamma}}}_{=:\overline{\mathbf{R}}\,\bar{\mathcal{A}}_2} + \delta Z_{\bar{\mathcal{A}}_2}$$

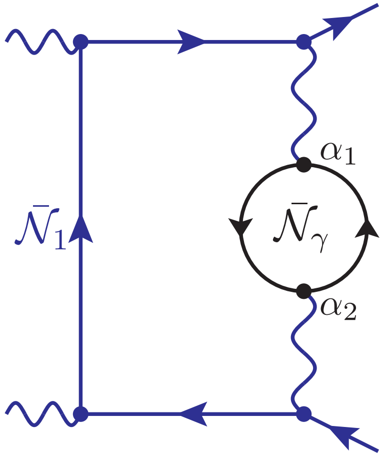
Splitting in pieces with 4-dim numerators
(what we can implement in OpenLoops)
$\Delta\gamma$: one-loop rational term insertion

$$\mathbf{R}\,\bar{\mathcal{A}}_2 = \underbrace{\mathcal{A}_2 + \sum_{\gamma} \left(\delta Z_{\bar{\gamma}} + \Delta\gamma\right) \cdot \mathcal{A}_2/_{\gamma}}_{=:\overline{\mathbf{R}}_4\,\mathcal{A}_2} + \delta Z_{\bar{\mathcal{A}}_2} + \Delta\mathcal{A}_2$$

Our task: Compute $\Delta\mathcal{A}_2 = \overline{\mathbf{R}}\,\bar{\mathcal{A}}_2 - \overline{\mathbf{R}}_4\,\mathcal{A}_2$ and show that it is rational.

The case with no global but one subdivergence

Use factorisation and split sub-diagram (chains 2+3) and rest (chain 1) into d-dim and 4-dim parts:



$$\bar{\mathbf{R}}\,\bar{\mathcal{A}}_2 = \int \mathrm{d}\bar{q}_1 \underbrace{\frac{\mathcal{N}_{1\,\alpha}(q_1) + \tilde{\mathcal{N}}_{1\,\tilde{\alpha}}(\bar{q}_1)}{\bar{D}_0^{(1)}\cdots\bar{D}_{N_1}^{(1)}}}_{\text{no divergence}} \underbrace{\left[\int \mathrm{d}\bar{q}_2 \frac{\bar{\mathcal{N}}_\gamma^{\bar{\alpha}}(\bar{q}_1,\bar{q}_2)}{\bar{D}_0^{(2)}\cdots\bar{D}_{N_2}^{(2)}\bar{D}_0^{(3)}\cdots\bar{D}_{N_3}^{(3)}} + t^{\bar{\alpha}}\delta Z_{\bar{\gamma}}\right]}_{\text{finite} \to \text{discard } \tilde{\mathcal{N}}_{1\,\tilde{\alpha}}}$$

$$= \int \mathrm{d}\bar{q}_1 \frac{\mathcal{N}_{1\,\alpha}(q_1)}{\bar{D}_0^{(1)}\cdots\bar{D}_{N_1}^{(1)}} \left[\int \mathrm{d}\bar{q}_2 \frac{\mathcal{N}_\gamma^{\alpha}(q_1,q_2)}{\bar{D}_0^{(2)}\cdots\bar{D}_{N_3}^{(3)}} + t^{\alpha}\delta Z_{\bar{\gamma}} + (\Delta\gamma)^{\alpha}\right] + \mathcal{O}(\varepsilon)$$

$\bar{\alpha} = (\bar{\alpha}_1\bar{\alpha}_2)$

$\tilde{\mathcal{N}} = \bar{\mathcal{N}} - \mathcal{N}$

$\bar{\alpha} = d$-dim index

$\alpha = 4$-dim index

t^{α} = tensor, e.g. $q_1^{\alpha_1}q_1^{\alpha_2} - q_1^2 g^{\alpha_1\alpha_2}$

$$\text{with } (\Delta\gamma)^{\alpha} = \int \mathrm{d}\bar{q}_2 \frac{\tilde{\mathcal{N}}_\gamma^{\alpha}(q_1,q_2)}{\bar{D}_0^{(2)}\cdots\bar{D}_{N_3}^{(3)}} + \left(t^{\bar{\alpha}} - t^{\alpha}\right)\delta Z_{\bar{\gamma}} = R_2(\gamma) + \frac{\Delta R_2(\gamma)}{\varepsilon}$$

Diagram can be computed with 4-dim numerators:

$$\bar{\mathbf{R}}\,\bar{\mathcal{A}}_2 = \mathcal{A}_2 + \mathcal{A}_2/_{\gamma}\left(\delta Z_{\bar{\gamma}} + \Delta\gamma\right) =: \mathbf{R}_4\,\mathcal{A}_2$$

- **No two-loop rational term, i.e.** $\Delta\mathcal{A}_2 = 0$
- One-loop rational terms $R_2(\gamma)$ needed only to order ε^0
- Additional $\tilde{q}_1^2$ term only in one-loop integrals → need to be computed

The case with a global divergence

- Define **for each chain** $i = 1, 2, 3$ the **maximum degree of divergence** of the full diagram ($X \leq 0$) and any sub-diagram constructed from chains $i, j \neq i$ (degree of sub-divergence X_{ij}):

$$Z_1 = \mathsf{Max}\,(X, X_{12}, X_{13})\,, \quad Z_2 = \mathsf{Max}\,(X, X_{12}, X_{23})\,, \quad Z_3 = \mathsf{Max}\,(X, X_{23}, X_{13})$$

- Use **exact decomposition of propagator denominators** on each chain i up to order $Z_i + 1$:

$$\frac{1}{D_0^{(i)}(\bar{q}_i) \cdots D_{N_i}^{(i)}(\bar{q}_i)} = \left[\mathbf{S}_{Z_i}^{(i)} + \mathbf{F}_{Z_i}^{(i)}\right] \frac{1}{D_0^{(i)}(\bar{q}_i) \cdots D_{N_i}^{(i)}(\bar{q}_i)}\,,$$

with $\mathbf{S}_{Z_i}^{(i)}$ selecting all terms contributing to a (sub-)divergence,

$$\mathbf{S}_Z^{(i)} \frac{1}{D_0^{(i)}(\bar{q}_i) \cdots D_{N_i}^{(i)}(\bar{q}_i)} = \sum_{n=0}^{Z_i} \frac{\mathsf{P}_n^{(i)}(\bar{q}_i, p_j^{(i)})}{(\bar{q}_i^2 - M^2)^{N+1+n}} \qquad \leftarrow \text{tadpoles with scale } M^2$$

and $\mathbf{F}_{Z_i}^{(i)}$ selecting all other terms, especially those with at least one original propagator $D_j^{(i)}$

- In particular, terms selected by $\mathbf{F}_{Z_i}^{(i)}$ have **no global divergence** $\Rightarrow$ **Apply previous case!**

The case with a global divergence

Exact decomposition ⇒ isolate divergent parts ⇒ Discard terms which cancel from calculation:

$$\Delta\mathcal{A}_2 = \left(\mathbf{S}^{(1)}_{Z_1} + \mathbf{F}^{(1)}_{Z_1}\right)\left(\mathbf{S}^{(2)}_{Z_2} + \mathbf{F}^{(2)}_{Z_2}\right)\left(\mathbf{S}^{(3)}_{Z_3} + \mathbf{F}^{(3)}_{Z_3}\right)\Delta\mathcal{A}_2 = \Delta\mathcal{A}_{2,\mathrm{rat}} + \sum_{i=1}^{3}\Delta\mathcal{A}^{(i)}_{2,\mathrm{sub}} + \Delta\mathcal{A}_{2,\mathrm{fin}}\,,$$

$$\begin{aligned} \text{with}\quad \Delta\mathcal{A}_{2,\mathrm{rat}} &= \mathbf{S}^{(1)}_{Z_1}\mathbf{S}^{(2)}_{Z_2}\mathbf{S}^{(3)}_{Z_3}\Delta\mathcal{A}_2\,, \\ \Delta\mathcal{A}^{(i)}_{2,\mathrm{sub}} &= \mathbf{F}^{(i)}_{Z_i}\mathbf{S}^{(j)}_{Z_j}\mathbf{S}^{(k)}_{Z_k}\Delta\mathcal{A}_2\,, \\ \Delta\mathcal{A}_{2,\mathrm{fin}} &= \mathbf{F}^{(1)}_{Z_1}\mathbf{F}^{(2)}_{Z_2}\mathbf{F}^{(3)}_{Z_3}\Delta\mathcal{A}_2 \\ &+ \sum_{i=1}^{3}\mathbf{S}^{(i)}_{Z_i}\mathbf{F}^{(j)}_{Z_j}\mathbf{F}^{(k)}_{Z_k}\Delta\mathcal{A}_2\,. \end{aligned}$$

← only one-scale tadpoles

← no global divergence, at most one sub-divergence ⇒ $\mathcal{O}(\varepsilon)$

← no global or sub-divergence ⇒ $\mathcal{O}(\varepsilon)$

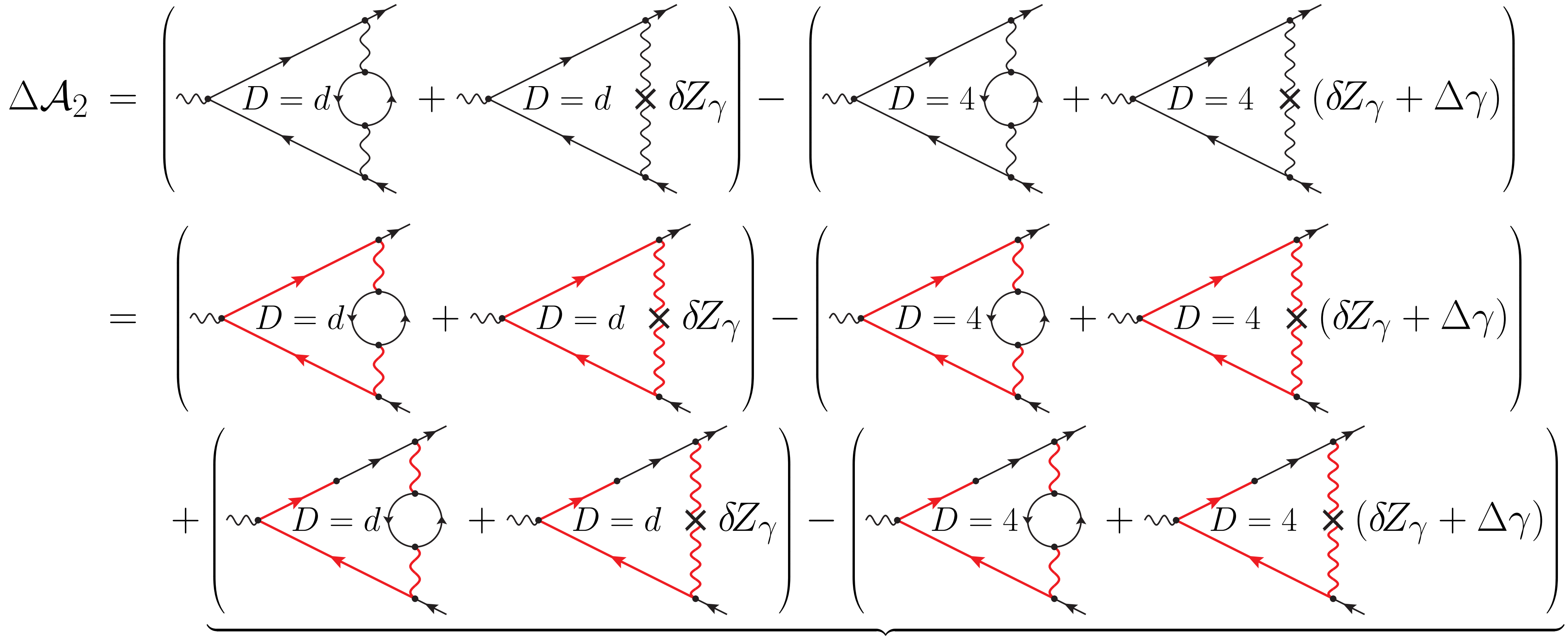
⇒ $\Delta\mathcal{A} = \Delta\mathcal{A}_{2,\mathrm{rat}}$ **is rational! Compute as in one-loop case from**

$$\begin{aligned} \Delta\mathcal{A}_2 &= \sum_{n_i=N_i+1}^{N_i+1+Z_i}\int\mathrm{d}\bar{q}_1\int\mathrm{d}\bar{q}_2\,\frac{\left(\bar{\mathcal{N}}(\bar{q}_1,\bar{q}_2)-\mathcal{N}(q_1,q_2)\right)\mathsf{P}_{n_1n_2n_3}(M^2,p_i^2,q_i\cdot p_j)}{(\bar{q}_1^2-M^2)^{n_1}\,(\bar{q}_2^2-M^2)^{n_2}\,(\bar{q}_3^2-M^2)^{n_3}} \\ &= \sum_{n_i=N_i+1}^{N_i+1+Z_i}\sum_{r=0}^{R}\sum_{s=0}^{S}\left(\bar{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r\bar{\nu}_1\cdots\bar{\nu}_s}-\mathcal{N}_{\mu_1\cdots\mu_r\nu_1\cdots\nu_s}\right)\mathcal{I}^{\bar{\mu}_1\cdots\bar{\mu}_r\bar{\nu}_1\cdots\bar{\nu}_s}_{n_1n_2n_3} \end{aligned}$$

with $\mathcal{I}^{\bar{\mu}_1\cdots\bar{\mu}_r\bar{\nu}_1\cdots\bar{\nu}_s}_{n_1n_2n_3} = \int\mathrm{d}\bar{q}_1\int\mathrm{d}\bar{q}_2\,\frac{\bar{q}_1^{\bar{\mu}_1}\cdots\bar{q}_1^{\bar{\mu}_r}\,\bar{q}_2^{\bar{\nu}_1}\cdots\bar{q}_2^{\bar{\nu}_s}}{(\bar{q}_1^2-M^2)^{n_1}\,(\bar{q}_2^2-M^2)^{n_2}\,(\bar{q}_3^2-M^2)^{n_3}}$

Example: QED vertex correction

Let $D \in \{4, d\}$ be the numerator dimension. Decomposing chain 1 (with external fermion line):



$=0+\mathcal{O}(\varepsilon)$ (case without global divergence)

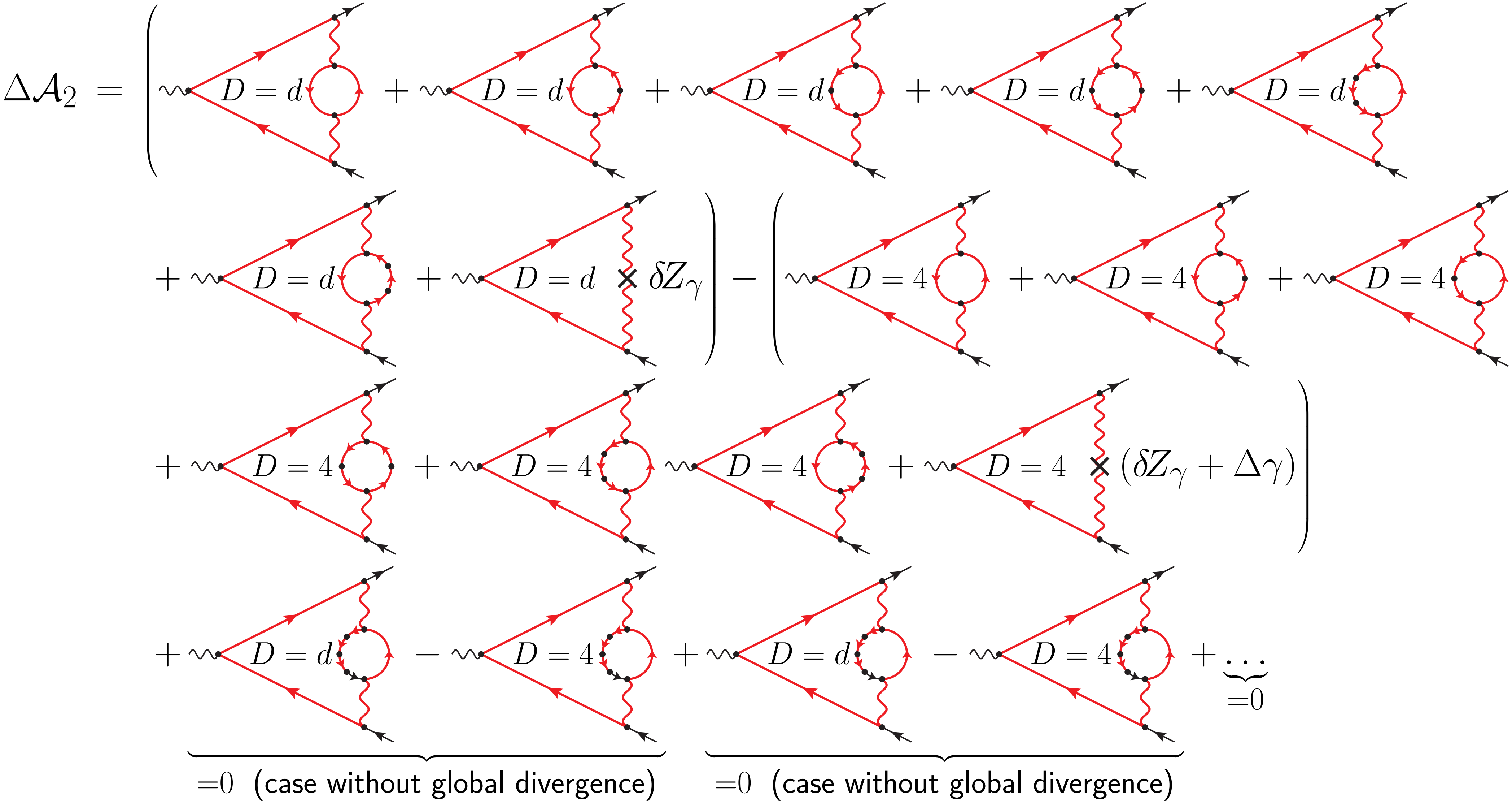
$+$ more terms with original propagator denominators along outer chain

$=0+\mathcal{O}(\varepsilon)$ (case without global divergence)

black lines: original propagators $\bar{D}_j^{(i)}$; red lines: factors in denominator decomposition $\propto (\bar{q}_i^2 - M^2)$.

Example: QED vertex correction

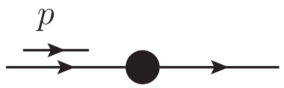
Decomposing chains 2,3 (photon self-energy sub-diagram):



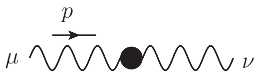
Only massive tadpoles contribute to difference $\Delta\mathcal{A}_2$.

Rational terms for QED in $\overline{MS}$ scheme ($\xi = 0$, $m = 0$)

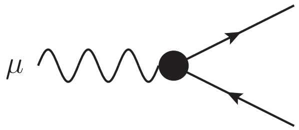
Calculation in the GEXCOM [Chetyrkin, M.Z.] framework: QGRAF [Noguira] $\to$ Q2E+EXP [Seidesticker, Harlander, Steinhauser] $\to$ FORM [Vermaseren] code $\to$ MATAD [Steinhauser]



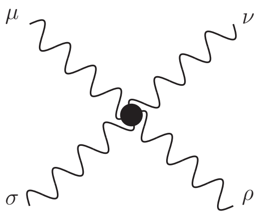
$$\Delta A_{1,e} = \frac{ie^2}{16\pi^2}\left[-1\right]\not{p}, \qquad \Delta A_{2,e} = \frac{ie^4}{(16\pi^2)^2}\left[\frac{19}{18\varepsilon} + \frac{247}{108}\right]\not{p}$$



$$\Delta A^{\mu\nu}_{1,\gamma} = \frac{ie^2}{16\pi^2}\left[\frac{2}{3}p^2 + \frac{2}{3\varepsilon}\tilde{p}^2\right]g^{\mu\nu}, \qquad \Delta A^{\mu\nu}_{2,\gamma} = \frac{ie^4}{(16\pi^2)^2}\left[(p^\mu p^\nu - g^{\mu\nu}p^2)\left(\frac{2}{3\varepsilon} - \frac{71}{18}\right) + g^{\mu\nu}p^2\left(-\frac{11}{12}\right)\right]$$



$$\Delta A^{\mu}_{1,ee\gamma} = \frac{ie^3}{16\pi^2}\left[-2\right]\gamma^\mu, \qquad \Delta A^{\nu}_{2,ee\gamma} = \frac{ie^5}{(16\pi^2)^2}\left[\frac{13}{9\varepsilon} + \frac{191}{27}\right]\gamma^\mu$$



$$\Delta A^{\mu\nu\rho\sigma}_{1,4\gamma} = \frac{ie^4}{16\pi^2}\left[\frac{4}{3}\right]t^{\mu\nu\rho\sigma}, \qquad \Delta A^{\mu\nu\rho\sigma}_{2,4\gamma} = \frac{ie^6}{(16\pi^2)^2}\left[-3\right]t^{\mu\nu\rho\sigma}$$

with $t^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}$

- $\Delta\mathcal{A}_l$ are polynomial in p and independent of M^2 $\Rightarrow$ Rational terms
- One-loop rational term for photon self-energy extended with term $\propto \tilde{p}^2/\varepsilon$
- Full dependence on ξ and electron mass m in upcoming paper

Summary and Outlook

- Renormalised diagrams in d-dimensions can be split into objects with 4-dimensional numerators:

$$\overline{\mathbf{R}}\,\bar{\mathcal{A}}_2 = \Big(\mathcal{A}_2 + \sum_{\gamma}\left(\delta Z_{\bar{\gamma}} + \Delta\gamma\right)\cdot \mathcal{A}_2/\gamma\Big) + \Delta\mathcal{A}_2$$

  with (extended) one-loop rational terms $\Delta\gamma$ and a two-loop rational term $\Delta\mathcal{A}_2$.
  $\Rightarrow$ **Numerical implementation in automated tools, e.g. OpenLoops, possible**

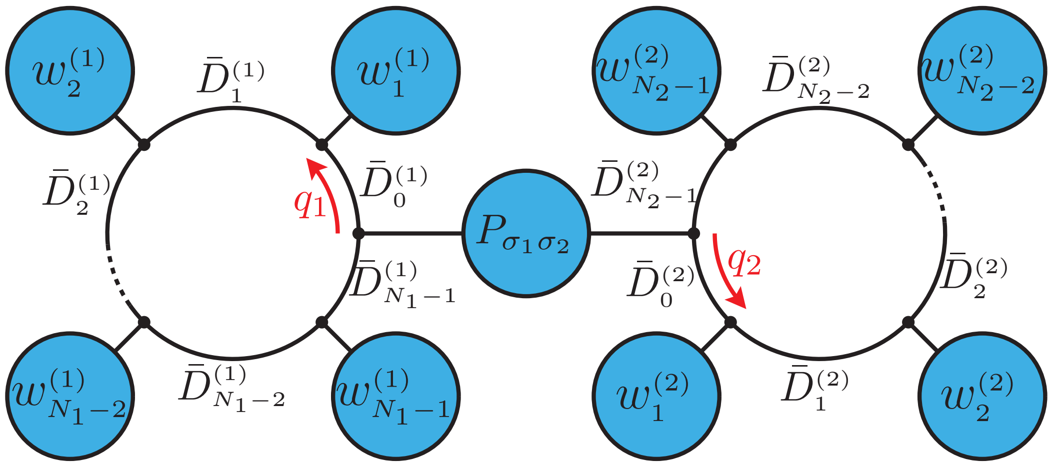
- We present a **generic method to compute $\Delta\mathcal{A}_l$** from tadpoles with one auxiliary scale M^2, which also serves as a **proof that $\Delta\mathcal{A}_l$ is rational**

- Full set of QED rational terms at two-loop level

- Further tasks:
  - Compute necessary one-loop integrals involving $\tilde{q}^2$ to order ε
  - Implementation in OpenLoops

Backup

Reducible two-loop diagrams



The one-loop procedure is directly applicable for reducible two-loop diagrams (two one-loop diagrams connected by a bridge P) due to the factorisation

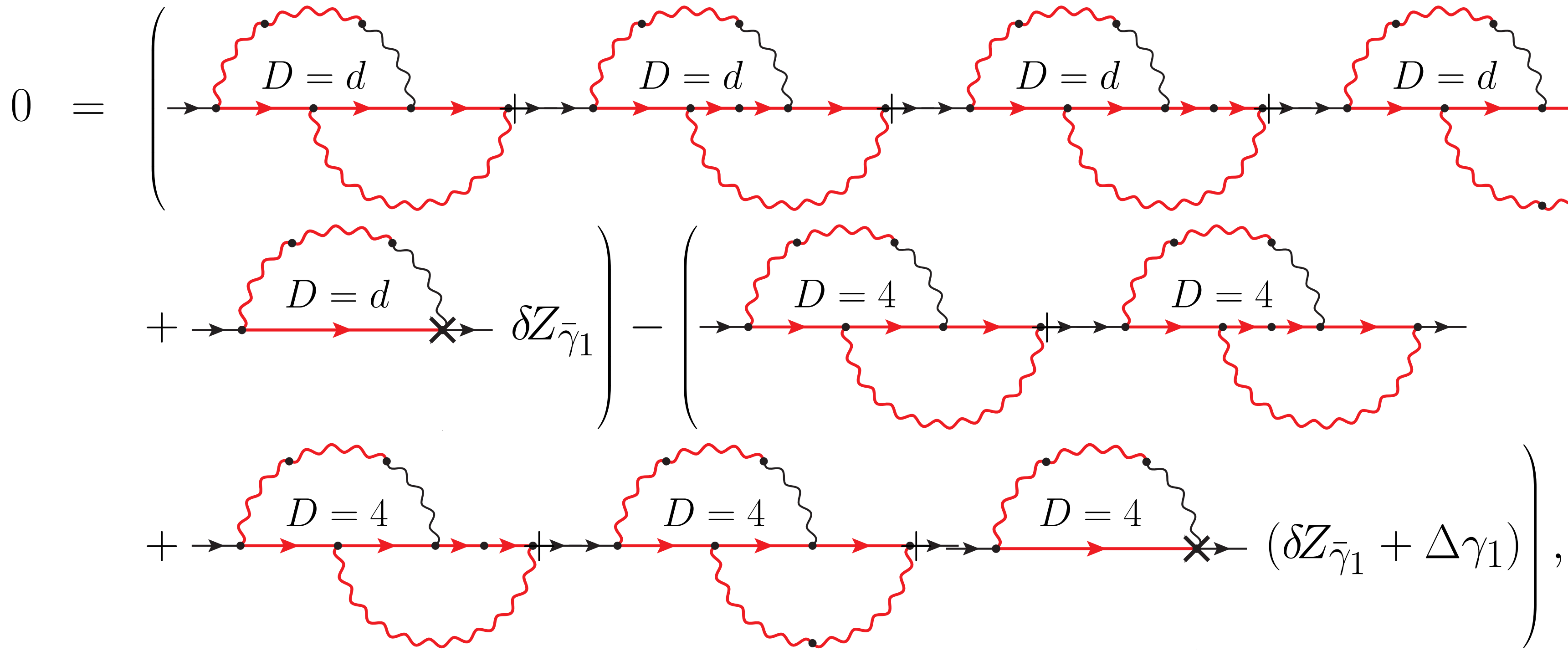
$$
\begin{aligned}
\mathbf{R}\bar{\mathcal{A}}_2 &= \mathbf{R}\left[\int \mathrm{d}\bar{q}_1 \frac{\mathrm{Tr}\left[\bar{\mathcal{N}}^{(1)}(\bar{q}_1)\right]^{\sigma_1}}{\bar{D}_0^{(1)}(\bar{q}_1)\cdots\bar{D}_{N_1}^{(1)}(\bar{q}_1)}\right] P_{\sigma_1\sigma_2}\mathbf{R}\left[\int \mathrm{d}\bar{q}_2 \frac{\mathrm{Tr}\left[\bar{\mathcal{N}}^{(2)}(\bar{q}_2)\right]^{\sigma_2}}{\bar{D}_0^{(2)}(\bar{q}_2)\cdots\bar{D}_{N_2}^{(2)}(\bar{q}_2)}\right] \\
&= \mathbf{R}_4\left[\int \mathrm{d}\bar{q}_1 \frac{\mathrm{Tr}\left[\mathcal{N}^{(1)}(q_1)\right]^{\sigma_1}}{\bar{D}_0^{(1)}(\bar{q}_1)\cdots\bar{D}_{N_1}^{(1)}(\bar{q}_1)}\right] P_{\sigma_1\sigma_2}\mathbf{R}_4\left[\int \mathrm{d}\bar{q}_2 \frac{\mathrm{Tr}\left[\mathcal{N}^{(2)}(q_2)\right]^{\sigma_2}}{\bar{D}_0^{(2)}(\bar{q}_2)\cdots\bar{D}_{N_2}^{(2)}(\bar{q}_2)}\right]
\end{aligned}
$$

and the fact that both terms $\mathbf{R}\left[\ldots\right]$ are finite.

Example with two sub-divergences

γ_1: upper photon-fermion loop; γ_2: lower photon-fermion loop.

Contributions with one original propagator $\bar{D}_j^{(i)}$ (black) in sub-diagram $\bar{\gamma}_2$:



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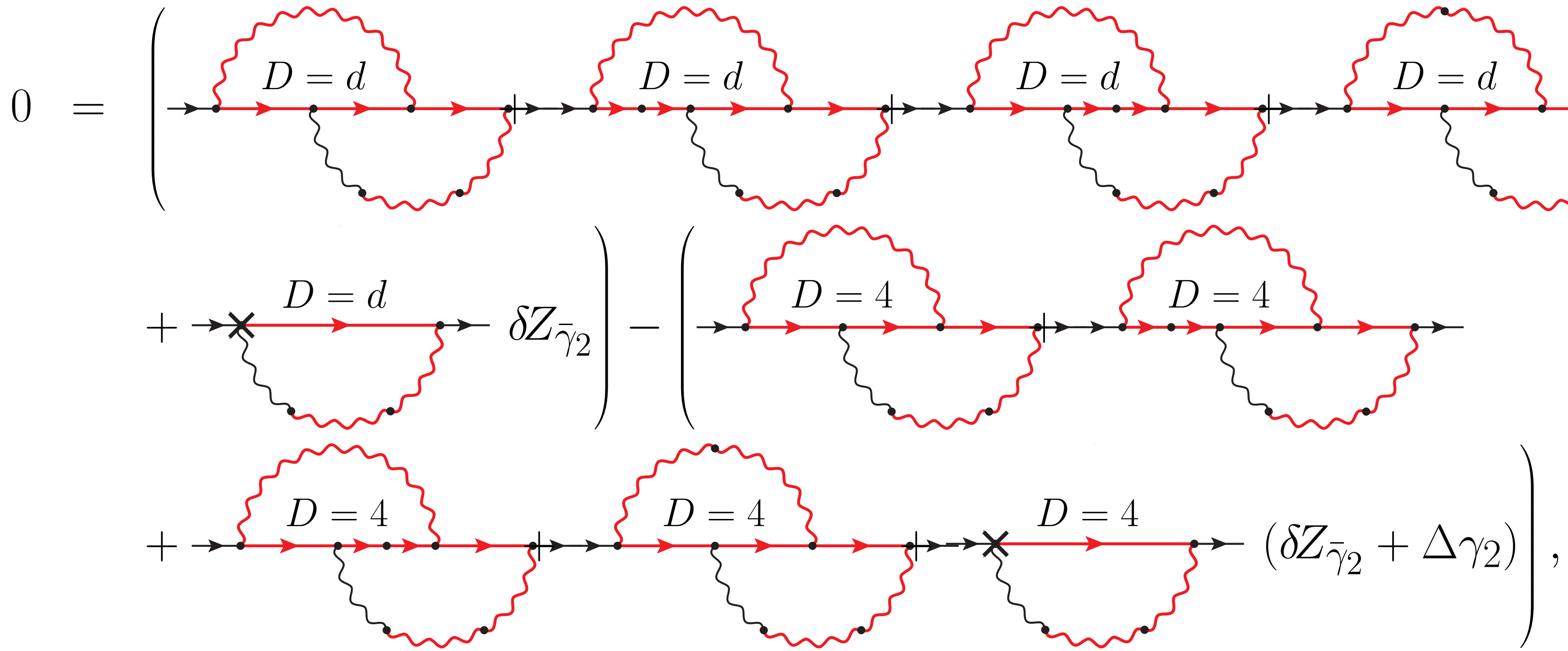
Contributions with one original propagator $\bar{D}_j^{(i)}$ (black) in sub-diagram $\bar{\gamma}_2$:



Contributions with one original propagator $\bar{D}_j^{(i)}$ (black) in both sub-diagrams $\bar{\gamma}_1, \bar{\gamma}_2$: