Syzygies for No-Numerator or No-Dot Relations Between Feynman Integrals

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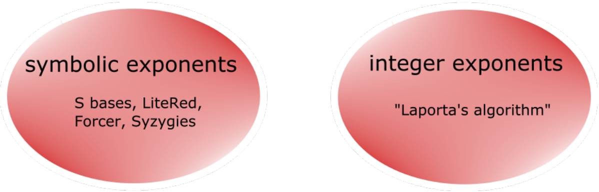
Multi-Loop Feynman integrals

$$I(\nu_1,\ldots,\nu_N) = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{1}{D_1^{\nu_1}\cdots D_N^{\nu_N}} \qquad \nu_i \in \mathbb{Z}, \quad D_1 = k_1^2 - m_1^2 \text{ etc.}$$

family of loop integrals:

- fulfill linear relations: integration-by-parts identities [Chetyrkin, Tkachov '81]
- systematic reduction to master integrals possible [Laporta '00]
- linear vector space with some finite basis
- specific basis choices:
  - ϵ-basis (uniform weight) for differential equations [Henn '13], see talk by [Stefan Weinzierl]
  - basis of finite integrals for direct integration (analyt., numeric.) [A.v.M., Panzer, Schabinger '16]
  - uniform weight + finite: [Schabinger '18]
- depending on application, ν_i will be different from 0 and 1:
  - numerators ($\nu_i < 0$): occur naturally in scattering amplitudes, ...
  - dots ($\nu_i > 1$): occur naturally for finite integrals, ...

Approaches to IBP reduction



- in real life: typically a combination of both approaches
- new focus: syzygy constructions [Gluza, Kadja, Kosower] and [Bitoun, Böhm, Bogner, Georgoudis, Ita, Klausen, Larsen, Lee, Panzer, Schabinger, Schulze, Zhang, Zeng and others], see talk by [Ben Page]

This talk:

1. improve integer part: finite fields + rational reconstruction
2. motivation for dotted integrals
3. improve symbolic part: avoid numerators
4. improve symbolic part: avoid dots

Main example: four-loop form factors

Original motivation: analytical calculation of $q\bar{q}\gamma$ and ggH to four-loop QCD
[AvM, Erik Panzer, Rober Schabinger]

- $O(50000)$ diagrams
- just one overall scale
- use R_ξ gauge (power one for props + ext pol)
- $O(10^9)$ integrals in diagrams, up to 6 irreducible scalar products
- $O(100)$ different 12-line topologies
- 10 integral families (18 indices)
- $O(300)$ master integrals
- $O(25000)$ sectors in reduction, $O(2000)$ non-shiftable
- up to $O(10^8 - 10^9)$ equations per sector

Methods useful also for further applications.

Part 1: finite fields and rational reconstruction

IBP reductions from finite field samples

A Finite Field Approach to IBPs [AvM, Schabinger '14]

1. finite field sampling
   - set variables to integer numbers
   - consider coefficients modulo a prime field $\mathbb{Z}_p$
2. solve finite field system
3. reconstruct rational solution from many such samples

finite field techniques:

- no intermediate expression swell by construction
- early discarding of redundant and auxiliary quantities
- great potential for parallelisation

established in computer sciences, more and more popular in physics:

- dense solver: [Kauers], filtering: `Ice` [Kant '13], integrand construction: [Peraro '16], public IBP codes: Kira [Maierhöfer, Usovitsch, Uwer '17], Fire [Smirnov, Chukharev '19], and private codes...
- see talks by [Tiziano Perraro] and [Ben Page]

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}}[x] & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

$$
\begin{array}{ccccccccc}
I_{\mathbb{Z}_p[x]} & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p[x]} & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_p, x_2} & \longrightarrow & O_{\mathbb{Z}_p, x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p, x3} & \longrightarrow & O_{\mathbb{Z}_p, x3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

note: multivariate case by iteration

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Package: finred
Author:  Andreas v. Manteuffel
```

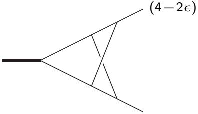
features:

- C++11 implementation for multivariate sparse matrices
- links `flint` library
- parallelisation: SIMD, threads, batch
- custom file formats with compression and checksums
- equation filtering: eliminate redundant rows
- pivot optimization
- plus lots of IBP specific features
- much faster than Reduze 2
- employed for construction and application of syzygies

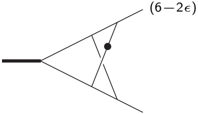
Part 2: motivation for dotted integrals

Proposal for singularity resolution [AvM, Panzer, Schabinger '14]

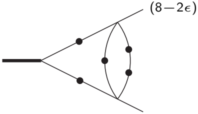
observation: always possible to decompose wrt basis of finite integrals



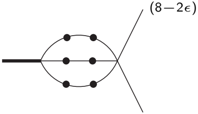
$$= -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2}$$



$$-\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4(1-\epsilon)^2(2-\epsilon)^2 q^2}$$



$$+\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)(1-\epsilon)^2(2-\epsilon)^2 q^2}$$



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basis consists of standard Feynman integrals, but

- in shifted dimensions
- with additional dots (propagators taken to higher powers)

Practical algorithm for basis construction

Algorithm: construction of finite basis

- systematic scan for finite integrals with dim-shifts and dots (with Reduze 2)
- IBP + dimensional recurrence for actual basis change

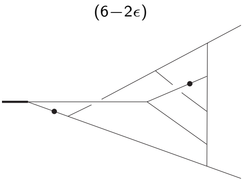
remarks:

- computationally expensive part shifted to IBP solver
- efficient, easy to automate
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon

Analytical integration @ 4-loops

[AvM, Panzer, Schabinger '15]

a non-planar 12-line topology @ 4-loops:



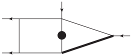
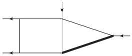
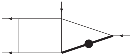
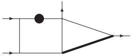
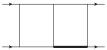
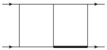
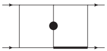
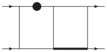
$$= \frac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \Big(24\zeta_2\zeta_3 + 20\zeta_5 - \frac{188}{105}\zeta_2^3 - 17\zeta_3^2 + 9\zeta_2^2\zeta_3$$

$$- 47\zeta_2\zeta_5 - 21\zeta_7 + \frac{6883}{2100}\zeta_2^4 + \frac{49}{2}\zeta_2\zeta_3^2 + \frac{1}{2}\zeta_3\zeta_5 - 9\zeta_{5,3}\Big)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

- only shallow ϵ expansion needed
- numerical result with `Fiesta` [A. Smirnov]: straight-forward confirmation
- high weight at leading order helps to decouple e.g. from cusp anomalous dimension

Numerical performance

- finite integrals render problematic double boxes numerically accessible [AvM, Schabinger '17]

finite	time	rel. err.	conventional	time	rel. err.
$(6-2\epsilon)$ 	201 s	2.34×10^{-4}	$(4-2\epsilon)$ 	384 s	8.12×10^{-4}
$(6-2\epsilon)$ 	150 s	4.83×10^{-4}	$(4-2\epsilon)$ 	56538 s	1.67×10^{-2}
$(6-2\epsilon)$ 	280 s	1.00×10^{-3}	$(4-2\epsilon)$ 	214135 s	8.29×10^{-3}
$(6-2\epsilon)$ 	294 s	1.21×10^{-3}	$(4-2\epsilon)$ 	3484378 s	30.9

timings with SecDec 3 in physical region

- used for top-quark mass dep. in HH [Heinrich et al], Hj [Jones et al], see talks by [Gudrun Heinrich], [Matthias Kerner]

Part 3: syzygies for no-numerator relations

Integration-by-parts (IBP) identities

in dimensional regularisation, integral over total derivative vanishes:

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(k_j^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right)$$

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(p_j^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right)$$

where p_j are external momenta, $\nu_i \in \mathbb{Z}$, $D_1 = k_1^2 - m_1^2$ etc.

problems of above construction:

- introduces many auxiliary integrals with additional dots and/or numerators
- sparse but still rather coupled system of equations
- here: how to avoid either numerators or dots

Syzygy Based IBPs Without Numerators

[Lee, Pomeransky '13] representation:

$$I(\nu_1,\ldots,\nu_N) = \mathcal{N}\left[\prod_{i=1}^{N}\int_0^\infty \mathrm{d}x_i x_i^{\nu_i-1}\right] G^{-d/2} \qquad \text{with } G = \mathcal{U}+\mathcal{F}$$

[Bitoun, Bogner, Klausen, Panzer '17]: define (twisted) Mellin Transform

$$\mathcal{M}\{f\}(\nu) := \left(\prod_{k=1} N\int_0^\infty \frac{x_k^{\nu_k-1}\mathrm{d}x_k}{\Gamma(\nu_k)}\right) f(x_1,\ldots,x_N)$$

Feynman integrals are Mellin transforms:

$$\tilde{I}(\nu) = \mathcal{M}\left\{G^{-d/2}\right\}(\nu)$$

with $\nu = (\nu_1,\ldots,\nu_N)$ and $\tilde{I}(\nu) = \Gamma[(L+1)d/2-\nu]I(\nu)$

Properties of Mellin transform

1. $\mathcal{M}\{\alpha f+\beta g\}(\nu) = \alpha\mathcal{M}\{f\}(\nu)+\beta\mathcal{M}\{g\}(\nu)$
2. $\mathcal{M}\{x_i f\}(\nu) = \nu_i\mathcal{M}\{f\}(\nu+e_i\}$
3. $\mathcal{M}\{-\partial_i f\}(\nu) = \mathcal{M}\{f\}(\nu-e_i\}$ (proof: partial integration)

Define shift operators

$$(\hat{i}^+ F)(\nu_1, \dots, \nu_N) = \nu_i F(\nu_1, \dots, \nu_i + 1, \dots, \nu_N)$$
$$(\hat{i}^- F)(\nu_1, \dots, \nu_N) = F(\nu_1, \dots, \nu_i - 1, \dots, \nu_N)$$

which form Weyl algebra, $[\hat{i}^+, \hat{j}^-] = \delta_{ij}$

[Lee '14; Bitoun, Bogner, Klausen, Panzer '17]: a differential operator P which annihilates $G^{-d/2}$

$$P\, G^{-d/2}$$

generates via the substitutions

$$x_i \to \hat{i}^+,$$
$$\partial_i \to -\hat{i}^-$$

a shift relation according to

$$\mathcal{M}\{P\, G^{-d/2}\} = 0$$

In fact, *every* shift relation is related in this way.

idea: construct annihilators:

$$\left[c_0 + \sum_{i=1}^{N} c_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{N} c_{ij} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \dots\right] G^{-d/2} = 0$$

new in this talk: annihilators beyond linear order

determine $c_0(x_1,\dots,x_N),\dots$ via syzygy equations:

$$c_0\left[-\frac{2}{d}G^2\right] + \sum_{i=1}^{N} c_i \left[G\frac{\partial G}{\partial x_i}\right] + \sum_{i,j=1}^{N} c_{ij}\left[G\frac{\partial^2 G}{\partial x_i \partial x_j} + \left(-\frac{d}{2}-1\right)\frac{\partial G}{\partial x_i}\frac{\partial G}{\partial x_j}\right] + \dots = 0$$

Syzygies generate linear relations for Feynman integrals:

$$\left(\left[c_0(\hat{1}^+,\dots,\hat{N}^+) - \sum_{i=1}^{N} c_i(\hat{1}^+,\dots,\hat{N}^+)\hat{i}^- + \sum_{i,j=1}^{N} c_{ij}(\hat{1}^+,\dots,\hat{N}^+)\hat{i}^-\hat{j}^- + \dots\right]\tilde{I}\right)(\nu_1,\dots,\nu_N) = 0$$

reminder: syzygies

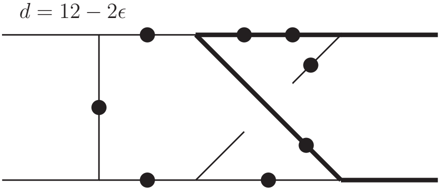
- suppose that for given polynomials $f = (f_1, f_2, \dots)$ one can find polynomials $s = (s_1, s_2, \dots)$ such that $\sum_i f_i s_i = 0$, then s is called a syzygy
- if s is a syzygy, then $s \cdot g$ is a syzygy for any polynomial g
- the (infinite) set of syzygies for f is a syzygy module

Computation

- computation of syzygies:
  - available in CAS like Singular
  - for complicated Feynman integrals: performance issues
- strategy:
  - observation: need only generating syzygies of low degree
  - use custom syzygy finder based on linear algebra (w/ `Finred`)
  - induced syzygies: via seed integrals
  - followed by "Laporta's reduction"
- basic linear algebra method: based on `LASyz` [Carbacas, Ding '11], see also [Schabinger '11]
  - homogenize system
  - compute generating syzygies up to some degree
  - represent explicit expressions in matrix
  - row reduce + nullspace → syzygies
  - remove syzygies induced by lower degree syzygies]
  - finite fields: benefit from late reconstruction
- result:
  - reductions of integrals with large number of dots possible
  - no auxiliary numerators need to be introduced

Example at 2-loops

reductions like this easily accessible now:



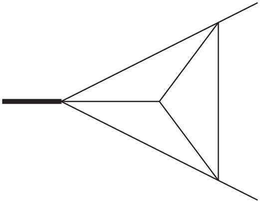
finite integral with improved convergence properties

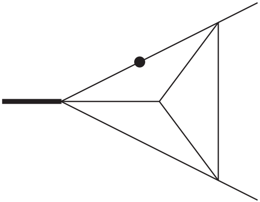
used in [Becchetti,, Bonciani, Casconi, Ferroglia, Lavacca, AvM '19], see talk by [Matteo Becchetti]

Question: are first order annihilators sufficient ?

Example: planar three-point function, 3 loops

- Relations with 1 derivative: leave two integrals unreduced

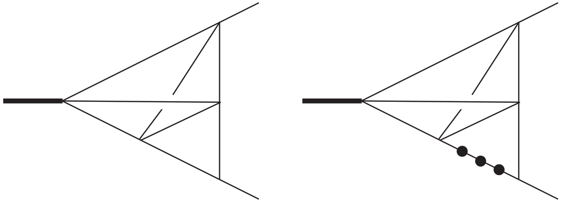




  - counting for integer powers, ignoring subsectors
  - tested up to 15 dots (degree of syzygies) in relations

- Relations with 2 derivatives: reduce one of the two integrals
  - degree 3 syzygies + seeds sufficient
  - in the same sector, in general with a different number of dots
  - in subsectors, possibly with a numerator (inverse propagator)
  - both, inverse propagators and sub-sub-sectors are key to completeness

- Relations with 1 derivative: a tower of integrals is not reduced:



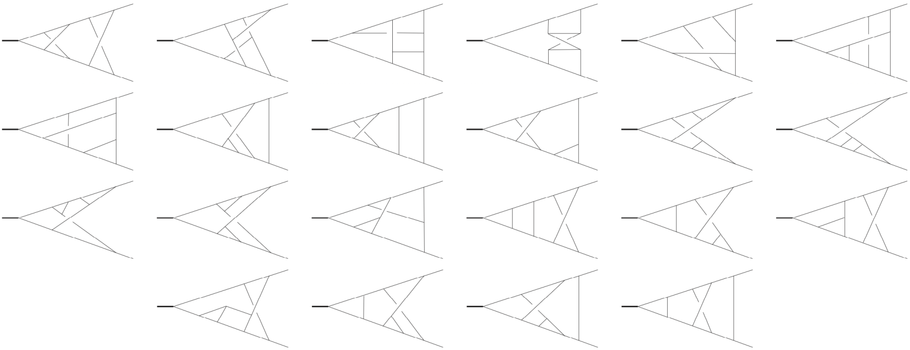
  - dots on a specific propagator unreduced
  - tested up to degree four syzygies + 17 dots for seeds

- Relations with 2 derivatives: reduce all but one integral
  - degree 3 syzygies + seeds sufficient

Recent results at four loops

Four loop form factors in QCD: [Henn, Smirnov, Smirnov, Steinhauser '16; Henn, Lee, Smirnov, Smirnov, Steinhauser '16, '17, '19; AvM, Schabinger '16, '19, '19]

New: contributions with two fermion loops to quark and gluon form factors:



46 twelve-line topologies, up to 6 ISPs, 163 master integrals

analytic results in terms of multiple zeta values [AvM, Schabinger '19]

details: see talk by [Robert Schabinger]

Part 4: syzygies for no-dot relations

Syzygy Based IBPs Without Dots

Baikov's parametric representation of Feynman integrals:

$$I(\nu_1,\ldots,\nu_N) = \mathcal{N}\int \mathrm{d}z_1\cdots\mathrm{d}z_m P^{\frac{d-L-E-1}{2}}\frac{1}{z_1^{\nu_1}\cdots z_N^{\nu_N}}$$

[Böhm, Georgoudis, Larsen, Schulze, Zhang '18]: useful for IBPs without dots

$$\begin{aligned}0 &= \int \mathrm{d}z_1\cdots\mathrm{d}z_m \sum_{i=1}^{m}\frac{\partial}{\partial z_i}\left(a_i P^{\frac{d-L-E-1}{2}}\frac{1}{z_1^{\nu_1}\cdots z_N^{\nu_N}}\right)\\ &= \int \mathrm{d}z_1\cdots\mathrm{d}z_m \sum_{i=1}^{N}\left(\frac{\partial a_i}{\partial z_i}+\frac{d-L-E-1}{2P}a_i\frac{\partial P}{\partial z_i}-\frac{\nu_i a_i}{z_i}\right)P^{\frac{d-L-E-1}{2}}\frac{1}{z_1^{\nu_1}\cdots z_N^{\nu_N}}\end{aligned}$$

explicit solutions to constraint:

$$\left(\sum_{i=1}^{N} a_i\frac{\partial P}{\partial z_i}\right)+bP=0 \qquad \text{(absence of dim. shifts)}$$

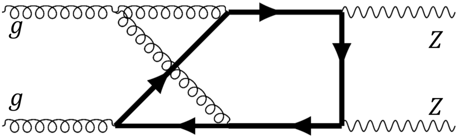
in addition, require for denominators of sector:

$$a_i = b_i z_i \qquad \text{(absence of dots)}$$

need intersection of two syzygy modules

No Dots with Basic Linear Algebra

- available CAS like Singular implement general syzygy intersection calculation
  but: performance can be issue for complicated problems
- alternative: basic linear algebra approach
  - explicit construction of syzygies for given degree
  - ordering to eliminate non-admissable terms: $a_i \neq b_i z_i$
  - row reduce with `Finred`
- multi-loop application: four-loop form factors ✓
- multi-scale application: finite top mass corrections to diboson production ✓



several top-level topologies, reduced up to rank 5 integrals, see talk by [Bakul Agarwal]

Conclusions

basis of finite integrals:

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations

reductions via finite field sampling:

- speeds up integration-by-parts reductions
- useful also in other contexts

reductions based on syzygies:

- custom syzygy finder with linear algebra (w/ finite fields)
- low degree generators for syzygies sufficient
- no numerators: need 2nd order derivatives
- no dots: fast method for module intersection