Calculating the static gravitational two-body potential to fifth post-Newtonian order with Feynman diagrams

Christian Sturm

Universität Würzburg
Institut für Theoretische Physik und Astrophysik
Theoretische Physik II (TP2)
Emil-Hilb-Weg 22
97074 Würzburg

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in collaboration with:

Stefano Foffa, Pierpaolo Mastrolia, Riccardo Sturani & William J. Torres Bobadilla

I. Introduction

Generalities & Background

- Newton's law → potential energy

$$\mathcal{V}(r) = -\frac{Gm_1 m_2}{r} + \ldots \text{ post-Newtonian corrections } \ldots$$

due to general relativity(GR) ↩

- Post-Newtonian(PN) expansion
  - static contributions: Newton's constant G ← here
  - non-static contrib.: velocities $\vec{v}$
- Expansion in powers of virial-related quantities

$$Gm/r \sim v^2$$

$$n\text{PN}: n = i + j - 1, \quad i, j: \text{powers of } G, v^2$$

(weak coupling, small velocities)

- Deviations from Newton potential observable, contribute to perihelion precession of mercury
- Potential ↔ energy of binary system
  energy loss ↔ gravitational waves ↔ inspiralling

Introduction

Generalities & Background, Motivation

- Detection of gravitational waves
  - indirect: Hulse-Taylor binary,...
  - direct: LIGO, Virgo,...
- Future experiments (KAGRA, Einstein Telescope, LISA,...)
  ↪ expect increase of sensitivity of factor ∼10
  ↪ precise theory calculations desirable
- Accurate theory predictions ↔ precise measurement
  ↪ test GR ↩
- Different approaches to solve grav. 2-body problem
  Here: Effective field theory approach (EFT) Goldberger, Rothstein
  ↪ methods of particle physics applicable,
  calculation of Feynman diagrams,...

  other approaches:
  ADM Hamiltonian formalism, numerical GR, post-Minkowskian,...

Introduction

EFT, action

- Action $S = S_{\text{pp}} + S_{\text{bulk}}$
- Point particle action

spinless point masses

$$S_{\text{pp}} = -\sum_{i=1,2} m_i \int \sqrt{-g_{\mu\nu}(x_i)\,dx_i^\mu dx_i^\nu}$$

point masses m_i of binary system, $g_{\mu\nu}$ metric

- Einstein-Hilbert action & gauge fixing term

$$S_{\text{bulk}} = 2\Lambda^2 \int d^{d+1}x\sqrt{-g}\left[R(g) - \frac{1}{2}\Gamma_\mu\Gamma^\mu\right]$$

$\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}$, Christoffel symbol $\Gamma^\mu_{\alpha\beta}$, $\Lambda^{-2} = 32\pi G L^{d-3}$

- Kaluza-Klein parametrization of metric tensor

Kol, Smolkin; Blanchet, Damour

$$g_{\mu\nu} = e^{2\phi/\Lambda}\begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d\phi/\Lambda}(\delta_{ij} + \sigma_{ij}/\Lambda) - A_iA_j/\Lambda^2 \end{pmatrix}$$

with massless scalar field ϕ, vector field A_i, tensor field σ_{ij}

- Static case: A_i do not contribute

Introduction

EFT approach

- Two-body effective action $\leftrightarrow$ integrate out gravity fields

$$\exp\left[\mathrm{i}S_{\text{eff}}\right] = \int D\phi D\sigma_{ij} \exp\left[\mathrm{i}(S_{\text{pp}} + S_{\text{bulk}})\right]$$

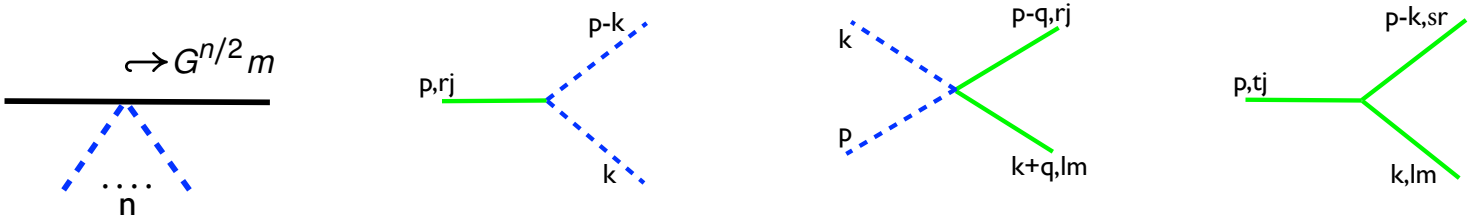
- Expand functional integration perturbatively
  ... Feynman diagrams
- Gravitational modes ϕ, σ $\rightarrow$ internal lines
  emitted/absorbed by point particles
  $\hookrightarrow$ nondynamical sources
- Take <u>classical</u> contribution
  $\hookrightarrow$ no quantum corrections

Vertices & propagators

Static case

- Propagators:

- Vertices:



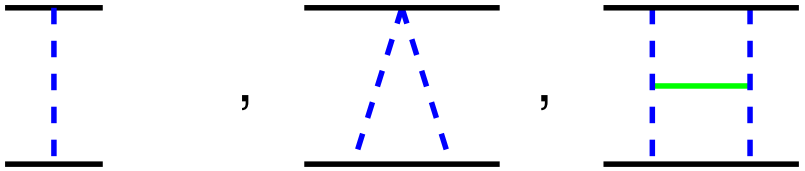
bulk vertices contain 2 or 0 ϕ's

momenta: p, k, q indices: r, j, l, m, t, s

non-dynamical sources: black lines scalar field: blue dashed lines

tensor field: green lines

- Typical Feynman graphs: N, 1PN, 2PN, ...



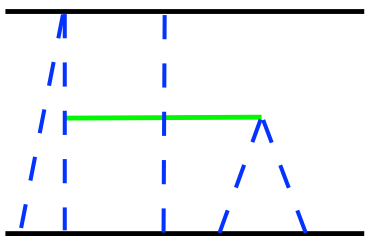
,...

- Momentum space $\overset{\text{Fourier-trafo}}{\longleftrightarrow}$ position space

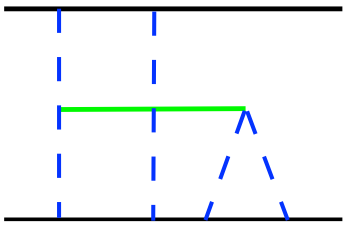
II. Calculation

Factorization property, static graphs, odd PN orders

- Characterization of static gravity-graphs:
  – Factorizable graphs, contain at least one $m\phi^n$-vertex with $n > 1$
  – Prime graphs, contain only vertices of the type $m\phi$
  Examples:



factorizable



prime

- Theorem:

$\boxed{\text{Static prime graphs exist only at even } (2n)\text{-PN orders}}$

equivalently

Foffa, Mastrolia, Sturani, C.S., Torres Bobadilla

$\boxed{\text{Static graphs at odd } (2n{+}1)\text{-PN orders are factorizable}}$

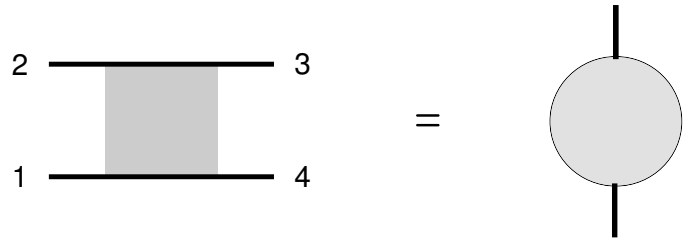
↪ Static odd-PN graphs can be obtained recursively from lower PN order results

Diagrams

Selfenergies

- Gravity diagram $\leftrightarrow$ selfenergy mapping:
  $\hookrightarrow$ allowed calc. static 4PN contr. using HET methods Foffa, Mastrolia, Sturani, C.S. '16

  Momentum conservation: $p_1 + p_2 = p_3 + p_4$



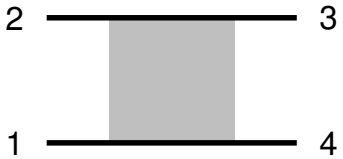
  Sources (black lines): static, do not propagate

  $\Rightarrow p_3 - p_2 = \mathrm{p} = p_1 - p_4$ (momentum transfer)

  any gravity-amplitude of $\mathcal{O}(G^\ell)$ can be mapped onto $(\ell - 1)$-loop 2-point funct.

- Fourier transform: contribution to two-body potential $\mathcal{V}$:

$$\mathcal{V} = \mathrm{i} \lim_{d \to 3} \int_p e^{\mathrm{i} p \cdot r}$$



  $\hookrightarrow$ additional integration over momentum transfer p

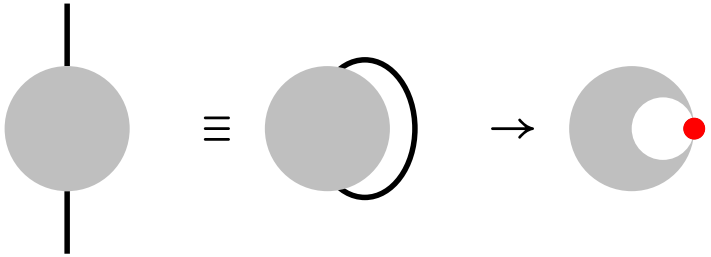
Factorization

Visualized diagrammatically

Foffa, Mastrolia, Sturani, C.S., Torres Bobadilla

- Integration on p can be seen as additional loop integration:

$$\int_p e^{ip\cdot r}$$



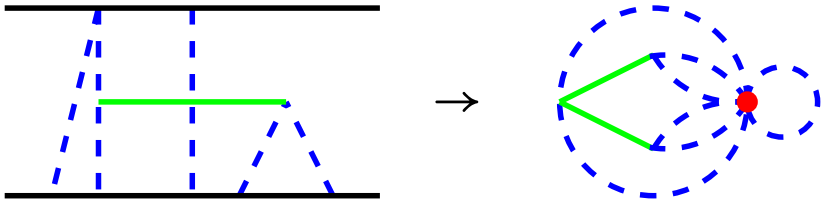
a) Joining external legs into a propagator-like line
$\hookrightarrow$ ℓ-loop vacuum diagram

b) Suggestive diagrammatic representation:
Pinching internal black line $\longrightarrow$ dot "•"

$\rightarrow$ factorization becomes apparent diagrammatically

- Example: Pinching of factorizable EFT-diagrams

$$\int_p e^{ip\cdot r}$$



$\leftrightarrow$ 4PN $\times$ 0PN

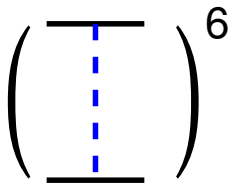
$\longrightarrow$ product of factorized "vacuum diagrams"

Static 5PN contribution

Four different cases: N^6, $N^3 \times 2PN$, $N \times 4PN$, $2PN \times 2PN$

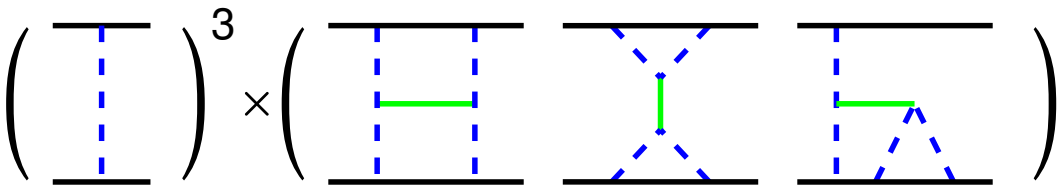
1

11 diagrams of six Newtonian factors:



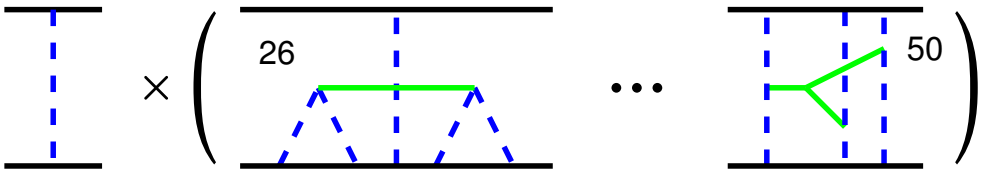
2

49 diagrams as products of 3 Newtonian graphs and 2PN prime graphs:



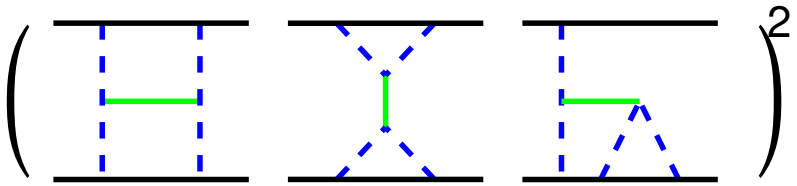
3

79 diagrams as products of 1 Newtonian graph and a 4PN prime graph:



4

15 diagrams as products of two 2PN prime graphs:



(Display only lower PN diagrams from which they originate)

Combinatorics

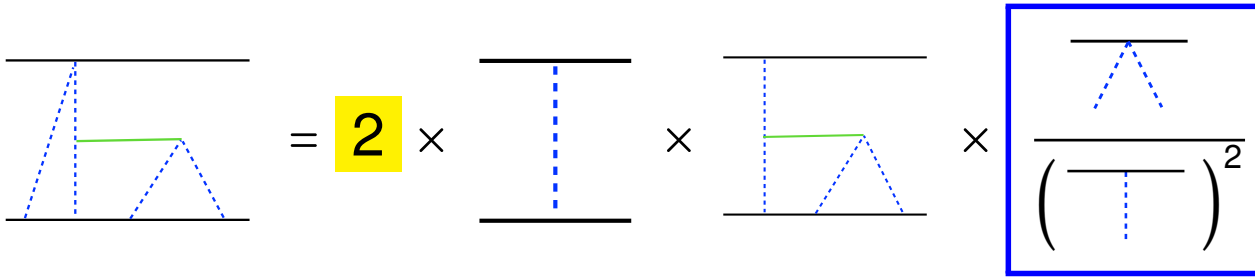
- 154 five-loop diagrams of which 30 vanish
- Contribution of n-PN factorizable diagrams to potential:

$$\mathcal{V}_n^{\text{factorizable}} = \mathcal{C} \times (\mathcal{V}_{L,n_1} \times \mathcal{V}_{R,n_2}) \times \boxed{\mathcal{K}}, \qquad n = n_1 + n_2 + 1$$

  – Combinatoric factor: $\mathcal{C} = \mathcal{C}_n^{\text{factorizable}} / (\mathcal{C}_{L,n_1} \times \mathcal{C}_{R,n_2})$
  – Factor $\mathcal{K}$ accounts for new vertex arising from sewing

- Diagrammatically, 3PN example:

$$\mathcal{V}_{3\text{PN}}^{\text{factorizable}} = \mathcal{C} \times \mathcal{V}_{L,0\text{PN}} \times \mathcal{V}_{R,2\text{PN}} \times \boxed{\mathcal{K}}$$

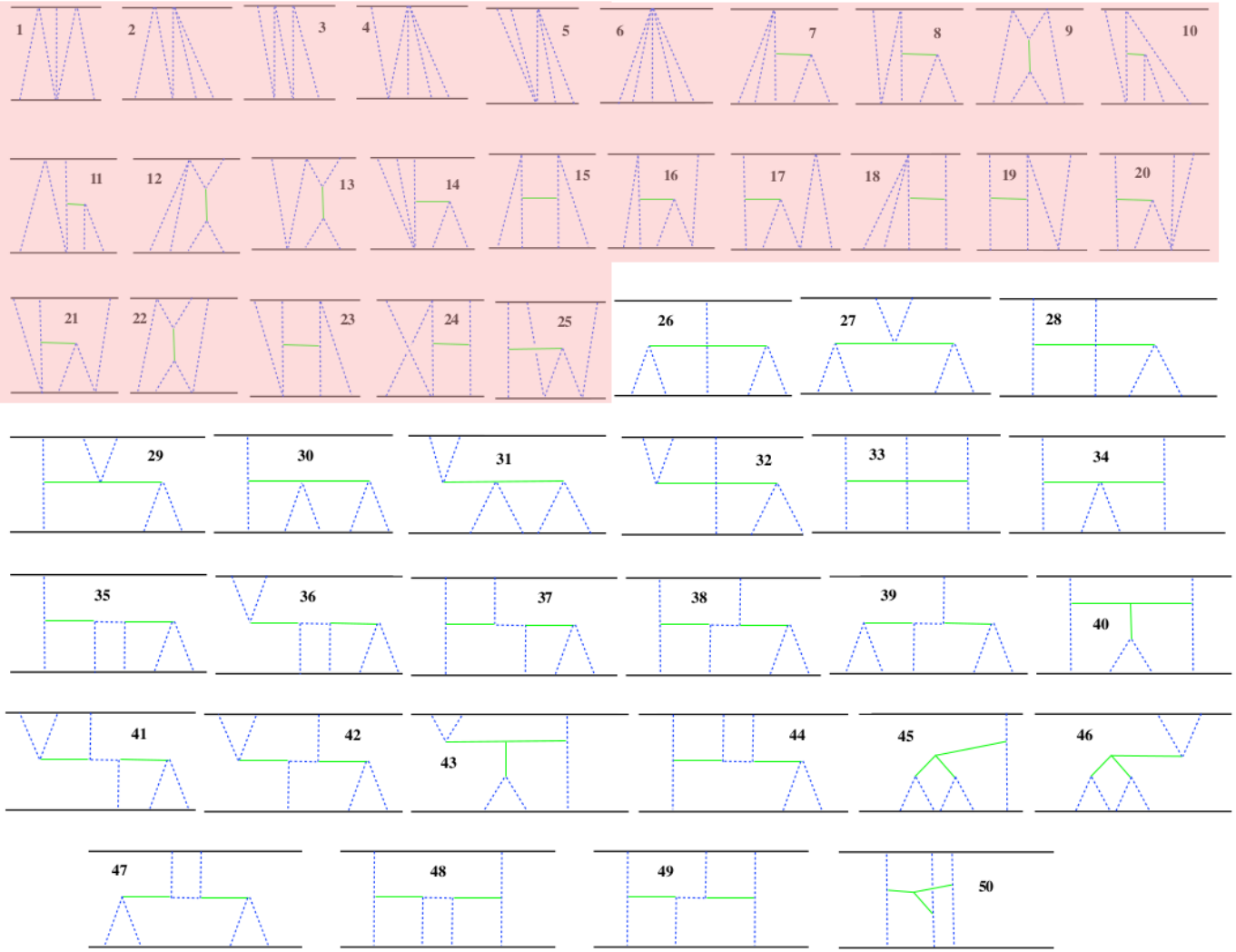


$$\frac{G^4 m_1^4 m_2}{3r^4} = 2 \times \left(-\frac{G m_1 m_2}{r}\right) \times \left(-\frac{G^3 m_1^3 m_2}{3r^3}\right) \times \boxed{\frac{G m_2/2}{\left(\sqrt{G} m_2\right)^2}}$$

- No integral needs to be computed!
  but even PN orders needed: 2PN, 4PN ← available

Static 4PN

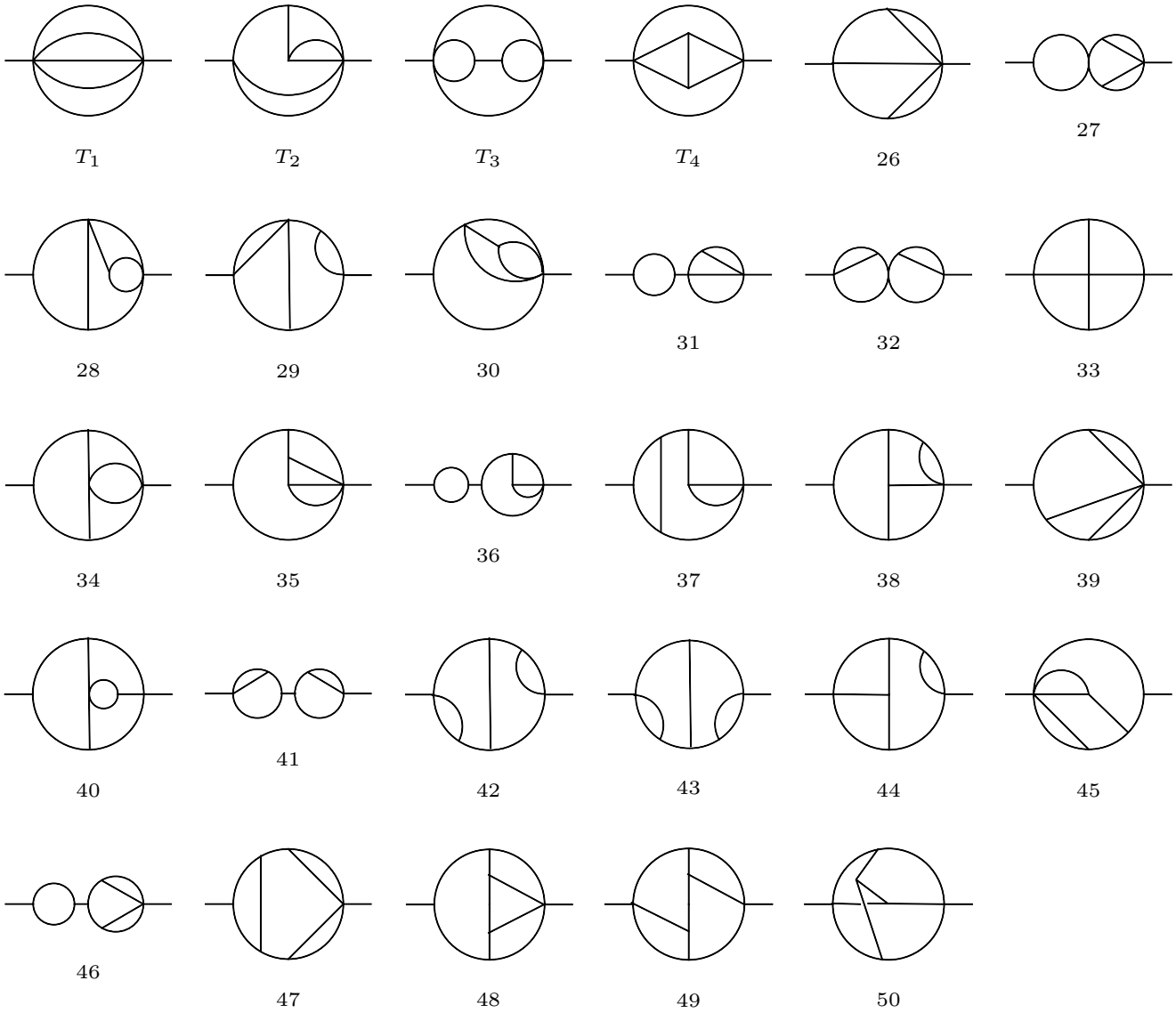
50 diagrams in the EFT approach



$\Rightarrow$ Factorization also applicable to large part of even PN orders! $\Longrightarrow$ here 50%

Static 4PN

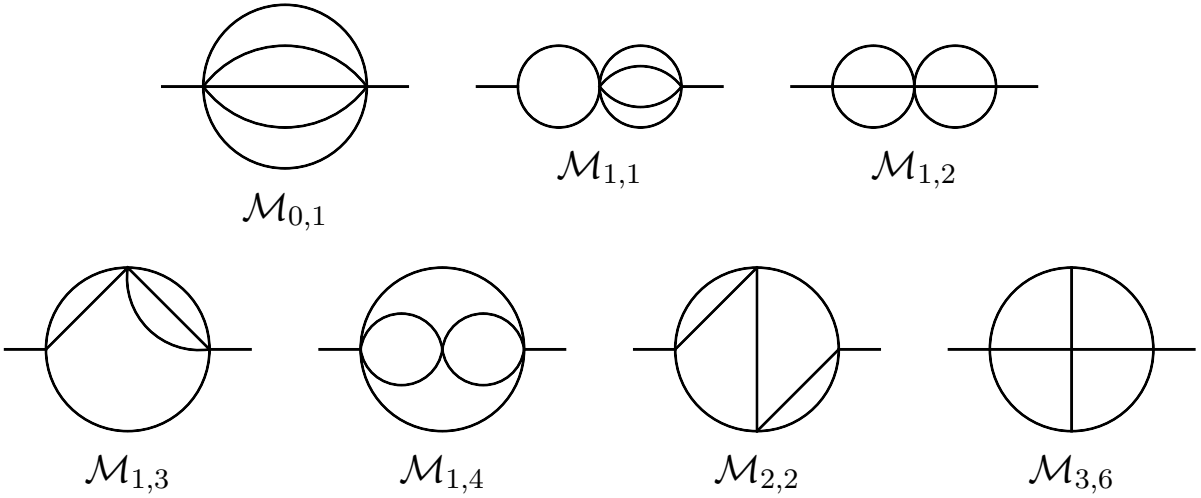
29 diagrams corresponding to 4-loop 2-point functions



Static 4PN

7 master integrals

- IBP reduction with Laporta's algorithm to master integrals
  $\hookrightarrow$ 2 independent reductions: in-house code / Reduze
  Manteuffel, Studerus
- 7 master integrals in $d = 3 + \varepsilon$:



- First five $\mathcal{M}_{0,1}$, $\mathcal{M}_{1,1}$, $\mathcal{M}_{1,2}$, $\mathcal{M}_{1,3}$, $\mathcal{M}_{1,4}$ entirely expressible in closed form in d dimensions in terms of Γ-Fct.s
- $\mathcal{M}_{2,2}$ always multiplied with ε^n, $n > 0 \rightarrow$ does not contribute

Static 4PN

master integrals, $\mathcal{M}_{3,6}$

- $\mathcal{M}_{3,6}$: high precision numerical solution with SummerTime
  Lee, Mingulov
- Numerical result + PSLQ:

$$
\begin{aligned}
\mathcal{M}_{3,6}^{d=3+\varepsilon} &= \frac{c(\varepsilon)}{s^2}\Big[\\
&\quad 0.500000000000000000000000000000000000000000000000000000000000/\varepsilon^2 \\
&\quad -0.500000000000000000000000000000000000000000000000000000000000/\varepsilon \\
&\quad -3.588766483287943390881896208338493702695262524698300390566 11 \\
&\quad +15.62341561179455120672187512690825773840230657361477356893 17\,\varepsilon \\
&\quad +\mathcal{O}\left(\varepsilon^2\right)\Big] \\
&\overset{\text{PSLQ}}{\hat{=}} \frac{c(\varepsilon)}{s^2}\left[\frac{1}{2\varepsilon^2}-\frac{1}{2\varepsilon}-4+\frac{\pi^2}{24}-\varepsilon\left(9-\pi^2\left(\frac{13}{8}-\log 2\right)-\frac{77}{6}\zeta_3\right)+\mathcal{O}\left(\varepsilon^2\right)\right] \\
&\quad c(\varepsilon) = e^{2\varepsilon\gamma_E} s^{2\varepsilon}/(4\pi)^{4+2\varepsilon},\ s = p^2
\end{aligned}
$$

- Analytic result in expansion in ε Damour, Jaranowski ✓

III. Result

Static PN corrections

$$\mathcal{V}_{\text{static}}(r) =$$

$$-\frac{Gm_1m_2}{2r}$$ Newton — N

$$+\frac{1}{2}\frac{G^2m_1^2m_2}{r^2}$$ Einstein, Infeld, Hoffmann,... — 1PN

$$-\frac{1}{2}\frac{G^3m_1^3m_2}{r^3}-\frac{3}{2}\frac{G^3m_1^2m_2^2}{r^3}$$ Ohta, Okamura, Kimura, Hiida, Damour, Schäfer, Gilmore, Ross,... — 2PN

$$+\frac{3}{8}\frac{G^4m_1^4m_2}{r^4}+6\frac{G^4m_1^3m_2^2}{r^4}$$ Damour, Jaranowski, Schäfer, Blanchet, Faye, Itoh, Futamase, Esposito-Farese,... — 3PN

$$-\frac{3}{8}\frac{G^5m_1^5m_2}{r^5}-\frac{31}{3}\frac{G^5m_1^4m_2^2}{r^5}-\frac{141}{8}\frac{G^5m_1^3m_2^3}{r^5}$$ Damour, Jaranowski, Schäfer, Foffa, Porto, Rothstein, Sturani, Mastrolia, C.S., Bernard, Blanchet, Bohé, Faye, Marsat,... — 4PN

$$+\frac{5}{16}\frac{G^6m_1^6m_2}{r^6}+\frac{91}{6}\frac{G^6m_1^5m_2^2}{r^6}+\frac{653}{6}\frac{G^6m_1^4m_2^3}{r^6}$$ 5PN

↑ ***New*** Foffa, Mastrolia, Sturani, C.S., Torres Bobadilla
confirmed by Blümlein, Maier, Marquard

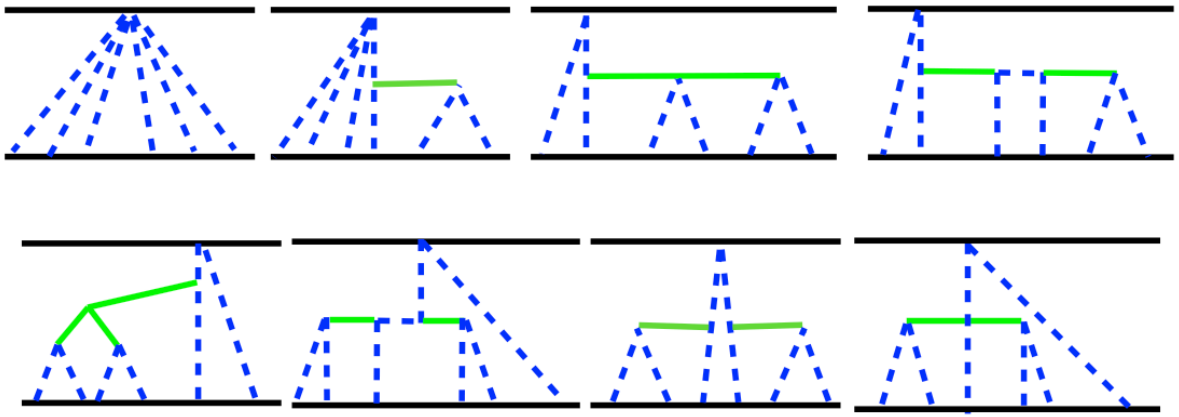
$$+\quad\cdots$$

$$+\,(m_1\leftrightarrow m_2)$$

(Coefficients are just rational numbers)

Checks of the calculation

- Verified factorization at all lower odd-PN orders ✓
- Independent calculations of combinatoric factors ✓
- Test-particle limit: ✓
  Consider extreme mass ratio limit $m_2 \ll m_1$
  ↪ body with mass m_2: test particle
  in Schwarzschild metric of body with mass m_1
  ↪ Term in highest power of m_1 verifiable, @ 5PN: $m_1^6 m_2$



5PN graphs contributing to the test-particle limit

Check of the calculation

- Effective Lagrangian in static limit $v_1 = 0 = v_2$

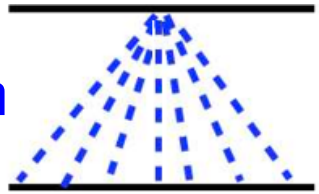
$$\mathcal{L}_{\text{static}}^{m_2 \ll m_1} = -m_2 \sqrt{\frac{1 - \frac{Gm_1}{r}}{1 + \frac{Gm_1}{r}}}$$

expansion in Gm_1/r

$$= \cdots \underbrace{\frac{Gm_1 m_2}{r}}_{N} \underbrace{- \frac{1}{2} \frac{G^2 m_1^2 m_2}{r^2}}_{1PN} \underbrace{+ \frac{1}{2} \frac{G^3 m_1^3 m_2}{r^3}}_{2PN} \underbrace{- \frac{3}{8} \frac{G^4 m_1^4 m_2}{r^4}}_{3PN} \underbrace{+ \frac{3}{8} \frac{G^5 m_1^5 m_2}{r^5}}_{4PN} \underbrace{- \frac{5}{16} \frac{G^6 m_1^6 m_2}{r^6}}_{5PN} \cdots$$

N ✓ $1PN$ ✓ $2PN$ ✓ $3PN$ ✓ $4PN$ ✓ $5PN$ ✓

(sign alternating)

- In general, nPN static terms: $G^n m_1^n m_2 / r^n$ predictable ✓
- For example 6PN: $+\frac{5}{16} \frac{G^7 m_1^7 m_2}{r^7}$ $\leftrightarrow$ 6-loop diagram  ,...

IV. Summary & Conclusion

- Studied gravitational two-body interaction at 4PN and 5PN in EFT approach to GR in static limit
- Problem can be mapped on 4- & 5-loop selfenergies
  ↪ can be solved with tools commonly used in HEP
- Established factorization property of static diagrams at odd-PN order
  ↪ Contributions determinable recursively from lower PN ones
  ↪ NO computation of any integral needed!!!
- First-time determination of static 5PN contribution
- Factorization property also applicable to large subset of even-PN diagrams ⇒ simplifies evaluation
  ⇒ powerful tool! ⇒ eases high-PN computations