Calculating the static gravitational two-body potential to fifth post-Newtonian order with Feynman diagrams

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I. Introduction
Generalities & Background

- Newton’s law $\rightarrow$ potential energy
  $$\mathcal{V}(r) = -\frac{Gm_1 m_2}{r} + \ldots \text{ post-Newtonian corrections } \ldots$$
  due to general relativity (GR) $\leftarrow$

- Post-Newtonian (PN) expansion
  - static contributions: Newton’s constant $G$ $\leftarrow$ here
  - non-static contrib.: velocities $\vec{v}$

- Expansion in powers of virial-related quantities
  $$Gm/r \sim v^2$$

  $n_{\text{PN}}: n = i + j - 1, \quad i, j: \text{powers of } G, v^2$$

  (weak coupling, small velocities)

- Deviations from Newton potential observable, contribute to perihelion precession of mercury

- Potential $\leftrightarrow$ energy of binary system
  energy loss $\leftrightarrow$ gravitational waves $\leftrightarrow$ inspiralling
Detection of gravitational waves
- indirect: Hulse-Taylor binary,...
- direct: LIGO, Virgo,...

Future experiments (KAGRA, Einstein Telescope, LISA,...)
↔ expect increase of sensitivity of factor $\sim 10$
↔ precise theory calculations desirable

Accurate theory predictions ↔ precise measurement
↔ test GR ↔

Different approaches to solve grav. 2-body problem
Here: Effective field theory approach (EFT) (Goldberger, Rothstein)
↔ methods of particle physics applicable,
calculation of Feynman diagrams,...
other approaches:
ADM Hamiltonian formalism, numerical GR, post-Minkowskian,...
Introduction

EFT, action

- **Action** $S = S_{pp} + S_{\text{bulk}}$

- **Point particle action**

  $$S_{\text{pp}} = - \sum_{i=1,2} m_i \int \sqrt{-g_{\mu\nu}(x_i)} \, dx_i^\mu \, dx_i^\nu$$

  point masses $m_i$ of binary system, $g_{\mu\nu}$ metric

- **Einstein-Hilbert action & gauge fixing term**

  $$S_{\text{bulk}} = 2\Lambda^2 \int d^{d+1}x \sqrt{-g} \left[ R(g) - \frac{1}{2} \Gamma_\mu \Gamma^\mu \right]$$

  $$\Gamma^\mu = g^{\alpha\beta} \Gamma^\mu_{\alpha\beta}, \text{Christoffel symbol} \quad \Gamma^\mu_{\alpha\beta}, \Lambda^{-2} = 32\pi GL^{d-3}$$

- **Kaluza-Klein parametrization of metric tensor**

  $$g_{\mu\nu} = e^{2\phi/\Lambda} \begin{pmatrix} -1 & A_i/\Lambda \\ A_i/\Lambda & e^{-c_d\phi/\Lambda} (\delta_{ij} + \sigma_{ij}/\Lambda) - A_i A_j/\Lambda^2 \end{pmatrix}$$

  with massless scalar field $\phi$, vector field $A_i$, tensor field $\sigma_{ij}$

- **Static case:** $A_i$ do not contribute
Two-body effective action \( \leftrightarrow \) integrate out gravity fields

\[
\exp [iS_{\text{eff}}] = \int D\phi D\sigma_{ij} \exp [i(S_{\text{pp}} + S_{\text{bulk}})]
\]

Expand functional integration perturbatively

\( \ldots \) Feynman diagrams

Gravitational modes \( \phi, \sigma \) \( \rightarrow \) internal lines
emitted/absorbed by point particles

\( \leftrightarrow \) nondynamical sources

Take \textit{classical} contribution

\( \leftrightarrow \) no quantum corrections
**Vertices & propagators**

**Static case**

- **Propagators:**
  \[ p, kl \]

- **Vertices:**
  \[ G^{n/2} m \]

  - bulk vertices contain 2 or 0 \( \phi \)'s
  - momenta: \( p, k, q \)
  - non-dynamical sources: black lines
  - scalar field: blue dashed lines
  - tensor field: green lines

- **Typical Feynman graphs:**
  \[ N, 1PN, 2PN, \ldots \]

- **Momentum space** ↔ **position space**
II. Calculation

Factorization property, static graphs, odd PN orders

- Characterization of static gravity-graphs:
  - Factorizable graphs, contain at least one $m\phi^n$-vertex with $n > 1$
  - Prime graphs, contain only vertices of the type $m\phi$

Examples:

![Factorizable and Prime Graphs]

Theorem:

- Static prime graphs exist only at even $(2n)$-PN orders
  equivalently
- Static graphs at odd $(2n+1)$-PN orders are factorizable

$\rightarrow$ Static odd-PN graphs can be obtained recursively from lower PN order results
Diagrams
Selfenergies

Gravity diagram ↔ selfenergy mapping:

Selfenergy mapping:
allowed calc. static 4PN contr. using HET methods

Foffa, Mastrolia, Sturani, C.S. '16

Momentum conservation: $p_1 + p_2 = p_3 + p_4$

Sources (black lines): static, do not propagate

$\Rightarrow p_3 - p_2 = p_1 - p_4$ (momentum transfer)

any gravity-amplitude of $O(G^\ell)$ can be mapped onto $(\ell - 1)$-loop 2-point funct.

Fourier transform: contribution to two-body potential $\mathcal{V}$:

$\mathcal{V} = i \lim_{d \to 3} \int_p e^{ip \cdot r}$

additional integration over momentum transfer $p$
Integration on $p$ can be seen as additional loop integration:

\[ \int_p e^{ip \cdot r} \]

a) Joining external legs into a propagator-like line
\[ \ell \)-loop vacuum diagram

b) Suggestive diagrammatic representation:
Pinching internal black line $\rightarrow$ dot "●"
\[ \rightarrow \text{factorization becomes apparent diagrammatically} \]

Example: Pinching of factorizable EFT-diagrams

\[ \int_p e^{ip \cdot r} \]

$\leftrightarrow$ product of factorized "vacuum diagrams"

\[ 4\text{PN} \times 0\text{PN} \]
Static 5PN contribution

Four different cases: $N^6$, $N^3 \times 2PN$, $N \times 4PN$, $2PN \times 2PN$

1. 11 diagrams of six Newtonian factors:

\[
\begin{array}{c}
\frac{ }{ }^6
\end{array}
\]

2. 49 diagrams as products of 3 Newtonian graphs and 2PN prime graphs:

\[
\begin{array}{c}
\frac{ }{ }^3 \times \left( \frac{ }{ } \right)
\end{array}
\]

3. 79 diagrams as products of 1 Newtonian graph and a 4PN prime graph:

\[
\begin{array}{c}
\frac{ }{ } \times \left( \frac{ }{ } \right)
\end{array}
\]

4. 15 diagrams as products of two 2PN prime graphs:

\[
\begin{array}{c}
\left( \frac{ }{ } \right)^2
\end{array}
\]

(Display only lower PN diagrams from which they originate)
Combinatorics

- 154 five-loop diagrams of which 30 vanish
- Contribution of $n$-PN factorizable diagrams to potential:

$$V_{n}^{\text{factorizable}} = C \times (V_{L,n_1} \times V_{R,n_2}) \times \mathcal{K}, \quad n = n_1 + n_2 + 1$$

- Combinatoric factor: $C = C_{n}^{\text{factorizable}} / (C_{L,n_1} \times C_{R,n_2})$
- Factor $\mathcal{K}$ accounts for new vertex arising from sewing

- Diagrammatically, 3PN example:

$$V_{3\text{PN}}^{\text{factorizable}} = C \times V_{L,0\text{PN}} \times V_{R,2\text{PN}} \times \mathcal{K}$$

$$\frac{G^4 m_1^2 m_2}{3r^4} = 2 \times \left( - \frac{G m_1 m_2}{r} \right) \times \left( - \frac{G^3 m_3^2 m_2}{3r^3} \right) \times \frac{G m_2/2}{(\sqrt{G} m_2)^2}$$

- No integral needs to be computed!
- but even PN orders needed: 2PN, 4PN  ← available
Static 4PN
50 diagrams in the EFT approach

⇒ Factorization also applicable to large part of even PN orders! ⇒ here 50%
Static 4PN

29 diagrams corresponding to 4-loop 2-point functions
Static 4PN

7 master integrals

- IBP reduction with Laporta’s algorithm to master integrals
  $\leftrightarrow$ 2 independent reductions: in-house code / Reduze
  Manteuffel, Studerus

- 7 master integrals in $d = 3 + \varepsilon$:

  - $M_{0,1}$
  - $M_{1,1}$
  - $M_{1,2}$
  - $M_{1,3}$
  - $M_{1,4}$
  - $M_{2,2}$
  - $M_{3,6}$

- First five $M_{0,1}, M_{1,1}, M_{1,2}, M_{1,3}, M_{1,4}$ entirely expressible in closed form in $d$ dimensions in terms of $\Gamma$-Fct.s

- $M_{2,2}$ always multiplied with $\varepsilon^n$, $n > 0 \rightarrow$ does not contribute
Static 4PN
master integrals, $\mathcal{M}_{3,6}$

- $\mathcal{M}_{3,6}$: high precision numerical solution with SummerTime
  
  Lee, Mingulov

- Numerical result + PSLQ:

  $\mathcal{M}_{3,6}^{d=3+\varepsilon} = \frac{c(\varepsilon)}{s^2} \left[ 0.50000000000000000000000000000000000000000000000000000000000/\varepsilon^2 \right. + \left. \frac{1}{2\varepsilon} \right]$

  $- 0.50000000000000000000000000000000000000000000000000000000000/\varepsilon$

  $- 3.5887664832879439088189620833849370269526252469830039056611$

  $+ 15.6234156117945512067218751269082577384023065736147735689317 \varepsilon$

  $+ O(\varepsilon^2)]$

  $\mathcal{M}_{3,6}^{\text{PSLQ}} = \frac{c(\varepsilon)}{s^2} \left[ \frac{1}{2\varepsilon} - \frac{1}{2\varepsilon} - 4 + \frac{\pi^2}{24} - \varepsilon \left( 9 - \pi^2 \left( \frac{13}{8} - \log 2 \right) - \frac{77}{6} \zeta_3 \right) + O(\varepsilon^2) \right]$

  $c(\varepsilon) = e^{2\varepsilon}\gamma_E s^{2\varepsilon}/(4\pi)^{4+2\varepsilon}, \ s = r^2$

- Analytic result in expansion in $\varepsilon$  Damour, Jaranowski ✓
III. Result

Static PN corrections

\[ V_{\text{static}}(r) = \]

\[- \frac{G m_1 m_2}{2r} \]

\[ + \frac{1}{2} \frac{G^2 m_1^2 m_2}{r^2} \]

\[- \frac{1}{2} G^3 m_1^3 m_2 \]

\[ - \frac{3}{2} \frac{G^3 m_1^2 m_2^2}{r^3} \]

\[ + \frac{3}{8} \frac{G^4 m_1^4 m_2}{r^4} + 6 \frac{G^4 m_1^3 m_2}{r^4} \]

\[- \frac{3}{8} G^5 m_1^5 m_2 \]

\[ - \frac{31}{3} \frac{G^5 m_1^4 m_2^2}{r^5} - \frac{141}{8} \frac{G^5 m_1^3 m_2^3}{r^5} \]

\[ + \frac{5}{16} \frac{G^6 m_1^6 m_2}{r^6} + \frac{91}{6} \frac{G^6 m_1^5 m_2^2}{r^6} + \frac{653}{6} \frac{G^6 m_1^4 m_2^3}{r^6} \]

+ \[ \ldots \]

+ \[ (m_1 \leftrightarrow m_2) \]

(Coefficients are just rational numbers)

\[ 1 \text{PN} \]

\[ 2 \text{PN} \]

\[ 3 \text{PN} \]

\[ 4 \text{PN} \]

\[ 5 \text{PN} \]

New \[ \text{Foffa, Mastrolia, Sturani, C.S., Torres Bobadilla} \]

confirmed by \[ \text{Blümlein, Maier, Marquard} \]
Checks of the calculation

- Verified factorization at all lower odd-PN orders ✓
- Independent calculations of combinatoric factors ✓
- Test-particle limit:
  Consider extreme mass ratio limit $m_2 \ll m_1$
  $\leftrightarrow$ body with mass $m_2$: test particle in Schwarzschild metric of body with mass $m_1$
  $\leftrightarrow$ Term in highest power of $m_1$ verifiable, @ 5PN: $m_1^6 m_2$

5PN graphs contributing to the test-particle limit
Check of the calculation

- Effective Lagrangian in static limit $v_1 = 0 = v_2$

$$L_{\text{static}}^{m_2 \ll m_1} = -m_2 \sqrt{\frac{1 - \frac{Gm_1}{r}}{1 + \frac{Gm_1}{r}}}$$

expansion in $Gm_1/r$

\[
\frac{Gm_1 m_2}{r} + \frac{1}{2} \frac{G^2 m_1^2 m_2}{r^2} + \frac{1}{2} \frac{G^3 m_1^3 m_2}{r^3} - \frac{3}{8} \frac{G^4 m_1^4 m_2}{r^4} + \frac{3}{8} \frac{G^5 m_1^5 m_2}{r^5} - \frac{5}{16} \frac{G^6 m_1^6 m_2}{r^6} \ldots
\]

(sign alternating)

- In general, $n$PN static terms: $G^n m_1^n m_2/r^n$ predictable

- For example 6PN: $+ \frac{5}{16} \frac{G^7 m_1^7 m_2}{r^7} \leftrightarrow 6$-loop diagram

In general, $n$PN static terms: $G^n m_1^n m_2/r^n$ predictable
IV. Summary & Conclusion

- Studied gravitational two-body interaction at 4PN and 5PN in EFT approach to GR in static limit
- Problem can be mapped on 4- & 5-loop selfenergies
  - can be solved with tools commonly used in HEP
- Established factorization property of static diagrams at odd-PN order
  - Contributions determinable recursively from lower PN ones
  - NO computation of any integral needed!!!
- First-time determination of static 5PN contribution
- Factorization property also applicable to large subset of even-PN diagrams ⇒ simplifies evaluation
  ⇒ powerful tool! ⇒ eases high-PN computations
Static gravitational 2-body potential to 5\textsuperscript{th} PN order