# Infrared structure of $\mathcal{N}$=4 SYM and Leading Transcendentality principle in Gauge theory

Amlan Chakraborty
IMSc, Chennai, India
in collaboration with V. Ravindran, P. Banerjee, P.K. Dhani and S. Seth(JHEP 1904(2019) 058)

**Radcor 2019,Avignon**

September 10, 2019

## Motivation

- To understand the universal structures of amplitudes and Xsection
- To study at factorisation properties of them in a model independent manner
- $N = SYM$ serves as a theoretical laboratory to unravel the rich infrared Structure of not only QCD but also a wide class of non-Abelian gauge theories
- Direct comparison of observable quantities with QCD

# Goal

- The QCD Splitting functions are fully known to 3-loops and partially at 4-loops in planar limit

  (Altareli et al., Vermaseren, Moch, Vogt, Curci, Furmanski, Petronzio, Badger, Glover)

- From the QCD Splitting function is it possible to predict same quantities for $N = 4SYM$ ?
- KLOV calculated the *NNLO* anomolous dimentions for Wilson operators in Mellin space.

  (Kotikov,Lipatov,Onishchenko,Velizhanin)

## Goal

- We applied an alternative method to derive the Splitting functions and extend it to inclusive Xsection level for various channels with different colorless final states in $\mathcal{N} = 4$.
- Understand the universal properties (Ex: factorisation) and $N^3LO$ SV inclusive Xsection calculation in $\mathcal{N} = 4SYM$ .
- Compare with QCD observables

# Outline

- Brief introduction to N = 4 SYM
- Infrared structure
- Sudakov Form factor
- Splitting functions
- General factorisation structure for inclusive SV Xsection

- Cousin of QCD with simpler UV and IR Structure
- UV Conformal i.e. $\beta-$function vanishes to all order in perturbation Theory
- Certain composite operators (example : Konishi) are not protected by SUSY current conservation requires UV renormalization
- Consists of 4 majorana fermion($\lambda_n$) ,1 gluon($A^\mu$), 3 scalars($\phi_i$) and 3 pseudo-scalars($\chi_i$) in adjoint representaion of SU(N) group

## Transcendental Weight

- The weight of transcendentality, $\tau$, of a function, $f$, is defined as the number of iterated integrals required to define the function $f$.
- $\tau(\log) = 1\,, \tau(\mathrm{Li}_n) = n\,, \tau(\zeta_n) = n$ and also we define $\tau(f_1 f_2) = \tau(f_1) + \tau(f_2)$.
- Algebraic factors are assigned weight zero and dimensional regularisation parameter $\epsilon$ to -1.

## Operators taken for the computation

- The strategy for the computation was to evaluate certain cross sections, whose final state contains heavy particle sharing effective vertices with certain operator in N=4 SYM.
- We have taken half-BPS and Energy Momentum tensor operator, $T_{\mu\nu}^{\mathcal{N}=4\,\mathrm{SYM}}$ and Konishi given by the following Lagrangian

$$\mathcal{L}_{\text{int}} = \mathcal{L}^{\text{BPS}} + \mathcal{L}^{\text{T}} + \mathcal{L}^{\text{K}}$$

where

$$\mathcal{L}^{\text{BPS}} = J_{rt}^{\text{BPS}} \mathcal{O}_{rt}^{\text{BPS}}, \quad \mathcal{L}^{\text{T}} = J_{\mu\nu}^{T} T_{\mathcal{N}=4\,\mathrm{SYM}}^{\mu\nu}, \quad \mathcal{L}^{\text{K}} = J^{\text{Kon}} \mathcal{O}^{\text{Kon}}$$

$$\mathcal{O}_{rt}^{\mathrm{BPS}} = \phi_r^a \phi_t^a - \frac{1}{3} \delta_{rt} \phi_s^a \phi_s^a. \quad \mathcal{O}^{\mathrm{Kon}} = \phi_s^a \phi_s^a.$$

# Infrared Structure

- Expand the multiloop amplitude in d=4+$\epsilon$
- Infrared divergences consists of Soft and Collinear divergences
- Overlapping soft + collinear divergences at each loop order imply leading poles are $1/\epsilon^{2L}$ at L loops

## Infrared Structure

- Pole terms are predictable

**Soft-Collinear Factorization**

$$\mathcal{M}_n = S(k_i, \mu, \alpha_s(\mu), \epsilon)[\prod_{i=1}^{n} J_i(\mu, \alpha_s(\mu), \epsilon)] \times h_n(k_i, \mu, \alpha_s(\mu), \epsilon)$$

- S = soft function(only depends on color of $i^{th}$ particle)
- J = jet or collinear function (color diagonal depends on $i^{th}$ spin)
- H = hard function(finite after UV renormalization)

## Sudakov Form Factor

- Final state colorless particle
- UV renormalised form factor satisfies a renormalisation group equation -

$$\frac{d}{d \ln Q^2} \ln \mathcal{F}_f^\rho = \frac{1}{2} \left[ K_f^\rho + G_f^\rho \right]$$

## Sudakov Form Factor

$$\frac{d}{d \ln Q^2} \ln \mathcal{F}_f^\rho = \frac{1}{2}\left[K_f^\rho + G_f^\rho\right]$$

- The $Q^2$ independent $K_f^\rho\left(a,\epsilon\right)$ contains all the poles in $\epsilon$, whereas $G_f^\rho\left(a, Q^2/\mu^2,\epsilon\right)$ involves only the finite terms in $\epsilon \to 0$.
- Clearly dictates UV/IR factorisation
- Leading Transcendental(LT) part of Born divided Sudakov form factor in N = 4 SYM (BPS, Konishi...) is equal to LT term of QCD form factor ($gg \to H$, $b\overline{b} \to H$) after Ca=Cf=nf=N scaling

Sudakov,Mueller,Sen,Sterman,Henn,Vermaseren etal....

# Sudakov Form Factor

- Inspired from Sudakov Form factor Structure in QCD one can propose the general solution for $N = 4$ SYM

$$\ln \mathcal{F}^{\rho}_{f}(a, Q^2, \mu^2, \epsilon) = \sum_{j=1}^{\infty} a^j \left(\frac{Q^2}{\mu^2}\right)^{j\frac{\epsilon}{2}} \mathcal{L}^{\rho}_{f,j}(\epsilon)$$

with

$$\mathcal{L}^{\rho}_{f,j}(\epsilon) = \frac{1}{\epsilon^2}\left\{-\frac{2}{j^2}A_j\right\} + \frac{1}{\epsilon}\left\{\frac{1}{j}G^{\rho}_{f,j}(\epsilon)\right\}$$

Ahmed, Rana, Dhani, Banerjee, Ravindran.

## Sudakov Form Factor

$$\mathcal{L}^{\rho}_{f,j}(\epsilon) = \frac{1}{\epsilon^2}\left\{-\frac{2}{j^2}A_j\right\} + \frac{1}{\epsilon}\left\{\frac{1}{j}G^{\rho}_{f,j}(\epsilon)\right\}$$

$A = \sum_{j=1}^{\infty} a^j A_j$ are the cusp anomolous dimensions in $\mathcal{N} = 4$ SYM.

$$G^{\rho}_{f,j}(\epsilon) = 2\left(B_j - \gamma^{\rho}_j\right) + f_j + \sum_{k=1}^{\infty} \epsilon^k g^{\rho,k}_{f,j}$$

where, $B = \sum_{j=1}^{\infty} a^j B_j$ and $f = \sum_{j=1}^{\infty} a^j f_j$ are the collinear and soft anomolous dimensions in $\mathcal{N} = 4$ SYM and $\gamma^{\rho}_j$ is overall UV anomalus dimention for effective operator taken

**Anomolous dimention of Wilson operator -**

- Mellin moments of the splitting functions are the UV anomolous dimention of certain Wilson operator
- Wilson operator can be found by applying OPE(operator product expantion ) in DIS channel
- They are the operator definition of nonperturbative parton distribution functions (PDF)

**Anomolous dimention of wilson operator -**

- Their evolution is controled by the splitting functions and universal quantities of the theory.
- Interpreted as the probabilty distribution functions of the partons.
- parton distribution function satisty a RG equation namely DGLAP equation

Vermaseren,Moch,Vogt,Lipatov,Altareli,Parisi,Curci,Furmanski,Petronzio ...

## Evaluating the Splitting Functions

- We followed a different method to obtain splitting functions taking it as unknown quantity
- Extract by demanding the inclusive Xsection as finite after mass factorisation with one massive final state.

## Evaluating the Splitting Functions

- Due to KLN theorem in non-Abelian gauge theory, all final state soft and collinear poles goes away when all of the final states summed up.
- Initial state collinear divergences still remains.

**remedy**

- use mass factorisation prescription at Xsection level to remove initial state divergences at the factorisation scale $\mu_F$

$$\hat{\Delta}^i_{ab}(z, Q^2, 1/\epsilon) = \sum_{c,d=\lambda,\phi,\chi,g} \Gamma_{ca}(z, \mu_F^2, 1/\epsilon) \otimes \Gamma_{db}(z, \mu_F^2, 1/\epsilon) \otimes \Delta^i_{cd}(z, Q^2, \mu_F^2)$$

$$
\begin{aligned}
\hat{\Delta}^{i}_{ab}(z, Q^2, 1/\epsilon) &= \sum_{c,d=\lambda,\phi,\chi,g} \Gamma_{ca}(z, \mu_F^2, 1/\epsilon) \otimes \\
&\quad \Gamma_{db}(z, \mu_F^2, 1/\epsilon) \otimes \Delta^{i}_{cd}(z, Q^2, \mu_F^2)
\end{aligned}
$$

- $\hat{\Delta} \equiv \hat{\sigma}/z$
- $\hat{\sigma}_{ab}$ = UV renormalized cross section
- $\Delta_{lm}$ = finite when $\epsilon \to 0$
- $\Gamma_{ij}$ = massfactorisation kernel

## Splitting Functions

$$\Gamma_{ab}(z,\mu_F^2,1/\epsilon) = \sum_{k=0}^{\infty} a_s^k(\mu_F^2)\Gamma_{ab}^{(k)}(z,\mu_F^2,1/\epsilon)$$

$$\Gamma_{ab}^{(0)} = \delta_{ab}\delta(1-z)$$

$$\Gamma_{ab}^{(1)} = \frac{1}{\epsilon}P_{ab}^{(0)}(z)$$

$$\Gamma_{ab}^{(2)} = \frac{1}{\epsilon^2}\left(\frac{1}{2}P_{ac}^{(0)} \otimes P_{cb}^{(0)} + \beta_0 P_{ab}^{(0)}\right) + \frac{1}{\epsilon}\left(\frac{1}{2}P_{ab}^{(1)}\right)$$

- $P_{ab}^{(i)}$ are the Altarelli-Parisi splitting functions. The symbol $\otimes$ stands for the convolution

$$(f \otimes g)(z) \equiv \int_z^1 \frac{dx}{x} f(x) g\left(\frac{z}{x}\right)$$

## Evaluating the Splitting Functions

- Expanding the unrenormalised coefficient function and the mass factorised one in powers of strong coupling constant as

$$\hat{\Delta}^{i}_{ab} = \sum_{k=0}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{Q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Delta}^{i,(k)}_{ab}$$

$$\Delta^{i}_{ab} = \sum_{k=0}^{\infty} a_s^k(\mu_F^2) \Delta^{i,(k)}_{ab}$$

- Mass factorisation theorem guarantees that all initial collinear divergences can be absorbed into $\Gamma_{ab}$.
- By demanding finiteness of the $\Delta^{i}_{ab}$ we can find the splitting functions order by order in Perturbation theory .

## DGLAP evolution equation for wilson operator

$$\frac{df_c(x,Q)}{d\ln Q} = \sum_{n=1}^{\infty} a_s^n \int_x^1 \frac{dy}{y} \sum_{b=\{\lambda,g,\phi,\chi\}} P_{cb}^{(n-1)}(x/y) f_b(y,Q).$$

- The Spitting functions ,$P^{(n-1)}(x)$ satisfies the following momentum conservation relations

$$\sum_{n=1}^{\infty} \int_0^1 dx\, x \sum_{b=\{\lambda,g,\phi,\chi\}} P_{b\lambda}^{(n-1)} = 0$$

$$\sum_{n=1}^{\infty} \int_0^1 dx\, x \sum_{b=\{\lambda,g,\phi,\chi\}} P_{bg}^{(n-1)} = 0$$

$$\sum_{n=1}^{\infty} \int_0^1 dx\, x \sum_{b=\{\lambda,g,\phi,\chi\}} P_{b\phi}^{(n-1)} = 0$$

# DGLAP evolution equation for wilson operator

$$\sum_{n=1}^{\infty} \int_0^1 dx\, x \sum_{b=\{\lambda,g,\phi,\chi\}} P_{b\chi}^{(n-1)} = 0$$

- Following identities are satisfied at each order of perturbation theory
- A crucial cross check of our calculation after expicitly evaluating the splitting functions

## Universal Properties Of the Splitting functions and similarities with QCD

We find that the both LO and NLO splitting functions satisfy the following relations:

$$\sum_{a=\lambda,g,\phi,\chi} P^{(i)}_{a\lambda} = \sum_{a=\lambda,g,\phi,\chi} P^{(i)}_{ag} = \sum_{a=\lambda,g,\phi,\chi} P^{(i)}_{a\phi} = \sum_{a=\lambda,g,\phi,\chi} P^{(i)}_{a\chi} = I^{(i)}(x),$$

where

$$I^{(0)}(x) = 8\left[\frac{1}{(1-x)_+} + \frac{1}{x}\right],$$

$$
\begin{aligned}
I^{(1)}(x) &= 24\zeta_3\delta(1-x) \\
&+ 32\frac{1}{x}\left[\mathrm{Li}_2(-x)+\log(x)\log(1+x)-\log(x)\log(1-x)\right] \\
&+ \frac{1}{(1-x)_+}\left[-32\log(x)\log(1-x)+8\log^2(x)-16\zeta_2\right] \\
&+ \frac{1}{1+x}\Big[-32\mathrm{Li}_2(-x)-32\log(x)\log(1+x)+8\log^2(x) \\
&-16\zeta_2\Big].
\end{aligned}
$$

- All of them have Uniform Transcendentality!

## Universal Properties Of the Splitting functions and similarities with QCD

$$\sum_{a=\lambda,g,\phi,\chi} \int_0^1 dx\, xP_{ab}^{(i)} = \int_0^1 dx\, xI^{(i)}(x) = 0\,,$$

$$i = 0, 1 \text{ and } b = \{\lambda, g, \phi, \chi\}.$$

- Both at NLO and NNLO, only the diagonal splitting functions contain "$+$ *distributions*"
- NNLO level, terms proportional to $\delta(1-x)$ start contributing to diagonal splitting functions as in N = 4 SYM the collinear anomolous dimention is zero in NLO level.

# Universal Properties Of the Splitting functions and similarities with QCD

In the limit x $\to$ 1, the diagonal splitting functions take the form

$$P_{aa}^{(i)}(x) = 2A_{i+1}\frac{1}{(1-x)_+} + 2B_{i+1}\delta(1-x) + R_{aa}^{(i)}(x),$$

where $A_{i+1}$ and $B_{i+1}$ are the cusp and collinear anomolous dimensions respectively. $R_{aa}^{(i)}(x)$ is the regular function as $x \to 1$.

- $A_1 = 4, A_2 = -8\zeta_2$
- $B_1 = 0, B_2 = 12\zeta_3$
- $A_i$ and $B_i$ are Leading Transcendental wrt QCD ones
- Match with those of earlier Sudakov Form Factor of *BPS* operator(Van Neerven,Henn,Gehrmann,Brandhuber)
- Matches with KLOV prediction in Mellin space

  (Kotikov,Lipatov,Onishchenko,Velizhanin)

.

## General factorisation equation of soft virtual crossection

- Due to Gauge and renormalisation group invariance Soft plus Virtual Cross-section for the process $a + a \to$ colorless final state for QCD factorises

$$\Delta^{I,SV}_{aa} = \left(Z^I(a,\epsilon)\right)^2 |\hat{F}^I_{aa}(Q^2,\epsilon)|^2 \delta(1-z) \otimes \mathcal{C}\exp\left(2\Phi^I_{aa}(z,Q^2,\epsilon)\right) \otimes \Gamma^{-1}_{aa}(z,\mu_F^2,\epsilon) \otimes \Gamma^{-1}_{aa}(z,\mu_F^2,\epsilon).$$

(Varmaseren etal,Ravindran ...)

$Z^I$ = Overall UV renormalisation constant of the final state operator
$\Phi^I_{aa}(z,Q^2,\epsilon)$ = Process independent soft function if the probing particle is colorless (Higgs,Weak Bosons)

## General factorisation equation of soft virtual crossection in $N = 4SYM$

- The factorisation is true for $N = 4SYM$ too!
- Valid for any colorless operator in $N = 4SYM$ ( checked for Konishi, $BPS$, Stress tensor) !
- Process independent soft distribution function $\Phi^{I}_{aa}$ is equal to LT part of QCD after Ca=Cf=nf=N scaling
- $\Phi^{I}_{aa}$ contains contributions only from soft radiations!

(AC etal.)

## Infrared Structure of $\Phi^I_{aa}$

- Satisfy Form Factor like $\bar{K} + \bar{G}$ equation

$$\frac{d}{dQ^2}\Phi^I_{aa} = \frac{1}{2}\left[\bar{K} + \bar{G}\right]$$

- $\bar{K}$ contains all the poles and $\bar{G}$ is finite in the limit $\epsilon \to 0$
- After solving one get upto third order

$$\Phi^I_{aa} = \sum_{i=0}^{\infty} a^i \left(\frac{q^2(1-z)^2}{\mu_F^2}\right)^{i\epsilon/2}\left(\frac{1}{1-z}\right)\left[\frac{2A_i}{i\epsilon} - f_i + \overline{\mathcal{G}}^I_{ia}(\epsilon)\right]$$

where

$$f_1 = 0\,, \qquad f_2 = -28\zeta_3 \qquad f_3 = \frac{176}{3}\zeta_2\zeta_3 + 192\zeta_5.$$

(AC etal.)

## Results and Comparison with QCD

LT part of the Soft virtual cross section of Different operator in $N = 4SYM$ upto third order

$$
\begin{aligned}
\Delta_{aa}^{I,(0),\mathrm{SV}} &= \delta(1-z)\,, \\
\Delta_{aa}^{I,(1),\mathrm{SV}} &= 8\zeta_2\delta(1-z) + 16\mathcal{D}_1(z)\,, \\
\Delta_{aa}^{I,(2),\mathrm{SV}} &= -\frac{4}{5}\zeta_2^2\delta(1-z) + 312\zeta_3\mathcal{D}_0(z) - 160\zeta_2\mathcal{D}_1(z) + 128\mathcal{D}_3(z)
\end{aligned}
$$

# Results and Comparison with QCD

$$\Delta_{aa}^{I,(3),\mathrm{SV}} = \left[-\frac{8012}{3}\zeta_6\right]\delta(1-z) + \left[11904\zeta_5 - \frac{23200}{3}\zeta_2\zeta_3\right]\mathcal{D}_0 + \left[-\frac{9856}{5}\zeta_2^2\right]\mathcal{D}_1 + 11584\zeta_3\mathcal{D}_2 + \left[-3584\zeta_2\right]\mathcal{D}_3 + 512\mathcal{D}_5$$

(AC etal.)

## Results and Comparison with QCD

- Leading Transcendental Part of Soft Vitual inclusive Higgs cross section after seting $C_a = C_F = n_f =$ N, we find that is matching with LT part of different SV inclusive cross section (Anastasiou etal,Melnikov,Kilgore,Harlander,Ravindran,Smith,Van Neerven)
- Matches with the calculation of $N^3LO$ SV cross section of BPS opeartor via SCET method (Von Manteuffel etal.)

# Conclusions

- We have calculated the second-order splitting functions in $N = 4$ SYM
- We have calculated second order IR safe inclusive Cross section for BPS, stress tensor and Konishi as final state and demonstrated the universal factorisation property.
- We have predicted $N^3LO$ SV inclusive cross section for different operators
- Compared different part of the factorised cross sections with QCD one in the light of Leadng Transcendentality.

## Future directions

- Calculation of three loop splitting functions in N=4 SYM and comparison with QCD
- Factorisation properties of cross sections involving color particles more than two external states in $N = 4SYM$
- Direct Comparison of different componets of the factorised cross sections of in $N = 4SYM$ with QCD one in the light of Leading Transcendentality.

THANK YOU