NNLO local analytic subtraction for final state radiation

Sandro Uccirati
University of Torino - INFN Torino

In collaboration with:

Lorenzo Magnea, Ezio Maina, Giovanni Pelliccioli,
Chiara Signorile-Signorile, Paolo Torrielli

RADCOR 2019

LHC is …

- a hadron machine → QCD-based processes
- a high-energy machine → complex processes
- entering a high-precision phase → theory must follow
- searching new physics → must control SM background

High precision computation in QCD needed

- PDFs, resumation, parton shower, hadronization and …
- … fixed order computations

Ambitious goal: Automatic NNLO QCD computations

- Loop computations and …
- … cancellation of soft and collinear singularities → this talk

Well established subtraction schemes at NLO

- Frixione-Kunst-Signer (FKS) subtraction — Frixione, Kunszt, Signer
- Catani-Seymour (CS) Dipole subtraction — Catani, Seymour
- Nagy-Soper subtraction — Nagy, Soper

Many methods available at NNLO

- Antenna subtraction — Gehrmann De Ridder, Gehrmann, Glover, Heinrich, et al.
- Sector-improved residue subtraction — Czakon et al.; Melnikov et al.
- Colourful subtraction — Del Duca, Duhr, Kardos, Somogyi, Troscanyi, et al.
- qT-slicing — Catani, Grazzini, et al.
- N-jettiness slicing — Boughezal, Petriello, et al.
- Projection to Born — Cacciari, Salam, Zanderighi, et al.
- Sector decomposition — Anastasiou, Binoth, et al.
- ε-prescription — Frixione, Grazzini
- Unsubtraction — Rodrigo et al.
- Geometric — Herzog

Rationale of our approach

Search for a “minimal” subtraction procedure at NNLO suited for analytical integration:

- We have well established methods at NLO:
  - Frixione-Kunst-Signer (FKS) subtraction — Frixione, Kunszt, Signer 9512328; Frixione 9706545
  - Catani-Seymour (CS) Dipole subtraction — Catani, Seymour 9605323; Catani et al. 0201036
  - Nagy-Soper subtraction — Nagy, Soper, 0308127
- Understand their basic features
- Try to find a simpler subtraction at NLO, by merging them
- Then generalize to NNLO

Subtraction procedure at NLO

$$\frac{d\sigma_{\mathrm{NLO}} - d\sigma_{\mathrm{LO}}}{dX} = \int d\Phi_n \Big(V + I \Big) \delta_{X_n} + \int d\Phi_{n+1} \Big(R\, \delta_{X_{n+1}} - K\, \delta_{X_n} \Big)$$

Subtraction procedure at NLO

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n \Big(V + I \Big) \delta_{X_n} + \int d\Phi_{n+1} \Big(R\, \delta_{X_{n+1}} - K\, \delta_{X_n} \Big)$$

- Divide phase space with sector functions $\mathcal{W}_{ij}$

FKS SECTOR FUNCTIONS

Subtraction procedure at NLO

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n \Big(V + I \Big) \delta_{X_n} + \int d\Phi_{n+1} \Big(R\, \delta_{X_{n+1}} - K\, \delta_{X_n} \Big)$$

- Divide phase space with sector functions $\mathcal{W}_{ij}$ FKS SECTOR FUNCTIONS
- Identify counterterms through IR limits $\mathbf{S}_i, \ \mathbf{C}_{ij}$

Subtraction procedure at NLO

$$\frac{d\sigma_{\mathrm{NLO}} - d\sigma_{\mathrm{LO}}}{dX} = \int d\Phi_n \Big(V + I \Big) \delta_{X_n} + \int d\Phi_{n+1} \Big(R\, \delta_{X_{n+1}} - K\, \delta_{X_n} \Big)$$

- Divide phase space with sector functions $\mathcal{W}_{ij}$ FKS SECTOR FUNCTIONS
- Identify counterterms through IR limits $\mathbf{S}_i,\ \mathbf{C}_{ij}$
- Counterterms are sums of terms, each with its remapped momenta

$\{k\} \ \to \ \{\bar{k}\}^{(abc)}$ CATANI-SEYMOUR REMAPPING

Subtraction procedure at NLO

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n \Big(V + I \Big) \delta_{X_n} + \int d\Phi_{n+1} \Big(R\, \delta_{X_{n+1}} - K\, \delta_{X_n} \Big)$$

- Divide phase space with sector functions $\mathcal{W}_{ij}$ FKS SECTOR FUNCTIONS
- Identify counterterms through IR limits $\mathbf{S}_i,\ \mathbf{C}_{ij}$ → $\overline{\mathbf{S}}_i,\ \overline{\mathbf{C}}_{ij}$
- Counterterms are sums of terms, each with its remapped momenta

$\{k\} \to \{\bar{k}\}^{(abc)}$

CATANI-SEYMOUR REMAPPING

Subtraction procedure at NLO

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n \Big(V + I \Big) \delta_{X_n} + \int d\Phi_{n+1} \Big(R\, \delta_{X_{n+1}} - K\, \delta_{X_n} \Big)$$

- Divide phase space with sector functions $\mathcal{W}_{ij}$ FKS SECTOR FUNCTIONS
- Identify counterterms through IR limits $\mathbf{S}_i, \ \mathbf{C}_{ij}$ → $\overline{\mathbf{S}}_i, \ \overline{\mathbf{C}}_{ij}$
- Counterterms are sums of terms, each with its remapped momenta

$\{k\} \ \to \ \{\bar{k}\}^{(abc)}$ CATANI-SEYMOUR REMAPPING

- Phase space reparametrized differently for each term of the sum

$d\Phi_{n+1}(\{k\}) = d\Phi_n \left(\{\bar{k}\}^{(abc)} \right) d\Phi_1(\bar{s}_{bc}; y, z, \phi)$ CATANI-SEYMOUR REPARAMETRIZATION

Subtraction procedure at NLO

$$\frac{d\sigma_{\mathrm{NLO}} - d\sigma_{\mathrm{LO}}}{dX} = \int d\Phi_n \Big(V + I \Big) \delta_{X_n} + \int d\Phi_{n+1} \Big(R\, \delta_{X_{n+1}} - K\, \delta_{X_n} \Big)$$

- Divide phase space with sector functions $\mathcal{W}_{ij}$ FKS SECTOR FUNCTIONS
- Identify counterterms through IR limits $\mathbf{S}_i, \ \mathbf{C}_{ij}$ → $\overline{\mathbf{S}}_i, \ \overline{\mathbf{C}}_{ij}$
- Counterterms are sums of terms, each with its remapped momenta

$\{k\} \ \rightarrow \ \{\bar{k}\}^{(abc)}$ CATANI-SEYMOUR REMAPPING

- Phase space reparametrized differently for each term of the sum

$d\Phi_{n+1}(\{k\}) = d\Phi_n \left(\{\bar{k}\}^{(abc)} \right) d\Phi_1(\bar{s}_{bc}; y, z, \phi)$ CATANI-SEYMOUR REPARAMETRIZATION

- Integrate analytically each term after getting rid of the sector functions

AS IN FKS

Subtraction procedure at NLO

$$\frac{d\sigma_{\text{NLO}} - d\sigma_{\text{LO}}}{dX} = \int d\Phi_n \Big(V + I\Big)\delta_{X_n} + \int d\Phi_{n+1}\Big(R\,\delta_{X_{n+1}} - \overline{K}\delta_{X_n}\Big)$$

- Divide phase space with sector functions $\mathcal{W}_{ij}$ — FKS SECTOR FUNCTIONS
- Identify counterterms through IR limits $\mathbf{S}_i,\ \mathbf{C}_{ij}$ → $\overline{\mathbf{S}}_i,\ \overline{\mathbf{C}}_{ij}$
- Counterterms are sums of terms, each with its remapped momenta

$\{k\} \to \{\bar{k}\}^{(abc)}$ — CATANI-SEYMOUR REMAPPING

- Phase space reparametrized differently for each term of the sum

$d\Phi_{n+1}(\{k\}) = d\Phi_n\left(\{\bar{k}\}^{(abc)}\right) d\Phi_1(\bar{s}_{bc}; y, z, \phi)$ — CATANI-SEYMOUR REPARAMETRIZATION

- Integrate analytically each term after getting rid of the sector functions

AS IN FKS

Subtraction procedure at NLO

$$\overline{K} = \sum_{i,\, j\neq i} \left[\left(\mathbf{S}_i \mathcal{W}_{ij}\right) \left(\overline{\mathbf{S}}_i R\right) + \left(\mathbf{C}_{ij} \mathcal{W}_{ij}\right) \left(\overline{\mathbf{C}}_{ij} R\right) - \left(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}\right) \left(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} R\right) \right]$$

$$\overline{\mathbf{S}}_i\, R \;=\; -\mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i}} \frac{s_{cd}}{s_{ic} s_{id}} B_{cd}\left(\{\bar{k}\}^{(icd)}\right) \qquad \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij}\, R \;=\; 2\mathcal{N}_1 C_{f_j} \frac{s_{jr}}{s_{ij} s_{ir}} B\left(\{\bar{k}\}^{(ijr)}\right)$$

$$\overline{\mathbf{C}}_{ij}\, R \;=\; \frac{\mathcal{N}_1}{s_{ij}} \left[P_{ij}\, B\left(\{\bar{k}\}^{(ijr)}\right) + Q_{ij}^{\mu\nu}\, B_{\mu\nu}\left(\{\bar{k}\}^{(ijr)}\right)\right]$$

$$I \;=\; I_{\rm s} + I_{\rm hc}$$

$$I_{\rm s} \;=\; \frac{\alpha_{\rm S}}{2\pi} \left(\frac{\mu^2}{s}\right)^{\epsilon} \left[B(\{\bar{k}\}) \left(\sum_c C_{f_c}\right) \left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + 6 - \frac{7}{2}\zeta_2\right) + \sum_{c,\, d\neq c} B_{cd}(\{\bar{k}\}) \ln \frac{\bar{s}_{cd}}{s} \left(\frac{1}{\epsilon} + 2 - \frac{1}{2} \ln \frac{\bar{s}_{cd}}{s}\right) + \mathcal{O}(\epsilon)\right]$$

$$I_{\rm hc} \;=\; -\frac{\alpha_{\rm S}}{2\pi} \left(\frac{\mu^2}{s}\right)^{\epsilon} \sum_p B(\{\bar{k}\}) \left[\delta_{f_p g} \frac{C_A + 4 T_R N_f}{6} \left(\frac{1}{\epsilon} + \frac{8}{3} - \ln \frac{\bar{s}_{pr}}{s}\right) + \delta_{f_p \{q,\bar{q}\}} \frac{C_F}{2} \left(\frac{1}{\epsilon} + 2 - \ln \frac{\bar{s}_{pr}}{s}\right) + \mathcal{O}(\epsilon)\right]$$

Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\,\delta_{X_n} + \int d\Phi_{n+1} RV\,\delta_{X_{n+1}} + \int d\Phi_{n+2}\,RR\,\delta_{X_{n+2}}$$

VV and RV have poles in ϵ, RV and RR diverge in phase space

Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\, \delta_{X_n} + \int d\Phi_{n+1} RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2}\, RR\, \delta_{X_{n+2}}$$

VV and RV have poles in ϵ, RV and RR diverge in phase space

Counterterms $K^{(1)}, K^{(12)}, K^{(2)}, K^{(RV)}$

$$\begin{aligned}\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = & \int d\Phi_n \Big(VV \Big)\, \delta_{X_n} \\ & + \int d\Phi_{n+1} \Big[\Big(RV \Big)\, \delta_{X_{n+1}} + \Big(- K^{(\mathbf{RV})}\Big)\, \delta_{X_n}\Big] \\ & + \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \Big(K^{(\mathbf{2})} + K^{(\mathbf{12})}\Big)\, \delta_{X_n}\Big]\end{aligned}$$

(RR-$K^{(1)}$-$K^{(2)}$-$K^{(12)}$), (RV-$K^{(RV)}$) converge in phase space

Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\,\delta_{X_n} + \int d\Phi_{n+1} RV\,\delta_{X_{n+1}} + \int d\Phi_{n+2}\,RR\,\delta_{X_{n+2}}$$

VV and RV have poles in ϵ, RV and RR diverge in phase space

Counterterms $K^{(1)}$,$K^{(12)}$,$K^{(2)}$,$K^{(RV)}$ and their integrals $I^{(1)}$, $I^{(12)}$, $I^{(2)}$, $I^{(RV)}$

$$\begin{aligned}\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} =& \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right)\delta_{X_n} \\ &+ \int d\Phi_{n+1}\left[\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right)\delta_{X_n}\right] \\ &+ \int d\Phi_{n+2}\left[RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right)\delta_{X_n}\right]\end{aligned}$$

(RR-$K^{(1)}$-$K^{(2)}$-$K^{(12)}$), (RV-$K^{(RV)}$) converge in phase space

$$\int d\Phi_{n+2}\,K^{(\mathbf{1})}\,\delta_{X_{n+1}} = \int d\Phi_{n+1}\,I^{(\mathbf{1})}\delta_{X_{n+1}} \qquad \int d\Phi_{n+2}\,K^{(\mathbf{12})}\,\delta_{X_n} = \int d\Phi_{n+1}\,I^{(\mathbf{12})}\delta_{X_n}$$

$$\int d\Phi_{n+2}\,K^{(\mathbf{2})}\,\delta_{X_n} = \int d\Phi_n\,I^{(\mathbf{2})}\,\delta_{X_n} \qquad \int d\Phi_{n+2}\,K^{(\mathbf{RV})}\,\delta_{X_n} = \int d\Phi_{n+1}\,I^{(\mathbf{RV})}\,\delta_{X_n}$$

Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\, \delta_{X_n} + \int d\Phi_{n+1} RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2}\, RR\, \delta_{X_{n+2}}$$

VV and RV have poles in ϵ, RV and RR diverge in phase space

Counterterms $K^{(1)}$, $K^{(12)}$, $K^{(2)}$, $K^{(RV)}$ and their integrals $I^{(1)}$, $I^{(12)}$, $I^{(2)}$, $I^{(RV)}$

$$\begin{aligned}\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = & \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right) \delta_{X_n} \\ & + \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})}\right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right) \delta_{X_n}\right] \\ & + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right) \delta_{X_n}\right]\end{aligned}$$

(RR-$K^{(1)}$-$K^{(2)}$-$K^{(12)}$), (RV-$K^{(RV)}$) and ($I^{(1)}$+$I^{(12)}$) converge in phase space

$$\int d\Phi_{n+2}\, K^{(\mathbf{1})}\, \delta_{X_{n+1}} = \int d\Phi_{n+1}\, I^{(\mathbf{1})} \delta_{X_{n+1}} \qquad \int d\Phi_{n+2}\, K^{(\mathbf{12})}\, \delta_{X_n} = \int d\Phi_{n+1}\, I^{(\mathbf{12})} \delta_{X_n}$$

$$\int d\Phi_{n+2}\, K^{(\mathbf{2})}\, \delta_{X_n} = \int d\Phi_n\, I^{(\mathbf{2})}\, \delta_{X_n} \qquad \int d\Phi_{n+2}\, K^{(\mathbf{RV})}\, \delta_{X_n} = \int d\Phi_{n+1}\, I^{(\mathbf{RV})}\, \delta_{X_n}$$

(V + $I^{(2)}$ + $I^{(RV)}$), (RV + $I^{(1)}$) and ($K^{(RV)}$ - $I^{(12)}$) are finite in ϵ

Subtraction procedure at NNLO

- Divide the phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl}$$

$$\sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i)^{\alpha}(w_{ij})^{\beta}} \frac{1}{(\mathcal{E}_k + \delta_{kj}\mathcal{E}_i)w_{kl}} \qquad \alpha > \beta > 1$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl}$$

$$\sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i)^\alpha (w_{ij})^\beta} \frac{1}{(\mathcal{E}_k + \delta_{kj}\mathcal{E}_i) w_{kl}} \qquad \alpha > \beta > 1$$

* Primary limits in the sectors:

$$\mathcal{W}_{ijjk} \; : \quad \mathbf{S}_i \,, \quad \mathbf{C}_{ij} \,, \quad \mathbf{S}_{ij} \,, \quad \mathbf{C}_{ijk} \,, \quad \mathbf{SC}_{ijk}$$

$$\mathcal{W}_{ijkj} \; : \quad \mathbf{S}_i \,, \quad \mathbf{C}_{ij} \,, \quad \mathbf{S}_{ik} \,, \quad \mathbf{C}_{ijk} \,, \quad \mathbf{SC}_{ijk} \,, \quad \mathbf{CS}_{ijk}$$

$$\mathcal{W}_{ijkl} \; : \quad \mathbf{S}_i \,, \quad \mathbf{C}_{ij} \,, \quad \mathbf{S}_{ik} \,, \quad \mathbf{C}_{ijkl} \,, \quad \mathbf{SC}_{ikl} \,, \quad \mathbf{CS}_{ijk}$$

$$\mathbf{SC}_{ikl}(f) = \mathbf{C}_{kl}\big(\mathbf{S}_i(f)\big) \qquad \mathbf{CS}_{ijk}(f) = \mathbf{S}_k\big(\mathbf{C}_{ij}(f)\big)$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl}$$

$$\sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i)^\alpha (w_{ij})^\beta} \frac{1}{(\mathcal{E}_k + \delta_{kj}\mathcal{E}_i) w_{kl}} \qquad \alpha > \beta > 1$$

* Single soft and single collinear limits

$$\mathbf{S}_i\, \mathcal{W}_{ijkl} = \left(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{kl}$$

$$\mathbf{S}_i \mathbf{C}_{ij}\, \mathcal{W}_{ijkl} = \left(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{kl}$$

$$\mathcal{W}_{ij}^{(\alpha\beta)} = \frac{\frac{1}{\mathcal{E}_i^\alpha w_{ij}^\beta}}{\sum_{i,j\neq i} \frac{1}{\mathcal{E}_i^\alpha w_{ij}^\beta}}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ijjk} = \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{[ij]k}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ijkj} = \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{k[ij]}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ijkl} = \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{kl}$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl}$$

$$\sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i)^{\alpha}(w_{ij})^{\beta}} \frac{1}{(\mathcal{E}_k + \delta_{kj}\mathcal{E}_i)w_{kl}} \qquad \alpha > \beta > 1$$

* Double soft and double collinear limits

$$\sum_{j\neq i, l\neq i,k} \mathbf{S}_{ik}\, \mathcal{W}_{ijkl} + \sum_{j\neq k, l\neq i,k} \mathbf{S}_{ik}\, \mathcal{W}_{kjil} = 1$$

$$\mathbf{C}_{ijk}\, \left(\mathcal{W}_{ikkj} + \mathcal{W}_{ijkj}\right) + (\text{perm. of } i, j, k) = 1$$

$$\mathbf{S}_{ik}\mathbf{C}_{ijk}(\mathcal{W}_{ikkj} + \mathcal{W}_{ijkj} + \mathcal{W}_{kiij} + \mathcal{W}_{kjij}) = \mathbf{C}_{ijk}\mathbf{S}_{ik}(\mathcal{W}_{ikkj} + \mathcal{W}_{ijkj} + \mathcal{W}_{kiij} + \mathcal{W}_{kjij}) = 1$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits

Sector $\mathcal{W}_{ijjk}$

$\mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{S}_{ij}, \mathbf{C}_{ijk}, \mathbf{SC}_{ijk}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk}) \times \times(1-\mathbf{SC}_{ijk}) RR\, \mathcal{W}_{ijjk} = RR\, \mathcal{W}_{ijjk} - K^{(\mathbf{1})}_{ijjk} - K^{(\mathbf{2})}_{ijjk} - K^{(\mathbf{12})}_{ijjk}$$

→ finite

where the candidates for counterterms are

$$K^{(\mathbf{1})}_{ijjk} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big] RR\, \mathcal{W}_{ijjk}$$

$$K^{(\mathbf{2})}_{ijjk} = \Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\Big] RR\, \mathcal{W}_{ijjk}$$

$$K^{(\mathbf{12})}_{ijjk} = -\Big\{\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij})\Big] + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\Big\} RR\, \mathcal{W}_{ijjk}$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits

Sector $\mathcal{W}_{ijjk}$

$\mathbf{S}_i$, $\mathbf{C}_{ij}$, $\mathbf{S}_{ij}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk}) \times \times(1-\mathbf{SC}_{ijk}) RR\,\mathcal{W}_{ijjk} = RR\,\mathcal{W}_{ijjk} - K^{(\mathbf{1})}_{ijjk} - K^{(\mathbf{2})}_{ijjk} - K^{(\mathbf{12})}_{ijjk}$$

→ finite

where the candidates for counterterms are

cancel in $K^{(\mathbf{2})} + K^{(\mathbf{12})}$

$$K^{(\mathbf{1})}_{ijjk} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big] RR\,\mathcal{W}_{ijjk}$$

$$K^{(\mathbf{2})}_{ijjk} = \Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\Big] RR\,\mathcal{W}_{ijjk}$$

$$K^{(\mathbf{12})}_{ijjk} = -\Big\{\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij})\Big] + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\Big\} RR\,\mathcal{W}_{ijjk}$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits

Sector $\mathcal{W}_{ijkj}$

$\mathbf{S}_i$, $\mathbf{C}_{ij}$, $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, $\mathbf{CS}_{ijk}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \times \times(1-\mathbf{SC}_{ijk})(1-\mathbf{CS}_{ijk}) RR\,\mathcal{W}_{ijkj} = RR\,\mathcal{W}_{ijkj} - K^{(\mathbf{1})}_{ijkj} - K^{(\mathbf{2})}_{ijkj} - K^{(\mathbf{12})}_{ijkj}$$

→ finite

where the candidates for counterterms are

cancel in $K^{(\mathbf{2})} + K^{(\mathbf{12})}$

$$K^{(\mathbf{1})}_{ijkj} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big] RR\,\mathcal{W}_{ijkj}$$

$$K^{(\mathbf{2})}_{ijkj} = \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk} + \mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big] RR\,\mathcal{W}_{ijkj}$$

$$K^{(\mathbf{12})}_{ijkj} = -\Big\{\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik})\Big] + (\mathbf{SC}_{ijk} + \mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big\} RR\,\mathcal{W}_{ijkj}$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits

Sector $\mathcal{W}_{ijkl}$

$\mathbf{S}_i\,, \mathbf{C}_{ij}\,, \mathbf{S}_{ik}\,, \mathbf{C}_{ijkl}, \mathbf{SC}_{ikl}\,, \mathbf{CS}_{ijk}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl}) \times \times(1-\mathbf{SC}_{ikl})(1-\mathbf{CS}_{ijk})RR\,\mathcal{W}_{ijkl} = RR\,\mathcal{W}_{ijkl} - K^{(\mathbf{1})}_{ijkl} - K^{(\mathbf{2})}_{ijkl} - K^{(\mathbf{12})}_{ijkl}$$

→ finite

where the candidates for counterterms are

cancel in $K^{(\mathbf{2})} + K^{(\mathbf{12})}$

$$K^{(\mathbf{1})}_{ijkl} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]RR\,\mathcal{W}_{ijkl}$$

$$K^{(\mathbf{2})}_{ijkl} = \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl} + \mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})\Big]RR\,\mathcal{W}_{ijkl}$$

$$K^{(\mathbf{12})}_{ijkl} = -\Big\{\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik})\Big] + (\mathbf{SC}_{ikl} + \mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})\Big\}RR\,\mathcal{W}_{ijkl}$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits

Momenta in $K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl}$ do not satisfy mass-shell condition and momenta conservation

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits

Momenta in $K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl}$ do not satisfy mass-shell condition and momenta conservation

$$K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl} \longrightarrow \boxed{\overline{K}^{(\mathbf{1})}_{ijkl}, \overline{K}^{(\mathbf{2})}_{ijkl}, \overline{K}^{(\mathbf{12})}_{ijkl}}$$

remapped momenta **in matrix elements** and partially **in IR kernels**

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits

Momenta in $K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl}$ do not satisfy mass-shell condition and momenta conservation

$$K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl} \longrightarrow \boxed{\overline{K}^{(\mathbf{1})}_{ijkl}, \overline{K}^{(\mathbf{2})}_{ijkl}, \overline{K}^{(\mathbf{12})}_{ijkl}}$$

remapped momenta **in matrix elements** and partially **in IR kernels**

They must satisfy:

$$\mathbf{L}_1\, \overline{K}^{(\mathbf{1})}_{ijkl} = K^{(\mathbf{1})}_{ijkl} \qquad \mathbf{L}_1 \in \{\mathbf{S}_i, \mathbf{C}_{ij}\}$$

$$\mathbf{L}_2\, \overline{K}^{(\mathbf{2})}_{ijkl} = K^{(\mathbf{2})}_{ijkl} \qquad \mathbf{L}_2 \in \{\mathbf{S}_{ik}, \mathbf{C}_{ijkl}, \mathbf{SC}_{ikl}, \mathbf{CS}_{ijk}\}$$

$$\mathbf{L}_{12}\, \overline{K}^{(\mathbf{12})}_{ijkl} = K^{(\mathbf{12})}_{ijkl} \qquad \mathbf{L}_{12} \in \{\mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{S}_{ik}.\mathbf{C}_{ijkl}, \mathbf{SC}_{ikl}, \mathbf{CS}_{ijk}\}$$

such that:

$$\boxed{RR\, \mathcal{W}_{ijkl} - \overline{K}^{(\mathbf{1})}_{ijkl} - \overline{K}^{(\mathbf{2})}_{ijkl} - \overline{K}^{(\mathbf{12})}_{ijkl} \longrightarrow \text{finite}}$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta

We use the properties of the sector functions

$$K^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \left[\left(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} RR \right) + \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \right) - \left(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \right) \right] \mathcal{W}_{kl}$$

$$\overline{K}^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \left[\left(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} \right) \left(\overline{\mathbf{S}}_i RR \right) \overline{\mathcal{W}}_{kl} + \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \right) \left(\overline{\mathbf{C}}_{ij} RR \right) \overline{\mathcal{W}}_{kl} - \left(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \right) \left(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} RR \right) \overline{\mathcal{W}}_{kl} \right]$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta

We use the properties of the sector functions

$$K^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \Big[\Big(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} RR \Big) + \Big(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \Big) - \Big(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \Big) \Big] \mathcal{W}_{kl}$$

$$\overline{K}^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \Big[\Big(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{S}}_i RR \big)\, \overline{\mathcal{W}}_{kl} + \Big(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{C}}_{ij} RR \big)\, \overline{\mathcal{W}}_{kl} - \Big(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} RR \big)\, \overline{\mathcal{W}}_{kl} \Big]$$

$$-\mathcal{N}_1 \sum_{\substack{a\neq i \\ b\neq i}} \mathcal{I}_{ab}^{(i)}\, R_{ab}\Big(\{\bar{k}\}^{(iab)} \Big) \overline{\mathcal{W}}_{kl}^{(iab)}$$

$$\frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij}\, R\Big(\{\bar{k}\}^{(ijr)} \Big) + Q_{ij}^{\mu\nu}\, R_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)} \Big) \Big] \overline{\mathcal{W}}_{kl}^{(ijr)}$$

NLO sector functions with remapped momenta

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta

We use the properties of the sector functions

$$K^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \Big[\Big(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} RR \Big) + \Big(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \Big) - \Big(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \Big) \Big] \mathcal{W}_{kl}$$

$$\overline{K}^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \Big[\Big(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{S}}_i RR \big)\, \overline{\mathcal{W}}_{kl} + \Big(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{C}}_{ij} RR \big)\, \overline{\mathcal{W}}_{kl} - \Big(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} RR \big)\, \overline{\mathcal{W}}_{kl} \Big]$$

$$-\mathcal{N}_1 \sum_{\substack{a\neq i \\ b\neq i}} \mathcal{I}_{ab}^{(i)}\, R_{ab}\Big(\{\bar{k}\}^{(iab)} \Big)\, \overline{\mathcal{W}}_{kl}^{(iab)}$$

$$\frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij}\, R\Big(\{\bar{k}\}^{(ijr)} \Big) + Q_{ij}^{\mu\nu}\, R_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)} \Big) \Big]\, \overline{\mathcal{W}}_{kl}^{(ijr)}$$

single remapping

NLO sector functions with remapped momenta

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta

Examples of double remappings

double remapped momenta

$$\overline{\mathbf{S}}_{ij}\, RR = \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d}} \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{ef}^{(j)}\, B_{cdef}\Big(\{\bar{k}\}^{(icd,jef)}\Big) + 4 \sum_{e\neq i,j,c,d} \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{ed}^{(j)}\, B_{cded}\Big(\{\bar{k}\}^{(icd,jed)}\Big)$$

$$+ 2\, \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{cd}^{(j)}\, B_{cdcd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \Big(\mathcal{I}_{cd}^{(ij)} - \frac{1}{2}\mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\mathcal{I}_{dd}^{(ij)}\Big) B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big)\Bigg]$$

$$\overline{\mathbf{C}}_{ijk}\, RR = \frac{\mathcal{N}_1^2}{s_{ijk}^2} \Big[P_{ijk}\, B\Big(\{\bar{k}\}^{(ijkr)}\Big) + Q_{ijk}^{\mu\nu}\, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijkr)}\Big)\Big]$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum

$\boxed{\overline{\mathbf{S}}_{ij} RR}$ $\boxed{B_{cdef} \text{ term}}$

$$d\Phi_{n+2}(\{k\}) = d\Phi_n\left(\{\bar{k}\}^{(icd,jef)}\right) d\Phi_1(s_{icd}; y, z, \phi)\, d\Phi_1\left(\bar{s}_{jef}^{(icd)}; y', z', \phi'\right)$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum

$\overline{\mathbf{S}}_{ij} RR$ $\quad$ B_{cdef} term

$$d\Phi_{n+2}(\{k\}) = d\Phi_n\left(\{\bar{k}\}^{(icd,jef)}\right) d\Phi_1(s_{icd}; y, z, \phi)\, d\Phi_1\left(\bar{s}_{jef}^{(icd)}; y', z', \phi'\right)$$

$\overline{\mathbf{S}}_{ij} RR$ $\quad$ B_{cd} term

$a, b, c, d = i, j, c, d$

$\overline{\mathbf{C}}_{ijk} RR$

$a, b, c, d = i, j, k, r$

$$d\Phi_{n+2}(\{k\}) = d\Phi_n\left(\{\bar{k}\}^{(abcd)}\right) d\Phi_2(s_{abcd}; y, z, \phi, y', z', x')$$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

We use the properties of the sector functions

$$I^{(\mathbf{1})} = \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathcal{W}}_{kl} \left[\sum_i \int d\Phi_1 \overline{\mathbf{S}}_i RR + \sum_{i,\, j>i} \int d\Phi_1 \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j) RR \right]$$

trivial integration

- Remapped sector functions sum to 1
- Are kept to combine with sectors of RV

Similar for $I^{(\mathbf{12})}$

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(\mathbf{2})} = \int d\Phi_2 \left\{ \sum_{\substack{i,j\neq i \\ k\neq i, l\neq i,k}} \overline{\mathbf{S}}_{ik} \mathcal{W}_{ijkl} + \sum_{\substack{i,j\neq i \\ k\neq i,j}} \overline{\mathbf{C}}_{ijk} \left[(1-\overline{\mathbf{S}}_{ij}) RR\, \mathcal{W}_{ijjk} + (1-\overline{\mathbf{S}}_{ik}) RR\, \mathcal{W}_{ijkj} \right] + \dots \right\}$$

- "Pure" double-unresolved part
- In all subtraction scheme the more difficult part to be integrated

- Basically products of single unresolved integrals
- Trivial integration

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(\mathbf{2})} = \int d\Phi_2 \Big\{ \sum_{\substack{i,j\neq i \\ k\neq i, l\neq i,k}} \overline{\mathbf{S}}_{ik} \mathcal{W}_{ijkl} + \sum_{\substack{i,j\neq i \\ k\neq i,j}} \overline{\mathbf{C}}_{ijk} \Big[(1-\overline{\mathbf{S}}_{ij}) RR\, \mathcal{W}_{ijjk} + (1-\overline{\mathbf{S}}_{ik}) RR\, \mathcal{W}_{ijkj} \Big] + \dots \Big\}$$

We use the properties of the sector functions

$$\sum_{\substack{i,j\neq i \\ k\neq i, l\neq i,k}} \bar{\mathbf{S}}_{ik}\, RR\, \mathcal{W}_{ijkl} = \sum_{i,k>i} \bar{\mathbf{S}}_{ik}\, RR$$

$$\sum_{\substack{i,j\neq i \\ k\neq i}} \overline{\mathbf{C}}_{ijk} \Big[(1-\overline{\mathbf{S}}_{ij}) RR\, \mathcal{W}_{ijjk} + (1-\overline{\mathbf{S}}_{ik}) RR\, \mathcal{W}_{ijkj} \Big] = \sum_{\substack{i,j>i \\ k>j}} \overline{\mathbf{C}}_{ijk} \,(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk})\, RR$$

Integration of the double-unresolved part of $I^{(2)}$

$$I^{(2)} = \sum_{i,j>i} \int d\Phi_2\, \overline{\mathbf{S}}_{ij} RR + \sum_{\substack{i,j>i \\ k>j}} \int d\Phi_2\, \overline{\mathbf{C}}_{ijk}(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}) RR + \dots$$

Catani, Grazzini
hep-ph: 9908523

Kernels to integrate

2 soft kernels	$\mathcal{I}_{cd}^{(ij)}[q\bar{q}]$	$\mathcal{I}_{cd}^{(ij)}[gg]$	
5 collinear kernels	$P_{ijk}[qq'\bar{q}']$	$P_{ijk}[qq\bar{q}]$	$P_{ijk}[q\bar{q}g]$
	$P_{ijk}[qgg]$	$P_{ijk}[ggg]$	

- Rational functions of six invariants s_{ab}, s_{ac}, s_{bc}, s_{cd}, s_{ad}, s_{bd}
- The possible denominators are only:

s_{ab}, s_{ac}, s_{bc}, s_{cd}, s_{ad}, s_{bd},

$s_{ac}+s_{bc}$, $s_{ad}+s_{bd}$, $s_{ad}+s_{cd}$, $s_{bd}+s_{cd}$

Integration of the double-unresolved part of $I^{(\mathbf{2})}$

- The integrala of the kernels are symmetric under:
  - the permutation of the four momenta k_a, k_c, k_b, k_d
  - the following permutations of invariants:

$$s_{ab} \leftrightarrow s_{cd} \qquad s_{ac} \leftrightarrow s_{bd} \qquad s_{ad} \leftrightarrow s_{bc}$$

- We can reduce the denominators to:

$$\begin{aligned}
s_{ab} &= y'\, y\, s_{abcd} \\
s_{ac} &= z'(1-y')\, y\, s_{abcd} \\
s_{bc} &= (1-y')(1-z')\, y\, s_{abcd} \\
s_{cd} &= (1-y')(1-y)(1-z)\, s_{abcd} \\
s_{bd} &= (1-y)\left[y'z'(1-z)+(1-z')z+2(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right] s_{abcd} \\
s_{ac}+s_{bc} &= (1-y')\, y\, s_{abcd}, \\
s_{ad}+s_{bd} &= (y'+z-y'\,z)\,(1-y)\, s_{abcd}, \\
s_{ab}+s_{bc} &= (1-z'+z'y')\, y\, s_{abcd}.
\end{aligned}$$

- The integral measure is:

$$\int d\Phi_2(p^2;y,z,\phi,y',z',x') = G_2\,(p^2)^{2-2\epsilon}\int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^1 dy \int_0^1 dz\,[x'(1-x')]^{-\epsilon-1/2} \left[y'z'(1-y')^2(1-z')y^2z(1-y)^2(1-z)\right]^{-\epsilon} y(1-y)(1-y')$$

Integration of the double-unresolved part of $I^{(\mathbf{2})}$

- Using the properties of the hypergeometric function ${}_2F_1$, we are left with integrals of the following types:

$$\int_0^1 dt\,(1-t)^\mu\, t^\nu\; {}_2F_1(n_1, n_2-\epsilon, n_3-2\epsilon, 1-t)$$

$$\int_0^1 dt \int_0^1 du\,(1-t)^\mu\, t^\nu\, (1-u)^\rho\, u^\sigma\, {}_2F_1(n_1, n_2-\epsilon, n_3-2\epsilon, 1-tu)$$

$$n_1, n_2, n_3 \in \mathbb{N}, \qquad n_1 \geq 1, \qquad n_3 \geq n_1+1, n_2$$

$$\mu, \nu, \rho, \sigma = n + m\,\epsilon, \qquad n, m \in \mathbb{Z}, \qquad n \geq -1$$

- All integrals could be written in terms of the hypergeometric functions

$${}_2F_1(a,b,c,1), \qquad {}_3F_2(a,b,c,1) \qquad {}_4F_3(a,b,c,1)$$

and then expanded in ϵ

- We have expanded the ${}_2F_1$ in ϵ and then integrated in t and u

All integrals checked against a numerical computation without using symmetries

Integration of the double-unresolved part of $I^{(\mathbf{2})}$

Results for the integrated kernels

$$A = \frac{1}{(4\pi)^4}\left(\frac{s_{abcd}\, e^{\gamma_E}}{4\pi}\right)^{-2\epsilon}$$

$$\int d\Phi_2\, I^{(ij)}_{cd}[q\bar{q}] \;=\; A\left\{\frac{2}{3}\frac{1}{\epsilon^3} + \frac{28}{9}\frac{1}{\epsilon^2} + \left[\frac{416}{27} - \frac{7}{9}\pi^2\right]\frac{1}{\epsilon} + \frac{5260}{81} - \frac{104}{27}\pi^2 - \frac{76}{9}\zeta(3)\right\} \qquad c \neq d$$

$$\int d\Phi_2\, I^{(ij)}_{cc}[q\bar{q}] \;=\; A\left\{-\frac{2}{3}\frac{1}{\epsilon^2} - \frac{16}{9}\frac{1}{\epsilon} - \frac{212}{27} + \pi^2\right\}$$

$$\int d\Phi_2\, I^{(ij)}_{cd}[gg] \;=\; A\left\{\frac{2}{\epsilon^4} + \frac{35}{3}\frac{1}{\epsilon^3} + \left[\frac{481}{9} - \frac{8}{3}\pi^2\right]\frac{1}{\epsilon^2} + \left[\frac{6218}{27} - \frac{269}{18}\pi^2 - \frac{154}{3}\zeta(3)\right]\frac{1}{\epsilon} + \frac{76912}{81} - \frac{3775}{54}\pi^2 - \frac{2050}{9}\zeta(3) - \frac{23}{60}\pi^4\right\}$$

$$\int d\Phi_2\, I^{(ij)}_{cc}[gg] \;=\; A\left\{-\frac{2}{3}\frac{1}{\epsilon^2} - \frac{10}{9}\frac{1}{\epsilon} - \frac{164}{27} + \pi^2\right\}$$

$$\int d\Phi_2\, P_{ijk}[qq'\bar{q}'] \;=\; A\left\{-\frac{1}{3}\frac{1}{\epsilon^3} - \frac{31}{18}\frac{1}{\epsilon^2} + \left[-\frac{889}{108} + \frac{\pi^2}{2}\right]\frac{1}{\epsilon} - \frac{23941}{648} + \frac{31}{12}\pi^2 + \frac{80}{9}\zeta(3)\right\}$$

$$\int d\Phi_2\, P_{ijk}[qq\bar{q}] \;=\; A\left\{\left[-\frac{13}{8} + \frac{1}{4}\pi^2 - \zeta(3)\right]\frac{1}{\epsilon} - \frac{227}{16} + \pi^2 + \frac{17}{2}\zeta(3) - \frac{11}{120}\pi^4\right\}$$

$$\int d\Phi_2\, P^{(\mathrm{ab})}_{ijk}[gq\bar{q}] \;=\; A\left\{-\frac{2}{3}\frac{1}{\epsilon^3} - \frac{31}{9}\frac{1}{\epsilon^2} + \left[-\frac{889}{54} + \pi^2\right]\frac{1}{\epsilon} - \frac{23833}{324} + \frac{31}{6}\pi^2 + \frac{160}{9}\zeta(3)\right\}$$

$$\int d\Phi_2\, P^{(\mathrm{nab})}_{ijk}[gq\bar{q}] \;=\; A\left\{-\frac{2}{3}\frac{1}{\epsilon^3} - \frac{41}{12}\frac{1}{\epsilon^2} + \left[-\frac{1675}{108} + \frac{17}{18}\pi^2\right]\frac{1}{\epsilon} - \frac{5404}{81} + \frac{1063}{216}\pi^2 + \frac{139}{9}\zeta(3)\right\}$$

$$\int d\Phi_2\, P^{(\mathrm{ab})}_{ijk}[ggq] \;=\; A\left\{\frac{2}{\epsilon^4} + \frac{7}{\epsilon^3} + \left[\frac{251}{8} - 3\pi^2\right]\frac{1}{\epsilon^2} + \left[\frac{2125}{16} - \frac{21}{2}\pi^2 - \frac{154}{3}\zeta(3)\right]\frac{1}{\epsilon} + \frac{17607}{32} - \frac{753}{16}\pi^2 - \frac{548}{3}\zeta(3) + \frac{13}{20}\pi^4\right\}$$

$$\int d\Phi_2\, P^{(\mathrm{nab})}_{ijk}[ggq] \;=\; A\left\{\frac{1}{2}\frac{1}{\epsilon^4} + \frac{8}{3}\frac{1}{\epsilon^3} + \left[\frac{905}{72} - \frac{2}{3}\pi^2\right]\frac{1}{\epsilon^2} + \left[\frac{11773}{216} - \frac{89}{24}\pi^2 - \frac{65}{6}\zeta(3)\right]\frac{1}{\epsilon} + \frac{295789}{1296} - \frac{845}{48}\pi^2 - \frac{2191}{36}\zeta(3) + \frac{19}{240}\pi^4\right\}$$

$$\int d\Phi_2\, P_{ijk}[ggg] \;=\; A\left\{\frac{5}{2}\frac{1}{\epsilon^4} + \frac{21}{2}\frac{1}{\epsilon^3} + \left[\frac{853}{18} - \frac{11}{3}\pi^2\right]\frac{1}{\epsilon^2} + \left[\frac{5450}{27} - \frac{275}{18}\pi^2 - \frac{188}{3}\zeta(3)\right]\frac{1}{\epsilon} + \frac{180739}{216} - \frac{1868}{27}\pi^2 - \frac{1555}{6}\zeta(3) + \frac{41}{60}\pi^4\right\}$$

Summary of subtraction at NNLO

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} =$$

$$\int d\Phi_n \left(\quad VV \quad + \quad I^{(\mathbf{2})} \quad + \quad I^{(\mathbf{RV})} \quad \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(\quad RV \quad + \quad I^{(\mathbf{1})} \quad \right) \delta_{X_{n+1}} + \left(\quad I^{(\mathbf{12})} \quad - \quad \overline{K}^{(\mathbf{RV})} \quad \right) \delta_{X_n} \right]$$

$$+ \int d\Phi_{n+2} \left[\quad RR\, \delta_{X_{n+2}} \quad - \quad \overline{K}^{(\mathbf{1})}\, \delta_{X_{n+1}} - \left(\quad \overline{K}^{(\mathbf{2})} \quad + \quad \overline{K}^{(\mathbf{12})} \quad \right) \delta_{X_n} \right]$$

Summary of subtraction at NNLO

$$\frac{d\sigma_{\mathrm{NNLO}} - d\sigma_{\mathrm{NLO}}}{dX} =$$

$$\int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - \overline{K}^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - \boxed{\overline{K}^{(\mathbf{1})}} \delta_{X_{n+1}} - \left(\overline{K}^{(\mathbf{2})} + \overline{K}^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

Summary of subtraction at NNLO

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} =$$

$$\int d\Phi_n \Big(\; VV \; + \; I^{(\mathbf{2})} \; + \; I^{(\mathbf{RV})} \; \Big) \, \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \Big[\Big(\; RV \; + \; I^{(\mathbf{1})} \; \Big) \, \delta_{X_{n+1}} + \Big(\; I^{(\mathbf{12})} \; - \; \overline{K}^{(\mathbf{RV})} \; \Big) \, \delta_{X_n} \Big]$$

$$+ \int d\Phi_{n+2} \Big[\; RR \, \delta_{X_{n+2}} \; - \; \overline{K}^{(\mathbf{1})} \, \delta_{X_{n+1}} - \Big(\; \overline{K}^{(\mathbf{2})} \; + \; \overline{K}^{(\mathbf{12})} \; \Big) \, \delta_{X_n} \Big]$$

$$\overline{K}^{(\mathbf{1})} = \sum_{i,\, j \neq i} \sum_{\substack{k \neq i \\ l \neq i,k}} \left[\left(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} \right) \left(\overline{\mathbf{S}}_i RR \right) \overline{\mathcal{W}}_{kl} + \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \right) \left(\overline{\mathbf{C}}_{ij} RR \right) \overline{\mathcal{W}}_{kl} - \left(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \right) \left(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} RR \right) \overline{\mathcal{W}}_{kl} \right]$$

Summary of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} =$$

$$\int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + \boxed{I^{(\mathbf{1})}} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - \overline{K}^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - \overline{K}^{(\mathbf{1})}\, \delta_{X_{n+1}} - \left(\overline{K}^{(\mathbf{2})} + \overline{K}^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

Summary of subtraction at NNLO

$$I^{(\mathbf{1})} \quad = \quad I_{\mathrm{s}}^{(\mathbf{1})} + I_{\mathrm{hc}}^{(\mathbf{1})}$$

$$\begin{aligned} I_{\mathrm{s}}^{(\mathbf{1})} \quad = \quad & \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon} \sum_{k,\, l\neq k} \Bigg[\overline{\mathcal{W}}_{kl}\, R(\{\bar{k}\}) \left(\sum_{c} C_{f_c}\right) \left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + 6 - \frac{7}{2}\zeta_2\right) \\ & + \sum_{c,\, d\neq c} \overline{\mathcal{W}}_{kl}\, R_{cd}(\{\bar{k}\}) \ln\frac{\bar{s}_{cd}}{s} \left(\frac{1}{\epsilon} + 2 - \frac{1}{2}\ln\frac{\bar{s}_{cd}}{s}\right) + \mathcal{O}(\epsilon) \Bigg] \end{aligned}$$

$$\begin{aligned} I_{\mathrm{hc}}^{(\mathbf{1})} \quad = \quad & -\frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon} \sum_{p} \sum_{k,\, l\neq k} \overline{\mathcal{W}}_{kl}\, R(\{\bar{k}\}) \\ & \left[\delta_{f_p g}\frac{C_A + 4 T_R N_f}{6}\left(\frac{1}{\epsilon} + \frac{8}{3} - \ln\frac{\bar{s}_{pr}}{s}\right) + \delta_{f_p\{q,\bar{q}\}}\frac{C_F}{2}\left(\frac{1}{\epsilon} + 2 - \ln\frac{\bar{s}_{pr}}{s}\right) + \mathcal{O}(\epsilon)\right] \end{aligned}$$

Summary of subtraction at NNLO

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} =$$

$$\int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - \overline{K}^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - \overline{K}^{(\mathbf{1})} \delta_{X_{n+1}} - \left(\boxed{\overline{K}^{(\mathbf{2})} + \overline{K}^{(\mathbf{12})}} \right) \delta_{X_n} \right]$$

Summary of subtraction at NNLO

$$
\begin{aligned}
&\overline{K}^{(\mathbf{2})} + \overline{K}^{(\mathbf{12})} \;=\\
&\quad \sum_{i,\,k>i} \left[\mathbf{S}_{ik} \left(\sum_{j\neq i} \sum_{l\neq i,k} \mathcal{W}_{ijkl} + \sum_{j\neq k} \sum_{l\neq i,k} \mathcal{W}_{kjil} \right) \right] \overline{\mathbf{S}}_{ik}\, RR \\
&+ \sum_{i,\,j>i} \left\{ \sum_{k>j} \left[\mathbf{C}_{ijk} \sum_{abc\,\in\,\pi(ijk)} \left(\mathcal{W}_{abbc} + \mathcal{W}_{abcb}\right) \right] \overline{\mathbf{C}}_{ijk}\, RR - \left[\mathbf{S}_{ij}\, \mathbf{C}_{ijk} \sum_{ab\,\in\,\pi(ij)} \left(\mathcal{W}_{abbk} + \mathcal{W}_{akbk}\right) \right] \overline{\mathbf{S}}_{ij}\, \overline{\mathbf{C}}_{ijk}\, RR \right\} \\
&+ \sum_{i,\,j>i} \sum_{\substack{k>i\\ k\neq j}} \sum_{\substack{l>k\\ l\neq j}} \sum_{\substack{ab\,\in\,\pi(ij)\\ cd\,\in\,\pi(kl)}} \left\{ \left[\mathbf{C}_{ijkl} (\mathcal{W}_{abcd} + \mathcal{W}_{cdab}) \right] \overline{\mathbf{C}}_{ijkl}\, RR - \left[\mathbf{S}_{ac}\, \mathbf{C}_{abcd} (\mathcal{W}_{abcd} + \mathcal{W}_{cdab}) \right] \overline{\mathbf{S}}_{ac}\, \overline{\mathbf{C}}_{ijkl}\, RR \right\} \\
&- \sum_{i,\,j>i} \sum_{k\neq i,j} \left[\mathbf{C}_{ij} \left(\mathcal{W}_{ij}^{(\alpha\beta)} + \mathcal{W}_{ji}^{(\alpha\beta)} \right) \right] \\
&\quad \left\{ \left[\mathbf{C}_{jk} \left(\overline{\mathcal{W}}_{jk} + \overline{\mathcal{W}}_{kj} \right) \right] \overline{\mathbf{C}}_{ij}\, \overline{\mathbf{C}}_{ijk} + \sum_{l\neq i,j,k} \left(\mathbf{C}_{kl}\, \overline{\mathcal{W}}_{kl} \right) \overline{\mathbf{C}}_{ijkl} + \left(\mathbf{S}_j \overline{\mathcal{W}}_{jk} \right) \overline{\mathbf{C}}_{ij}\, \overline{\mathbf{S}}_{ij} - \left(\mathbf{S}_j\, \mathbf{C}_{jk}\, \overline{\mathcal{W}}_{jk} \right) \overline{\mathbf{C}}_{ij}\, \overline{\mathbf{S}}_{ij}\, \overline{\mathbf{C}}_{ijk} \right\} \\
&+ \sum_{i,\,j\neq i} \sum_{k\neq i,j} \left[\mathbf{S}_i\, \mathbf{C}_{ij}\, \mathcal{W}_{ij}^{(\alpha\beta)} \right] \left\{ \left[\mathbf{C}_{jk} \left(\overline{\mathcal{W}}_{jk} + \overline{\mathcal{W}}_{kj} \right) \right] \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, \overline{\mathbf{C}}_{ijk} + \left(\mathbf{S}_j\, \overline{\mathcal{W}}_{jk} \right) \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, \overline{\mathbf{S}}_{ij} \right. \\
&\qquad\qquad \left. + \sum_{l\neq i,j,k} \left(\mathbf{S}_k\, \mathbf{C}_{kl}\, \overline{\mathcal{W}}_{kl} \right) \overline{\mathbf{S}}_i\, \overline{\mathbf{S}}_{ik}\, \overline{\mathbf{C}}_{ijkl} - \left(\mathbf{S}_j\, \mathbf{C}_{jk}\, \overline{\mathcal{W}}_{jk} \right) \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, \overline{\mathbf{S}}_{ij}\, \overline{\mathbf{C}}_{ijk} \right\} RR \\
&- \sum_{i,\,j\neq i} \sum_{\substack{k\neq i\\ k>j}} \left[\mathbf{S}_i\, \mathbf{C}_{ijk} \left(\mathcal{W}_{ij}^{(\alpha\beta)} + \mathcal{W}_{ik}^{(\alpha\beta)} \right) \right] \\
&\qquad \left\{ \left[\mathbf{C}_{jk} \left(\overline{\mathcal{W}}_{jk} + \overline{\mathcal{W}}_{kj} \right) \right] \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ijk} - \left(\mathbf{S}_j\, \mathbf{C}_{jk}\, \overline{\mathcal{W}}_{jk} \right) \overline{\mathbf{S}}_i\, \overline{\mathbf{S}}_{ij}\, \overline{\mathbf{C}}_{ijk} - \left(\mathbf{S}_k\, \mathbf{C}_{jk}\, \overline{\mathcal{W}}_{kj} \right) \overline{\mathbf{S}}_i\, \overline{\mathbf{S}}_{ik}\, \overline{\mathbf{C}}_{ijk} \right\} RR \\
&- \sum_{i} \sum_{\substack{k\neq i\\ l\neq i,k}} \left[\mathbf{S}_i \sum_{j\neq i} \mathcal{W}_{ij}^{(\alpha\beta)} \right] \left(\mathbf{S}_k\, \overline{\mathcal{W}}_{kl} \right) \overline{\mathbf{S}}_i\, \overline{\mathbf{S}}_{ik}\, RR
\end{aligned}
$$

Summary of subtraction at NNLO

$$\frac{d\sigma_{\mathrm{NNLO}} - d\sigma_{\mathrm{NLO}}}{dX} =$$

$$\int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(\boxed{I^{(\mathbf{12})}} - \overline{K}^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - \overline{K}^{(\mathbf{1})} \delta_{X_{n+1}} - \left(\overline{K}^{(\mathbf{2})} + \overline{K}^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

Summary of subtraction at NNLO

$$I^{(\mathbf{12})} = I_{\mathrm{s}}^{(\mathbf{12})} + I_{\mathrm{hc}}^{(\mathbf{12})}$$

$$I_{\mathrm{s}}^{(\mathbf{12})} = \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon} \sum_{k,\,l\neq k} \Bigg\{ \left[\left(\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl}(1-\overline{\mathbf{S}}_k)\right)\overline{\mathcal{W}}_{kl}\, R(\{\bar{k}\})\right]\left(\sum_c C_{f_c}\right)\left(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}+6-\frac{7}{2}\zeta_2\right)$$
$$+\sum_{c,\,d\neq c}\left[\left(\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl}(1-\overline{\mathbf{S}}_k)\right)\overline{\mathcal{W}}_{kl}\, R_{cd}(\{\bar{k}\})\right]\ln\frac{\bar{s}_{cd}}{s}\left(\frac{1}{\epsilon}+2-\frac{1}{2}\ln\frac{\bar{s}_{cd}}{s}\right)+\mathcal{O}(\epsilon)\Bigg\}$$

$$I_{\mathrm{hc}}^{(\mathbf{12})} = -\frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\sum_p \sum_{k,\,l\neq k}\left[\left(\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl}(1-\overline{\mathbf{S}}_k)\right)\overline{\mathcal{W}}_{kl}\, R(\{\bar{k}\})\right]$$
$$\left[\delta_{f_p g}\frac{C_A+4T_R N_f}{6}\left(\frac{1}{\epsilon}+\frac{8}{3}-\ln\frac{\bar{s}_{pr}}{s}\right)+\delta_{f_p\{q,\bar{q}\}}\frac{C_F}{2}\left(\frac{1}{\epsilon}+2-\ln\frac{\bar{s}_{pr}}{s}\right)+\mathcal{O}(\epsilon)\right]$$

Summary of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} =$$

$$\int d\Phi_n \left(VV + \boxed{I^{(\mathbf{2})}} + I^{(\mathbf{RV})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - \overline{K}^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - \overline{K}^{(\mathbf{1})}\, \delta_{X_{n+1}} - \left(\overline{K}^{(\mathbf{2})} + \overline{K}^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

Summary of subtraction at NNLO

$$I^{(\mathbf{2})} = I_{\rm ss}^{(\mathbf{2})} + I_{\rm hcc}^{(\mathbf{2})} + I_{\rm cc4}^{(\mathbf{2})} + I_{\rm sc3}^{(\mathbf{2})}$$

Summary of subtraction at NNLO

$$I^{(2)} = I^{(2)}_{\rm ss} + I^{(2)}_{\rm hcc} + I^{(2)}_{\rm cc4} + I^{(2)}_{\rm sc3}$$

$$\begin{aligned}
I^{(2)}_{\rm ss} = \left(\frac{\alpha_{\rm s}}{4\pi}\right)^2 \left(\frac{\mu^2}{s}\right)^{2\epsilon} \Bigg\{ & \Bigg[2\Big(\sum_{a,b} C_{f_a} C_{f_b}\Big) I^{\rm ss}_{C_f C_f} + 8\Big(\sum_a C^2_{f_a}\Big) I^{\rm ss}_{C^2_f} \\
& - \Big(\sum_a C_{f_a}\Big)\Big(N_f\, T_R\, I^{\rm ss}_{C_f T_R} - \frac{C_A}{2} I^{\rm ss}_{C_f C_A}\Big)\Bigg] B(\{\bar{k}\}) \\
& + 2\sum_{c,d\neq c}\Bigg[-2\Big(\sum_a C_{f_a}\Big) I^{\rm ss}_{C_f B_{cd}} + N_f\, T_R\, I^{\rm ss}_{T_R B_{cd}} - \frac{C_A}{2} I^{\rm ss}_{C_A B_{cd}}\Bigg] B_{cd}(\{\bar{k}\}) \\
& + 2\sum_{c,d\neq c} I^{\rm ss}_{B_{cdcd}} B_{cdcd}(\{\bar{k}\}) + 4\sum_{c,d\neq c}\sum_{e\neq d} I^{\rm ss}_{B_{cded}} B_{cded}(\{\bar{k}\}) \\
& + \sum_{c,d\neq c}\sum_{e,f\neq e} I^{\rm ss}_{B_{cdef}} B_{cdef}(\{\bar{k}\}) + \mathcal{O}(\epsilon) \Bigg\}
\end{aligned}$$

$I^{(\mathbf{2})}_{\rm ss}$

$$
\begin{aligned}
I^{\rm ss}_{C_f C_f} &= \frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \left(16 - \frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2} + \left(60 - \frac{14}{3}\pi^2 - \frac{50}{3}\zeta(3)\right)\frac{1}{\epsilon} + 216 - \frac{56}{3}\pi^2 - \frac{200}{3}\zeta(3) + \frac{29}{120}\pi^4 \\
I^{\rm ss}_{C_f^2} &= \left(1 - \frac{\pi^2}{6}\right)\frac{1}{\epsilon^2} + \left(10 - \frac{2}{3}\pi^2 - 6\,\zeta(3)\right)\frac{1}{\epsilon} + 68 - 4\,\pi^2 - 24\,\zeta(3) - \frac{7}{72}\pi^4 \\
I^{\rm ss}_{C_f T_R} &= \frac{2}{3}\frac{1}{\epsilon^3} + \frac{34}{9}\frac{1}{\epsilon^2} + \left(\frac{464}{27} - \frac{7}{9}\pi^2\right)\frac{1}{\epsilon} + \frac{5896}{81} - \frac{131}{27}\pi^2 - \frac{76}{9}\zeta(3) \\
I^{\rm ss}_{C_f C_A} &= \frac{2}{\epsilon^4} + \frac{35}{3}\frac{1}{\epsilon^3} + \left(\frac{487}{9} - \frac{8}{3}\pi^2\right)\frac{1}{\epsilon^2} \\
&\quad + \left(\frac{6248}{27} - \frac{269}{18}\pi^2 - \frac{154}{3}\zeta(3)\right)\frac{1}{\epsilon} + \frac{77404}{81} - \frac{3829}{54}\pi^2 - \frac{2050}{9}\zeta(3) - \frac{23}{60}\pi^4 \\
I^{\rm ss}_{C_f B_{cd}} &= \ln\frac{\bar{s}_{cd}}{s}\left[-\frac{1}{\epsilon^3} - \frac{4}{\epsilon^2} - \left(20 - \frac{11}{6}\pi^2\right)\frac{1}{\epsilon} - 100 + \frac{22}{3}\pi^2 + \frac{122}{3}\zeta(3)\right. \\
&\quad \left. + \frac{1}{2}\ln\frac{\bar{s}_{cd}}{s}\left(\frac{1}{\epsilon^2} + \frac{4}{\epsilon} + 20 - \frac{11}{6}\pi^2\right) - \frac{1}{6}\ln^2\frac{\bar{s}_{cd}}{s}\left(\frac{1}{\epsilon} + 4\right) + \frac{1}{24}\ln^3\frac{\bar{s}_{cd}}{s}\right] \\
I^{\rm ss}_{T_R B_{cd}} &= \ln\frac{\bar{s}_{cd}}{s}\left[-\frac{2}{3}\frac{1}{\epsilon^2} - \frac{34}{9}\frac{1}{\epsilon} - \frac{464}{27} + \frac{7}{9}\pi^2 + \ln\frac{\bar{s}_{cd}}{s}\left(\frac{2}{3}\frac{1}{\epsilon} + \frac{34}{9}\right) - \frac{4}{9}\ln^2\frac{\bar{s}_{cd}}{s}\right. \\
I^{\rm ss}_{C_A B_{cd}} &= \ln\frac{\bar{s}_{cd}}{s}\left[-\frac{2}{\epsilon^3} - \frac{35}{3}\frac{1}{\epsilon^2} - \left(\frac{487}{9} - \frac{8}{3}\pi^2\right)\frac{1}{\epsilon} - \frac{6248}{27} + \frac{269}{18}\pi^2 + \frac{154}{3}\zeta(3)\right. \\
&\quad \left. + \ln\frac{\bar{s}_{cd}}{s}\left(\frac{2}{\epsilon^2} + \frac{35}{3}\frac{1}{\epsilon} + \frac{487}{9} - \frac{8}{3}\pi^2\right) - \frac{2}{3}\ln^2\frac{\bar{s}_{cd}}{s}\left(\frac{2}{\epsilon} + \frac{35}{3}\right) + \frac{2}{3}\ln^3\frac{\bar{s}_{cd}}{s}\right] \\
I^{\rm ss}_{B_{cdcd}} &= -4\Big(1 - \zeta(3)\Big)\left(\frac{1}{\epsilon} - 2\ln\frac{\bar{s}_{cd}}{s}\right) - 40 - \frac{\pi^2}{3} + 12\,\zeta(3) + \frac{13}{36}\pi^4 \\
I^{\rm ss}_{B_{cded}} &= \ln\frac{\bar{s}_{cd}}{s}\ln\frac{\bar{s}_{ed}}{s}\left(1 - \frac{\pi^2}{6}\right), \\
I^{\rm ss}_{B_{cdef}} &= \ln\frac{\bar{s}_{cd}}{s}\ln\frac{\bar{s}_{ef}}{s}\left[\frac{1}{\epsilon^2} + \frac{4}{\epsilon} + 16 - \frac{7}{6}\pi^2 - \frac{1}{2}\left(\ln\frac{\bar{s}_{cd}}{s} + \ln\frac{\bar{s}_{ef}}{s}\right)\left(\frac{1}{\epsilon} + 4\right)\right. \\
&\quad \left. + \frac{1}{6}\left(\ln^2\frac{\bar{s}_{cd}}{s} + \ln^2\frac{\bar{s}_{ef}}{s}\right) + \frac{1}{4}\ln\frac{\bar{s}_{cd}}{s}\ln\frac{\bar{s}_{ef}}{s}\right]
\end{aligned}
$$

$(\{\bar{k}\})$

Summary of subtraction at NNLO

$$I^{(\mathbf{2})} \quad = \quad I^{(\mathbf{2})}_{\rm ss} + I^{(\mathbf{2})}_{\rm hcc} + I^{(\mathbf{2})}_{\rm cc4} + I^{(\mathbf{2})}_{\rm sc3}$$

Summary of subtraction at NNLO

$$I^{(\mathbf{2})} = I^{(\mathbf{2})}_{\mathrm{ss}} + I^{(\mathbf{2})}_{\mathrm{hcc}} + I^{(\mathbf{2})}_{\mathrm{cc4}} + I^{(\mathbf{2})}_{\mathrm{sc3}}$$

$$I^{(\mathbf{2})}_{\mathrm{hcc}} = 2\left(\frac{\alpha_{\mathrm{s}}}{4\pi}\right)^2\left(\frac{\mu^2}{s}\right)^{2\epsilon}\sum_p \Bigg\{ \delta_{f_p g}\left[N_f\, C_F\, T_R\, I^{\mathrm{hcc}}_{C_F g} + N_f\, C_A\, T_R\, I^{\mathrm{hcc}}_{C_A} + C_A^2\, I^{\mathrm{hcc}}_{C_A^2}\right] + \delta_{f_p\{q,\bar{q}\}}\, C_F\left[N_f\, T_R\, I^{\mathrm{hcc}}_{C_F q} + C_F\, I^{\mathrm{hcc}}_{C_F^2} + C_A\, I^{\mathrm{hcc}}_{C_F C_A}\right] + \mathcal{O}(\epsilon)\Bigg\} B(\{\bar{k}\})$$

$$I^{\rm hcc}_{C_F g} = -\frac{4}{3}\frac{1}{\epsilon^3} - \frac{62}{9}\frac{1}{\epsilon^2} + \left(-\frac{889}{27} + 2\,\pi^2\right)\frac{1}{\epsilon} - \frac{23833}{162} + \frac{31}{3}\,\pi^2 + \frac{320}{9}\,\zeta(3)$$
$$+ 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{4}{3}\frac{1}{\epsilon^2} + \frac{62}{9}\frac{1}{\epsilon} + \frac{889}{27} - 2\,\pi^2\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{4}{3}\frac{1}{\epsilon} - \frac{62}{9}\right) + \frac{16}{9}\,\ln^3\frac{\bar{s}_{pr}}{s}$$

$$I^{\rm hcc}_{C_A} = -\frac{2}{\epsilon^3} - \frac{89}{9}\frac{1}{\epsilon^2} + \left(-\frac{1211}{27} + 3\,\pi^2\right)\frac{1}{\epsilon} - \frac{5240}{27} + \frac{89}{6}\,\pi^2 + \frac{160}{3}\,\zeta(3)$$
$$+ 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{2}{\epsilon^2} + \frac{89}{9}\frac{1}{\epsilon} + \frac{1211}{27} - 3\,\pi^2\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{2}{\epsilon} - \frac{89}{9}\right) - \frac{8}{3}\,\ln^3\frac{\bar{s}_{pr}}{s}$$

$$I^{\rm hcc}_{C_A^2} = -\frac{5}{6}\frac{1}{\epsilon^3} - \frac{77}{18}\frac{1}{\epsilon^2} + \left(-16 + \frac{11}{12}\pi^2 - \zeta(3)\right)\frac{1}{\epsilon} - \frac{16943}{324} + \frac{61}{12}\,\pi^2 + \frac{56}{9}\zeta(3) - \frac{3}{40}\,\pi^4$$
$$+ 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{5}{6}\frac{1}{\epsilon^2} + \frac{77}{18}\frac{1}{\epsilon} + 16 - \frac{11}{12}\pi^2 + \zeta(3)\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{5}{6}\frac{1}{\epsilon} - \frac{77}{18}\right) - \frac{20}{18}\,\ln^3\frac{\bar{s}_{pr}}{s}$$

$$I^{\rm hcc}_{C_F q} = \frac{1}{6}\frac{1}{\epsilon^2} + \left(\frac{13}{36} + \frac{\pi^2}{9}\right)\frac{1}{\epsilon} - \frac{119}{216} + \frac{17}{108}\,\pi^2 + \frac{14}{3}\,\zeta(3)$$
$$+ 2\,\ln\frac{\bar{s}_{pr}}{s}\left(-\frac{1}{6}\frac{1}{\epsilon} - \frac{13}{36} - \frac{\pi^2}{9}\right) + \frac{1}{3}\,\ln^2\frac{\bar{s}_{pr}}{s}$$

$$I^{\rm hcc}_{C_F^2} = -\frac{2}{\epsilon^3} - \frac{37}{4}\frac{1}{\epsilon^2} + \left(-\frac{333}{8} + \frac{7}{2}\,\pi^2 - 6\,\zeta(3)\right)\frac{1}{\epsilon} - \frac{2815}{16} + \frac{127}{8}\,\pi^2 + \frac{187}{3}\,\zeta(3) - \frac{31}{60}\,\pi^4$$
$$+ 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{2}{\epsilon^2} + \frac{37}{4}\frac{1}{\epsilon} + \frac{333}{8} - \frac{7}{2}\,\pi^2 + 6\,\zeta(3)\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{2}{\epsilon} - \frac{37}{4}\right) + \frac{8}{3}\,\ln^3\frac{\bar{s}_{pr}}{s}$$

$$I^{\rm hcc}_{C_F C_A} = -\frac{1}{2}\frac{1}{\epsilon^3} - \frac{23}{12}\frac{1}{\epsilon^2} + \left(-\frac{365}{72} - \frac{7}{36}\,\pi^2 + 5\,\zeta(3)\right)\frac{1}{\epsilon} - \frac{3089}{432} - \frac{163}{216}\,\pi^2 - \frac{49}{3}\,\zeta(3) + \frac{53}{120}\,\pi^4$$
$$+ 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{1}{2}\frac{1}{\epsilon^2} + \frac{23}{12}\frac{1}{\epsilon} + \frac{365}{72} + \frac{7}{36}\,\pi^2 - 5\,\zeta(3)\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{1}{2}\frac{1}{\epsilon} - \frac{23}{12}\right) + \frac{2}{3}\,\ln^3\frac{\bar{s}_{pr}}{s}$$

Summary of subtraction at NNLO

$$I^{(\mathbf{2})} = I^{(\mathbf{2})}_{\rm ss} + I^{(\mathbf{2})}_{\rm hcc} + I^{(\mathbf{2})}_{\rm cc4} + I^{(\mathbf{2})}_{\rm sc3}$$

Summary of subtraction at NNLO

$$I^{(\mathbf{2})} = I^{(\mathbf{2})}_{\mathrm{ss}} + I^{(\mathbf{2})}_{\mathrm{hcc}} + I^{(\mathbf{2})}_{\mathrm{cc4}} + I^{(\mathbf{2})}_{\mathrm{sc3}}$$

$$I^{(\mathbf{2})}_{\mathrm{cc4}} = 2\left(\frac{\alpha_{\mathrm{S}}}{4\pi}\right)^2\left(\frac{\mu^2}{s}\right)^{2\epsilon}\sum_{p,t\neq p}\Bigg\{ \delta_{f_p g}\delta_{f_t g}\left[C_A^2\, I^{\mathrm{cc4}}_{C_A^2} + N_f\, C_A\, T_R\, I^{\mathrm{cc4}}_{C_A T_R} + N_f^2\, T_R^2\, I^{\mathrm{cc4}}_{T_R^2}\right]$$
$$+\,\delta_{f_p g}\delta_{f_t\{q,\bar q\}}\left[C_A\, C_F\, I^{\mathrm{cc4}}_{C_A C_F} + N_f\, C_F\, T_R\, I^{\mathrm{cc4}}_{C_F T_R}\right]$$
$$+\,\delta_{f_p\{q,\bar q\}}\delta_{f_t\{q,\bar q\}}\, C_F^2\, I^{\mathrm{cc4}}_{C_F^2} + \mathcal{O}(\epsilon)\Bigg\} B(\{\bar k\})$$

$$I^{cc4}_{C_A^2} = \frac{7}{12}\frac{1}{\epsilon^3} + \frac{28}{9}\frac{1}{\epsilon^2} + \left(\frac{3187}{216} - \frac{7}{8}\pi^2\right)\frac{1}{\epsilon} + \frac{85351}{1296} - \frac{14}{3}\pi^2 - \frac{140}{9}\zeta(3)$$
$$+2\ln\frac{\bar{s}_{pt}}{s}\left(-\frac{7}{12}\frac{1}{\epsilon^2} - \frac{28}{9}\frac{1}{\epsilon} - \frac{3187}{216} + \frac{7}{8}\pi^2\right) + 2\ln^2\frac{\bar{s}_{pt}}{s}\left(\frac{7}{12}\frac{1}{\epsilon} + \frac{28}{9}\right) - \frac{7}{9}\ln^3\frac{\bar{s}_{pt}}{s}$$

$$I^{cc4}_{C_A T_R} = \frac{7}{3}\frac{1}{\epsilon^3} + \frac{116}{9}\frac{1}{\epsilon^2} + \left(\frac{6793}{108} - \frac{7}{2}\pi^2\right)\frac{1}{\epsilon} + \frac{185653}{648} - \frac{58}{3}\pi^2 - \frac{560}{9}\zeta(3)$$
$$+2\ln\frac{\bar{s}_{pt}}{s}\left(-\frac{7}{3}\frac{1}{\epsilon^2} - \frac{116}{9}\frac{1}{\epsilon} - \frac{6793}{108} + \frac{7}{2}\pi^2\right) + 2\ln^2\frac{\bar{s}_{pt}}{s}\left(\frac{7}{3}\frac{1}{\epsilon} + \frac{116}{9}\right) - \frac{28}{9}\ln^3\frac{\bar{s}_{pt}}{s}$$

$$I^{cc4}_{T_R^2} = \frac{4}{9}\frac{1}{\epsilon^2} + \frac{77}{27}\frac{1}{\epsilon} + \frac{2369}{162} - \frac{2}{3}\pi^2 + 2\ln\frac{\bar{s}_{pt}}{s}\left(-\frac{4}{9}\frac{1}{\epsilon} - \frac{77}{27}\right) + \frac{8}{9}\ln^2\frac{\bar{s}_{pt}}{s}$$

$$I^{cc4}_{C_A C_F} = \frac{7}{3}\frac{1}{\epsilon^3} + \frac{23}{2}\frac{1}{\epsilon^2} + \left(\frac{3925}{72} - \frac{7}{2}\pi^2\right)\frac{1}{\epsilon} + \frac{105199}{432} - \frac{69}{4}\pi^2 - \frac{560}{9}\zeta(3)$$
$$+2\ln\frac{\bar{s}_{pt}}{s}\left(-\frac{7}{3}\frac{1}{\epsilon^2} - \frac{23}{2}\frac{1}{\epsilon} - \frac{3925}{72} + \frac{7}{2}\pi^2\right) + 2\ln^2\frac{\bar{s}_{pt}}{s}\left(\frac{7}{3}\frac{1}{\epsilon} + \frac{23}{2}\right) - \frac{28}{9}\ln^3\frac{\bar{s}_{pt}}{s}$$

$$I^{cc4}_{C_F T_R} = \frac{7}{3}\frac{1}{\epsilon^3} + \frac{40}{3}\frac{1}{\epsilon^2} + \left(\frac{2351}{36} - \frac{7}{2}\pi^2\right)\frac{1}{\epsilon} + \frac{64427}{216} - 20\pi^2 - \frac{560}{9}\zeta(3)$$
$$+2\ln\frac{\bar{s}_{pt}}{s}\left(-\frac{7}{3}\frac{1}{\epsilon^2} - \frac{40}{3}\frac{1}{\epsilon} - \frac{2351}{36} + \frac{7}{2}\pi^2\right) + 2\ln^2\frac{\bar{s}_{pt}}{s}\left(\frac{7}{3}\frac{1}{\epsilon} + \frac{40}{3}\right) - \frac{28}{9}\ln^3\frac{\bar{s}_{pt}}{s}$$

$$I^{cc4}_{C_F^2} = \frac{7}{4}\frac{1}{\epsilon^3} + \frac{17}{2}\frac{1}{\epsilon^2} + \left(\frac{161}{4} - \frac{21}{8}\pi^2\right)\frac{1}{\epsilon} + 180 - \frac{51}{4}\pi^2 - \frac{140}{3}\zeta(3)$$
$$+2\ln\frac{\bar{s}_{pt}}{s}\left(-\frac{7}{4}\frac{1}{\epsilon^2} - \frac{17}{2}\frac{1}{\epsilon} - \frac{161}{4} + \frac{21}{8}\pi^2\right) + 2\ln^2\frac{\bar{s}_{pt}}{s}\left(\frac{7}{4}\frac{1}{\epsilon} + \frac{17}{2}\right) - \frac{7}{3}\ln^3\frac{\bar{s}_{pt}}{s}$$

Summary of subtraction at NNLO

$$I^{(\mathbf{2})} \quad = \quad I^{(\mathbf{2})}_{\mathrm{ss}} + I^{(\mathbf{2})}_{\mathrm{hcc}} + I^{(\mathbf{2})}_{\mathrm{cc4}} + I^{(\mathbf{2})}_{\mathrm{sc3}}$$

Summary of subtraction at NNLO

$$I^{(\mathbf{2})} = I^{(\mathbf{2})}_{\mathrm{ss}} + I^{(\mathbf{2})}_{\mathrm{hcc}} + I^{(\mathbf{2})}_{\mathrm{cc4}} + I^{(\mathbf{2})}_{\mathrm{sc3}}$$

$$\begin{aligned}
I^{(\mathbf{2})}_{\mathrm{sc3}} = \; & 4\left(\frac{\alpha_{\mathrm{S}}}{4\pi}\right)^2\left(\frac{\mu^2}{s}\right)^{2\epsilon}\sum_p \Bigg\{\delta_{f_p g}(C_A+4N_fT_R)\Bigg[\Bigg(\Big(2C_A-\sum_c C_{f_c}\Big)I^{sc3}_{g,C_A}+C_{f_r}I^{sc3}_{g,r}\Bigg)B(\{\bar{k}\})+I^{sc3}_{g,r}B_{pr}(\{\bar{k}\}) \\
& + \sum_{c\neq p,r} I^{sc3}_{g,B_{cr}}B_{cr}(\{\bar{k}\}) + \sum_{c\neq p}\sum_{d\neq c,p} I^{sc3}_{g,B_{cd}}B_{cd}(\{\bar{k}\})\Bigg] \\
& + \delta_{f_p\{q,\bar{q}\}}\, C_F\Bigg[\Bigg(\Big(2\,C_F-\sum_c C_{f_c}\Big)I^{sc3}_{q,C_F}+C_{f_r}I^{sc3}_{q,r}\Bigg)B(\{\bar{k}\})+I^{sc3}_{q,r}B_{pr}(\{\bar{k}\}) \\
& + \sum_{c\neq p,r} I^{sc3}_{q,B_{cr}}B_{cr}(\{\bar{k}\}) + \sum_{c\neq p}\sum_{d\neq c,p} I^{sc3}_{q,B_{cd}}B_{cd}(\{\bar{k}\})\Bigg] + \mathcal{O}(\epsilon)\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
I^{sc3}_{g,C_A} &= \frac{1}{\epsilon^3} + \frac{14}{3}\,\frac{1}{\epsilon^2} + \left(\frac{172}{9} - \frac{7}{6}\,\pi^2\right)\frac{1}{\epsilon} + \frac{1964}{27} - \frac{49}{9}\,\pi^2 - \frac{50}{3}\,\zeta(3) \\
&\quad + \ln\frac{\bar{s}_{pr}}{s}\left(-\frac{1}{\epsilon^2} - \frac{14}{3}\,\frac{1}{\epsilon} - \frac{172}{9} + \frac{7}{6}\,\pi^2\right) + \frac{1}{2}\,\ln^2\frac{\bar{s}_{pr}}{s}\left(\frac{1}{\epsilon} + \frac{14}{3}\right) - \frac{1}{6}\,\ln^3\frac{\bar{s}_{pr}}{s} \\
I^{sc3}_{g,r} &= \left(-2 + \frac{1}{3}\,\pi^2\right)\left(\frac{1}{\epsilon} - \ln\frac{\bar{s}_{pr}}{s}\right) - \frac{64}{3} + \frac{14}{9}\,\pi^2 + 12\,\zeta(3) \\
I^{sc3}_{g,B_{cr}} &= -\ln\frac{\bar{s}_{cr}}{s}\left(2 - \frac{1}{3}\,\pi^2\right) \\
I^{sc3}_{g,B_{cd}} &= \ln\frac{\bar{s}_{cd}}{s}\left[-\frac{1}{\epsilon^2} - \frac{14}{3}\,\frac{1}{\epsilon} - \frac{172}{9} + \frac{7}{6}\,\pi^2 + \frac{1}{2}\,\ln\frac{\bar{s}_{cd}}{s}\left(\frac{1}{\epsilon} + \frac{14}{3}\right) + \ln\frac{\bar{s}_{pr}}{s}\left(\frac{1}{\epsilon} + \frac{14}{3}\right)\right. \\
&\qquad \left. - \frac{1}{6}\,\ln^2\frac{\bar{s}_{cd}}{s} - \frac{1}{2}\,\ln\frac{\bar{s}_{cd}}{s}\,\ln\frac{\bar{s}_{pr}}{s} - \frac{1}{2}\,\ln^2\frac{\bar{s}_{pr}}{s}\right] \\
I^{sc3}_{q,C_F} &= \frac{1}{\epsilon^3} + \frac{4}{\epsilon^2} + \left(16 - \frac{7}{6}\,\pi^2\right)\frac{1}{\epsilon} + 60 - \frac{14}{3}\,\pi^2 - \frac{50}{3}\,\zeta(3) \\
&\quad + \ln\frac{\bar{s}_{pr}}{s}\left(-\frac{1}{\epsilon^3} - \frac{4}{\epsilon^2} - 16 + \frac{7}{6}\,\pi^2\right) + \frac{1}{2}\,\ln^2\frac{\bar{s}_{pr}}{s}\left(\frac{1}{\epsilon^3} + \frac{4}{\epsilon^2}\right) - \frac{1}{6}\,\ln^3\frac{\bar{s}_{pr}}{s} \\
I^{sc3}_{q,r} &= \left(-2 + \frac{1}{3}\,\pi^2\right)\left(\frac{1}{\epsilon} - \ln\frac{\bar{s}_{pr}}{s}\right) - 20 + \frac{4}{3}\,\pi^2 + 12\,\zeta(3) \\
I^{sc3}_{q,B_{cr}} &= -\ln\frac{\bar{s}_{cr}}{s}\left(2 - \frac{1}{3}\,\pi^2\right) \\
I^{sc3}_{q,B_{cd}} &= \ln\frac{\bar{s}_{cd}}{s}\left[-\frac{1}{\epsilon^2} - \frac{4}{\epsilon} - 16 + \frac{7}{6}\,\pi^2 + \frac{1}{2}\,\ln\frac{\bar{s}_{cd}}{s}\left(\frac{1}{\epsilon} + 4\right) + \ln\frac{\bar{s}_{pr}}{s}\left(\frac{1}{\epsilon} + 4\right)\right. \\
&\qquad \left. - \frac{1}{6}\,\ln^2\frac{\bar{s}_{cd}}{s} - \frac{1}{2}\,\ln\frac{\bar{s}_{cd}}{s}\,\ln\frac{\bar{s}_{pr}}{s} - \frac{1}{2}\,\ln^2\frac{\bar{s}_{pr}}{s}\right]
\end{aligned}
$$

$I^{(2)}_{\mathrm{sc3}} =$ … $_r(\{\bar{k}\})$ … $\})$ … $\epsilon)\,\}$

Summary of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} =$$

$$\int d\Phi_n \left(VV + I^{(\mathbf{2})} + \boxed{I^{(\mathbf{RV})}} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - \boxed{\overline{K}^{(\mathbf{RV})}} \right) \delta_{X_n} \right]$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - \overline{K}^{(\mathbf{1})}\, \delta_{X_{n+1}} - \left(\overline{K}^{(\mathbf{2})} + \overline{K}^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

Summary of subtraction at NNLO

$$\overline{K}^{(\mathbf{RV})} = \sum_{k,\, l\neq k} \left[\left(\mathbf{S}_k \mathcal{W}_{kl} \right) \left(\overline{\mathbf{S}}_k RV \right) + \left(\mathbf{C}_{kl} \mathcal{W}_{kl} \right) \left(\overline{\mathbf{C}}_{kl} RV \right) - \left(\mathbf{S}_k \mathbf{C}_{kl} \mathcal{W}_{kl} \right) \left(\overline{\mathbf{S}}_k \overline{\mathbf{C}}_{kl} RV \right) \right]$$

Somogyi-Trócsányi
hep-ph: 0609043

$$I^{(\mathbf{RV})} = \int d\Phi_1\, \overline{K}^{(\mathbf{RV})} = I_{\mathrm{s}}^{(\mathbf{RV})} + I_{\mathrm{hc}}^{(\mathbf{RV})}$$

- Integration of $I_{\mathrm{s}}^{(\mathbf{RV})}$ and $I_{\mathrm{hc}}^{(\mathbf{RV})}$ completed
- Under investigation the tripole contribution in $I_{\mathrm{s}}^{(\mathbf{RV})}$

Summary

The procedure

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

can be extended beyond NLO !!

- Generates universal local integrable counterterms
- Exploits the freedom in defining them
- The counterterms are as close as possible to the IR limits

Outlook

- Check the cancellation of the poles with VV
- Compute counterterms with initial state hadrons

  - Basically implement Catani-Seymour remappings
  - It does not seem more difficult then the final state (at NLO essentially done)

- Complete the implementation in a Monte Carlo generator
- Consider the massive case

  - Less singularities, but …
  - … more involved remappings, i.e. integration

Back-up slides

Subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IR limits
- Counterterms are sums of terms, each with its remapped momenta

Examples of double remappings

a, b, c, d, e, f are different

$$\bar{k}_c^{(acd,bef)} = \bar{k}_c^{(acd)} \qquad \bar{k}_d^{(acd,bef)} = \bar{k}_d^{(acd)}$$

$$\bar{k}_e^{(acd,bef)} = \bar{k}_b^{(acd)} + \bar{k}_e^{(acd)} - \frac{\bar{s}_{be}^{(acd)}}{\bar{s}_{bf}^{(acd)} + \bar{s}_{ef}^{(acd)}} \bar{k}_f^{(acd)} \qquad \bar{k}_f^{(acd,bef)} = \frac{\bar{s}_{bef}^{(acd)}}{\bar{s}_{bf}^{(acd)} + \bar{s}_{ef}^{(acd)}} \bar{k}_f^{(acd)}$$

$$\{\bar{k}\}^{(acd,bed)} \equiv \{\bar{k}\}^{(acd,bde)} \equiv \{\bar{k}\}^{(acd,bec)} \equiv \{\bar{k}\}^{(acd,bce)} \equiv \{\bar{k}\}^{(abcde)}$$

$$\bar{k}_c^{(abcde)} = \bar{k}_c^{(acd)} \qquad \bar{k}_d^{(abcde)} = \frac{\bar{s}_{bed}^{(acd)}}{\bar{s}_{bd}^{(acd)} + \bar{s}_{ed}^{(acd)}} \bar{k}_d^{(acd)} \qquad \bar{k}_e^{(abcde)} = \bar{k}_b^{(acd)} + \bar{k}_e^{(acd)} - \frac{\bar{s}_{be}^{(acd)}}{\bar{s}_{bd}^{(acd)} + \bar{s}_{ed}^{(acd)}} \bar{k}_d^{(acd)}$$

$$\{\bar{k}\}^{(acd,bcd)} \equiv \{\bar{k}\}^{(acd,bdc)} \equiv \{\bar{k}\}^{(abcd)}$$

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad} + s_{bd} + s_{cd}} k_d \qquad \bar{k}_d^{abcd} = \frac{s_{abcd}}{s_{ad} + s_{bd} + s_{cd}} k_d$$