Generalized hypergeometric functions and intersection theory for Feynman integrals

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- There is a well-known coaction on multiple polylogarithms (MPLs). [Goncharov, Brown]

- We've conjectured a corresponding coaction on Feynman diagrams. See talk by James Matthew.

- Natural compatibility of coaction with discontinuities and differential equations is potentially useful for computation.

- Now: we propose a corresponding coaction on hypergeometric functions, of the form

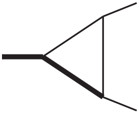
$$\Delta\left(\int_{\gamma}\omega\right)=\sum_{i,j}c_{ij}\int_{\gamma}\omega_i\otimes\int_{\gamma_j}\omega$$

  where the coefficients c_{ij} can be derived from intersection theory.

Feynman diagrams are generalized hypergeometric functions

The simplest Feynman diagrams are generalized hypergeometric functions expanded around integer parameters, giving MPL expansions in ϵ.

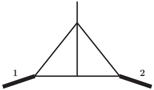
Examples:



$$\sim \quad {}_2F_1\left(1, 1+\epsilon; 2-\epsilon; \frac{p^2}{m^2}\right)$$

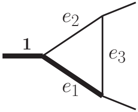
$$\sim \quad {}_2F_1\left(1+2\epsilon, 1+\epsilon, 1-\epsilon, \frac{p^2}{m^2}\right)$$

$$\sim \quad F_4\left(1+\epsilon, 1, 1-\epsilon, 1-\epsilon; \frac{m_1^2}{m_2^2}, \frac{p^2}{m_2^2}\right)$$



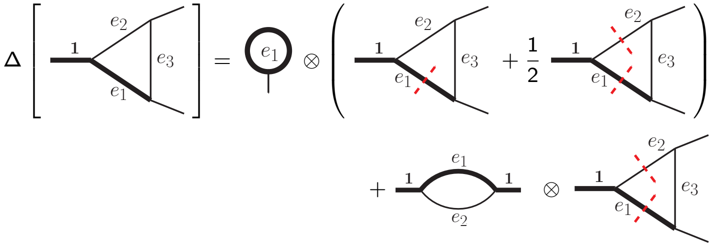
$$\sim \quad {}_3F_2\left(1-\epsilon, 1, 1-2\epsilon; 1+\epsilon, 2-2\epsilon; 1-\frac{p_1^2}{p_2^2}\right)$$

Feynman diagram example with ${}_2F_1$



$$= \frac{e^{\gamma_E \epsilon}\Gamma(1+\epsilon)}{\epsilon(1-\epsilon)}(m^2)^{-1-\epsilon}\,{}_2F_1\left(1, 1+\epsilon; 2-\epsilon; \frac{p^2}{m^2}\right)$$

$$= \frac{1}{p^2}\left[\frac{\log\left(\frac{m^2}{m^2-p^2}\right)}{\epsilon} + \mathrm{Li}_2\left(\frac{p^2}{m^2}\right) + \log^2\left(1-\frac{p^2}{m^2}\right) + \log(m^2)\log\left(1-\frac{p^2}{m^2}\right)\right] + \mathcal{O}(\epsilon)$$



$$\begin{aligned}
\Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\
\Delta(\log^2 z) &= 1 \otimes \log^2 z + 2\log z \otimes \log z + \log^2 z \otimes 1 \\
\Delta(\mathrm{Li}_2(z)) &= 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 + \mathrm{Li}_1(z) \otimes \log z
\end{aligned}$$

How to construct a coaction on ${}_2F_1$?

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

- Check the conditions defining a coaction, notably coassociativity: $(\Delta \otimes \mathrm{id})(\Delta) = (\mathrm{id} \otimes \Delta)(\Delta) \equiv \Delta(\Delta)$
- Coaction on MPLs follows same pattern of (contour preserved) $\otimes$ (integrand preserved), with a relation between the interior contour and integrand
- Seen from diagrams: factors on RHS are simpler/similar: $\Delta({}_2F_1) \sim {}_2F_1 \otimes {}_2F_1$
  More precisely, second factor is a single-valued/nonmotivic version, e.g. "mod $i\pi$." That's why we say "coaction" and not "coproduct."

How should we choose the integrands ω_i and contours γ_j, and find the coefficients c_{ij}?

At first, we tried to select just the right integrands ω_i and "dual" contours γ_j so that there were no coefficients: $\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$

We got pretty far with ${}_{p+1}F_p$ and Appell F_1, F_2, F_3, F_4 by trial and error...

...but some issues with F_4 led us to hunt for math results online, where we found that we'd been starting to recreate intersection theory, and where we found helpful papers from Japan.

See the talks by Mandal and Frellesvig for other applications of intersection theory in physics, and their slides introducing the topic more fully.

Example: ${}_2F_1$

Gauss's original hypergeometric function. Series definition:

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \,, \qquad \text{where } (x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)}. $$

Euler's integral representation:

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 du\, u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta} $$

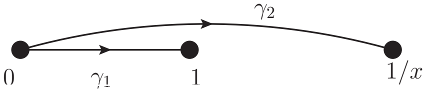
Let

$$ \int_\gamma \omega \equiv \int_0^1 du\, u^{n_0 + a_0 \epsilon} (1-u)^{n_1 + a_1 \epsilon} (1-ux)^{n_x + a_x \epsilon} $$

with $a_0, a_1, a_x \in \mathbb{C}^*$ are generic, and $n_i \in \mathbb{Z}$.

${}_2F_1$ basic picture

- Need a larger set of integrals related to $\int_\gamma \omega$ in which to express the coaction: additional integrands ω_i and contours γ_j.
- Stokes' theorem/IBP reduces ω to a basis of two forms ω_1, ω_2 with integer-shifted exponents ("contiguous relations").
- Contours can also be extended to a set of two by integrating between different branch points: $\gamma_1 = [0,1], \quad \gamma_2 = [0, 1/x]$.



- Choose integrands with dlog singularities at the branch points, $\omega_1 = a_1 \epsilon u^{a_0 \epsilon}(1-u)^{-1+a_1\epsilon}(1-ux)^{a_x \epsilon}, \omega_2 = x a_x \epsilon u^{a_0\epsilon}(1-u)^{a_1\epsilon}(1-ux)^{-1+a_x\epsilon}$, such that the rational terms (residues) of $\int_{\gamma_j} \omega_i$ are δ_{ij}.
- Then the coaction is $\Delta\left(\int_\gamma \omega\right) = \int_\gamma \omega_1 \otimes \int_{\gamma_1} \omega + \int_\gamma \omega_2 \otimes \int_{\gamma_2} \omega$.

${}_2F_1$ basic picture

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} $$

Finally, replace gamma function prefactors using
$\Delta(\Gamma(n+a\epsilon)) = \Gamma(1+a\epsilon) \otimes \Gamma(n+a\epsilon)$.

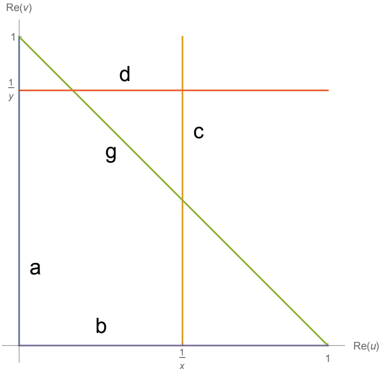
Coaction on ${}_2F_1$:

$$ \begin{aligned} \Delta\left({}_2F_1(\alpha, \beta; \gamma; x)\right) = {}& {}_2F_1(1+a\epsilon, b\epsilon; 1+c\epsilon; x) \otimes {}_2F_1(\alpha, \beta; \gamma; x) \\ & - \frac{b\epsilon}{1+c\epsilon}\, {}_2F_1(1+a\epsilon, 1+b\epsilon; 2+c\epsilon; x) \\ & \otimes \frac{\Gamma(1-\beta)\Gamma(\gamma)}{\Gamma(1-\beta+\alpha)\Gamma(\gamma-\alpha)} x^{1-\alpha}\, {}_2F_1\left(\alpha, 1+\alpha-\gamma; 1-\beta+\alpha; \frac{1}{x}\right) \end{aligned} $$

where $\alpha = n_\alpha + a\epsilon$, $\beta = n_\beta + b\epsilon$ and $\gamma = n_\gamma + c\epsilon$.

Appell F_3 basic picture

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}$$
$$\int_0^1 dv \int_0^{1-v} du\, u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'}$$



Number of independent γ_j = number of bounded chambers = 4.
Choose $\gamma_{abg} = \gamma, \gamma_{bcg}, \gamma_{cdg}, \gamma_{adg}$.

Appell F_3 basic picture

$$\begin{aligned} \omega &= \Phi \cdot u^{n_a} v^{n_b} (1-ux)^{n_c} (1-vy)^{n_d} (1-u-v)^{n_g}\, du \wedge dv \\ \Phi &= u^{a\epsilon} v^{b\epsilon} (1-ux)^{c\epsilon} (1-vy)^{d\epsilon} (1-u-v)^{g\epsilon} \end{aligned}$$

Basis of integrands:

$\omega_{ab} = \Phi\, d\log u \wedge d\log v$

$\omega_{bc} = \Phi\, d\log(1-ux) \wedge d\log v$

$\omega_{cd} = \Phi\, d\log(1-ux) \wedge d\log(1-vy)$

$\omega_{da} = \Phi\, d\log u \wedge d\log(1-vy)$

Read off the duality matrix by intersections (residues in 2 variables).

int	ab	bc	cd	ad
abg	$\frac{1}{ab\epsilon^2}$	0	0	0
bcg	0	$\frac{1}{bc\epsilon^2}$	0	0
cdg	0	0	$\frac{1}{cd\epsilon^2}$	0
adg	0	0	0	$\frac{1}{ad\epsilon^2}$

Coaction is

$$\Delta\left(\int_\gamma \omega\right) = ab\epsilon^2 \int_{\gamma_{abg}} \omega_{ab} + bc\epsilon^2 \int_{\gamma_{bcg}} \omega_{bc} + cd\epsilon^2 \int_{\gamma_{cdg}} \omega_{cd} + ad\epsilon^2 \int_{\gamma_{adg}} \omega_{ad}.$$

Twisted co/homology and intersection numbers: the key points

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

- Our starting point is the integral $\int_\gamma \omega$, so ω is a cohomology class and γ is a homology class.
- What are the co/homology groups?

  They can be defined systematically when $\omega = d\mathbf{u} \prod_k P_k(\mathbf{u})^{\alpha_k}$, where the $P_k(\mathbf{u})$ are polynomials in the integration variables $\mathbf{u}$, and when γ has its boundary on the zeros of $\prod_k P_k(\mathbf{u})$.

  [Aomoto-Kita book]
- Then, take $\{\omega_i\}$ to be a basis of the cohomology group, and $\{\gamma_j\}$ to be a basis of the homology group.

  In our examples where the $P_k(\mathbf{u})$ are mostly linear, this can be done in a natural way. We further associate "canonical forms" to the contours $\{\gamma_j\}$ and then compute intersection numbers with the $\{\omega_i\}$.

Principal example: ${}_2F_1$

$$\int_\gamma \omega \equiv \int_0^1 du\, u^{n_0+a_0\epsilon}(1-u)^{n_1+a_1\epsilon}(1-ux)^{n_x+a_x\epsilon}$$
$$\omega \equiv \Phi\varphi$$

$\varphi = u^{n_0}(1-u)^{n_1}(1-ux)^{n_x}\,du$ is a single-valued differential form.

$\Phi = u^{a_0\epsilon}(1-u)^{a_1\epsilon}(1-ux)^{a_x\epsilon}$ is a multi-valued function,
from which we construct the twist 1-form $d\log\Phi$:

$$d\log\Phi = a_0\frac{du}{u} - a_1\frac{du}{1-u} - xa_x\frac{du}{1-ux}$$

Then Stokes' theorem implies $\int_\gamma \Phi\varphi = \int_\gamma \Phi(\varphi + \nabla_\Phi\xi)$, so φ is a twisted cohomology class.

Contours γ also get a "twist" carrying the information of the choice of branch of Φ.

Intersection numbers

- The intersection number pairing of cohomology classes (forms) is defined by

$$\langle \varphi_i, \psi_j \rangle_\Phi = \frac{1}{(2\pi i)^2} \int \iota_\Phi(\varphi_i) \wedge \psi_j \,,$$

  where ι_Φ denotes a compactification away from the branch points.

- Residue formulas are more practical for computation. For dlog forms in one variable, we have

$$\langle \varphi_i, \psi_j \rangle_\Phi = \sum_{u_p} \frac{\mathrm{Res}_{u=u_p}\, \varphi_i \;\; \mathrm{Res}_{u=u_p}\, \psi_j}{\mathrm{Res}_{u=u_p}\, d\log\Phi} \,,$$

  where the u_p are the poles of $d\log\Phi$. Produces the simple expressions seen above. [Matsumoto, Mizera; see talks by Mandal & Frellesvig for more general versions]

- Invert the matrix of intersection numbers to read off the coefficients c_{ij} that we seek for the coaction formula.

To find bases of co/homology $\{\varphi_i\}$ and $\{\gamma_j\}$:

- Compute the dimension r.
- Try a set of r forms $\{\varphi_i\}$ and use intersection numbers to check linear independence.
- Try a set of r cycles $\{\gamma_j\}$. Construct their canonical forms, and use intersection numbers to check linear independence.

Intersection theory for bases of co/homology

To find bases of co/homology $\{\varphi_i\}$ and $\{\gamma_j\}$:

- Compute the dimension r.
  Given by Morse theory/Euler characteristic [cf. Lee-Pomeransky]. In simple cases, r is the number of solutions to $d \log \Phi = 0$.

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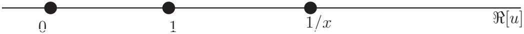
- Try a set of r cycles $\{\gamma_j\}$. Construct their canonical forms, and use intersection numbers to check linear independence.
  The canonical form of γ_j is a unique differential form with log singularities precisely at the boundary of γ_j.
  When the factors $P_k(\mathbf{u})$ are linear, a natural basis is given by "bounded chambers."

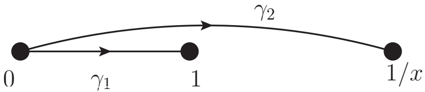
Principal example: ${}_2F_1$

$$\begin{aligned}\Phi &= u^{a_0\epsilon}(1-u)^{a_1\epsilon}(1-ux)^{a_x\epsilon}\\ d\log\Phi &= a_0\frac{du}{u} - a_1\frac{du}{1-u} - xa_x\frac{du}{1-ux}\end{aligned}$$

- The equation $d\log\Phi = 0$ has 2 solutions.
- We also see two bounded chambers in real coordinates;



Choose two independent contours with boundaries at the branch points, for example $\gamma_1 = [0,1], \quad \gamma_2 = [0, 1/x]$.



From these contours, construct their associated "canonical" dlog forms

$$\psi_1 = d\log\frac{u-1}{u}, \qquad \psi_2 = d\log\frac{u-1/x}{u},$$

- It is always possible to use these same forms as the basis of twisted cohomology: $\varphi_1 = \psi_1, \; \varphi_2 = \psi_2$.

Principal example: ${}_2F_1$

With

$$\varphi_1 = \psi_1 = d\log\frac{u-1}{u}\,, \qquad \varphi_2 = \psi_2 = d\log\frac{u-1/x}{u}\,,$$

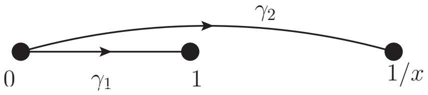
the intersection matrix is

$$\langle\varphi_1,\psi_1\rangle_\Phi = \frac{1}{a_0\epsilon} + \frac{1}{a_1\epsilon} \qquad \langle\varphi_1,\psi_2\rangle_\Phi = \frac{1}{a_0\epsilon}$$

$$\langle\varphi_2,\psi_1\rangle_\Phi = \frac{1}{a_0\epsilon} \qquad \langle\varphi_2,\psi_2\rangle_\Phi = \frac{1}{a_0\epsilon} + \frac{1}{a_x\epsilon}$$

which can be inverted to produce a coaction formula.

In this case with dlog forms and linear polynomials, it's no coincidence that you can just read off overlaps as intersections.



All of our other examples work in the same way.

The hypergeometric coaction

- Feynman integrals typically have degeneracies in the exponents – not allowed by the mathematics, but we have been able to take these limits as needed, preserving the general structure.

- What are those integrals $\int_{\gamma_j} \omega$?

- What if the polynomials $P_k(\mathbf{u})$ are nonlinear?

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- What are those integrals $\int_{\gamma_j} \omega$?
  They are the same class of function. Integration over a different contour can sometimes be mapped back to the usual representation by a change of variables. For Appell F_2 and F_4, we used specific results by Goto and Matsumoto to see $\Delta F_2 \sim F_2 \otimes F_2$ and $\Delta F_4 \sim F_4 \otimes F_4$ explicitly.

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- What if the polynomials $P_k(\mathbf{u})$ are nonlinear?
  Seen for example in F_4. No problem since we can still construct canonical dlog forms and then compute intersection numbers. Regions fall into the class of "positive geometries." [Arkani-Hamed, Bai, Lam]

Summary

- We propose a coaction on generalized hypergeometric functions,

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

where the coefficients c_{ij} can be derived from intersection theory, provided $\omega = d\mathbf{u} \prod_k P_k(\mathbf{u})^{\alpha_k}$ and γ has its boundary on the zeros of $\prod_k P_k(\mathbf{u})$.

- When the exponents α_k are expanded around integer values, we claim that this coaction is compatible with the coaction on the MPLs in the Laurent expansion. Checked to several orders for ${}_{p+1}F_p$, Appell F_1, F_2, F_3, F_4, Lauricella F_D.

- A version of this coaction has recently been proven for ${}_2F_1$ and Lauricella F_D [Brown, Dupont 1907.06603].

- This coaction supports our examples of Feynman-diagrammatic coaction to all orders, despite having some exponents with integer values.