# Five-Point Two-Loop Amplitudes Beyond the Planar Limit

Ben Page

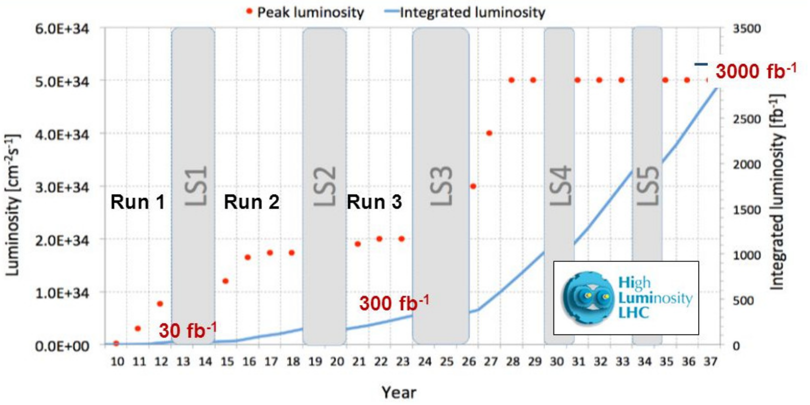
IPhT CEA Saclay

RadCor 2019
Avignon, $9^{\text{th}}$ – $13^{\text{th}}$ September 2019

In collaboration with S. Abreu, L. Dixon, E. Herrmann, and M. Zeng

based on [1807.11522], [1812.08941], [1901.08563]

# The LHC Potential: Precision Era




- LHC will be operating for two more decades
- ATLAS and CMS collect large proton collision data sets

# Why These Amplitudes?

5-point?

- High multiplicity pheno: $\alpha_s$, Higgs couplings.

Non-planar?

- Z, Higgs, Photon + jets not planar at leading $N_c$:

$\mathcal{N}=4$, $\mathcal{N}=8$ SUSY?

- Low tensor rank warmup for QCD corrections.

| process | known | desired |
|---|---|---|
| $pp \to 2\,\text{jets}$ | $\text{N}^2\text{LO}_{\text{QCD}}$<br>$\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$ | |
| $pp \to 3\,\text{jets}$ | $\text{NLO}_{\text{QCD}}$ | $\text{N}^2\text{LO}_{\text{QCD}}$ |
| ⋮ | ⋮ | ⋮ |
| $pp \to H+2j$ | $\text{NLO}_{\text{HEFT}}\otimes\text{LO}_{\text{QCD}}$<br>$\text{N}^3\text{LO}^{(\text{VBF}^*)}_{\text{QCD}}$ (incl.)<br>$\text{N}^2\text{LO}^{(\text{VBF}^*)}_{\text{QCD}}$<br>$\text{NLO}^{(\text{VBF})}_{\text{EW}}$ | $\text{N}^2\text{LO}_{\text{HEFT}}\otimes\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$<br>$\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}^{(\text{VBF})}_{\text{EW}}$ |
| ⋮ | ⋮ | ⋮ |
| $pp \to V+j$ | $\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$ | hadronic decays |
| $pp \to V+2j$ | $\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$<br>$\text{NLO}_{\text{EW}}$ | $\text{N}^2\text{LO}_{\text{QCD}}$ |
| $pp \to V+b\bar{b}$ | $\text{NLO}_{\text{QCD}}$ | $\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$ |
| ⋮ | ⋮ | ⋮ |
| $pp \to \gamma\gamma+j$ | $\text{NLO}_{\text{QCD}}$ | $\text{N}^2\text{LO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$ |

Les Houches Wishlist [1803.07977]

# Computational Challenges

Workflow:

- IBP Reduce amplitudes to master integrals.
- Compute master integrals with differential equations.

Challenge 1: Large linear systems in Laporta method for IBPs

- "Unitarity compatible" IBP vectors control propagator powers.

Challenge 2: Resulting relations huge and hard to handle.

- Write ansatz for result and numerically fit over finite fields.

[Schabinger, von Manteuffel '14; Peraro '16]

[See talks by Vasily, Dima and Tiziano]

# Unitarity Compatible IBPs

- IBP relations have auxilliary double propagator terms.

$$0 = \int \prod_{l=1,2} d^D \ell_l \frac{\partial}{\partial \ell_j^\nu} \left[ \frac{u_j^\nu}{\prod_{\text{props } k} \rho_k} \right].$$

- Can control doubling with unitarity compatible vectors.

$$u_i^\nu \frac{\partial}{\partial \ell_i^\nu} \rho_j = f_j \rho_j.$$

[Gluza, Kajda and Kosower '11]

- A "syzygy" equation.

[See Andreas' talk]

$$u_k^\nu = u_{ka}^{\text{loop}}(\alpha, \rho) \ell_a^\nu + u_{kb}^{\text{ext}}(\alpha, \rho) p_b^\nu$$

[Abreu, Febres-Cordero, Ita, B.P., Zeng '17]

- Write ansatz for $u_k^\nu$, find polynomial solutions in $\rho, \alpha$.
- Compute generating set of solutions using SINGULAR.

$$u_k^\nu(\rho, \alpha) \frac{\partial}{\partial \ell_k^\nu} \begin{pmatrix} \rho_{j(1)} \\ \rho_{j(2)} \\ \vdots \\ \rho_{j(|\Gamma|)} \end{pmatrix} - \begin{pmatrix} f_{j(1)} \rho_{j(1)} \\ f_{j(2)} \rho_{j(2)} \\ \vdots \\ f_{j(|\Gamma|)} \rho_{j(|\Gamma|)} \end{pmatrix} = 0\,,$$

Related [Georgoudis, Larsen, Zhang '16]

# Ansätze and Numerical Methods

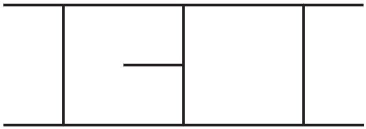
- Write ansatz for result.
- Sample ansatz over finite fields. [Schabinger, von Manteuffel '14], [Peraro '16]
- Rationally reconstruct coefficients back to $\mathcal{Q}$.
- Fixed size intermediate algebra.
- Trivially parallelizable.

$$c(\vec{p}) = \sum_{i=0}^{n} a_i g_i(\vec{p})$$

$$\begin{pmatrix} c(\vec{p}_0) \\ \vdots \\ c(\vec{p}_n) \end{pmatrix} = \begin{pmatrix} g_0(\vec{p}_0) & \cdots & g_n(\vec{p}_0) \\ \vdots & \cdots & \vdots \\ g_0(\vec{p}_n) & \cdots & g_n(\vec{p}_n) \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$$

$$\mathbb{F}_p = \{0, \ldots, p-1\}$$

# Master Integrals



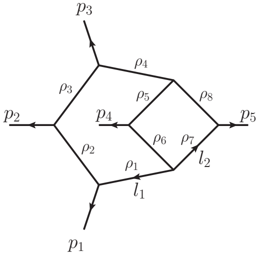
# Controlling Propagator Powers in DEs

[Abreu, B.P., Zeng '18]

- Derivative raises propagator powers.

$$\frac{\partial}{\partial x}\int \frac{N}{\rho_0\cdots\rho_n} = \sum_j \int \frac{N_j}{\rho_0\cdots\rho_j^2\cdots\rho_n}$$

- Choose routing well ⇒ only subset doubled.




- Hexabox case - $\rho_2, \rho_3$.
- Compute vectors that let only these double.

$$u_i^\nu \frac{\partial}{\partial \ell_i^\nu}\rho_j = f_j \rho_j, \quad j \neq 2, 3.$$

- Simplified linear systems.

# Differential Equation as Ansatz

[Abreu, B.P., Zeng '18]

- Canonical DE form very constrained ⇒ ansatz!
- $M_j$ are free parameters.
- 31 numerical reductions over single finite field.
- Requires knowledge of alphabet and pure basis.
- Here, full alphabet in maximal-cut DEs!

$$d\vec{I} = \epsilon \left( \sum_{j=1}^{31} d\log(W_j) M_j \right) \cdot \vec{I}$$

$$W_1 = s_{12}, \quad W_2 = s_{23},$$
$$W_6 = s_{12} - s_{23}, \ldots W_{31} = \mathrm{tr}_5.$$

[Chicherin, Henn, Mitev '17]

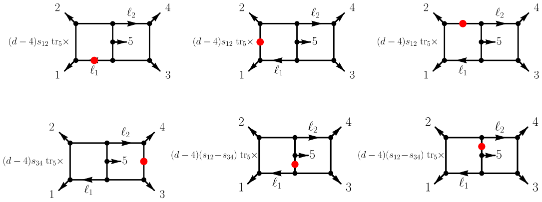
$$M_j \in \mathcal{Q}^{108,108}$$

$$M_j^{k,l} \sim O(10)$$

# Pure Double Pentagons

[Abreu, Dixon, Herrmann, B.P., Zeng '18]

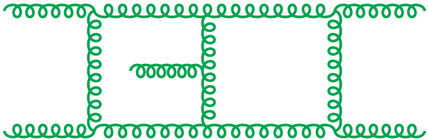
- Final ingredient for amplitudes.
- Even under $\mathrm{tr}_5 \to -\mathrm{tr}_5$ all contained in $\mathcal{N} = 4$ numerators.
- Odd integrals all found in 6d with dots for parity odd.




Alternative basis [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18]

- Construction of DE in $< 24$ CPU hours (Mathematica).
- Build symbol level solution using first entry condition.

# Amplitudes



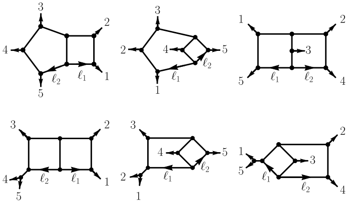
# Integrands and Double Copy

- Known *d*-dimensional integrand for $\mathcal{N} = 4$.
  [Carrasco, Johansson, '11]

- Express in trace basis.

- $A^{\mathrm{SLST}}$ expressible in terms of $A^{\mathrm{ST}}$ and $A^{\mathrm{DT}}$.
  [Edison, Naculich '12]

- Gravity integrand built through double copy.



$$\mathcal{A}_5^{(2)} = \sum_{S_5/(S_3\times Z_2)} \frac{\mathrm{tr}_5[15](\mathrm{tr}[234]-\mathrm{tr}[432])}{N_c}\mathcal{A}^{\mathrm{DT}}[15|234]$$

$$+\sum_{S_5/D_5}(\mathrm{tr}[12345]-\mathrm{tr}[54321])\left(\mathcal{A}^{\mathrm{ST}}[12345]+\frac{\mathcal{A}^{\mathrm{SLST}}[12345]}{N_c^2}\right).$$

$$\mathrm{Gravity} \sim \mathrm{YM}^2.$$

[Bern, Carrasco, Johansson, '08]

# Amplitude Level Ansatz

[Abreu, Dixon, Herrmann, B.P., Zeng '18]

- UT combination of rational factors and polylogarithms.

$$A = \sum_i r_i(\vec{p}) G_i(\vec{p})$$

- Fix $r_i(\vec{p})$ with Ansatz + numerical IBP reduction.

$$G_k \sim \{\log(x), \mathrm{Li}_2(x), \}$$

- Linear dependence of evaluations counts $m$. (useful in QCD, [Vasily's talk])

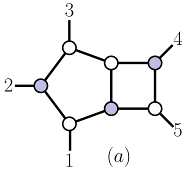
$$r_k(\vec{p}) = \sum_{j=1}^{m} c_{i,j} f_j(\vec{p})$$

- What's a good basis of the rational factors, $f_{ij}$?

$$S[A] = \sum_i r_i(\vec{p}) S[G_i(\vec{p})]$$

# The Hunt for Rational Factors [Abreu, Dixon, Herrmann, B.P., Zeng '18]

- Idea - Use leading singularities of amplitudes. [Cachazo '08]
- Maximal codimension residues of integrand.
- Coefficients from IBPs are combinations of leading singularities. [Kosower, Larsen '11]
- Parke-Taylor for $\mathcal{N} = 4$. [Arkani-Hamed et al '14]
- Need $d$-dimensional leading singularity for $\mathcal{N} = 8$.




$$\mathrm{LS}_{\mathrm{SYM}} = \left\{ \frac{\delta^{(8)}(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + 5 \text{ perms} \right\}$$

$$\mathrm{LS}_{\mathrm{SUGRA}} = \left\{ \frac{[12][34][45][51]}{\langle 12\rangle\langle 13\rangle\langle 24\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle} + 39 \text{ perms}, \frac{s_{12}[12][23][34][45][51]}{\mathrm{tr}_5\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + 4 \text{ perms} \right\}$$

$$\mathrm{LS}_{SUGRA} = \mathrm{LS}_{SYM}^2 \mathcal{J}$$

# 2-Loop 5-Point Amplitude for $\mathcal{N} = 4$ SYM

[Abreu, Dixon, Herrmann, B.P, Zeng '18]

See also [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, '18]

- Need only six numerical evaluations for new $g^{\mathrm{DT}}$.

$$\mathcal{A}_{\mathrm{ST}}[12345] = \mathrm{PT}[12345]\ \mathrm{M}^{\mathrm{BDS}}_{(2)}\,, \quad \mathcal{A}_{\mathrm{DT}}[15|234] = \sum_{\sigma(234)\in S_3} \mathrm{PT}[1\sigma_2\sigma_3\sigma_4 5]\ g^{\mathrm{DT}}_{\sigma_2\sigma_3\sigma_4}\,,$$

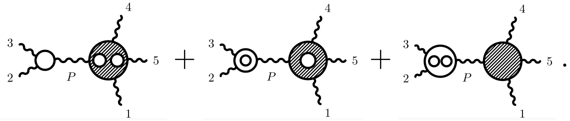
- First two-loop five-point amplitude beyond the planar limit.
- Uniform transcendental result - weight 4.
- 20 unexpected relations, for example 10 permutations of

$$\begin{aligned} &g[12345] + g[12453] + g[12534] + g[21345] + g[21453] + g[21534] \\ -&g[12435] - g[12543] - g[12354] - g[21435] - g[21543] - g[21354] = 0. \end{aligned}$$

# Checks

- Leading colour matches BDS ansatz. [Bern, Dixon, Smirnov, 2005]
- Matches IR poles. [Catani, 1998; Bern, Dixon, Kosower, 2004; Aybat, Dixon, Sterman, 2006]
- Matches collinear limits. [Bern, Dixon, Kosower '04]

$\mathcal{A}_5^{(2)} \xrightarrow{2||3}$




- Matches soft limits. [Dixon, Zhu, et al., in progress]

# The 5-Point Amplitude for $\mathcal{N} = 8$ SYM

See also [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, '19]

[Abreu, Dixon, Herrmann, B.P, Zeng '19]

- UT, as suggested by integrand singularities.
  [Bourjaily, Herrmann, Trnka '18]
- Subtract soft divergences to construct remainder.
- Don't exponentiate $\mathcal{A}^{(1)}$.
- $d$-dimensional LS drop out in remainder $R_5^{(2)}$.
- $R_5^{(2)}$ is symmetrization of single function $h$.

$$R_5^{(2)} \equiv \mathcal{A}_5^{(2)} - \Big(\sum_{i<j} s_{ij} \log s_{ij}\Big)\left[\frac{1}{2\epsilon^2}\Big(\sum_{i<j} s_{ij} \log s_{ij}\Big)\mathcal{A}_5^{(0)} + \frac{1}{\epsilon}\mathcal{A}_5^{(1),0} + \mathcal{A}_5^{(1),1}\right].$$

$$r_0 = \frac{[23][34][45][51]}{\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 24\rangle\langle 25\rangle}.$$

$$R_5^{(2)} = \sum_{\sigma \in S_5/Z_2} r_0(\sigma)\, h(\sigma).$$

# Conclusions

- We computed the two-loop, five-point amplitudes in $\mathcal{N} = 4/8$.
- Differential equation construction greatly simplified by numerical ansatz approach and unitarity compatible IBPs.
- Amplitude basis structures built from leading singularities.
- First step towards non-planar QCD corrections.