QFT with FDR

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Outline

1. *Introduction*
2. *Renormalizable theories*

   B. Page, R.P. Eur.Phys.J. C79 (2019) no.4, 361
3. *Effective theories*

   R.P., arXiv:1902.01767

UV subtraction at the integrand level

A suitable linear integral operator $\int [d^4q]$ can be defined:

$$I^1_{\rm FDR} = \int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2} = -i\pi^2 \ln \frac{M^2}{\mu_R^2}$$

$\Uparrow$ FDR integral

$\Uparrow$ Renormalization scale

Regularization and Renormalization at once

- It can be generalized to more loops, $I^\ell_{\rm FDR} = \sum_{k=0}^{\ell} c_k \ln^k(\mu_R^2)$
- μ_R cannot depend on kinematics

Two QFT core tenets respected by FDR

1. Gauge invariance $(\bar{q}^2 := q^2 - \mu^2)$

    - FDR integrals are invariant under the shift $q \to q + p \ \forall p$
    - Cancellations between numerators and propagators

$$\int [d^4q] \frac{\bar{q}^2}{\bar{q}^2(\bar{q}^2 - M^2)^2} = \int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2}$$

$\Rightarrow$ One can prove graphical WI in QFT

2. Unitarity of $S = I + iT$

$\Rightarrow$ $i(T - T^\dagger) = -T^\dagger T$ can be enforced

The fate of the renormalization scale

Given $\mathcal{L}(p_1, \ldots, p_m)$ and

$$\tilde{p}_i(\mu_R) := p_i^{\mathrm{TH},\ell-\mathrm{loop}}(\mathcal{O}_1^{\mathrm{EXP}}, \ldots, \mathcal{O}_m^{\mathrm{EXP}}, \mu_R)$$

1. Renormalizable Lagrangians:

$$\frac{d\mathcal{O}^{\mathrm{TH},\ell-\mathrm{loop}}\left(\tilde{p}_1(\mu_R), \ldots, \tilde{p}_m(\mu_R), \mu_R\right)}{d\mu_R} = 0$$

No $\mathcal{L} = \mathcal{L}_R + \Delta\mathcal{L}_{Couterterms}$

2. Nonrenormalizable (effective) Lagrangians:

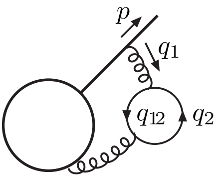
$$\frac{d\mathcal{O}^{\mathrm{TH},\ell-\mathrm{loop}}\left(\tilde{p}_1(\mu_R), \ldots, \tilde{p}_m(\mu_R), \mu_R\right)}{d\mu_R} \neq 0$$

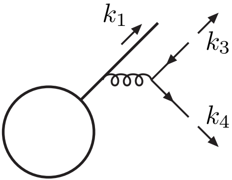
No $\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}_{High.\ Dim.\ Operators}$

In both cases $\mathcal{L}$ **untouched**

Renormalizable QFTs

NNLO final-state quark-pair corrections

$A^{(2)}_{n,\mathrm{IR}} =$ 

$A^{(0)}_{n+2,\mathrm{IR}} =$ 

$$\sigma_B \propto \int d\Phi_n \sum_{\mathrm{spin}} |A_n^{(0)}|^2$$

$$\sigma_V \propto \int d\Phi_n \sum_{\mathrm{spin}} \left\{ A_n^{(2)} (A_n^{(0)})^* + A_n^{(0)} (A_n^{(2)})^* \right\}$$

$$\sigma_R \propto \int d\Phi_{n+2} \sum_{\mathrm{spin}} \left\{ A_{n+2}^{(0)} (A_{n+2}^{(0)})^* \right\}$$

$$\boxed{\sigma^{NNLO} = \sigma_B + \sigma_V + \sigma_R}$$

Known IR finite results reproduced

$$\boxed{H \to b\bar{b} + jets}$$

$$\Gamma^{\rm NNLO}(y_b) = \Gamma_2^{(0)}(y_b)\left\{1 + a^2 C_F N_F \left(2\ln^2\frac{m^2}{s} - \frac{26}{3}\ln\frac{m^2}{s} + 8\zeta_3 + 2\pi^2 - \frac{62}{3}\right)\right\}$$

$$\boxed{\gamma^* \to jets}$$

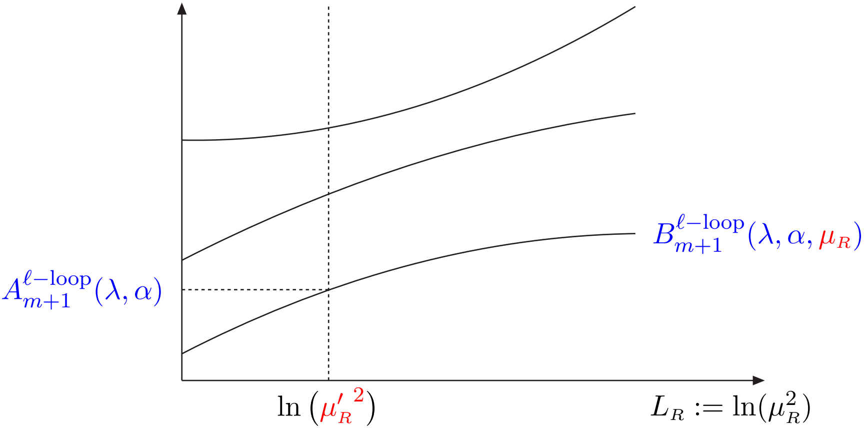
$$\sigma^{\rm NNLO}_{e^+e^- \to jets} = \sigma_2^{(0)}\left\{1 + a^2 C_F N_F\,(8\zeta_3 - 11)\right\}$$

$$a = \alpha_S/4\pi$$

Effective QFTs

Matching μ_R in effective QFTs

$$B_{m+1}(0,\alpha,\mu_R) = A_{m+1}(0,\alpha) \qquad \lambda_n = s_n/M_n^2 \quad (n = 1 \div N)$$



Under which conditions this is possible?

The all-order exact and effective amplitudes

$$A_{m+1}(\lambda,\alpha) = K(\alpha) + K(\alpha)\sum_{j=1}^{\infty} A_{0j}^{\{m_j\}}\lambda^{\{m_j\}} + K(\alpha)\sum_{i,j=1}^{\infty} A_{ij}^{\{m_j\}}\alpha^i\lambda^{\{m_j\}}$$

$$B_{m+1}(\lambda,\alpha,\mu_R) = K(\alpha) + K(\alpha)\sum_{\substack{i,j=1\\0\le k\le i}}^{\infty} B_{ijk}^{\{m_j\}}\alpha^i\lambda^{\{m_j\}}L_R^k$$

- $K(\alpha) = B_{m+1}(0,\alpha,\mu_R) = A_{m+1}(0,\alpha)$
- Sum over *assignments* $\{m_j\}$ understood, e.g. if $j=2$

  $$A_{02}^{\{m_2\}}\lambda^{\{m_2\}} = A_{02}^{(2,0)}\lambda_1^2 + A_{02}^{(0,2)}\lambda_2^2 + A_{02}^{(1,1)}\lambda_1\lambda_2 \quad (N=2)$$
- B_{m+1} depends on λ only through loops, unlike A_{m+1}

The all-order exact and effective amplitudes

$$A_{m+1}(\lambda,\alpha) = K(\alpha) + K(\alpha) \boxed{\sum_{j=1}^{\infty} A_{0j}^{\{m_j\}} \lambda^{\{m_j\}}} + K(\alpha) \sum_{i,j=1}^{\infty} A_{ij}^{\{m_j\}} \alpha^i \lambda^{\{m_j\}}$$

$$B_{m+1}(\lambda,\alpha,\mu_R) = K(\alpha) + K(\alpha) \sum_{\substack{i,j=1 \\ 0\le k\le i}}^{\infty} B_{ijk}^{\{m_j\}} \alpha^i \lambda^{\{m_j\}} L_R^k$$

- $K(\alpha) = B_{m+1}(0,\alpha,\mu_R) = A_{m+1}(0,\alpha)$
- Sum over *assignments* $\{m_j\}$ understood, e.g. if $j=2$

$$A_{02}^{\{m_2\}} \lambda^{\{m_2\}} = A_{02}^{(2,0)} \lambda_1^2 + A_{02}^{(0,2)} \lambda_2^2 + A_{02}^{(1,1)} \lambda_1 \lambda_2 \quad (N=2)$$

- B_{m+1} depends on λ only through loops, unlike A_{m+1}

A perturbative solution at order ℓ

$$A_{m+1}(\lambda,\alpha) = K(\alpha) + K(\alpha)\sum_{j=1}^{\infty} A_{0j}^{\{m_j\}}\lambda^{\{m_j\}} + K(\alpha)\sum_{i,j=1}^{\infty} A_{ij}^{\{m_j\}}\alpha^i\lambda^{\{m_j\}}$$

$$B_{m+1}(\lambda,\alpha,\mu_R) = K(\alpha) + K(\alpha)\sum_{\substack{i,j=1\\0\le k\le i}}^{\infty} B_{ijk}^{\{m_j\}}\alpha^i\lambda^{\{m_j\}}L_R^k$$

- The matching $\quad B_{m+1}^{\ell-\text{loop}}(\lambda,\alpha,\mu_R) = A_{m+1}^{\ell-\text{loop}}(\lambda,\alpha)$

  is obtained by setting $\quad L_R = \sum_{i=\boxed{-1}}^{\ell-1} X_i\alpha^i$

  with $i,j \le \ell$ in A_{m+1} and $i \le (\ell+1)$, $j \le \ell$ in B_{m+1}

- Matching compatible with FDR

only when kinematics independent solutions X_i' for the X_i exist

The $4N$ conditions for the $\ell = 1$ matching

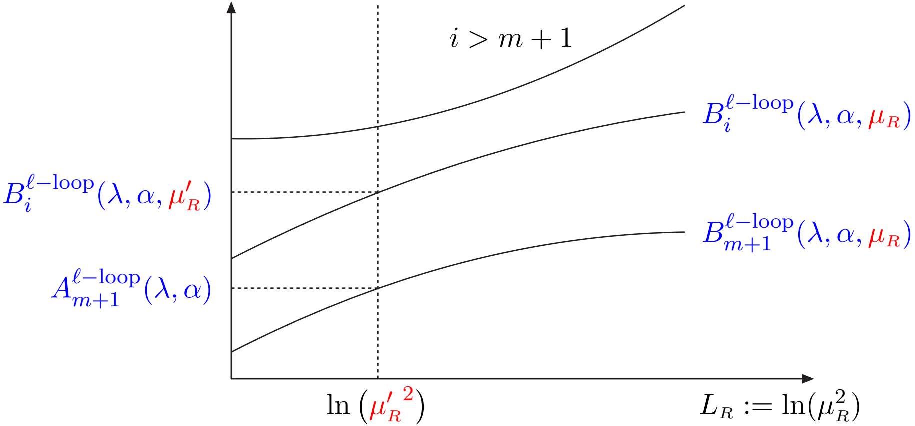
- $$\begin{cases} A_{01}^{\{m_1\}} - B_{111}^{\{m_1\}} X'_{-1} - B_{212}^{\{m_1\}} (X'_{-1})^2 = 0 \\ A_{11}^{\{m_1\}} - B_{110}^{\{m_1\}} - B_{111}^{\{m_1\}} X'_0 - B_{211}^{\{m_1\}} X'_{-1} - 2B_{212}^{\{m_1\}} X'_{-1} X'_0 = 0 \end{cases} \quad \forall \{m_1\}$$

- $$\frac{\partial X'_{-1}}{\partial s_n} = \frac{\partial X'_0}{\partial s_n} = 0$$

If they are all obeyed the constant solution is

$$\ln\left({\mu'_R}^2\right) = \frac{X'_{-1}}{\alpha} + X'_0$$

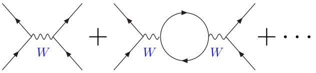
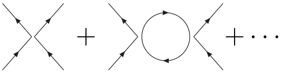
Predictivity of the effective QFTs



- Is $B_i^{\ell-\text{loop}}(\lambda, \alpha, \mu'_R) = A_i^{\ell-\text{loop}}(\lambda, \alpha)$?
- **Conjecture:** yes if $B_i^{\ell-\text{loop}}(0, \alpha, \mu'_R) = A_i^{\ell-\text{loop}}(0, \alpha)$

Matching high-energy e.w. fermion loops onto $\mathcal{L}^{\text{FERMI}}$

$$\mathcal{L}^{\text{EFF}}_{\text{INT}} = \mathcal{L}^{\text{QED}}_{\text{INT}} + \mathcal{L}^{\text{FERMI}} \qquad \boxed{\mathcal{L}^{\text{FERMI}} = -\frac{g^2}{8M^2} J^{\dagger}_{c\alpha} J^{\alpha}_{c} - \frac{g^2}{8M^2} J_{n\alpha} J^{\alpha}_{n}}$$

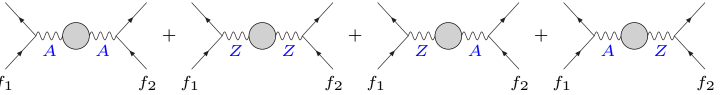
$A_{m+1,c}(\lambda, \alpha_{EM})$	$B_{m+1,c}(\lambda, \alpha_{EM}, \mu_R)$
	

$$\boxed{\ln\left({\mu'_R}^2\right) = \frac{\pi \hat{s}^2_\theta}{\alpha_{EM}} + K_1}$$ at **any** ℓ and **any** $\lambda = p^2/\hat{M}^2$

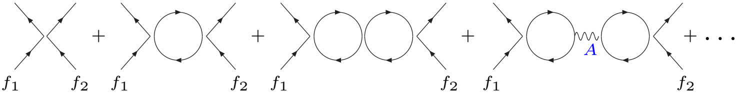
$$K_1 := \frac{1}{2} + \frac{\ln m_e^2 + \ln m_\mu^2 + \ln m_\tau^2}{12} + \frac{\ln m_u^2 + \ln m_c^2 + \ln m_t^2}{6} + \frac{\ln m_d^2 + \ln m_s^2 + \ln m_b^2}{12}$$

Independent $A_{i,n}(\lambda, \alpha_{EM})$ and $B_{i,n}(\lambda, \alpha_{EM}, \mu_R'^2)$

$A_{i,n} =$



$B_{i,n} =$



- By construction $B_{i,n}(\mathbf{0}, \alpha_{EM}, \mu_R'^2) = A_{i,n}(\mathbf{0}, \alpha_{EM})$
- $\boxed{B_{i,n}(\boldsymbol{\lambda}, \alpha_{EM}, \mu_R'^2) = A_{i,n}(\boldsymbol{\lambda}, \alpha_{EM})}$

  $\forall$ massless $f_{1,2}$ at **any** ℓ and **any** $\lambda = p^2/\hat{M}^2$
- Our conjecture is verified!

s^2_θ, g^2, M^2 at the solution

Bare parameters are μ_R dependent finite quantities in FDR

When $\mu_R \to \mu'_R$

- $s^2_\theta \to 0$ (lepton universality)
- $\frac{1}{g^2} \to 0$ (Landau pole of $g^2(\mu_R)$)
- $\frac{1}{M^2} \to 0$ (Landau pole of $M^2(\mu_R)$)

Fixed at all loop orders in terms of G_F, $\alpha_{EM}(0)$, $R_{e\nu}(0)$

Conclusion

1. **FDR** can be used to compute higher-order corrections in renormalizable QFTs
2. Under certain circumstances, loop corrections computed in high-energy renormalizable QFTs can be matched onto low-energy nonrenormalizable $\mathcal{L}_{\text{NR}}$ **without** modifying $\mathcal{L}_{\text{NR}}$
3. This is possible **only** if UV infinities are handled *à la* **FDR**
4. For instance, $\mathcal{L}_{\text{NR}} = \mathcal{L}^{\text{QED}}_{\text{INT}} + \mathcal{L}^{\text{FERMI}}$ can be used without modifications to reproduce the exact electroweak interactions between two massless fermion lines induced by one-fermion-loop resummed gauge boson propagators
5. This is the **first-ever** example of $\mathcal{L}_{\text{NR}}$ consistently made predictive at all loop orders and energies without replacing $\mathcal{L}_{\text{NR}} \to \mathcal{L}_{\text{NR}} + \sum_i C_i O_i$

Backup slides

"Vacuum" subtraction

1. $J(q^2) = \dfrac{1}{(q^2 - M^2)^2}$
2. $q^2 \stackrel{\mathrm{GP}}{\to} \bar{q}^2 := q^2 - \mu^2$
3. $J(q^2) \stackrel{\mathrm{GP}}{\to} \bar{J}(\bar{q}^2) := \dfrac{1}{(\bar{q}^2 - M^2)^2}$

$$\frac{1}{(\bar{q}^2 - M^2)^2} = \left[\frac{1}{\bar{q}^4}\right] + \left(\frac{M^2}{\bar{q}^2(\bar{q}^2 - M^2)^2} + \frac{M^2}{\bar{q}^4(\bar{q}^2 - M^2)}\right)$$

$\Uparrow$

Vacuum

"Vacuum" subtraction

1. $J(q^2) = \dfrac{1}{(q^2 - M^2)^2}$
2. $q^2 \stackrel{\text{GP}}{\to} \bar{q}^2 := q^2 - \mu^2$
3. $J(q^2) \stackrel{\text{GP}}{\to} \bar{J}(\bar{q}^2) := \dfrac{1}{(\bar{q}^2 - M^2)^2}$

$$\frac{1}{(\bar{q}^2 - M^2)^2} = \cancel{\left[\frac{1}{\bar{q}^4}\right]} + \left(\frac{M^2}{\bar{q}^2(\bar{q}^2 - M^2)^2} + \frac{M^2}{\bar{q}^4(\bar{q}^2 - M^2)}\right)$$

$\Uparrow$

Vacuum

$$\int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2} := \lim_{\mu \to 0} \int d^4q \left(\frac{M^2}{\bar{q}^2(\bar{q}^2 - M^2)^2} + \frac{M^2}{\bar{q}^4(\bar{q}^2 - M^2)}\right)$$

and $\mu = \mu_R$ after the asymptotic $\mu \to 0$ limit

Examples of two-loop vacua

- Global vacua $(q_{12} := q_1 + q_2)$:

$$\left[\frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_{12}^2}\right], \quad \left[\frac{1}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2}\right], \quad \left[\frac{1}{\bar{q}_1^4}\right]\left[\frac{1}{\bar{q}_2^4}\right]$$

- Sub-vacua:

$$\frac{M^4}{(\bar{q}_1^2 - M^2)\bar{q}_1^4}\left[\frac{1}{\bar{q}_2^2}\right], \quad \frac{M^4}{(\bar{q}_1^2 - M^2)^2\bar{q}_1^2}\left[\frac{1}{\bar{q}_2^4}\right]$$

Examples of two-loop vacua

- Global vacua $(q_{12} := q_1 + q_2)$:

$$\xcancel{\left[\frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_{12}^2}\right]}, \quad \xcancel{\left[\frac{1}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2}\right]}, \quad \xcancel{\left[\frac{1}{\bar{q}_1^4}\right]\left[\frac{1}{\bar{q}_2^4}\right]}$$

- Sub-vacua:

$$\xcancel{\frac{M^4}{(\bar{q}_1^2 - M^2)\bar{q}_1^4}\left[\frac{1}{\bar{q}_2^2}\right]}, \quad \xcancel{\frac{M^4}{(\bar{q}_1^2 - M^2)^2\bar{q}_1^2}\left[\frac{1}{\bar{q}_2^4}\right]}$$

A two-loop example

$$(\bar{D}_i := \bar{q}_i^2 - m_i^2)$$

$$
\begin{aligned}
\frac{1}{\bar{D}_1\bar{D}_2\bar{D}_{12}} &= \left[\frac{1}{\bar{q}_1^2\bar{q}_2^2\bar{q}_{12}^2}\right] \\
&+ m_1^2\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_{12}^2}\right] + \frac{m_1^4}{(\bar{D}_1\bar{q}_1^4)}\left[\frac{1}{\bar{q}_2^4}\right] - m_1^4\frac{q_1^2 + 2(q_1\cdot q_2)}{(\bar{D}_1\bar{q}_1^4)\bar{q}_2^4\bar{q}_{12}^2} \\
&+ m_2^2\left[\frac{1}{\bar{q}_1^2\bar{q}_2^4\bar{q}_{12}^2}\right] + \frac{m_2^4}{(\bar{D}_2\bar{q}_2^4)}\left[\frac{1}{\bar{q}_1^4}\right] - m_2^4\frac{q_2^2 + 2(q_1\cdot q_2)}{\bar{q}_1^4(\bar{D}_2\bar{q}_2^4)\bar{q}_{12}^2} \\
&+ m_{12}^2\left[\frac{1}{\bar{q}_1^2\bar{q}_2^2\bar{q}_{12}^4}\right] + \frac{m_{12}^4}{(\bar{D}_{12}\bar{q}_{12}^4)}\left[\frac{1}{\bar{q}_1^4}\right] - m_{12}^4\frac{q_{12}^2 - 2(q_1\cdot q_{12})}{\bar{q}_1^4\bar{q}_2^2(\bar{D}_{12}\bar{q}_{12}^4)} \\
&+ \frac{m_1^2m_2^2}{(\bar{D}_1\bar{q}_1^2)(\bar{D}_2\bar{q}_2^2)\bar{q}_{12}^2} + \frac{m_1^2m_{12}^2}{(\bar{D}_1\bar{q}_1^2)\bar{q}_2^2(\bar{D}_{12}\bar{q}_{12}^2)} + \frac{m_2^2m_{12}^2}{\bar{q}_1^2(\bar{D}_2\bar{q}_2^2)(\bar{D}_{12}\bar{q}_{12}^2)} \\
&+ \frac{m_1^2m_2^2m_{12}^2}{(\bar{D}_1\bar{q}_1^2)(\bar{D}_2\bar{q}_2^2)(\bar{D}_{12}\bar{q}_{12}^2)}
\end{aligned}
$$

Three-loop logarithmic vacua

$$\left[\frac{1}{\bar{q}_1^2\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{13}^2((q_2-q_3)^2-\mu^2)}\right]$$

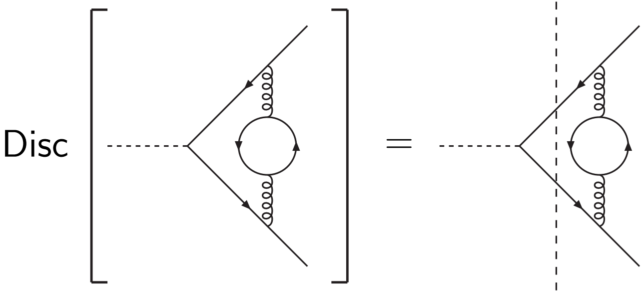
$$\left[\frac{1}{\bar{q}_1^2\bar{q}_3^2\bar{q}_2^4\bar{q}_{12}^2\bar{q}_{23}^2}\right]$$

$$\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{123}^2}\right]$$

$$\left[\frac{1}{\bar{q}_1^4\bar{q}_2^4\bar{q}_3^2\bar{q}_{123}^2}\right]$$

$$\left[\frac{1}{\bar{q}_1^6\bar{q}_2^2\bar{q}_3^2\bar{q}_{123}^2}\right]$$

Unitarity



Sub-Integration Consistency (SIC)

In any multi-loop Feynman diagram the divergent sub-diagrams must be treated consistently with the lower loop calculations

- Off-shell, B. Page, R.P. JHEP 1511 (2015) 183
- On-shell, B. Page, R.P. Eur.Phys.J. C79 (2019) no.4, 361