The four-loop slope of the Dirac form factor

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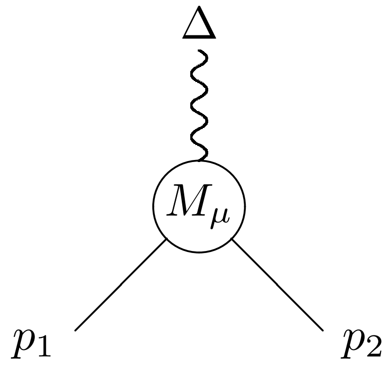
RADCOR 2019
Avignon
9-13 Sep 2019

- Form factors F_1 and F_2: the slope
- The slope at 1, 2, 3 loops
- The slope at 4 loops: numerical value
- The slope at 4 loops: analytical fit
- The slope at 4 loops: useful for comparison with experiments?
- Calculations (issues)



electron-photon vertex

$$M_\mu = F_1(t)\gamma_\mu + F_2(t)\frac{\sigma_{\mu\nu}}{2m}\Delta_\nu \qquad p_1^2 = p_2^2 = -m^2 \quad t = -\Delta^2$$

Dirac Pauli

$F_1(0) \equiv 1$ charge conservation

$F_2(0) = \dfrac{g-2}{2}$ anomalous magnetic moment

$F_1(t) = 1 + F_1'(0)t + O(t^2) + \ldots$ $F_1'(0) \neq 0$ $\rightarrow$ Slope

- $F_2(0) = (g-2)/2$ anomalous magnetic moment
- $F_1'(0) \rightarrow$ contribute to the energy level shift in the hydrogen atom (Lamb shift) (together with many other radiative corrections)

Perturbative expansion

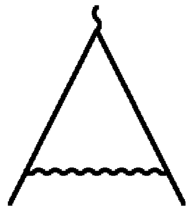
$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

r-loop contribution to the shift of the energy level nS

$$\Delta f(n\mathrm{S}, r-loop) = \frac{4(Z\alpha)^4 mc^2/h}{n^3}\left(\frac{m_r}{m}\right)^3\left[A_r\left(\frac{\alpha}{\pi}\right)^r\right] \qquad (r > 1)$$

No contribution from $F_1'(0)$ to the shift of states with $l > 0$

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$



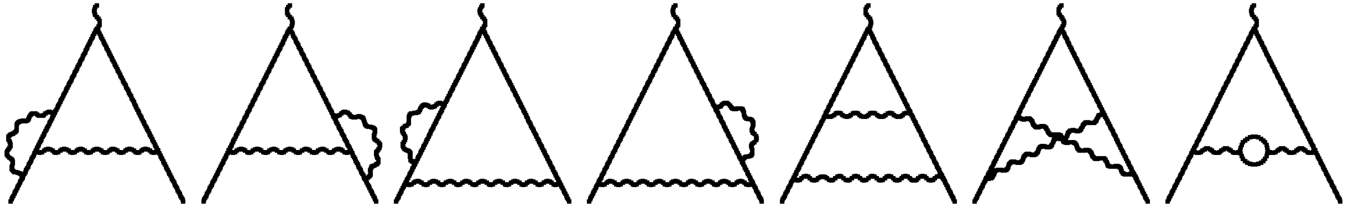
1 diagram

$$A_1 = -\frac{1}{8} + \frac{1}{6\epsilon} \qquad \text{in dimensional regularization}$$

The infrared divergence due to the mass-shell condition of the electron taking into account off mass-shell effects in the expression of ΔE A_1 must be replaced with

$$A_1 \to A_1' = -\tfrac{1}{3}\ln\tfrac{\Delta\epsilon}{m} - \tfrac{1}{8} + \tfrac{5}{18} \qquad (\Delta\epsilon = \text{Bethe energy})$$

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$



7 diagrams

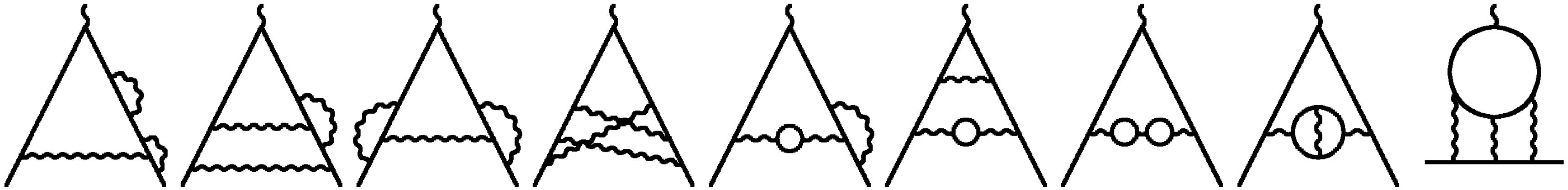
$$\begin{aligned} A_2 = &\ -\frac{4819}{5184} - \frac{49}{432}\pi^2 + \frac{1}{2}\pi^2 \ln 2 - \frac{3}{4}\zeta(3) \\ = &\ 0.469\,941\,487\,459\,992\ldots \end{aligned}$$

The two-loop coefficient was computed analytically by Barbieri, Mignaco and Remiddi in 1970.

2-loop g-2 coefficient:

$$\begin{aligned} C_2 = &\ \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) \\ = &\ -0.328\,478\,965\,579\,\ldots \end{aligned}$$

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$



72 diagrams

$$A_3 = -\frac{17}{24}\pi^2\zeta(3) + \frac{25}{8}\zeta(5) - \frac{217}{9}\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{103}{1080}\pi^2\ln^2 2 + \frac{3899}{25920}\pi^4 - \frac{2929}{288}\zeta(3) + \frac{41671}{360}\pi^2\ln 2 - \frac{454979}{38880}\pi^2 - \frac{77513}{186624}$$
$$= 0.171\ 720\ 018\ 909\ 775\ldots$$

Calculated analytically by Melnikov and Ritbergen in 1999

3-loop g-2 coefficient:

$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2\ln^2 2}{24}\right] - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$
$$= 1.181\ 241\ 456\ 587\ 200\ 006\ \ldots$$

loop	$F_1'(0)$	$F_2(0)$
1	∞	0.5
2	0.469941487459	−0.328478965579
3	0.171720018909	1.181241456587
4	??????????????	−1.912245764926
5		6.737(159)

positive?negative? alternating signs

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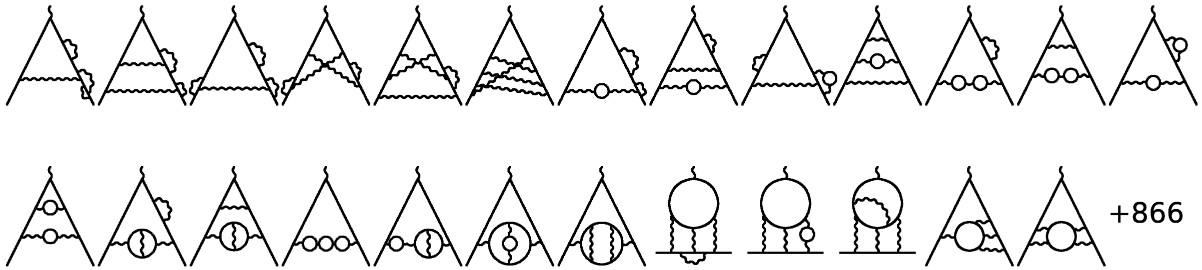
5		6.737(159)

	positive	alternating signs

only 12 digits?

The slope at 4 loops: 1100 digits

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$



891 diagrams

$A_4 =$

0.88654567394644314583682173061031535939042403266006474536805590932084031646562892745483648632417733686
93512758747218307996875923974888466826147611753011917584831446774752672980326917402719214651539325519844
79310049501962453137211937294671608006342998095842536958494506068383665985141387321894210012394882759515
38237865372203883496448560075689857616877564102719779603910290276615122356406105399227905150277608224
59236950433275703613350935251764763992516822679359645249285456658218441028674547644077579921118603788315
35011980067778515074780212674247904052222473302950218310742901990299162768291602289058991164264634498789
87630727082848364358743478002455415372434008969514716831155386425591883520934780665126748875033459025
99182245563613125124119880615415537621337112284846277684867421928289686568115480303537276007873036210930
59264752959892234017835732828971749623991833527848841324243696992642213640320068440006124235298158339663
32566753158241741448217616597381276692161976675095050740649309561361958988024564511635456757162309441
73884811565020098334847940590188785421700667378220853053541953188378610075518116 3...

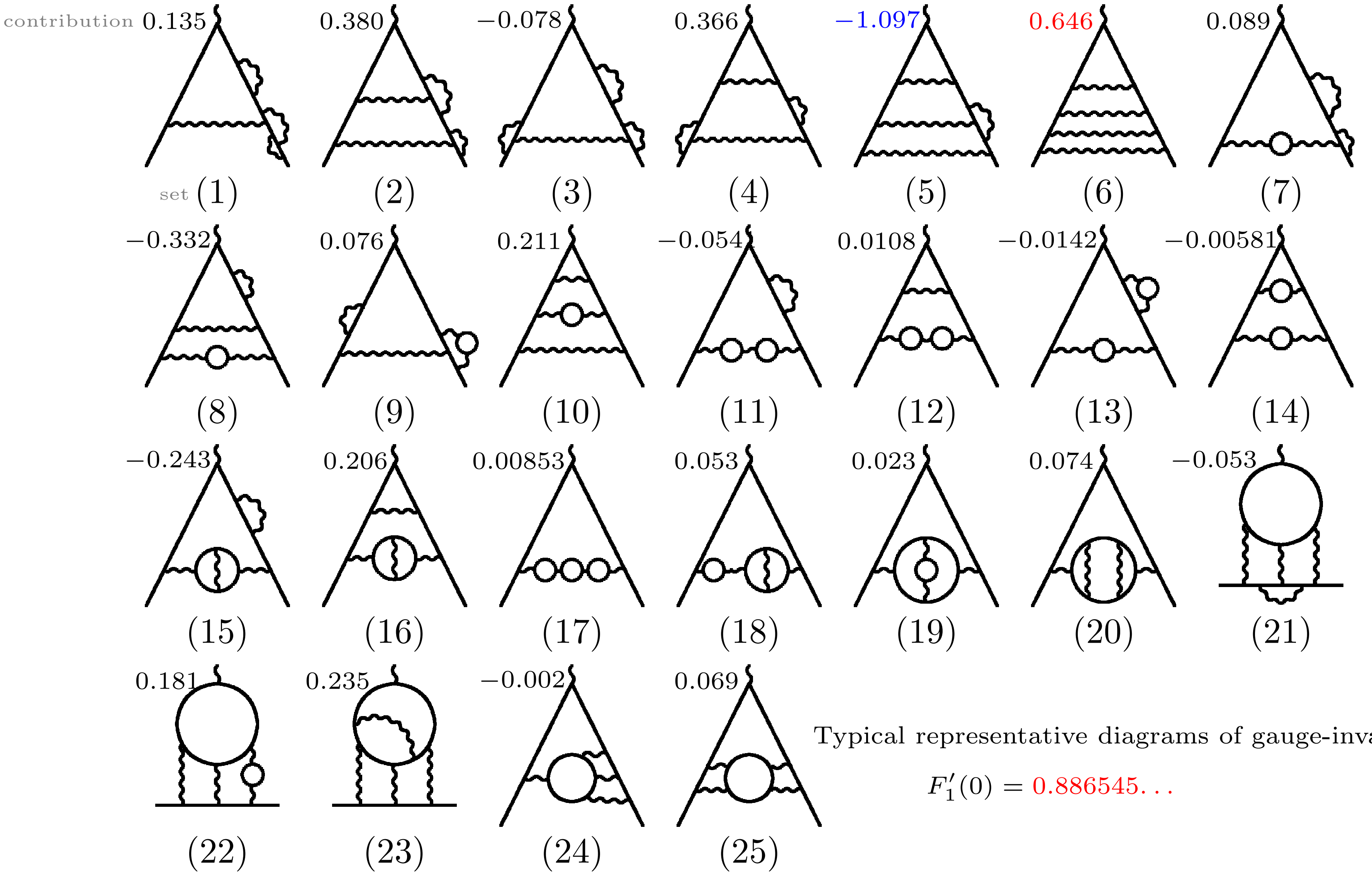
(S.L. 2019)

- High precision needed to fit analytical ansatz by using the PSLQ algorithm.

4-loop g-2 coefficient $C_4 =$

-1.912245764926445574152647167439830054060873390658725345171329848006038443980651706142760892 7...

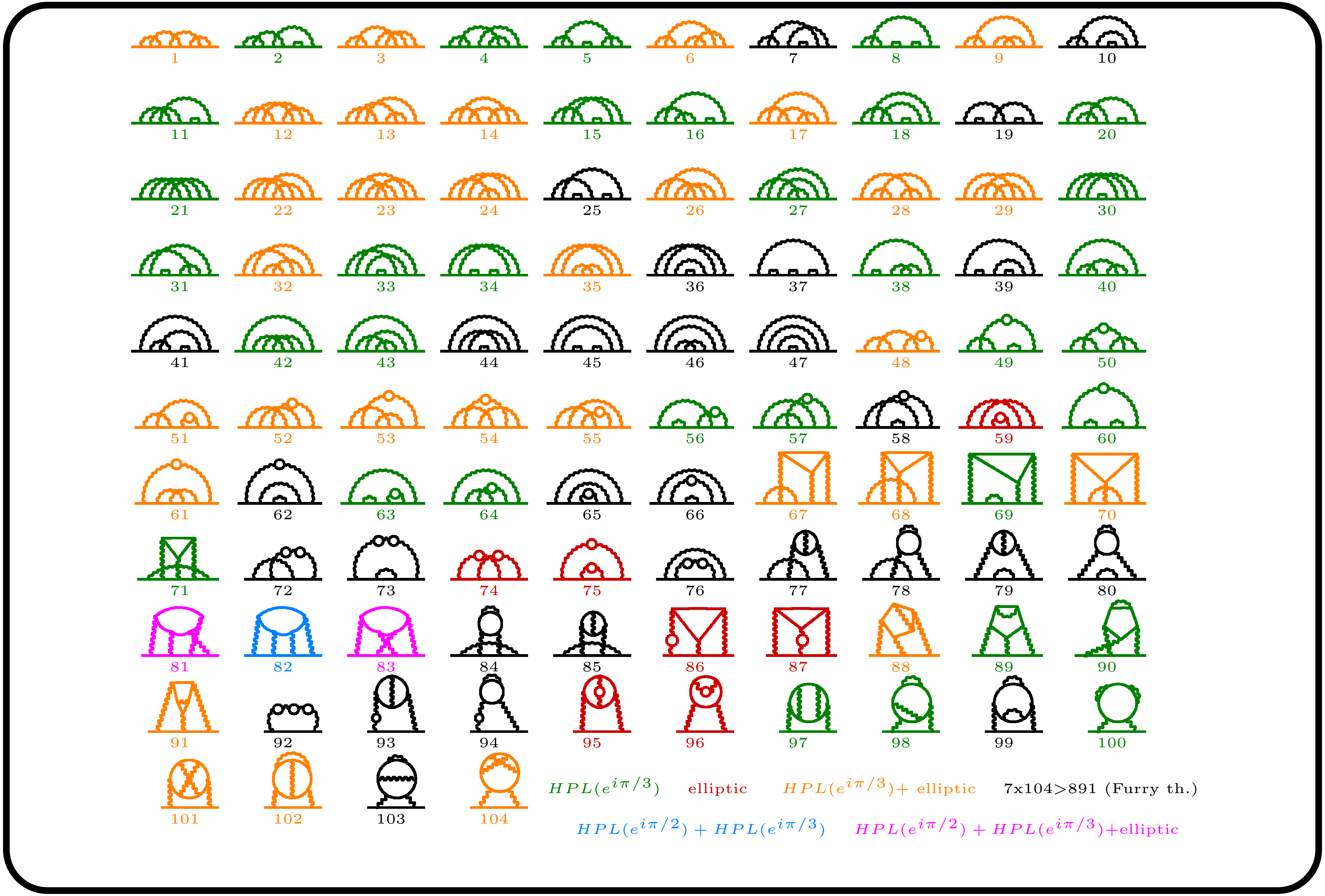
-1.02570009098000242831582543682 9...$= A_4 + C_4$?



Typical representative diagrams of gauge-invariant sets

$$F_1'(0) = 0.886545\ldots$$

A coloured view of the 891 diagrams arranged in insertions into 104 self-mass diagrams



The slope at 4 loops: analytical fit

- 891 vertex diagrams (from 104 self-mass diagrams)
- 334 master integrals (the same of 4-loop g-2)
- M.I. calculated numerically with precision 1100-9600 digits
- M.I. fitted analytically
- Once the slope is reduced to M.I., one uses the values/fits already known.
- Unsurprisingly, the structure is very similar to that of 4-loop g-2, *with one piece more*

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

- T: part containing "normal" trascendentals
- V: parts containing (Im, Re) HPL of $\exp(i\pi/3)$
- W: parts containing (Im, Re) HPL of $\exp(i\pi/2)$
- E: part containing elliptic constants written as one-dim integrals
- U: part containing unknown elliptic constants

$$
\begin{aligned}
A_4 =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) + \frac{4572662443}{12247200}\ln^2 2\,\zeta(2) - \frac{1449791143}{3061800}\Big(a_4 + \frac{1}{24}\ln^4 2\Big) + \frac{90355973}{134400}\zeta(5) \\
&+ \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 - \frac{68168}{135}\Big(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\Big) - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 + \frac{18215}{27}a_5\ln 2 \\
&- \frac{18215}{27}\zeta(5)\ln 2 + \frac{402152509}{189000}a_4\zeta(2) + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 - \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 - \frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7) \\
&- \frac{1267114025}{387072}\zeta(4)\zeta(3) - \frac{427145}{504}a_4\zeta(3) - \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) + \frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 + \frac{607282}{189}a_6\ln 2 \\
&- \frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 + \frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 + \frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2 \\
&+ \frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2 + \sqrt{3}\Big[-\frac{14186171}{194400}\mathrm{Cl}_4\Big(\frac{\pi}{3}\Big) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,-1}\Big(e^{i\frac{\pi}{3}}\Big) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,1}\Big(e^{i\frac{2\pi}{3}}\Big) \\
&+ \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,1,-1}\Big(e^{i\frac{2\pi}{3}}\Big) + \frac{458299}{36855}\mathrm{Im}H_{0,0,1,0,1,1}\Big(e^{i\frac{2\pi}{3}}\Big) + \frac{10540877}{442260}\mathrm{Im}H_{0,0,0,1,1,1}\Big(e^{i\frac{2\pi}{3}}\Big) + \frac{178619489}{3980340}\mathrm{Cl}_6\Big(\frac{\pi}{3}\Big) + \frac{1833196}{45927}a_4\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) - \frac{12563350487}{2579260320}\zeta(5)\pi \\
&+ \frac{533401067}{459270}\zeta(4)\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) + \frac{844343}{18900}\mathrm{Im}H_{0,1,1,-1}\Big(e^{i\frac{2\pi}{3}}\Big)\zeta(2) + \frac{844343}{28350}\mathrm{Im}H_{0,1,1,-1}\Big(e^{i\frac{\pi}{3}}\Big)\zeta(2) + \frac{458299}{21870}\zeta(3)\mathrm{Im}H_{0,1,-1}\Big(e^{i\frac{\pi}{3}}\Big) + \frac{458299}{14580}\zeta(3)\mathrm{Im}H_{0,1,1}\Big(e^{i\frac{2\pi}{3}}\Big) \\
&- \frac{263673944}{295245}\mathrm{Cl}_4\Big(\frac{\pi}{3}\Big)\zeta(2) - \frac{39924629}{6889050}\zeta(3)\zeta(2)\pi + \frac{844343}{1224720}\zeta(4)\pi\ln 2 - \frac{844343}{11340}\mathrm{Im}H_{0,1,-1}\Big(e^{i\frac{\pi}{3}}\Big)\zeta(2)\ln 2 - \frac{844343}{7560}\mathrm{Im}H_{0,1,1}\Big(e^{i\frac{2\pi}{3}}\Big)\zeta(2)\ln 2 + \frac{458299}{275562}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\ln^4 2 \\
&+ \frac{19130869}{367416}\zeta(2)\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\ln^2 2\Big] + \frac{212671}{2400}\Big(\mathrm{Re}H_{0,0,0,1,0,1}\Big(e^{i\frac{\pi}{3}}\Big) + \mathrm{Cl}_4\Big(\frac{\pi}{3}\Big)\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\Big) - \frac{1031987}{14400}\mathrm{Cl}_2^2\Big(\frac{\pi}{3}\Big)\zeta(2) - \frac{507}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\Big(e^{i\frac{\pi}{3}}\Big) \\
&- 507\mathrm{Re}H_{0,0,0,0,1,1,-1}\Big(e^{i\frac{\pi}{3}}\Big) + \frac{13689}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\Big(e^{i\frac{2\pi}{3}}\Big) + \frac{68445}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\Big(e^{i\frac{2\pi}{3}}\Big) + \frac{13689}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\Big(e^{i\frac{2\pi}{3}}\Big) - \frac{507}{4}\mathrm{Cl}_4\Big(\frac{\pi}{3}\Big)\mathrm{Im}H_{0,1,-1}\Big(e^{i\frac{\pi}{3}}\Big) \\
&- \frac{1521}{8}\mathrm{Cl}_4\Big(\frac{\pi}{3}\Big)\mathrm{Im}H_{0,1,1}\Big(e^{i\frac{2\pi}{3}}\Big) - \frac{24505}{176}\mathrm{Cl}_6\Big(\frac{\pi}{3}\Big)\pi - \frac{295}{4}\mathrm{Re}H_{0,1,0,1,-1}\Big(e^{i\frac{\pi}{3}}\Big)\zeta(2) - \frac{295}{2}\mathrm{Re}H_{0,0,1,1,-1}\Big(e^{i\frac{\pi}{3}}\Big)\zeta(2) - \frac{2655}{16}\mathrm{Re}H_{0,1,0,1,1}\Big(e^{i\frac{2\pi}{3}}\Big)\zeta(2) \\
&- \frac{2655}{8}\mathrm{Re}H_{0,0,1,1,1}\Big(e^{i\frac{2\pi}{3}}\Big)\zeta(2) - \frac{295}{4}\mathrm{Im}H_{0,1,-1}\Big(e^{i\frac{\pi}{3}}\Big)\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\zeta(2) - \frac{885}{8}\mathrm{Im}H_{0,1,1}\Big(e^{i\frac{2\pi}{3}}\Big)\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\zeta(2) - \frac{1117}{36}\zeta(2)\mathrm{Cl}_2\Big(\frac{\pi}{2}\Big) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\Big(\frac{\pi}{2}\Big) \\
&- 118\Big(4\mathrm{Re}H_{0,1,0,1,1}\Big(e^{i\frac{\pi}{2}}\Big)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\Big(e^{i\frac{\pi}{2}}\Big)\mathrm{Cl}_2\Big(\frac{\pi}{2}\Big)\zeta(2) - 2\mathrm{Cl}_4\Big(\frac{\pi}{2}\Big)\zeta(2)\pi + \mathrm{Cl}_2^2\Big(\frac{\pi}{2}\Big)\zeta(2)\ln 2\Big) + \sqrt{3}\Big[\pi\Big(+\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\Big) \\
&- \frac{11495611}{3265920}\pi f_2(0,0,1) + \pi\Big(\frac{751}{972}\ln 2\, f_2(0,0,1) - \frac{365478661}{24494400}f_2(0,2,0) + \frac{119022487}{5443200}f_2(0,1,1) - \frac{119022487}{14515200}f_2(0,0,2)\Big) - \frac{751}{729}\zeta(2)f_1(0,0,1) \\
&+ \pi\Big(-\frac{1735283}{497664}\zeta(2)f_2(0,0,1) + \frac{1105}{108}\ln 2\, f_2(0,0,2) - \frac{2210}{81}\ln 2\, f_2(0,1,1) + \frac{4420}{243}\ln 2\, f_2(0,2,0) - \frac{1104271}{497664}f_2(0,0,3) + \frac{272833}{41472}f_2(0,1,2) - \frac{4011005}{497664}f_2(0,2,1) \\
&+ \frac{8417635}{2239488}f_2(0,3,0) + \frac{157753}{248832}f_2(1,0,2) + \frac{354323}{248832}f_2(1,1,1) - \frac{298711}{124416}f_2(1,2,0) - \frac{157753}{497664}f_2(2,0,1) - \frac{98285}{248832}f_2(2,1,0)\Big)\Big] + \zeta(2)\Big(-\frac{4629335}{165888}f_1(0,0,2) \\
&+ \frac{112357}{1536}f_1(0,1,1) - \frac{99731}{1944}f_1(0,2,0) + \frac{157753}{41472}f_1(1,0,1)\Big) + \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c}
\end{aligned}
$$

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U \qquad \text{122 terms}$$

$$\begin{aligned}
T =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) \\
&+ \frac{4572662443}{12247200}\zeta(2)\ln^2 2 - \frac{1449791143}{3061800}\left(a_4 + \frac{1}{24}\ln^4 2\right) + \frac{90355973}{134400}\zeta(5) + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 \\
&- \frac{68168}{135}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 \\
&+ \frac{402152509}{189000}a_4\zeta(2) + \frac{18215}{27}a_5\ln 2 - \frac{18215}{27}\zeta(5)\ln 2 + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 \\
&- \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 - \frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7) - \frac{1267114025}{387072}\zeta(4)\zeta(3) - \frac{427145}{504}a_4\zeta(3) \\
&- \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) + \frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 + \frac{607282}{189}a_6\ln 2 \\
&- \frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 + \frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 \\
&+ \frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2 + \frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2
\end{aligned}$$

$a_n = \mathrm{Li}_n(1/2)$, $b_6 = H_{0,0,0,0,1,1}(1/2)$, $b_7 = H_{0,0,0,0,0,1,1}(1/2)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
T =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) \\
&+ \frac{4572662443}{12247200}\zeta(2)\ln^2 2 - \frac{1449791143}{3061800}t_4 + \frac{90355973}{134400}\zeta(5) + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 \\
&- \frac{68168}{135}t_5 - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 \\
&+ \frac{402152509}{189000}t_4\zeta(2) - \frac{18215}{27}t_{61} + \frac{26062}{27}t_{62} - \frac{7224951103}{1741824}\zeta(7) - \frac{1267114025}{387072}\zeta(4)\zeta(3) \\
&- \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{971827}{128}\zeta(6)\ln 2 - \frac{6242389}{6048}\zeta(3)\zeta(2)\ln^2 2 - \frac{427145}{504}t_4\zeta(3) + \frac{1420289}{180}t_5\zeta(2) \\
&+ \frac{256321}{756}t_{71} - \frac{116987}{63}t_{72} + \frac{104041}{20}t_{73}
\end{aligned}$$

$a_n = \mathrm{Li_n}(1/2)$, $b_6 = H_{0,0,0,0,1,1}(1/2)$, $b_7 = H_{0,0,0,0,0,1,1}(1/2)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$

Decomposition in constants valid for each diagram contribution to $F_1'(0)$ & $F_2(0)$

$$t_4 = a_4 + \frac{1}{24}\ln^4 2 \qquad t_5 = a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2 \qquad t_{62} = a_6 - \frac{1}{48}\zeta(2)\ln^4 2 + \frac{1}{720}\ln^6 2$$

$$t_{61} = b_6 - a_5\ln 2 + \zeta(5)\ln 2 + \frac{1}{6}\zeta(3)\ln^3 2 - \frac{1}{12}\zeta(2)\ln^4 2 + \frac{1}{144}\ln^6 2$$

$$t_{71} = d_7 - 2b_6\ln 2 + 4a_6\ln 2 + 2a_5\ln^2 2 - \frac{49}{32}\zeta^2(3)\ln 2 - \frac{95}{32}\zeta(5)\ln^2 2 + \frac{1}{8}\zeta(4)\ln^3 2 - \frac{1}{3}\zeta(3)\ln^4 2 + \frac{1}{12}\zeta(2)\ln^5 2 - \frac{\ln^7 2}{120}$$

$$t_{72} = b_7 - 3a_7 - a_6\ln 2 - \frac{1}{2}\zeta(5)\ln^2 2 + \frac{1}{48}\zeta(4)\ln^3 2 - \frac{1}{24}\zeta(3)\ln^4 2 + \frac{1}{120}\zeta(2)\ln^5 2 - \frac{\ln^7 2}{1680}$$

$$t_{73} = \left(a_4 - \frac{1}{4}\zeta(2)\ln^2 2 + \frac{7}{16}\zeta(3)\ln 2 + \frac{1}{24}\ln^4 2\right)\zeta(2)\ln 2$$

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$V_a = -\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}v_{61} + \frac{844343}{28350}v_{62} + \frac{178619489}{3980340}v_{63} - \frac{263673944}{295245}v_{64}$$

$$\mathrm{Cl}_n(\theta) = \mathrm{Im}\mathrm{Li}_n(e^{i\theta})$$

$$v_{61} = \mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{27}{26}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{207}{104}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10}{3}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{7}{4}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{21}{8}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{5}{72}\zeta(3)\zeta(2)\pi - \frac{5}{6}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{5}{36}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2$$

$$v_{62} = \zeta(2)\Big[\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{1}{6}\zeta(3)\pi + \frac{1}{108}\zeta(2)\pi\ln 2 - \frac{5}{2}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\ln 2 - \frac{15}{4}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\ln 2 + \frac{25}{12}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 - \frac{661}{1188}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\Big]$$

$$v_{63} = \mathrm{Cl}_6\left(\frac{\pi}{3}\right) - \frac{3}{4}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$$

$$v_{64} = \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{91}{66}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$$

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$V_b = \frac{212671}{2400} v_{65} - \frac{1031987}{14400}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) - \frac{507}{4} v_{71} - \frac{295}{4} v_{72}$$

$$v_{65} = \mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right),$$

$$v_{71} = \mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 4\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{27}{8}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{135}{16}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$- \frac{27}{2}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{145}{132}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi$$

$$v_{72} = \zeta(2)\Bigg[\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 2\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{9}{4}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$+ \frac{9}{2}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\Bigg]$$

The slope at 4 loops: analytical fit part 3

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$W_a = -\frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)$$

New

$$W_b = +\frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) - 472\upsilon_{73}$$

$$\upsilon_{73} = \zeta(2)\left(\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right) + \mathrm{Cl}_2\left(\frac{\pi}{2}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right) - \frac{1}{2}\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\pi + \frac{1}{4}\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\ln 2\right)$$

- $\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)$ appears in $F_1'(0)$, cancels out in $F_2(0)$

$\mathrm{Cl}_2\left(\frac{\pi}{2}\right)$ Catalan's constant $\beta_2 = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$

The slope at 4 loops: analytical fit part 4

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$E_a = \pi\left(\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) - \frac{11495611}{3265920}\pi f_2(0,0,1) - \frac{365478661}{24494400}e_{61} + \frac{119022487}{5443200}e_{62} - \frac{98285}{248832}e_{71} - \frac{157753}{497664}e_{72}$$

$$E_b = -\frac{751}{729}\zeta(2)f_1(0,0,1) + \frac{157753}{41472}e_{73} - \frac{99731}{1944}e_{74}$$

$$e_{71} = \pi\Big(f_2(2,1,0) + \frac{7}{3}f_2(1,2,0) - 2f_2(1,1,1) + \frac{40}{27}f_2(0,3,0) - \frac{7}{3}f_2(0,2,1) + f_2(0,1,2) - 30\ln 2 f_2(0,2,0) + 45\ln 2 f_2(0,1,1) - \frac{135}{8}\ln 2 f_2(0,0,2)\Big)$$

$$e_{72} = \pi\Big(f_2(2,0,1) + \frac{14}{3}f_2(1,2,0) - 2f_2(1,1,1) - 2f_2(1,0,2) - \frac{370}{27}f_2(0,3,0) + \frac{85}{3}f_2(0,2,1) - 22f_2(0,1,2) + 7f_2(0,0,3) + 11\zeta(2)f_2(0,0,1) - 20\ln 2 f_2(0,2,0) + 30\ln 2 f_2(0,1,1) - \frac{45}{4}\ln 2 f_2(0,0,2)\Big)$$

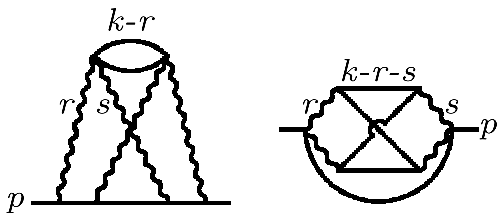
$$e_{73} = \zeta(2)\left(f_1(1,0,1) - f_1(0,1,1) + \frac{1}{4}f_1(0,0,2)\right)$$

$$e_{74} = \zeta(2)\left(f_1(0,2,0) - \frac{3}{2}f_1(0,1,1) + \frac{9}{16}f_1(0,0,2)\right)$$

other combinations: $e_{73} - \frac{e_{72}}{4\sqrt{3}}$

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$U = \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c}$$



(f, f', f'') (g, g', g'')

C_{8x} are the ϵ^0 coefficients of the $\epsilon-$expansion of the six master integrals (f, f', f'') and (g, g', g''), and have numerators respectively equal to $(1, p.k, (p.k)^2)$

- number of terms is 122 (one more that g-2).
- introducing 19 combinations of constant the number of terms becomes 57
- the basis with 57 term is valid for the decomposition of the contributions of each diagram to $F_1'(0)$ and $F_2(0)$.
- shorter basis useful for direct PSLQ fits of the numerical contribution of the diagrams
- Strong numerical cancellations, for example the term $-\frac{2749470791}{387072}\zeta(2)\zeta(5) = -12115.862$

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$

$$\Delta f(n\mathrm{S}, \text{4-loop } F_1'(0)) = \frac{4(Z\alpha)^4 mc^2/h}{n^3}\left(\frac{m_r}{m}\right)^3 \left[A_4\left(\frac{\alpha}{\pi}\right)^4\right] \approx \frac{36.11}{n^3}\mathrm{Hz}$$

Experimental result

$$f(1\mathrm{S}-2\mathrm{S}) = 2466\,061\,413\,187\,018 \pm 11 \text{ Hz} \qquad 4.4\times 10^{-15} \qquad \text{Matveev 2013}$$

- $\Delta f(1S) - \Delta f(2S) = 31.6$Hz comparable with the experimental error (11Hz)
- there are many other four-loop corrections to the shift
- some two- and three-loop corrections have still theoretical errors in the range $100 - 1000$Hz

The method used is the same used for 4-loop g-2:
extraction of contribution $\rightarrow$ algebraic reduction to master integrals $\rightarrow$ numerical calculation of master integrals

- The 891 vertex diagrams are the same of 4-loop g-2
- Therefore the master integrals are the same; I used the numerical values already calculated with 1100-9600 digits of precision.
- The big difference is the algebraic reduction to M.I.: the derivative generates Feynman integrals with one power more in the denominator and two power more in the numerator $\rightarrow$ huge sizes
- For g-2 each I.B.P. system contained tipically 5×10^6 identities with a size up to 100GB. Total size 3TB.
- For $F_1'(0)$ each I.B.P. system contains tipically 500×10^6 identities with a size up to 1TB. Total size 30TB.
- The number of identities of each system is similar to what I would expect in a calculation of 5-loop g-2.

Calculation: issues

- For the calculation I used my program `SYS`.
- Calculation was done mainly on clusters of the University of Zurich, on big-memory machines of the ITP of Zurich, with memory sizes from 384GB to 768GB (thank to Thomas Gehrmann for the use of these computers). Some smaller parts were done on the CloudVeneto infrastructure.
- A disk crash (in Zurich) destroyed completely 10TB of data (6 months of work). The disk system was working in RAID6 mode, able to survive to the simultaneous loss of two disks. *Two* broke at the same time, and a third one broke while the RAID was rebuilding.
- Parallelization of code with threads and MPI was heavily used.
- I did not detect calculation errors due to the hardware
- I *did* detect errors due to some (nasty!) software bug in the public library OpenMPI used for parallelization with full-thread support. Results of processes were misassigned one every few weeks.
- This kind of errors was found by discovering that identities supposed to be trivially zero (70% of the total) became monsters with millions of integrals.

- The 4-loop slope was calculated for the first time.
- The value is positive, and of the order of 1
- It allows to determine the size of the first $\alpha^4(Z\alpha)^4$ contribution to level shift.
- This shift is of the same order of magnitude of the experimental precision of measurements of 1S-2S frequency.
- The analytical structure is almost identical to the 4-loop g-2 (1 term more).
- The experience obtained is useful for the preparation of a calculation of the 5-loop g-2

The End