The two-loop five-gluon all-plus helicity amplitude

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S. Badger, D.C., T. Gehrmann, G. Heinrich, J.M. Henn, T. Peraro, P. Wasser, Y. Zhang, S. Zoia

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Towards an era of precision collider measurements

- Ever improving experimental precision at the LHC

- For many QCD processes Next-to-Leading Order approximation is insufficient, e.g. strong coupling from 3-jet/2-jet ratio:

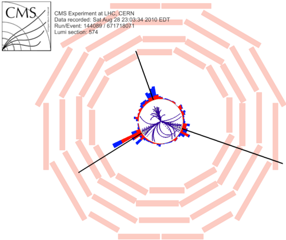
$$\alpha_s(M_Z) = 0.1148 \pm \underset{\text{(exp)}}{0.0014} \pm \underset{\text{(PDF)}}{0.0018} \pm \boxed{\underset{\text{(theory)}}{0.0050}}$$

[CMS collaboration, Eur.Phys.J C73 (2013)]

Large theoretical uncertainty!

- NNLO predictions are required to fully exploit the LHC data

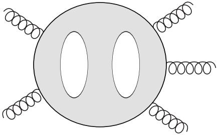
Multi-jet processes at NNLO



process	known	desired
$pp \to 2\,\text{jets}$	$\text{N}^2\text{LO}_{\text{QCD}}$ $\text{NLO}_{\text{QCD}}+\text{NLO}_{\text{EW}}$	
$pp \to 3\,\text{jets}$	NLO_{QCD}	$\text{N}^2\text{LO}_{\text{QCD}}$

Table I.2: Precision wish list: jet final states.

- State of the art: two-to-two processes at NNLO
- Multi-jet processes are important for phenomenology:
  - ◇ α_s determination
  - ◇ tests of Standard Model
  - ◇ search for new physics
- Three jets: double virtual corrections (two-loop five-particle amplitudes) are major bottleneck



from "Les Houches 2017. Standard Model Working Group Report"

Recent progress in calculation of the two-loop five-particle amplitudes

- **All QCD amplitudes in the planar limit are known analytically** [Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov '19] [Abreu, Dormans, Febres Cordero, Ita, Page '18] Previous numerical [Badger, Brønnum-Hansen, Hartanto, Peraro '17][Abreu, Cordero, Ita, Page, Zeng '17] [Abreu, Cordero, Ita, Page, Sotnikov '18][Badger, Brønnum-Hansen, Gehrmann, Hartanto, Henn, Lo Presti, Peraro '18] and analytical results [Gehrmann, Henn, Lo Presti '15][Dunbar, Perkins '16] [Badger, Brønnum-Hansen, Hartanto, Peraro '18] in the planar approximation [talk by Sotnikov]

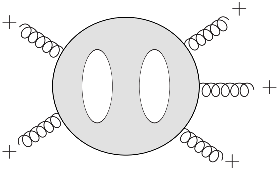
- **Full-color $\mathcal{N} = 4$ super-Yang-Mills and $\mathcal{N} = 8$ supergravity amplitudes** (at symbol level) [D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '18 '19][Abreu, Dixon, Herrmann, Page, Zeng '18 '19] [talk by Page]

- **Full-color five-gluon all-plus helicity amplitude** [Badger, D.C., Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia '19] [this talk]

$\Longrightarrow$ Very first complete analytic two-loop five-particle amplitude!

Towards all full-color two-loop five-parton QCD amplitudes

The all-plus helicity amplitude is extremely simple – a one-line formula

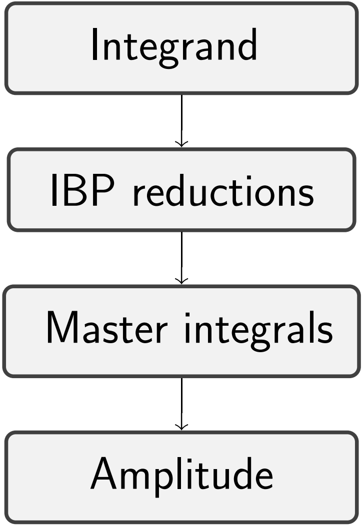


Other helicity configurations are more complicated, but

- The developed tools are indispensable for all QCD amplitudes

- We calculated all two-loop master integrals for massless five-particle scattering
  - ✓ analytic results
  - ✓ high-precision numerics
  - ✓ physical scattering region

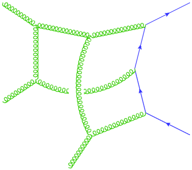
Amplitude calculation workflow

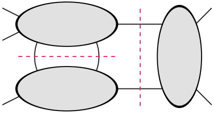


Efficient methods to construct integrands

Integrand → IBP reductions → Master integrals → Amplitude

- Feynman diagrams
- Unitarity [Bern, Dixon, Dunbar, Kosower '94], Generalized unitarity [Bern, Dixon, Kosower '98][Britto, Cachazo, Feng '05][Ossola, Papadopoulos, Pittau '06], Numerical unitarity [Ita '15][Abreu, Febres, Cordero, Ita, Jaquier, Page, Zeng '17]





Integrand of the full-color two-loop five-point all-plus amplitude [Badger, Mogull, Ochirov, O'Connell '15] contains numerators of degree five/six

Dramatic improvement of the Integration-By-Parts reduction due to finite-field arithmetics

Integrand → **IBP reductions** → Master integrals → Amplitude

Feynman integrals are not all independent. IBP-reduction to a finite number of **master integrals** $\Longrightarrow$ Heavy computational problem

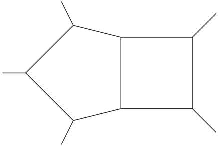
$$\mathcal{I}(s,\epsilon) = \sum_i c_i(s,\epsilon)\, g_i(s,\epsilon) \quad , \qquad D = 4 - 2\epsilon$$

master integrals

Finite fields and rational reconstruction significantly improve IBP reduction algorithms [von Manteuffel, Schabinger '15][Peraro '16 '19][Maierhoefer, Usovitsch '18][Smirnov, Chukharev '19]

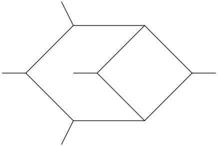
[talk by Peraro]

The master integral families for massless two-loop five-particle scattering



[Gehrmann, Henn, Lo Presti '15, '18]

[Papadopoulos, Tommasini, Wever '15]

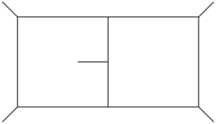


[D.C., Mitev, Henn '17]

[Boehm, Georgoudis, Larsen, Schoenemann, Zhang '18]

[Abreu, Dixon, Herrmann, Page, Zeng '18]

[D.C., Gehrmann, Henn, Lo Presti, Mitev, Wasser '18]



[Abreu, Dixon, Herrmann, Page, Zeng '18]

[D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '18]

All master integrals evaluate to pentagon functions

Kinematics of five-particle scattering

Massless particles: $p_i^2 = 0$

Mandelstam invariants:

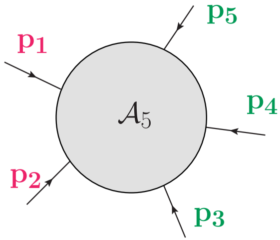
$$s_{ij} = (p_i + p_j)^2$$

Five independent:

$$s_{12},\ s_{23},\ s_{34},\ s_{45},\ s_{15}$$

One pseudo-scalar:

$$\epsilon_5 \equiv i\epsilon_{\mu_1\mu_2\mu_3\mu_4}\, p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_4^{\mu_4}$$



Physical scattering region **12** $\rightarrow$ **345**

$$s_{12}, s_{34}, s_{45}, s_{35} > 0$$

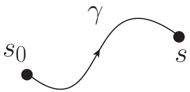
$$s_{13}, s_{14}, s_{15}, s_{23}, s_{24}, s_{25} < 0$$

$$(\epsilon_5)^2 < 0$$

Pentagon functions

- Proposed in [D.C., Mitev, Henn '17]
- Confirmed in [Abreu, Dixon, Herrmann, Page, Zeng '18] [D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '18]

Iterated integrals along path γ

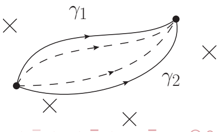


$$\int_\gamma d\log W_{i_1}(s)\,\ldots\,d\log W_{i_n}(s)$$

$\{W_i(s)\}_{i=1}^{31}$ – functions of energies and scattering angles

n – transcendental weight

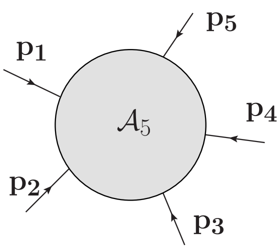
Homotopy invariance $\gamma_1 \sim \gamma_2 \;\Rightarrow\; \mathrm{I}(\gamma_1) = \mathrm{I}(\gamma_2)$



Pentagon alphabet

31-letter alphabet $\mathbb{A} = \left\{ W_j(s) \right\}_{j=1}^{31}$

W_1	$2\, p_1 \cdot p_2$	$+(4)$
W_6	$2\, p_4 \cdot (p_3 + p_5)$	$+(4)$
W_{11}	$2\, p_3 \cdot (p_4 + p_5)$	$+(4)$
W_{16}	$2\, p_1 \cdot p_3$	$+(4)$
W_{21}	$2\, p_3 \cdot (p_1 + p_4)$	$+(4)$
W_{26}	$\frac{\mathrm{tr}[(1-\gamma_5)\not{p}_1\not{p}_2\not{p}_4\not{p}_5]}{\mathrm{tr}[(1+\gamma_5)\not{p}_1\not{p}_2\not{p}_4\not{p}_5]}$	$+(4)$
W_{31}	ϵ_5	



- The alphabet splits into orbits of $\mathbb{Z}_5$
- Invariance under $\mathcal{S}_5$
- 26 parity-even and 5 parity-odd letters
- Zero loci of letters: branch points of the master integrals

Iterated integrals in terms of familiar functions

- One-fold integrals: Logarithms, e.g.

$$\log(s_{12}),\ \log(-s_{23}),\ldots$$

- Two-fold integrals: Dilogarithms, e.g.

$$s_{ij} \equiv (p_i + p_j)^2$$

$$\mathrm{Li}_2\left(1-\frac{s_{34}}{s_{12}}\right),\ \log(-s_{13})\log(s_{34}),\ \ldots$$

- Multi-fold integrals: Goncharov polylogarithms
  - ◇ Well-studied functions in math and HEP
  - ◇ Fast numerical routines (`GiNaC`) [Vollinga, Weinzierl '04]
- Planar sector: well-studied and fast numerical implementation [Gehrmann, Henn, Lo Presti '18]

pentagonfunctions.hepforge.org

Master integrals from differential equations

Change of master integral basis $\vec{f}$ enormoulsy simplifies DE [Henn '13]

$$d\vec{f}(s,\epsilon) = \epsilon\, d\widetilde{A}(s)\, \vec{f}(s,\epsilon)$$

$$d\widetilde{A}(s) = \sum_{i=1}^{31} a_i\, d\log W_i(s)$$

letters of the alphabet

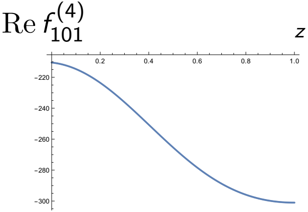
rational matrices

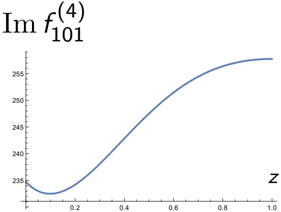
Solution has uniform transcendentality

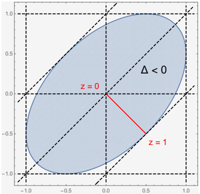
$$\vec{f}(s,\epsilon) = \mathrm{Pexp}\left(\epsilon \int_\gamma d\tilde{A}(s)\right)\vec{f}(s_0,\epsilon) \implies$$ Pentagon functions

- ✓ Construction of the canonical basis:
  - ⋄ Algorithm to find 4D dlog integrals [Wasser '16]
  - ⋄ D-dimensional leading singularities based on the Baikov parametrization
- ✓ Absence of spurious singularities $\Longrightarrow$ boundary constants $\vec{f}(s_0,\epsilon)$

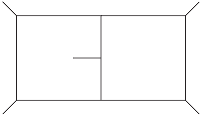
From analytic formulae to numeric values







Evaluate $f_{101}^{(4)}$ along the path in kinematic space

$f_{101} =$ 

$$= \sum_{w \geq 0} \frac{1}{\epsilon^{4-w}} f_{101}^{(w)}$$

$$s_{13} = 3$$

$$s_{23} = -1 + \frac{z}{1+z^2}$$

$$s_{34} = 1$$

$$s_{45} = 1$$

$$s_{15} = -1 - \frac{z}{1+z^2}\,, \quad 0 < z < 1$$

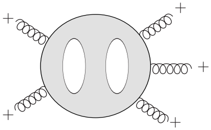
✓ High-precision evaluation (GiNaC)

✓ Checks using SecDec

Assembly of the amplitude

Integrand → IBP reductions → Master integrals → **Amplitude**

The amplitude is much simpler than the ingredients!



$= \text{Color} \otimes \epsilon \otimes$ Rational factors $\otimes$ Pentagon functions

Naive assembly of the amplitude is impossible because of the size and complexity of the ingredients. Additional steps are needed:

- IR-subtraction $\Longrightarrow$ hard function
- Basis of rational factors
- Rational reconstruction

Factorization of the Infrared divergences

$$\mathcal{A}(\epsilon) = \mathcal{Z}(\epsilon) \cdot \mathcal{A}^f(\epsilon)$$

Matrix in color space, captures all ϵ-poles

finite at $\epsilon \to 0$

Hard function is finite

$$\mathcal{H} = \lim_{\epsilon \to 0} \mathcal{A}^f(\epsilon)$$

- Simpler than the amplitude
- Truly new piece of information
- Relevant for cross sections

New result: non-planar two-loop hard function

$$\mathcal{H}^{(2)}_{\substack{\text{double}\\ \text{trace}}} = \sum_{\mathcal{S}_5/\Sigma} \mathrm{Tr}(12)\left[\mathrm{Tr}(345) - \mathrm{Tr}(543)\right] \sum_{\Sigma} \Bigg\{ 6\kappa^2 \left[\frac{\langle 24\rangle[14][23]}{\langle 12\rangle\langle 23\rangle\langle 45\rangle^2} + 9\,\frac{\langle 24\rangle[12][23]}{\langle 12\rangle\langle 34\rangle\langle 45\rangle^2} \right] + \kappa \frac{[15]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} \left[I_{234;15} + I_{243;15} - I_{324;15} - 4I_{345;12} - 4I_{354;12} - 4I_{435;12} \right] \Bigg\}$$

Finite part of the one-mass box function:

$$I_{123;45} = \mathrm{Li}_2\left(1 - \frac{s_{12}}{s_{45}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{23}}{s_{45}}\right) + \log^2\left(\frac{s_{12}}{s_{23}}\right) + \frac{\pi^2}{6}$$

Spinor-helicity variables: $\langle ij\rangle = \sqrt{s_{ij}}\, e^{i\varphi_{ij}}$ and $[ij] = \sqrt{s_{ij}}\, e^{-i\varphi_{ij}}$

Gluon spin dimension: $\kappa \equiv \frac{g^{\mu}{}_{\mu} - 2}{6}$

- Weight-1,3,4 iterated integrals canceled out
- Analytic continuation to other regions is straightforward $s_{ij} \to s_{ij} + i0$
- Correct factorization in the collinear limits

Summary

- First **analytic** result for full-color five-particle two-loop amplitude
  - ✓ Functional level
  - ✓ Nonplanar
- All master integrals for 3-jet production at NNLO are known **analytically**
- Pentagon functions describe all massless five-particle two-loop amplitudes