The two-loop five-gluon all-plus helicity amplitude

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Towards an era of precision collider measurements

- Ever improving experimental precision at the LHC

- For many QCD processes Next-to-Leading Order approximation is insufficient, e.g. strong coupling from 3-jet/2-jet ratio:

  \[ \alpha_s(M_Z) = 0.1148 \pm 0.0014 \pm 0.0018 \pm 0.0050 \]

  (exp) (PDF) (theory)


  Large theoretical uncertainty!

- NNLO predictions are required to fully exploit the LHC data
Multi-jet processes at NNLO

- State of the art: two-to-two processes at NNLO

- Multi-jet processes are important for phenomenology:
  - $\alpha_s$ determination
  - tests of Standard Model
  - search for new physics

- Three jets: double virtual corrections (two-loop five-particle amplitudes) are major bottleneck

<table>
<thead>
<tr>
<th>process</th>
<th>known</th>
<th>desired</th>
</tr>
</thead>
<tbody>
<tr>
<td>$pp \to 2$ jets</td>
<td>$N^2LO_{QCD}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$NLO_{QCD}+NLO_{EW}$</td>
<td></td>
</tr>
<tr>
<td>$pp \to 3$ jets</td>
<td>$NLO_{QCD}$</td>
<td>$N^2LO_{QCD}$</td>
</tr>
</tbody>
</table>

Table I.2: Precision wish list: jet final states.

Recent progress in calculation of the two-loop five-particle amplitudes

- **All QCD amplitudes in the planar limit are known analytically** [Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov ‘19] [Abreu, Dormans, Febres Cordero, Ita, Page ‘18] Previous numerical [Badger, Brønnum-Hansen, Hartanto, Peraro ‘17][Abreu, Cordero, Ita, Page, Zeng ‘17] [Abreu, Cordero, Ita, Page, Sotnikov ‘18][Badger, Brønnum-Hansen, Gehrmann, Hartanto, Henn, Lo Presti, Peraro ‘18] and analytical results [Gehrmann, Henn, Lo Presti ‘15][Dunbar, Perkins ‘16] [Badger, Brønnum-Hansen, Hartanto, Peraro ‘18] in the planar approximation [talk by Sotnikov]

- **Full-color $\mathcal{N} = 4$ super-Yang-Mills and $\mathcal{N} = 8$ supergravity amplitudes** (at symbol level) [D.C., Gehrmann, Henn, Wasser, Zhang, Zoia ‘18 ‘19][Abreu, Dixon, Herrmann, Page, Zeng ‘18 ‘19] [talk by Page]

- **Full-color five-gluon all-plus helicity amplitude** [Badger, D.C., Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia ‘19] [this talk]

$\implies$ Very first complete analytic two-loop five-particle amplitude!
Towards all full-color two-loop five-parton QCD amplitudes

The all-plus helicity amplitude is extremely simple – a one-line formula

Other helicity configurations are more complicated, but

- The developed tools are indispensable for all QCD amplitudes

- We calculated all two-loop master integrals for massless five-particle scattering
  - analytic results
  - high-precision numerics
  - physical scattering region
Amplitude calculation workflow

Integrand

IBP reductions

Master integrals

Amplitude
Efficient methods to construct integrands

Integrand $\rightarrow$ IBP reductions $\rightarrow$ Master integrals $\rightarrow$ Amplitude

- Feynman diagrams
- Unitarity [Bern, Dixon, Dunbar, Kosower '94], Generalized unitarity [Bern, Dixon, Kosower '98][Britto, Cachazo, Feng '05][Ossola, Papadopoulos, Pittau '06], Numerical unitarity [Ita '15][Abreu, Febres, Cordero, Ita, Jaquier, Page, Zeng '17]

Integrand of the full-color two-loop five-point all-plus amplitude [Badger, Mogull, Ochirov, O’Connell ’15] contains numerators of degree five/six
Dramatic improvement of the Integration-By-Parts reduction due to finite-field arithmetics

\[
\mathcal{I}(s, \epsilon) = \sum_i c_i(s, \epsilon) g_i(s, \epsilon) , \quad D = 4 - 2\epsilon
\]

Finite fields and rational reconstruction significantly improve IBP reduction algorithms \cite{vonManteuffel-Schabinger-2015, Peraro-2016-2019, Maierhoefer-Usovitsch-2018, Smirnov-Chukharev-2019} [talk by Peraro]
The master integral families for massless two-loop five-particle scattering

[Gehrmann, Henn, Lo Presti ’15, ’18]
[Papadopoulos, Tommasini, Wever ’15]

[D.C., Mitev, Henn ’17]
[Boehm, Georgoudis, Larsen, Schoenemann, Zhang ’18]
[Abreu, Dixon, Herrmann, Page, Zeng ’18]
[D.C., Gehrmann, Henn, Lo Presti, Mitev, Wasser ’18]

[Abreu, Dixon, Herrmann, Page, Zeng ’18]
[D.C., Gehrmann, Henn, Wasser, Zhang, Zoia ’18]

All master integrals evaluate to pentagon functions
Kinematics of five-particle scattering

Massless particles: \( p_i^2 = 0 \)

Mandelstam invariants:

\[ s_{ij} = (p_i + p_j)^2 \]

Five independent:

\[ s_{12}, s_{23}, s_{34}, s_{45}, s_{15} \]

One pseudo-scalar:

\[ \epsilon_5 \equiv i\epsilon_{\mu_1\mu_2\mu_3\mu_4} p_{1\mu_1} p_{2\mu_2} p_{3\mu_3} p_{4\mu_4} \]

Physical scattering region \( 12 \rightarrow 345 \)

\[
\begin{align*}
& s_{12}, s_{34}, s_{45}, s_{35} > 0 \\
& s_{13}, s_{14}, s_{15}, s_{23}, s_{24}, s_{25} < 0 \\
& (\epsilon_5)^2 < 0
\end{align*}
\]
Pentagon functions

- Proposed in [D.C., Mitev, Henn '17]
- Confirmed in [Abreu, Dixon, Herrmann, Page, Zeng '18]
  [D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '18]

Iterated integrals along path $\gamma$

$$\int_{\gamma} d \log W_{i_1}(s) \ldots d \log W_{i_n}(s)$$

$\{W_i(s)\}_{i=1}^{31}$ – functions of energies and scattering angles

$n$ – transcendental weight

Homotopy invariance $\gamma_1 \sim \gamma_2 \Rightarrow I(\gamma_1) = I(\gamma_2)$
### Pentagon alphabet

31-letter alphabet $\mathbb{A} = \left\{ W_j(s) \right\}_{j=1}^{31}$

<table>
<thead>
<tr>
<th>$W_j$</th>
<th>Expression</th>
<th>Sign</th>
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<tbody>
<tr>
<td>$W_1$</td>
<td>$2 \ p_1 \cdot p_2$</td>
<td>+(4)</td>
</tr>
<tr>
<td>$W_6$</td>
<td>$2 \ p_4 \cdot (p_3 + p_5)$</td>
<td>+(4)</td>
</tr>
<tr>
<td>$W_{11}$</td>
<td>$2 \ p_3 \cdot (p_4 + p_5)$</td>
<td>+(4)</td>
</tr>
<tr>
<td>$W_{16}$</td>
<td>$2 \ p_1 \cdot p_3$</td>
<td>+(4)</td>
</tr>
<tr>
<td>$W_{21}$</td>
<td>$2 \ p_3 \cdot (p_1 + p_4)$</td>
<td>+(4)</td>
</tr>
<tr>
<td>$W_{26}$</td>
<td>$\frac{\text{tr}[(1-\gamma_5)\ p_1\ p_2\ p_4\ p_5]}{\text{tr}[(1+\gamma_5)\ p_1\ p_2\ p_4\ p_5]}$</td>
<td>+(4)</td>
</tr>
<tr>
<td>$W_{31}$</td>
<td>$\epsilon_5$</td>
<td></td>
</tr>
</tbody>
</table>

- The alphabet splits into orbits of $\mathbb{Z}_5$
- Invariance under $S_5$
- 26 parity-even and 5 parity-odd letters
- Zero loci of letters: branch points of the master integrals
Iterated integrals in terms of familiar functions

- **One-fold integrals**: Logarithms, e.g.
  \[ \log(s_{12}), \log(-s_{23}), \ldots \]

- **Two-fold integrals**: Dilogarithms, e.g.
  \[ \text{Li}_2 \left( 1 - \frac{s_{34}}{s_{12}} \right), \log(-s_{13}) \log(s_{34}), \ldots \]

- **Multi-fold integrals**: Goncharov polylogarithms
  - Well-studied functions in math and HEP
  - Fast numerical routines (GiNaC) [Vollinga, Weinzierl '04]

- **Planar sector**: well-studied and fast numerical implementation [Gehrmann, Henn, Lo Presti '18]
Master integrals from differential equations

Change of master integral basis $\vec{f}$ enormously simplifies DE [Henn '13]

\[ df(s, \epsilon) = \epsilon d\tilde{A}(s) f(s, \epsilon) \]

Solution has uniform transcendentality

\[ \vec{f}(s, \epsilon) = \text{Pexp}\left(\epsilon \int_{\gamma} d\tilde{A}(s)\right) \vec{f}(s_0, \epsilon) \quad \rightarrow \quad \text{Pentagon functions} \]

✓ Construction of the canonical basis:

○ Algorithm to find 4D dlog integrals [Wasser '16]

○ D-dimensional leading singularities based on the Baikov parametrization

✓ Absence of spurious singularities $\implies$ boundary constants $\vec{f}(s_0, \epsilon)$
From analytic formulae to numeric values

$$\text{Re } f_{101}^{(4)} \quad z \quad \text{Im } f_{101}^{(4)}$$

$$f_{101} = \sum_{w \geq 0} \frac{1}{\epsilon^{4-w}} f_{101}^{(w)}$$

- High-precision evaluation (GiNaC)
- Checks using SecDec

Evaluate $f_{101}^{(4)}$ along the path in kinematic space

$$s_{13} = 3$$
$$s_{23} = -1 + \frac{z}{1 + z^2}$$
$$s_{34} = 1$$
$$s_{45} = 1$$
$$s_{15} = -1 - \frac{z}{1 + z^2}, \quad 0 < z < 1$$
Assembly of the amplitude

Integrand $\rightarrow$ IBP reductions $\rightarrow$ Master integrals $\rightarrow$ Amplitude

The amplitude is much simpler than the ingredients!

Naive assembly of the amplitude is impossible because of the size and complexity of the ingredients. Additional steps are needed:

• IR-subtraction $\Rightarrow$ hard function
• Basis of rational factors
• Rational reconstruction
Factorization of the Infrared divergences

\[ \mathcal{A}(\epsilon) = \mathcal{Z}(\epsilon) \cdot \mathcal{A}^f(\epsilon) \]

- Matrix in color space, captures all \( \epsilon \)-poles
- finite at \( \epsilon \rightarrow 0 \)

**Hard function is finite**

\[ \mathcal{H} = \lim_{\epsilon \rightarrow 0} \mathcal{A}^f(\epsilon) \]

- Simpler than the amplitude
- Truly new piece of information
- Relevant for cross sections
New result: non-planar two-loop hard function

\[ \mathcal{H}^{(2)}_{\text{double trace}} = \sum_{S_5 / \Sigma} \text{Tr}(12) [\text{Tr}(345) - \text{Tr}(543)] \sum_{\Sigma} \left\{ 6\kappa^2 \left[ \frac{\langle 24 \rangle [14][23]}{\langle 12 \rangle \langle 23 \rangle \langle 45 \rangle^2} + 9 \frac{\langle 24 \rangle [12][23]}{\langle 12 \rangle \langle 34 \rangle \langle 45 \rangle^2} \right] \right. \\
+ \kappa \frac{[15]^2}{\langle 23 \rangle \langle 34 \rangle \langle 42 \rangle} \left[ l_{234;15} + l_{243;15} - l_{324;15} - 4l_{345;12} - 4l_{354;12} - 4l_{435;12} \right] \right\} \]

Finite part of the one-mass box function:

\[ I_{123;45} = \text{Li}_2 \left( 1 - \frac{s_{12}}{s_{45}} \right) + \text{Li}_2 \left( 1 - \frac{s_{23}}{s_{45}} \right) + \log^2 \left( \frac{s_{12}}{s_{23}} \right) + \frac{\pi^2}{6} \]

Spinor-helicity variables: \( \langle ij \rangle = \sqrt{s_{ij}} e^{i\varphi_{ij}} \) and \( [ij] = \sqrt{s_{ij}} e^{-i\varphi_{ij}} \)

Gluon spin dimension: \( \kappa \equiv \frac{g_{\mu\mu} - 2}{6} \)

- Weight-1,3,4 iterated integrals canceled out
- Analytic continuation to other regions is straightforward \( s_{ij} \to s_{ij} + i0 \)
- Correct factorization in the collinear limits
Summary

• First **analytic** result for full-color five-particle two-loop amplitude
  
  ✓ Functional level
  
  ✓ Nonplanar

• All master integrals for 3-jet production at NNLO are known **analytically**

• Pentagon functions describe all massless five-particle two-loop amplitudes