Hadronic vacuum polarization contribution to the muon magnetic moment from lattice QCD

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SM prediction vs experiment

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{11}$	Ref.
QED [5 loops]	116584718.841 ± 0.034	[Aoyama et al '18]
HVP LO	6933 ± 25 [0.32%]	[KNT '18]
	6939 ± 40 [0.58%]	[DHMZ '19]
	6881 ± 41 [0.60%]	[Jegerlehner '17]
HVP NLO	-98.7 ± 0.9	[Kurz et al '14]
		[Kurz et al '14, Jegerlehner '16]
HVP NNLO	12.4 ± 0.1	[Kurz et al '14]
		[Jegerlehner '16]
HLbyL	105 ± 26 [25%]	[Prades et al '09]
	$54 \pm 14 \pm ??$ [??%]	[RBC '16]
Weak (2 loops)	153.6 ± 1.0	[Gnendiger et al '15]
SM Tot [0.31 ppm]	116591824 ± 36	[w/ KNT '18]
[0.41 ppm]	116591830 ± 48	[w/ DHMZ '19]
[0.42 ppm]	116591772 ± 49	[Jegerlehner '17]
Exp [0.54 ppm]	116592091 ± 63	[Bennett et al '06]
Exp − SM [3.7σ]	267 ± 72	[KNT '18]
[3.3σ]	261 ± 79	[DHMZ '19]
[4.0σ]	319 ± 80	[Jegerlehner '17]

a_μ present and future

- Today $a_\mu^{\rm exp} - a_\mu^{\rm SM} \simeq 3.3 \div 4.0\,\sigma$, w/ $\delta a_\mu^{\rm exp} \sim \delta a_\mu^{\rm SM}$
- SM error dominated by those of the HVP ($\sim 80\%$) and the HLbyL ($\sim 55\%$) contributions
- Fermilab E989 began in 2017 and aims for 0.14 ppm vs 0.54 ppm today
- J-PARC E34 to begin $\geq$ 2021 and aims for similar precision

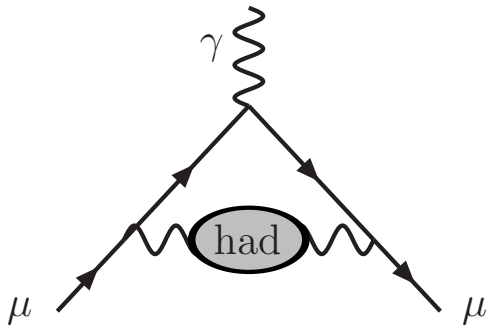
$\Rightarrow$ to fully leverage requires $\delta a_\mu^{\rm had} \leq 1.6 \times 10^{-10}$ vs 4.8×10^{-10} today

- If exp. & SM central values stay the same:
  - E989 alone (exp. error $/4$) $\rightarrow 5 \div 7\,\sigma$
  - E989 & equally precise theory $\rightarrow 12 \div 14\,\sigma$
- But before concluding presence of BSM physics
  $\Rightarrow$ need independent, first-principle determinations of these 2, nonperturbative hadronic contributions
  $\Rightarrow$ Lattice QCD
- Discrepancy is:

  $\sim$ $2\times$ electroweak contribution
  $\sim$ $3.6\times$ the HLbyL contribution
  $\sim$ 4% of the LO-HVP contribution

HVP contribution to a_μ from LQCD: introduction

Consider in Euclidean spacetime (Blum '02)



$$\Pi_{\mu\nu}(Q) = \int d^4x\, e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0)\rangle = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

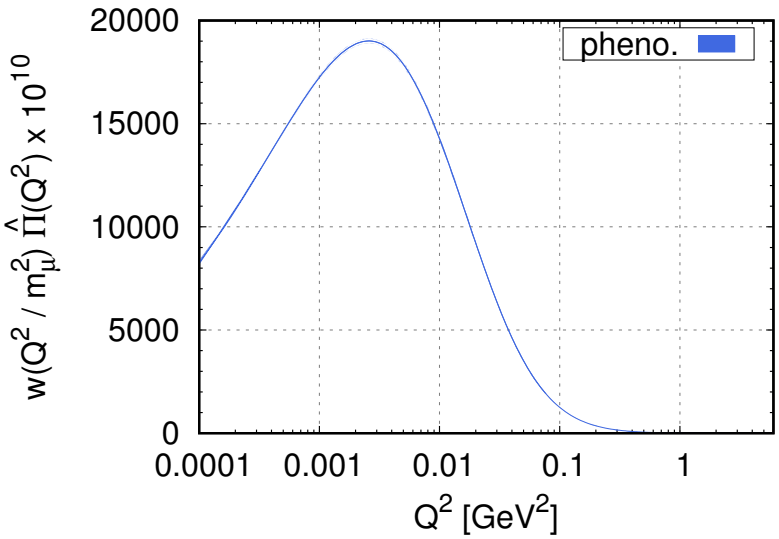
Then (Lautrup et al '69, Blum '02)

$$a_\ell^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dQ^2}{m_\ell^2}\, w(Q^2/m_\ell^2)\hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$

- Integrand peaked for $Q \sim (m_\ell/2) \sim 50\,\text{MeV}$ for $\ell = \mu$
- 80% of $a_\mu^{\text{LO-HVP}}$ comes from $Q \leq 300\,\text{MeV} \sim \Lambda_{\text{QCD}}$

$\Rightarrow$ $\hat{\Pi}(Q^2)$ must be computed very precisely for Q in QCD's nonperturbative domain



(HVP from Jegerlehner, "alphaQEDc17" (2017))

Low-Q^2 challenges in finite volume (FV)

A. $\Pi_{\mu\nu}(Q=0) \neq 0$ in FV $\Rightarrow \Pi_{\mu\nu}(Q)/Q^2 \overset{Q^2 \sim 0}{\sim} \Pi(Q^2) + \mathrm{cst}/Q^2$

B. On-shell renormalization requires $\Pi(0)$ which is problematic (see above)

C. Need $\hat{\Pi}(Q^2)$ interpolation due to $Q_{\min} = 2\pi/T \sim 135\,\mathrm{MeV} > \frac{m_\mu}{2} \sim 50\,\mathrm{MeV}$ for $T \sim 9\,\mathrm{fm}$

$$\Downarrow$$

- Compute on $T \times L^3$ lattice

$$C_L(t) = \frac{a^3}{3} \sum_{i=1}^{3} \sum_{\vec{x}} \langle J_i(x) J_i(0) \rangle$$

- Decompose ($C_L^{I=1} = \frac{9}{10} C_L^{ud}$)

$$\begin{aligned} C_L(t) &= C_L^{ud}(t) + C_L^{s}(t) + C_L^{c}(t) + C_L^{\mathrm{disc}}(t) \\ &= C_L^{I=1}(t) + C_L^{I=0}(t) \end{aligned}$$

- Define (Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...) (ad A, B, C)

$$\hat{\Pi}_L^f(Q^2) \equiv \Pi_L^f(Q^2) - \Pi_L^f(0) = \frac{1}{3} \sum_{i=1}^{3} \frac{\Pi_{ii,L}^f(0) - \Pi_{ii,L}^f(Q)}{Q^2} - \Pi_L^f(0) = 2a \sum_{t=0}^{T/2} \mathrm{Re}\left[\frac{e^{iQt}-1}{Q^2} + \frac{t^2}{2}\right] \mathrm{Re}\, C_L^f(t)$$

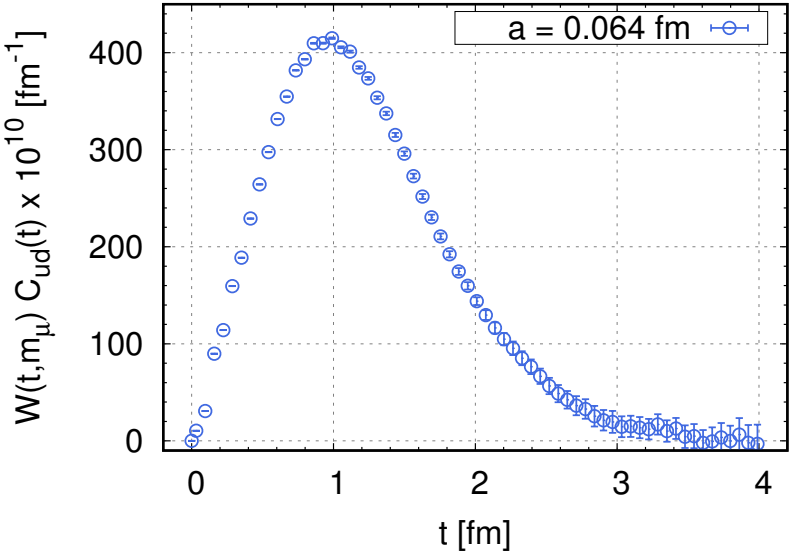
“Standard” lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

Combining everything, get $a_{\ell,f}^{\text{LO-HVP}}$ from $C_L^f(t)$:

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \leq Q^2_{\max}) = \lim_{a\to 0,\, L\to\infty,\, T\to\infty} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m_\ell^2}\right) \sum_{t=0}^{T/2} W(tm_\ell, Q^2_{\max}/m_\ell^2)\ \mathrm{Re}C_L^f(t)$$

where

$$W(\tau, x_{\max}) = \int_0^{x_{\max}} dx\, w(x) \left(\tau^2 - \frac{4}{x}\sin^2\frac{\tau\sqrt{x}}{2}\right)$$

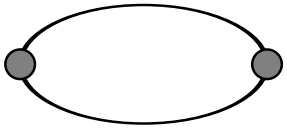


(BMWc ’17, 144×96^3, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

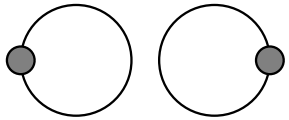
Simulation challenges

D. $\pi\pi$ contribution very important $\rightarrow$ must have physically light π

E. Two types of contributions



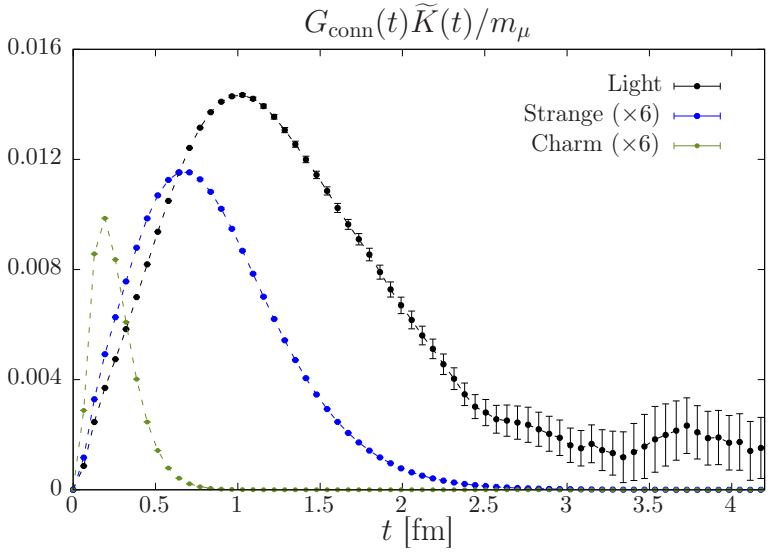
quark-connected (qc)



quark-disconnected (qd)

where qd contributions are $SU(3)_f$ and Zweig suppressed but very challenging

F. $C_L^{ud,\text{disc}}(t)$ have N/S $\sim \exp\left[(M_\rho - M_\pi)t\right]$ + need high-precision results



(Mainz '19)

$\rightarrow$ need very high statistics + many algorithmic improvements + rigorous bounds on large t behaviour $C_L^{ud,\text{disc}}(t)$

More simulation and other challenges

G. Need large volumes to control long-distance, 2π contributions to $C_L^{ud,\mathrm{disc}}(t)$ (Aubin et al '16)

H. Need controlled continuum limit $\rightarrow$ must have several $a \leq 0.1\,\mathrm{fm}$

I. Need $\hat{\Pi}(Q^2)$ for $Q^2 \in [0, +\infty[$, but $\frac{\pi}{a} \sim 10.\,\mathrm{GeV}$ for $a \sim 0.06\,\mathrm{fm}$
$\rightarrow$ match onto perturbation theory

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_\ell(Q_{\text{max}})\,\hat{\Pi}^f(Q_{\text{max}}^2) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

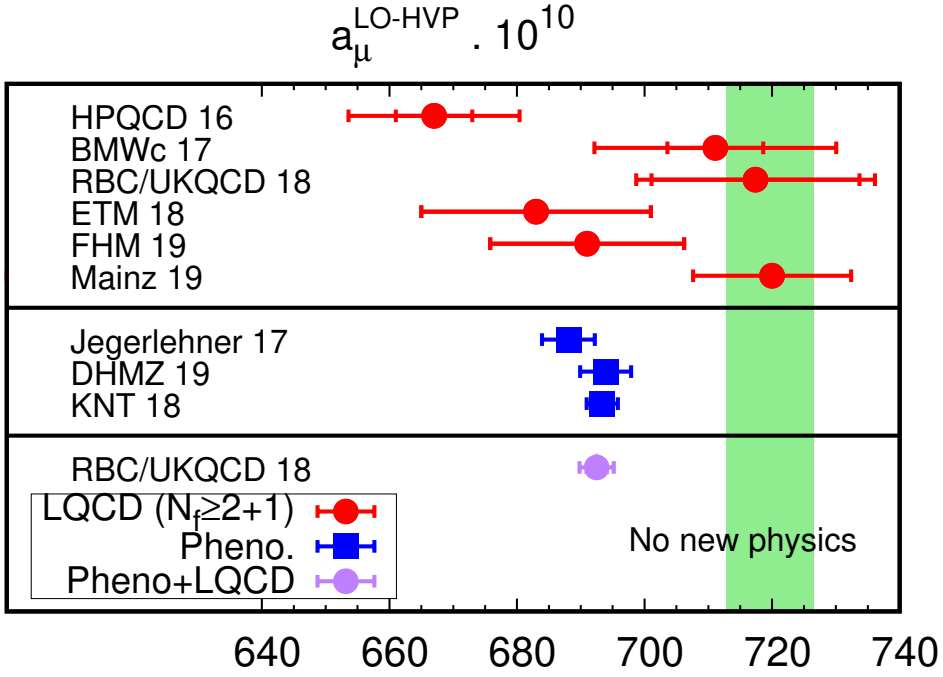
J. Include c quark in sea for higher precision and good matching onto perturbation theory

K. Large gauge ensembles typically generated with $m_u = m_d$ and $\alpha = 0$

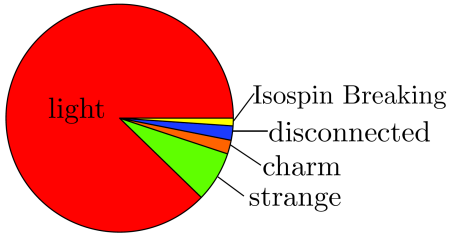
$\Rightarrow$ missing effects that are relevant at % level

$\rightarrow$ must be included

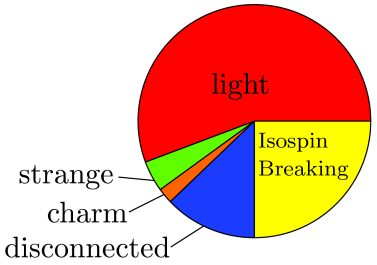
Current published situation: $a_\mu^{\text{LO-HVP}}$



Contributions to $a_\mu^{\text{LO-HVP}}$



Contributions to $\delta a_\mu^{\text{LO-HVP}} \sim 2.5\%$



(Gülpers, Lattice '19)

- Some lattice results suggest new physics others not but all compatible with phenomenology
- Differences originate mostly from treatment of noisy, large t behaviour of $C_L^{ud}(t)$ and of its FV effects
- Contributions from s, c and disc mainly agree w/ small contributions to total error
- Lattice errors $\gtrsim 2\%$ vs phenomenology errors $\sim 0.4\%$
- Must reduce lattice error to $< 1\%$ to have impact

BMWc @ end 2017: simulations and results

BMWc, Borsanyi et al, PRL121 (2018) 022002 (Editors' Suggestion)

15 high-statistics simulations w/ N_f=2+1+1 flavors of 4-stout staggered quarks:

- Bracketing physical m_{ud}, m_s, m_c
- 6 a's: 0.134 $\rightarrow$ 0.064 fm
- $L = 6.1 \div 6.6$ fm, $T = 8.6 \div 11.3$ fm
- Conserved EM current
- Close to 9M / 39M conn./disc. measurements w/ AMA

β	a [fm]	$T \times L$	#conf-conn	#conf-disc
3.7000	0.134	64 $\times$ 48	1000	1000
3.7500	0.118	96 $\times$ 56	1500	1500
3.7753	0.111	84 $\times$ 56	1500	1500
3.8400	0.095	96 $\times$ 64	2500	1500
3.9200	0.078	128 $\times$ 80	3500	1000
4.0126	0.064	144 $\times$ 96	450	-

Contrib.	$a_\mu^{\text{LO-HVP}} \times 10^{10}$
$I = 1$	583(7)(7)(0)(0)(5)(13)
$I = 0$	121(3)(4)(0)(0)(1)
Total	711(8)(8)(0)(0)(6)(13)(5)

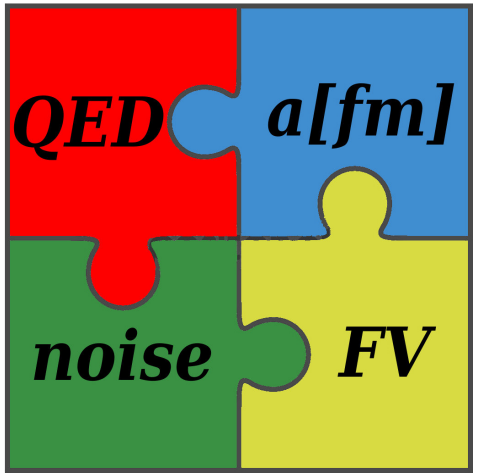
$\sigma_{\text{FV}} > \sigma_{\text{stat}} \sim \sigma_{\text{ana}} > \sigma_{\text{a-syst}} \sim \sigma_{\text{IB}}$

Error on total:

- Stat. = 1.1% (all from ud)
- $a \rightarrow 0$ syst. = 1.1% (mostly from ud)
- a syst. = 0.8%
- FV = 1.9% (all from ud) (NLO χPT)
- IB = 0.7% (Pheno.)
- Total = 2.7%

Path to HVP Nirvana

To reach sub-percent accuracy on lattice, must:



- reduce statistical error on ud contribution significantly
- correct FV effects reliably (preferably with LQCD)
- control quark-disconnected contributions
  $\rightarrow$ already done (BMWc '16 & '17, RBC/UKQCD '16)
- determine $m_u \neq m_d$ and EM corrections on lattice: only leading order in $(m_d - m_u)$ and α is needed
- determine a very precisely: $\frac{\sigma_{a_\mu^{\text{LO-HVP}}}}{a_\mu^{\text{LO-HVP}}} \sim 2 \times \frac{\sigma_a}{a}$
- staggered: must improve continuum extrapolation of $a_{\mu,ud}^{\text{LO-HVP}}$

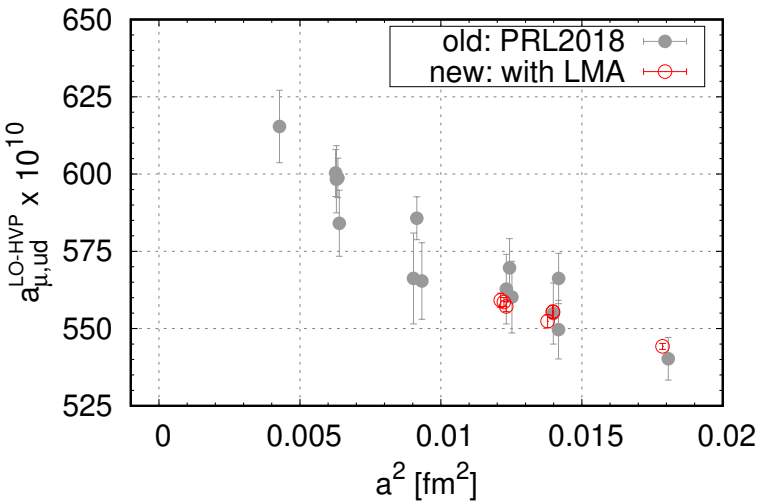
Noise reduction for $C_L^{ud/disc}(t)$ at large t: algorithms

Use all-mode averaging (AMA) (Blum et al '13) as in BMWc '16,'17 w/ low-mode averaging (LMA) (Giusti et al '03):

- compute many low modes and evs of Dirac operator on each configuration
- use to explicitly construct largest, long-distance contribution to correlator from all-to-all lattice points (instead of single to all)
- contributions of high modes obtained by inversion in orthognal complement

$\rightarrow$ allows a huge increase in statistics $\propto T \times L^3$

$\rightarrow$ implemented in RBC/UKQCD '18, Aubin et al '19 and BMWc (in progress)



(BWMc in progress, very numerically expensive part has yet to be done)

$\Rightarrow$ can reach few permil statistical precision at fixed lattice spacing and volume w/ an additional trick . . .

Noise reduction for $C_L^{ud/disc}(t)$ at large t: bounds

- In FV, spectrum is discrete, and spectral decomposition of EM current correlator is

$$C_L^{ud}(t) = \sum_{n\geq 1} |Z_n|^2 \varphi(E_n, t)$$

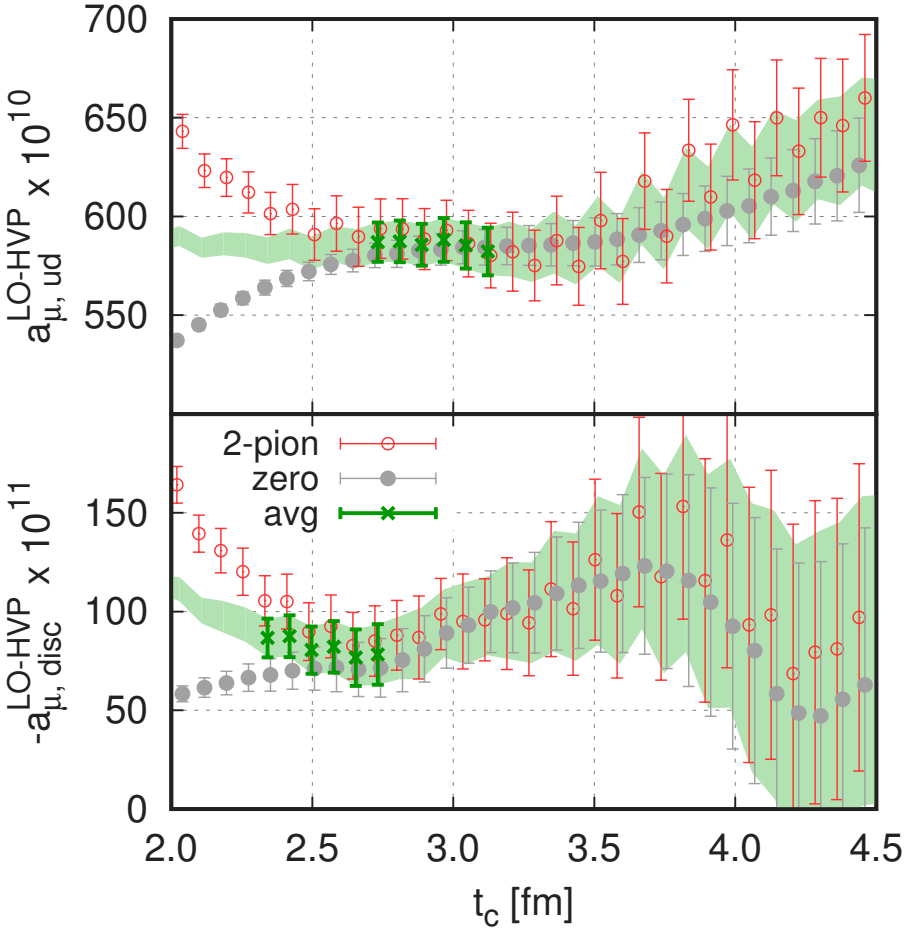
$$\varphi(E_n, t) = \cosh\left[E_n(T/2 - t)\right]$$

- Upper and lower bounds for $t > t_c$ (BMWc '16 & '17, RBC/UKQCD '18, Blum et al '19)

$$0 \leq C_L^{ud}(t_c)\frac{\varphi(E_c, t)}{\varphi(E_c, t_c)} \leq C_L^{ud}(t) \leq C_L^{ud}(t_c)\frac{\varphi(E_1, t)}{\varphi(E_1, t_c)}$$

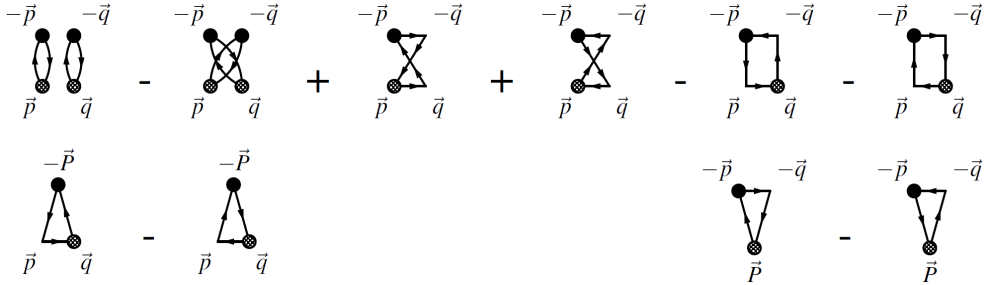
  - E_1 is FV ground state energy $\sim$ two back-to-back pions w/ one lattice unit of momentum
  - E_c is effective mass of $C_L^{ud}(t)$ at t_c

- Use lattice $C_L^{ud}(t)$ for $t \leq t_c$
- Use average of bounds for $t > t_c$ above meeting point
- Vary t_c for systematic error
- Similar bounds for $C_L^{disc}(t)$



Noise reduction: reconstruction of $C_L^{ud/disc}(t)$ at large t

- Determine E_n & $|Z_n|$ for lowest N states
- Requires:
  - constructing a matrix of correlation functions composed of $\geq N$ two-pion and rho-meson operators and j_μ

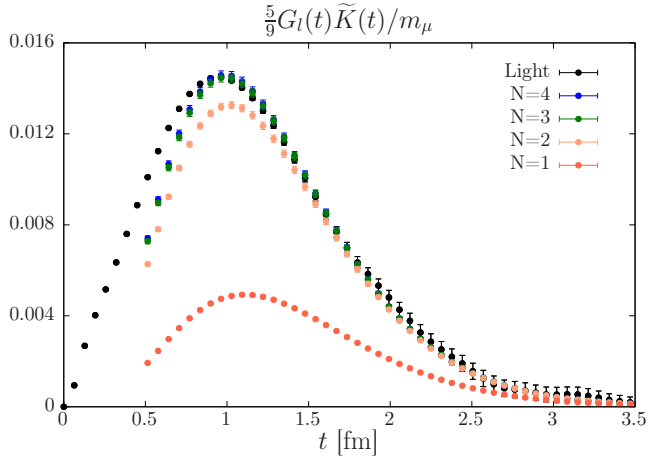


  - solving a GEVP to extract the desired E_n & $|Z_n|$
  - RBC/UKQCD (Lattice '19) checked possible contributions from four-pion states!

- Use to reconstruct large t tail of $C_L^{ud}(t)$

$$C_L^{ud}(t) \overset{t\to\infty}{\sim} \sum_{n=1}^{N} |Z_n|^2 \varphi(E_n, t)$$

- Can be used for improving bound method



(Mainz '19, also RBC/UKQCD (Lattice '18))

Noise reduction: improved bounds

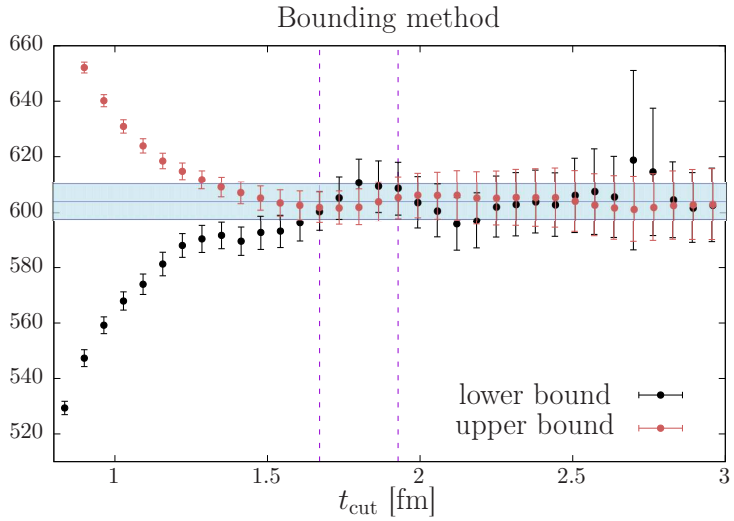
- Use N lowest states to improve bounds (Mainz '19, RBC/UKQCD Lattice '18)

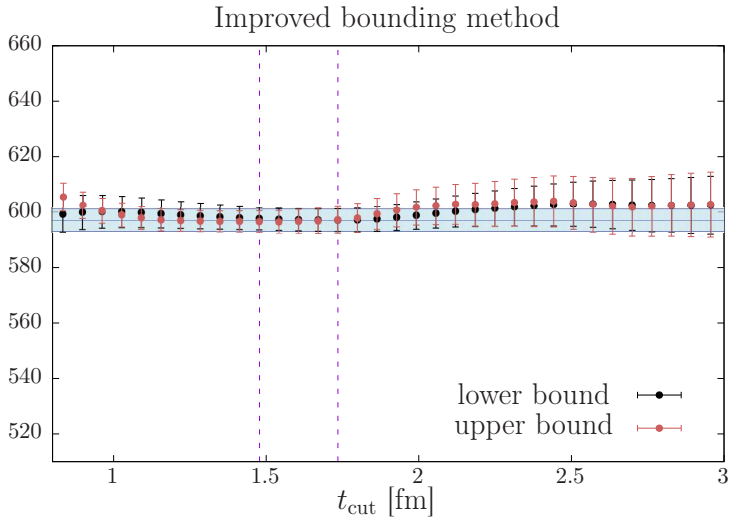
$$\Delta C_L^{ud}(t) = C_L^{ud}(t) - \sum_{n=1}^{N} |Z_n|^2 \varphi(E_n, t) \,, \qquad \varphi(E_n, t) = \cosh\left[E_n(T/2 - t)\right]$$

- Upper and lower bounds are now

$$0 \leq \Delta C_L^{ud}(t_c) \frac{\varphi(E_c, t)}{\varphi(E_c, t_c)} \leq \Delta C_L^{ud}(t) \leq \Delta C_L^{ud}(t_c) \frac{\varphi(E_{N+1}, t)}{\varphi(E_{N+1}, t_c)}$$

- Meet for earlier t_c and resulting $C_L^{ud}(t)|_{\text{new}} \equiv \Delta C_L^{ud}(t)|_{\text{bnd avg}} + \sum_{n=1}^{N} |Z_n|^2 \varphi(E_n, t)$ has reduced errors

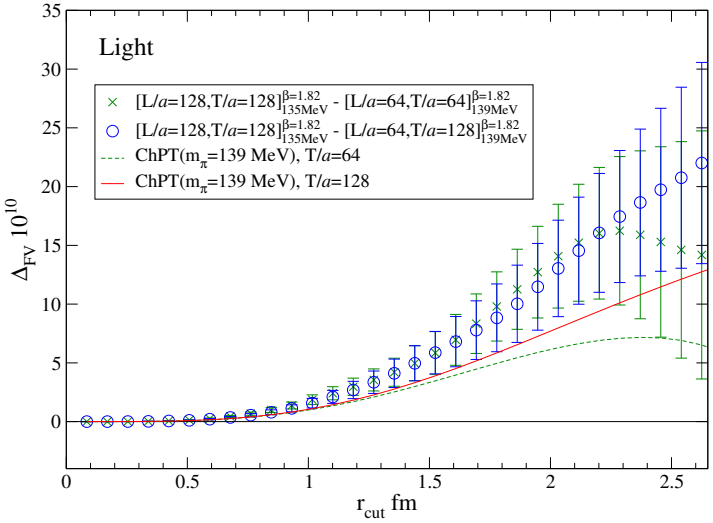




(Mainz '19)

Correction of FV effects

- Dominated by long-range, low-lying 2π states and surprisingly large ($\sim 3\%$ for $L \sim 6\,\mathrm{fm}$)
- According to NLO χPT (Aubin et al '16), correction is $\sim (13 \pm 13) \times 10^{-10}$ (BMWc '17) for $L \sim 6\,\mathrm{fm}$
- Modeling of low-lying 2π contributions (Meyer '11) using Lüscher for E_n and Lellouch-Lüscher for $|Z_n|$, using Gounaris-Sakurai phase shift, gives $\sim 20 \times 10^{-10}$
- NNLO χPT (Aubin et al '19, Bijnens et al '17) gives $\sim 18 \times 10^{-10}$
- $\sim 20 \times 10^{-10}$ confirmed by LQCD calculations in different volumes, w/ large errors (Shintani et al '19, RM123 '18, RBC/UKQCD Lattice '19)



(Shintani et al '19, $L = 5.4, 10.8\,\mathrm{fm}$)

- Combination of LQCD calculations of FV corrections and models will help reduce current FV uncertainty to few permil

Including isospin breaking on the lattice

$$S_{\mathrm{QCD+QED}} = S^{\mathrm{iso}}_{\mathrm{QCD}} + \frac{1}{2}(m_u - m_d)\int(\bar{u}u - \bar{d}d) + ie\int A_\mu j_\mu, \qquad j_\mu = \bar{q}Q\gamma_\mu q$$

- Separation into isospin limit results and corrections requires an unambiguous definition of this limit (scheme and scale)
- Must be included not only in calculation of $\langle j_\mu j_\nu \rangle$ correlator **BUT ALSO** of all quantities used to fix quark masses and scale

(1) operator insertion method (RM123 '12, '13, ...)

$$\langle \mathcal{O} \rangle_{\mathrm{QCD+QED}} = \langle \mathcal{O} \rangle^{\mathrm{iso}}_{\mathrm{QCD}} - \underbrace{\frac{1}{2}(m_u - m_d)\langle \mathcal{O} \int(\bar{u}u - \bar{d}d)\rangle^{\mathrm{iso}}_{\mathrm{QCD}}}_{(a)}$$

$$+ \underbrace{\frac{1}{2}e^2\langle \mathcal{O} \int_{xy} j_\mu(x) D_{\mu\nu}(x-y) j_\nu(y)\rangle^{\mathrm{iso}}_{\mathrm{QCD}}}_{(b)} + \text{hot}$$

(2) direct method (Eichten et al '97, BMWc '14, ...)

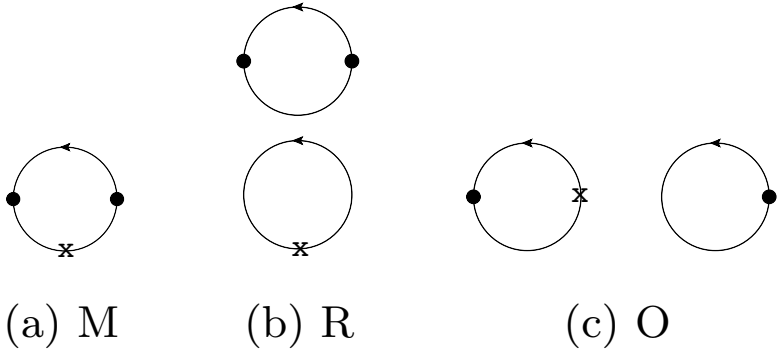
Include $m_u \neq m_d$ and QED directly in calculation of observables and generation of gauge configurations

(3) combinations of (1) & (2)

For valence effects, (2) is often easier, and for sea effects, (1) is

Strong isospin-breaking corrections

(1) perturbative approach



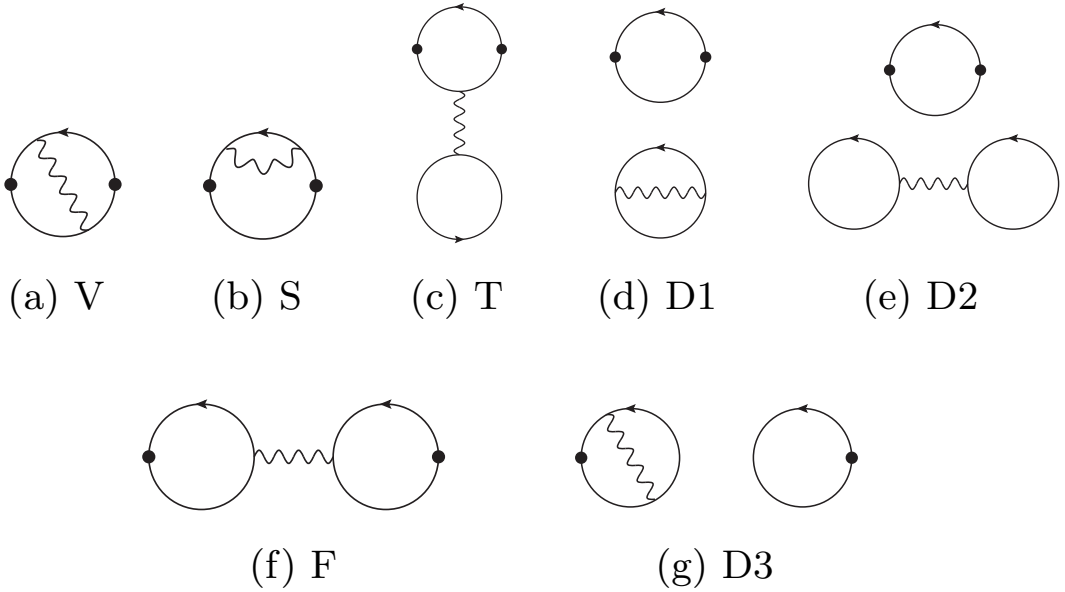
(RBC/UKQCD 18)

- Sea-quark corrections are $(m_d - m_u)^2 \Rightarrow R = 0$
- For the moment, only M is calculated
- But $1/N_c$ O will have to be calculated for $< 1\%$ accuracy
- $\Delta^{\text{SIB-M}} a_\mu^{\text{LO-HVP}} = 10.6(4.3)(6.8) \times 10^{-10}$ (RBC/UKQCD '18)
- $\Delta^{\text{SIB-M}} a_\mu^{\text{LO-HVP}} = 6.0(2.3) \times 10^{-10}$ (ETMC '19)

(2) direct approach (HPQCD '18)

- $\Delta^{\text{SIB-M}} a_\mu^{\text{LO-HVP}} = 7.7(3.7) \times 10^{-10}$, only valence
- $\Delta^{\text{SIB-M}} a_\mu^{\text{LO-HVP}} = 9.0(2.3) \times 10^{-10}$, w/ sea

QED corrections



(RBC/UKQCD 18)

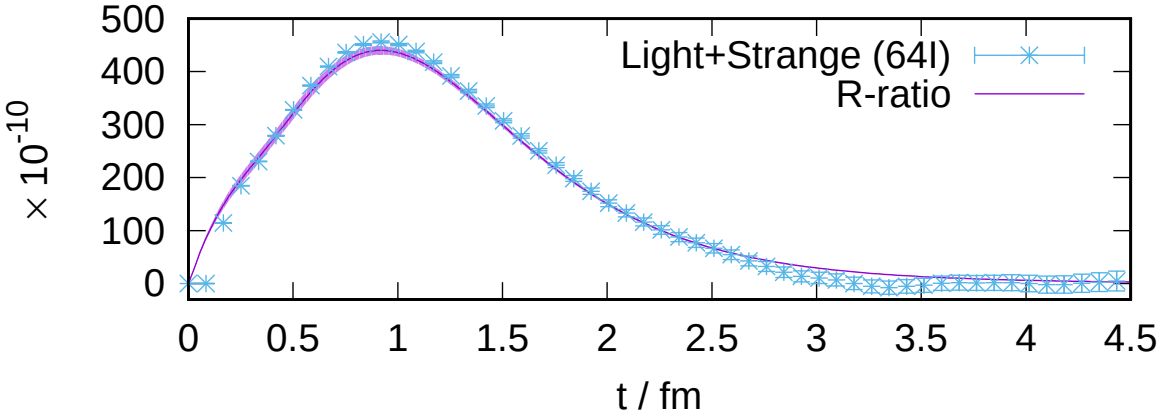
- For the moment, only V, S and F are calculated
- Other diagrams are $1/N_c$, but must be calculated for $< 1\%$ accuracy

Results

- $\Delta^{\text{QED-V+S}} a_\mu^{\text{LO-HVP}} = 5.9(5.7)(1.7) \times 10^{-10}$ (RBC/UKQCD '18)
- $\Delta^{\text{QED-V+S}} a_\mu^{\text{LO-HVP}} = 1.1(1.0) \times 10^{-10}$ (ETMC '19)
- $\Delta^{\text{QED-F}} a_\mu^{\text{LO-HVP}} = -6.9(2.1)(1.4) \times 10^{-10}$ (RBC/UKQCD '18)

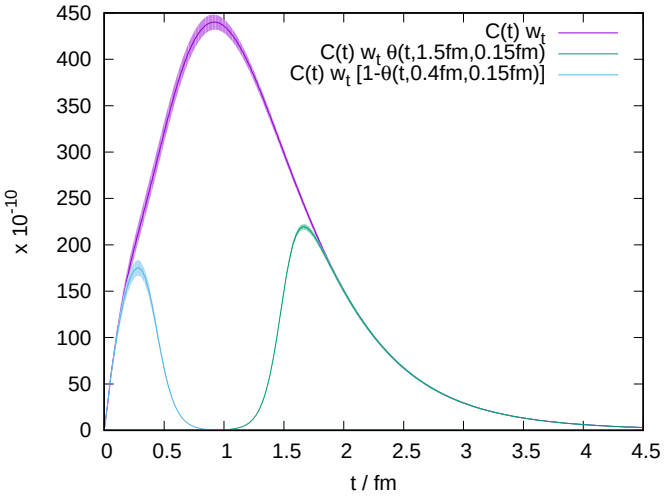
Time window: lattice + phenomenology

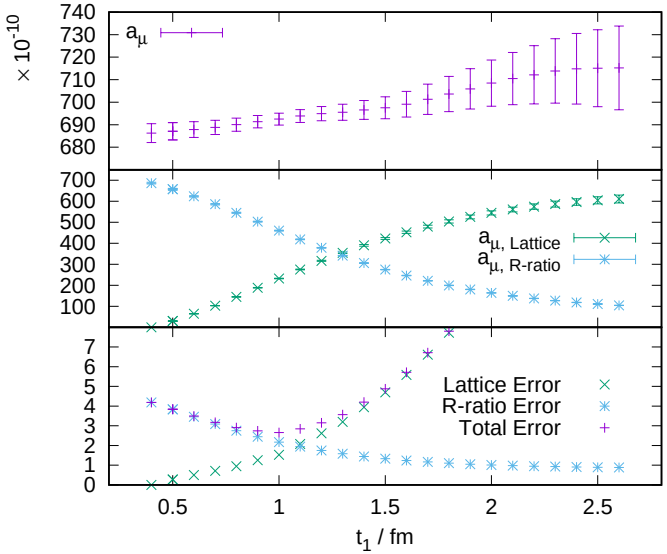
- $C(t)$ may be more precise in certain euclidean time ranges on lattice than in phenomenology



→ combine lattice with phenomenology to reduce error (RBC/UKQCD '18 + Jegerlehner '17)

$$a_\mu^{\text{LO-HVP}} = a_{\mu,\,\text{SD, pheno}}^{\text{LO-HVP}} + a_{\mu,\,\text{W, lat}}^{\text{LO-HVP}} + a_{\mu,\,\text{LD, pheno}}^{\text{LO-HVP}}$$





Conclusions

- Current error on lattice $a_\mu^{\text{LO-HVP}}$ is $\sim 18 \times 10^{-10}$
- Needs to be reduced by ~ 4 to match phenomenology and ~ 10, future a_μ measurements
- Error dominated by
  - Large distance noise on $C_L^{ud}(t)$
  - Uncontrolled FV corrections
  - Calculation of SID and QED corrections
  - Uncertainty on lattice spacing determination
  - Staggered: continuum extrapolation of $a_{\mu,ud}^{\text{LO-HVP}}$
- As shown, all of these issues are being aggressively addressed and progress is being made quickly
- $\lesssim 1\%$ results should be available w/in a year and phenomenology should be matched by time of final FNAL E989 results

$\rightarrow$ Combine LQCD and phenomenology to improve overall uncertainty (RBC/UKQCD '18), only if the two agree statistically with comparable errors

- If not, difference should be understood
- The muon $(g-2)$ Theory Initiative has had a number of meetings and is preparing a first white paper
  $\rightarrow$ anyone interested is welcome to participate

BACKUP

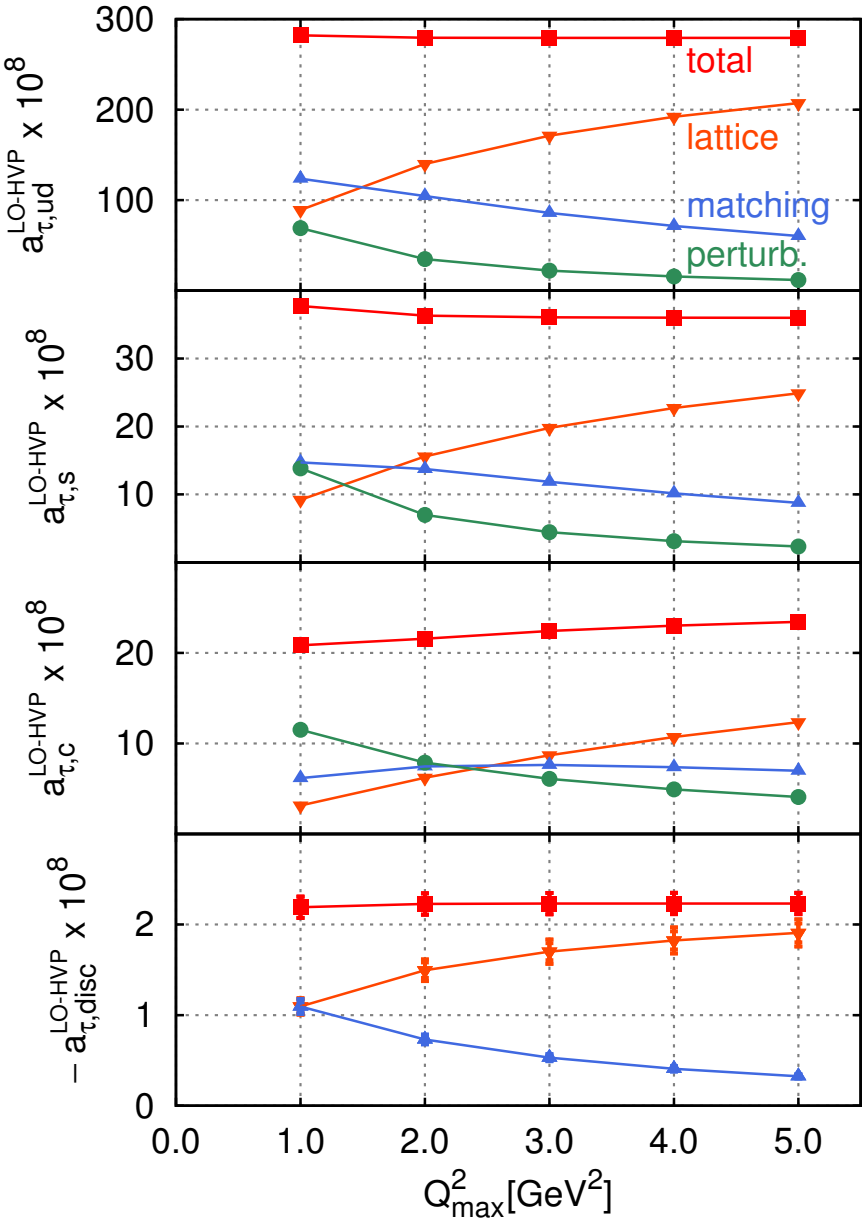
Matching to perturbation theory: ad I & J

Consider separation ($\ell = e,\ \mu,\ \tau$)

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_\ell(Q_{\text{max}})\, \hat{\Pi}^f(Q^2_{\text{max}}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

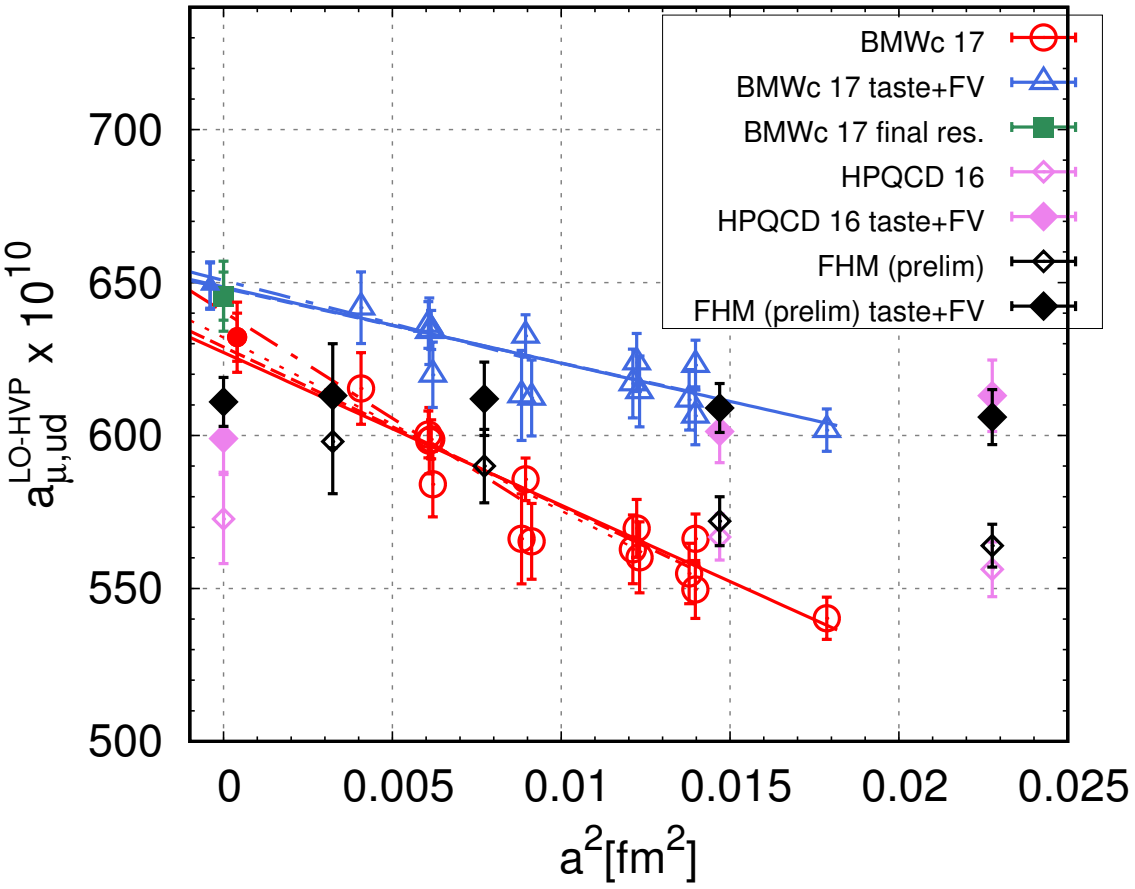
- Compute $\Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$ using $R_{\text{pert}}(s)$ to $O(\alpha_s^4)$ from Harlander et al '03
- Not relevant for $\ell = e, \mu$ but important for τ
- Perfect matching of continuum lattice results for $Q^2_{\text{max}} \geq 2\,\text{GeV}^2$

  $\rightarrow$ control $\hat{\Pi}(Q^2)$ up to $Q^2 \rightarrow \infty$
- Get matching systematic from considering $Q^2_{\text{max}} = 2$ and $5\,\text{GeV}^2$



Staggered continuum extrapolation of $a_{\mu,ud}^{\text{LO-HVP}}$

- Goldstone has more massive "taste" partners that dilute Golsdtone contribution to $a_{\mu,ud}^{\text{LO-HVP}}$
- "Effective" pion mass larger at larger a, e.g. $M_\pi^{\text{RMS}} \simeq 310\,\text{MeV}$ for $a = 0.134\,\text{fm}$
- Effect dissappears in $a \to 0$ limit
- $a \to 0$ extrapolation includes $M_\pi^{\text{RMS}} \to M_\pi^{\text{PDG}}$ extrapolation and is quite pronounced



FNAL/HPQCD/MILC 16 & prelim already include large t modeling of $C(t)$

LO χPT for FV effects

- FV effects are long-distance effects, determined by lightest states contributing to process
- Here $I = J = 1$, 2-π states
- Determine in χPT, to LO (Aubin et al 15), i.e.

$$C^{I=1}_{L,\text{LO-}\chi\text{PT}}(t) = \frac{1}{3L^3} \sum_{\vec{p}_{\text{free}}} \left(\frac{\vec{p}_{\text{free}}}{E_p^{\text{free}}} \right)^2 e^{-2E_p^{\text{free}} t}$$

with $E_p^{\text{free}} = \sqrt{M_\pi^2 + \vec{p}^2_{\text{free}}}$

- Then $C^{I=1}_{\infty,\text{LO-}\chi\text{PT}}(t) - C^{I=1}_{L,\text{LO-}\chi\text{PT}}(t)$ can be used to estimate FV effects
- Find, for $M_\pi \sim 135\,\text{MeV}$ and $L \sim 6\,\text{fm}$ (BMWc 17),

$$\Delta_{\text{FV}} a^{\text{LO-HVP}}_{\mu,I=1} \sim 2.3\% \times a^{\text{LO-HVP}}_{\mu,I=1}$$

$\rightarrow$ probably $O(50\%)$ too small (Della Morte et al 17, Shintani et al 19, Aubin et al 19 ...)

- Can do better

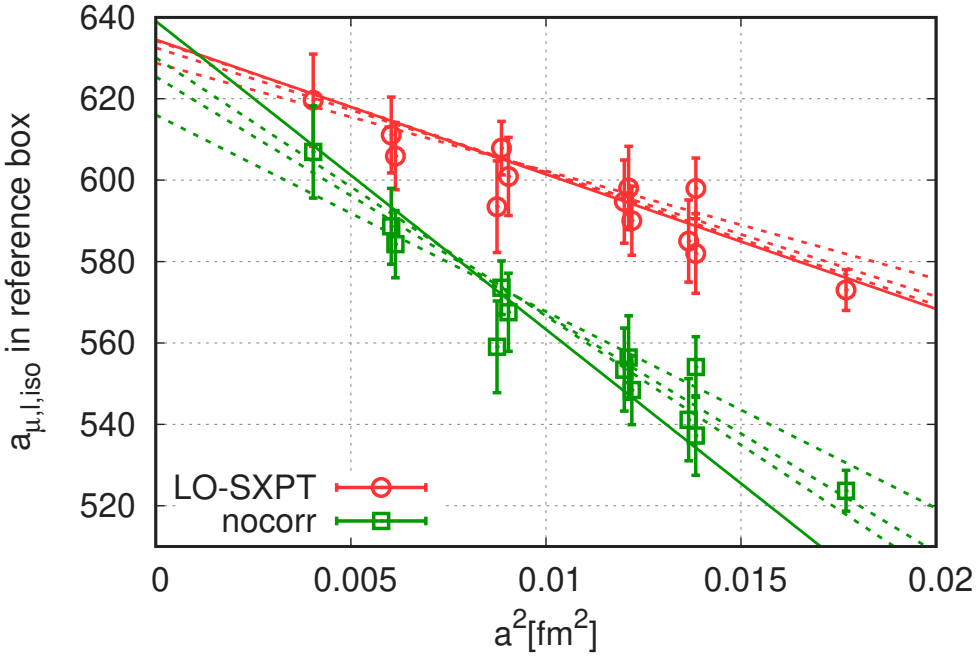
LO SχPT for taste effects

- Taste-breaking effects also mostly come from low-lying 2-π states
- Determine in SχPT, to LO (Aubin et al 15, HPQCD 17), i.e.

$$C^{I=1}_{L,\text{LO-S}\chi\text{PT}}(t, a^2\Delta^{\text{KS}}) = \frac{1}{3L^3}\sum_{\vec{p}_{\text{free}}}\sum_{j=0}^{4} w_j \left(\frac{\vec{p}_{\text{free}}}{E^{\text{free}}_{p,j}}\right)^2 e^{-2E^{\text{free}}_{p,j}t}$$

with $E^{\text{free}}_{p,j} = \sqrt{M^2_{\pi,j} + \vec{p}^2_{\text{free}}}$

- Then $C^{I=1}_{L,\text{LO-S}\chi\text{PT}}(t,0) - C^{I=1}_{L,\text{LO-S}\chi\text{PT}}(t, a^2\Delta^{\text{KS}})$ can be used to estimate taste effects



- Helps but is it possible to do better?

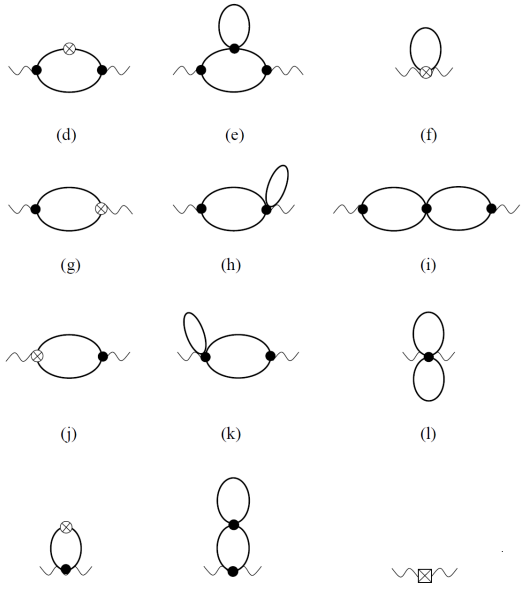
NLO SχPT

- At LO, the two π are free

  $\Rightarrow$ omits strong $\rho-\pi\pi$ coupling

  $\Rightarrow$ compute at NLO (Bijnens et al 99, Aubin et al 19)

- NLO includes LO 2-π rescattering and slope of $F_\pi(Q^2)$



- However NLO only obtained in continuum (Aubin et al 19)

  $\Rightarrow$ helps FV corrections: increase by $O(50\%)$ for $M_\pi \sim 135\,\mathrm{MeV}$ and $L \sim 6\,\mathrm{fm}$

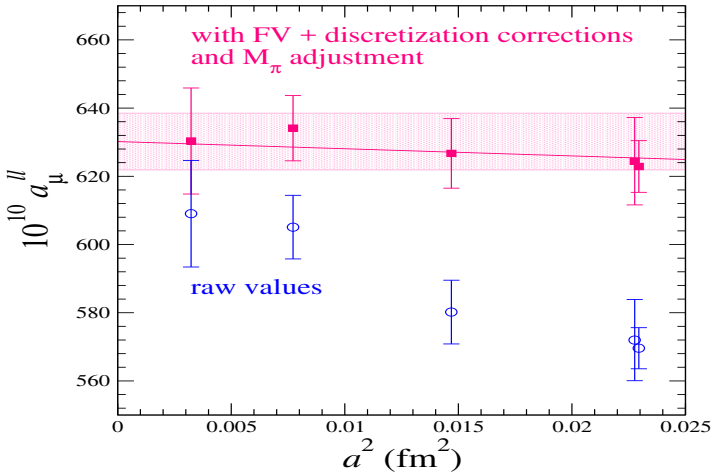
  $\Rightarrow$ does not improve taste corrections

Add point-like ρ to SXPT and model lattice ρ

- Construct field theory that couples γ-ρ-$\pi\pi$ (Jegerlehner 11)
- Gives much better description of $\Pi(Q^2)$ in continuum
- Add taste breaking to pion loop contributions and work out coupled system to one pion loop (HPQCD 17)

$\rightarrow$ gives FV and taste corrections similar to LO SχPT

$\Rightarrow$ further correct for taste breaking by measuring m_μ in units of lattice "M_ρ" from usual correlator fits (ETMC 11, HPQCD 17) or by modelling $C_L(t)$ at large t (FNAL/HPQCD/MILC 17)



Possible issues:

- ρ not treated as a resonance in M_ρ-rescaling
- reduction of number of (staggered) states in modelling of $C_L(t)$ is done by fitting, not e.g. by systematically eliminating taste copies

Phenomenology inspired FV corrections

Use (Meyer 11):

$$C^{I=1}_{L,\text{LO-}\chi\text{PT}}(t) = \frac{1}{3L^3}\sum_{\vec{p}_{\text{free}}}\left(\frac{\vec{p}_{\text{free}}}{E_p^{\text{free}}}\right)^2 e^{-2E_p^{\text{free}}t} \rightarrow C^{I=1}_{L,\text{LLGS}}(t) = \frac{1}{3L^3}\sum_i\sum_p |\langle 0|J_i|\pi^+(\vec{p})\pi^-(-\vec{p})\rangle|^2_L \;\; e^{-2E_p t}$$

with

- Lüscher to get $E_p = \sqrt{M_\pi^2 + p^2}$ in FV, where $p = |\vec{p}|$ is momentum carried by each of the two interacting π in FV
- Lellouch-Lüscher (LL) for interacting $|\langle 0|J_i|\pi^+(p)\pi^-(p)\rangle|_L$ in FV from free amplitude $\frac{p_i^{\text{free}}}{E_p^{\text{free}}}$
- 2-π, $\delta^{J=1}_{I=1}(p)$ from phenomenology

Then $C^{I=1}_{\infty,\text{LLGS}}(t) - C^{I=1}_{L,\text{LLGS}}(t)$ gives estimate of FV

$\rightarrow$ good for FV corrections: increase by $O(50\%)$ over LO χPT for $M_\pi \sim 135\,\text{MeV}$ and $L \sim 6\,\text{fm}$

(Della Morte et al 17, Shintani et al 19, Aubin et al 19 . . .)

Implementing simulations at different volumes to test model

Phenomenology inspired taste corrections

Here explore naive staggerization of phenomenological model

$$C^{I=1}_{L,\mathrm{LLGS}}(t) \to C^{I=1}_{L,\mathrm{SLLGS}}(t; a^2\Delta^{\mathrm{KS}}) = \frac{1}{3L^3}\sum_i \sum_p \sum_{j=0}^{4} w_j |\langle 0|J_i|\pi^+(\vec{p})\pi^-(-\vec{p})\rangle|^2_{j,L} \; e^{-2E_{p,j}t}$$

- Compare model to lattice data using a sliding window

$$a^{\text{LO-HVP}}_{\mu,I=1}(t_{\mathrm{win}}, \Delta t, \Delta, Q^2 \leq Q^2_{\mathrm{max}}) = \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m_\mu^2}\right) \sum_{t=0}^{T/2} \big[\Theta(t; t_{\mathrm{win}}, \Delta) - \Theta(t; t_{\mathrm{win}} + \Delta t, \Delta)\big] W(t m_\mu, Q^2_{\mathrm{max}}/m_\mu^2)\, \mathrm{Re}C^{I=1}_L(t)$$

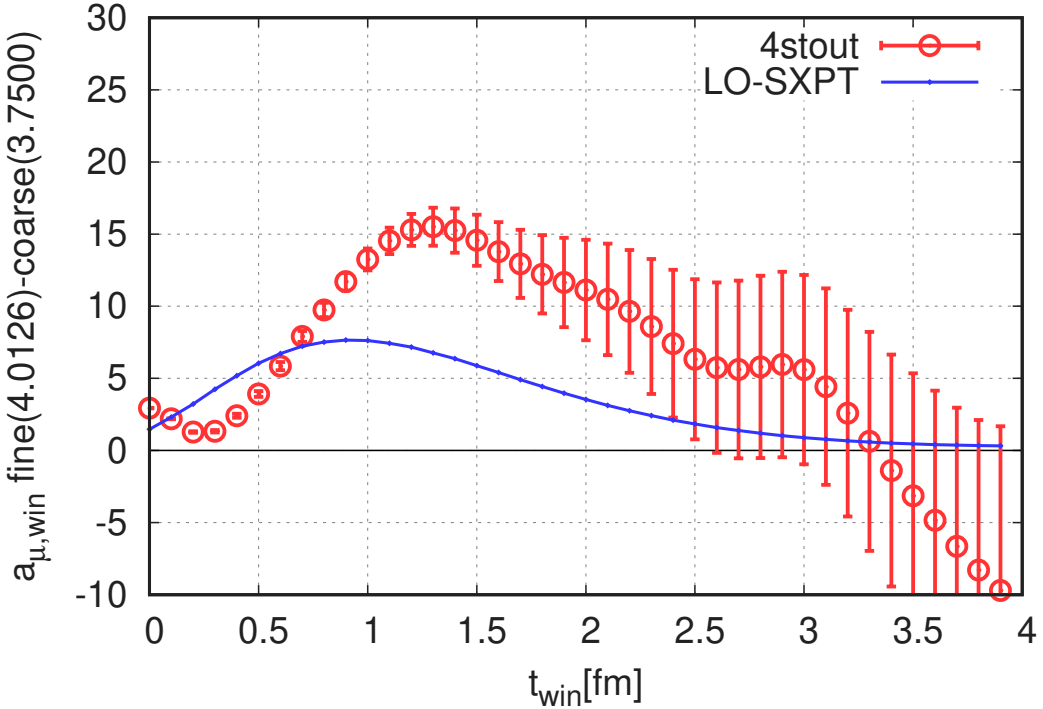
w/ $\Theta(t, t_0, \Delta) = [1 + \tanh[(t - t_0)]/\Delta]/2$ as in RBC/UKQCD 18

- Take $\Delta t = 0.5\,\mathrm{fm}$, $\Delta = 0.15\,\mathrm{fm}$ and slide window in steps of $0.1\,\mathrm{fm}$
- Implement using Gounaris-Sakurai (GS) model (Francis et al 13)

Sliding window: lattice vs LO-SχPT

$$\Delta^{\text{lat}}_{\text{taste}}(t_{\text{win}}) = a^{\text{LO-HVP}}_{\mu,l=1,\text{lat}}(t_{\text{win}}, L, a_{\text{fine}}) - a^{\text{LO-HVP}}_{\mu,l=1,\text{lat}}(t_{\text{win}}, L, a_{\text{coarse}})$$

$$\Delta^{\text{LO-S}\chi\text{PT}}_{\text{taste}}(t_{\text{win}}) = a^{\text{LO-HVP}}_{\mu,l=1,\text{LO-S}\chi\text{PT}}(t_{\text{win}}, L, (a^2\Delta^{\text{KS}})_{\text{fine}}) - a^{\text{LO-HVP}}_{\mu,l=1,\text{LO-S}\chi\text{PT}}(t_{\text{win}}, L, (a^2\Delta^{\text{KS}})_{\text{coarse}})$$

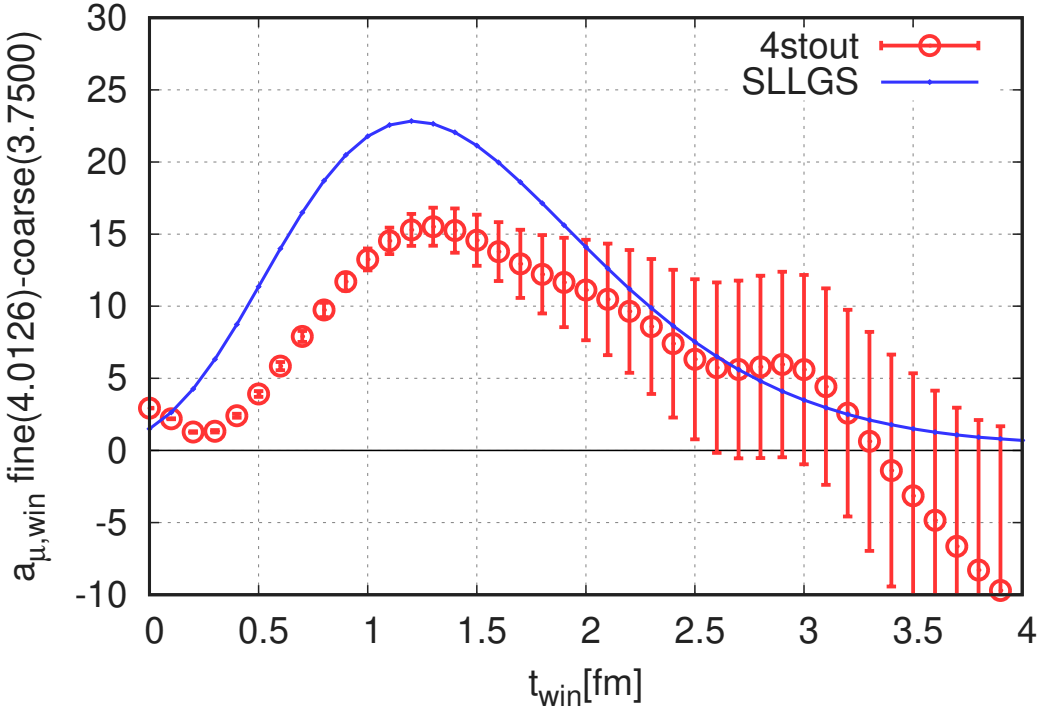


- LO SXPT describes taste-breaking corrections well for $t \gtrsim 2.0\,\text{fm}$
- Correct taste breaking in simulations with LO SXPT using either $t \geq 2.0\,\text{fm}$ or $t \geq 2.5\,\text{fm}$
- Use spread in continuum systematic error

Sliding window: lattice vs SLLGS - PRELIMINARY

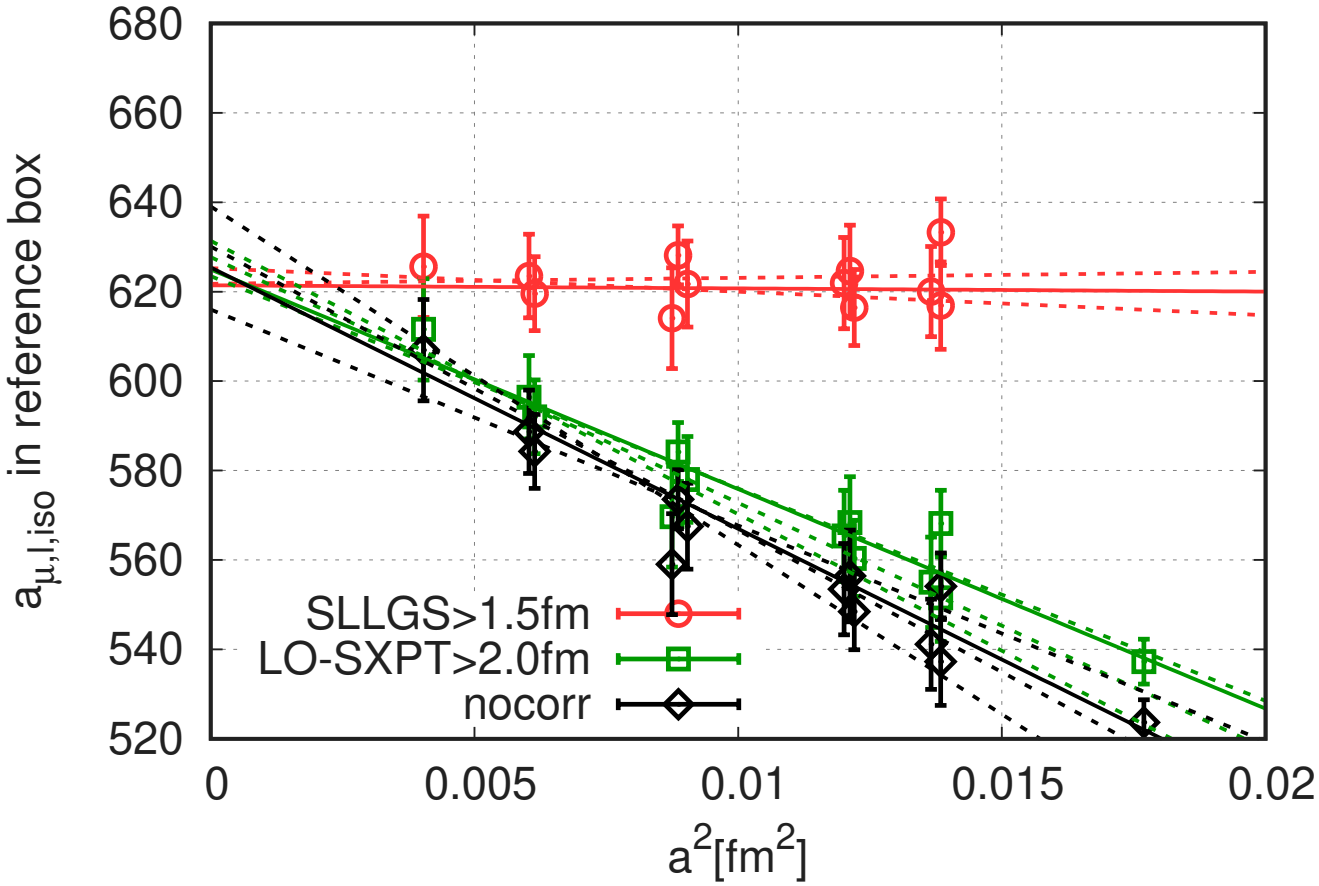
$$\Delta^{\text{lat}}_{\text{taste}}((t_{\text{win}}) = a^{\text{LO-HVP}}_{\mu,l=1,\text{lat}}(t_{\text{win}}, L, a_{\text{fine}}) - a^{\text{LO-HVP}}_{\mu,l=1,\text{lat}}(t_{\text{win}}, L, a_{\text{coarse}})$$

$$\Delta^{\text{SLLGS}}_{\text{taste}}(t_{\text{win}}) = a^{\text{LO-HVP}}_{\mu,l=1,\text{SLLGS}}(t_{\text{win}}, L, (a^2\Delta^{\text{KS}})_{\text{fine}}) - a^{\text{LO-HVP}}_{\mu,l=1,\text{SLLGS}}(t_{\text{win}}, L, (a^2\Delta^{\text{KS}})_{\text{coarse}})$$



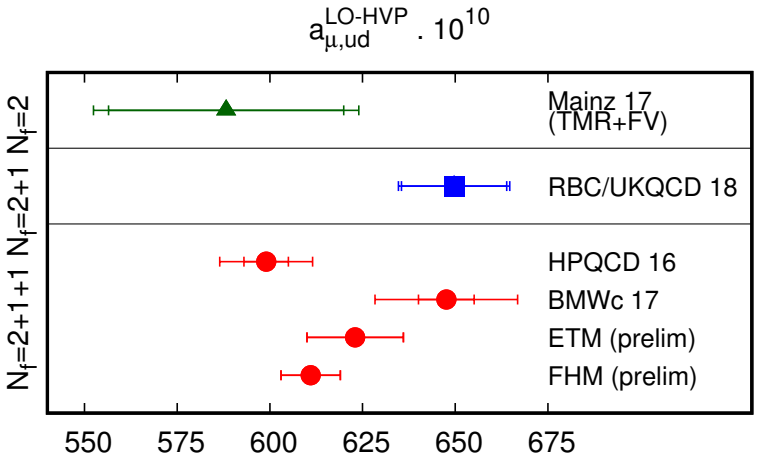
- SLLGS describes taste-breaking corrections well for $t \gtrsim 1.5\,\text{fm}$
- Correct taste breaking in simulations with SLLGS using either $t \geq 1.5\,\text{fm}$ or $t \geq 2.0\,\text{fm}$
- Use spread in continuum systematic error

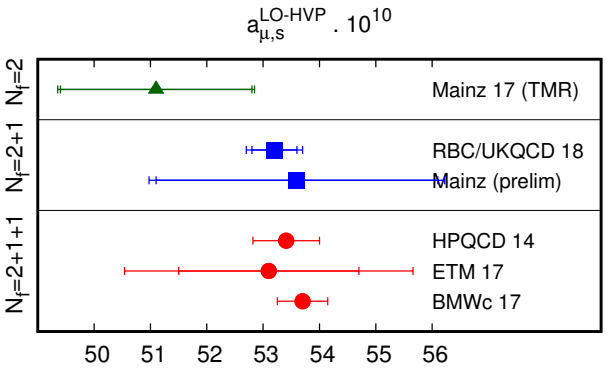
Continuum extrapolations of $a_{\mu,ud}^{\text{LO-HVP}}$ - PRELIMINARY

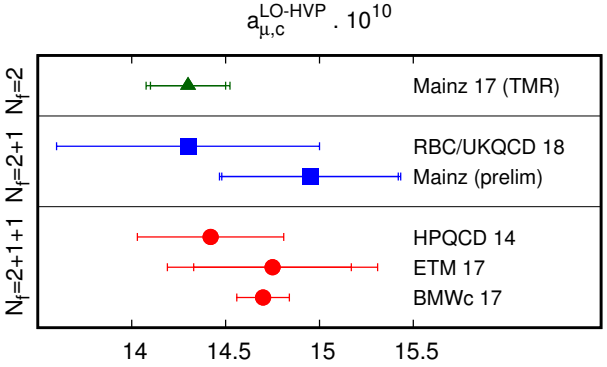


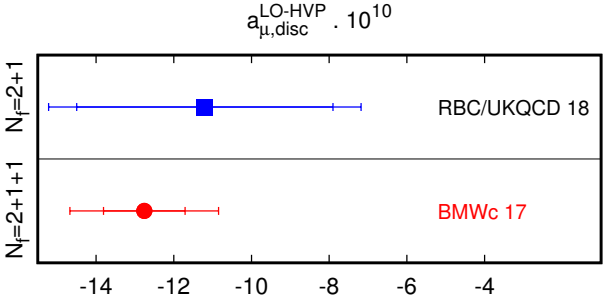
error [%] / correction	none	LO-SχPT	SLLGS
$\delta^{\text{stat}}(a_{\mu,ud}^{\text{LO-HVP}})$	1.5	1.5	1.6
$\delta^{a\text{-extrap}}(a_{\mu,ud}^{\text{LO-HVP}})$	2.2	1.0	0.6

More detailed comparison







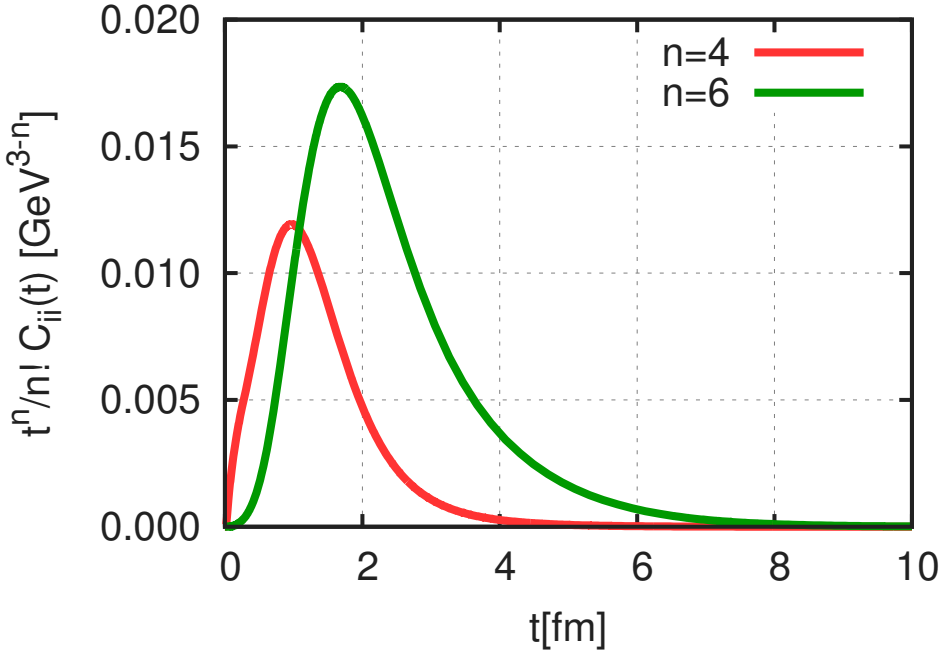


- BMWc '17 *ud* contribution is larger than $N_f = 2+1+1$ HPQCD '16 and $N_f = 2$ Mainz '17 results ($\sim 2.1/1.5\sigma$)
- BMWc '17 is only calculation performed directly at physical quark masses w/ 6 *a*'s to fully control continuum extrapolation
- $a_{\mu,s,c,\text{disc}}^{\text{LO-HVP}}$ already known with high enough precision for FNAL E989

Time moments

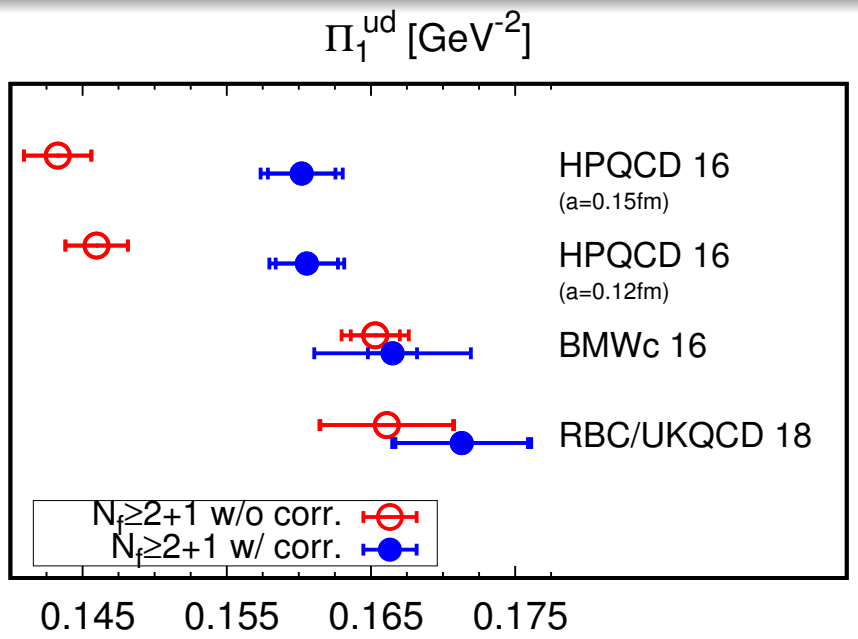
$$\Pi_n = \frac{1}{n!}\frac{\partial^n}{\partial Q^{2n}}\Pi(Q^2)|_{Q^2=0} = a\sum_t (-1)^n \frac{t^{2n+2}}{(2n+2)!}C(t)$$

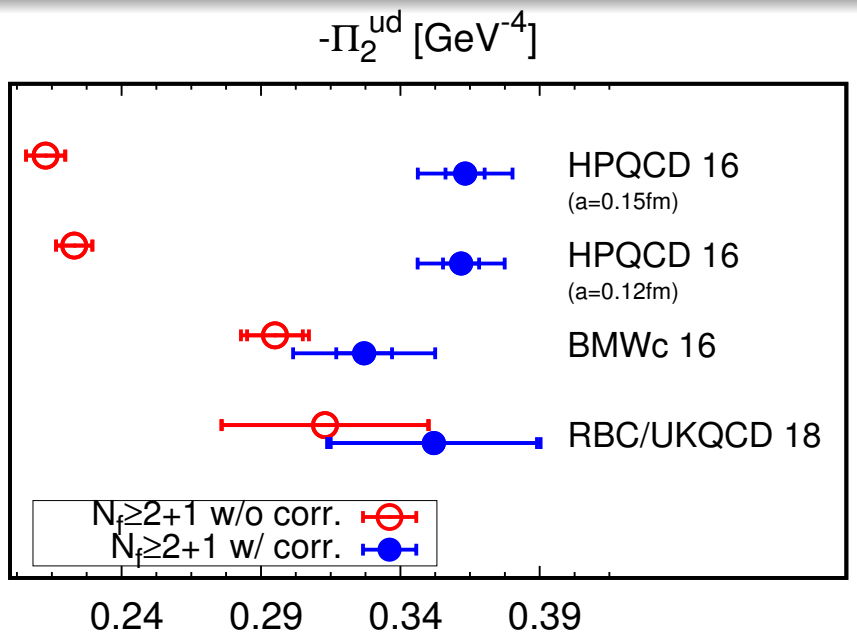
Larger n probe larger distances



$$\Pi_n(x) = \frac{5}{9}\Pi_n^{ud}(x) + \frac{1}{9}\Pi_n^{s}(x) + \frac{4}{9}\Pi_n^{c}(x) + \frac{1}{9}\Pi_n^{\text{disc}}(x)$$

Derivatives of $\Pi(Q^2)$ at $Q^2 = 0$: *ud* contribution

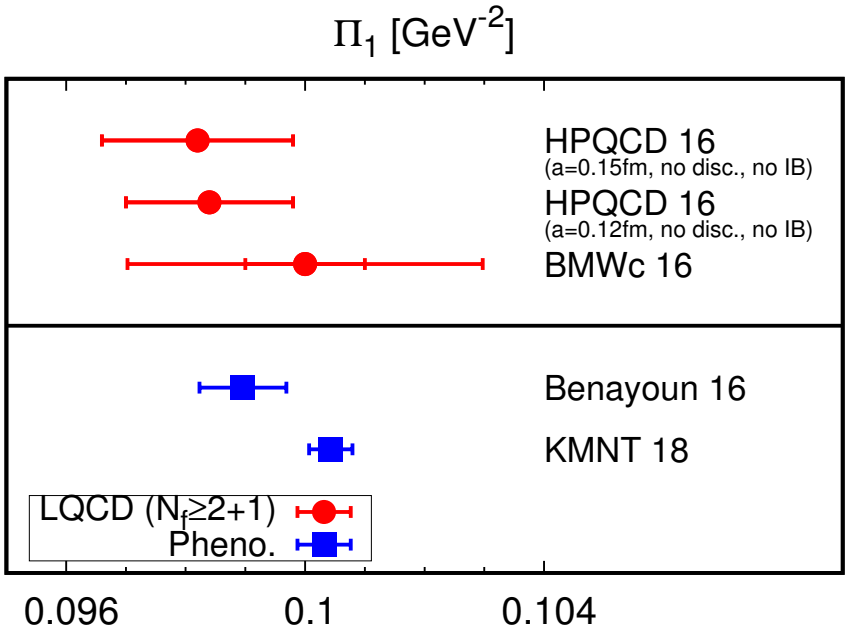


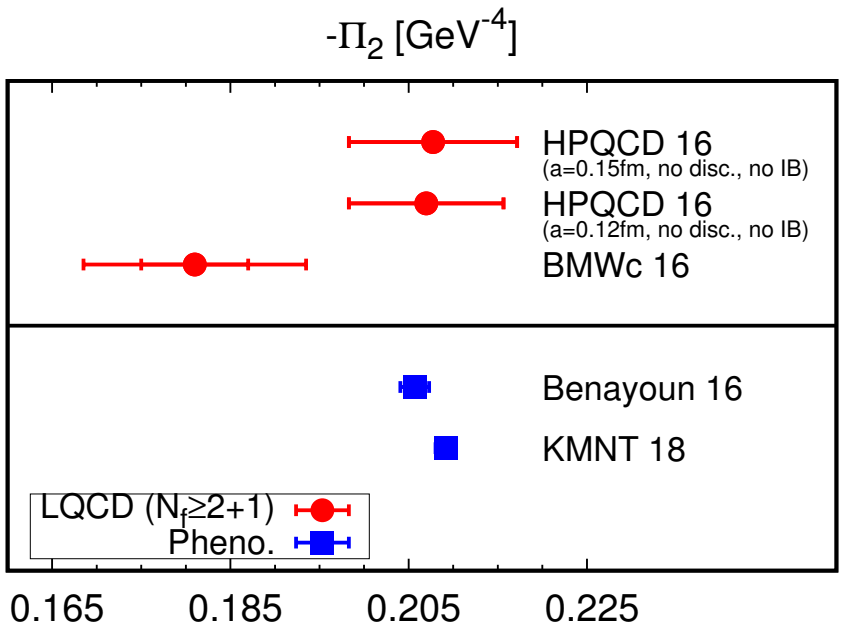


- In Padé picture (and probably generally) larger $\Pi_1 \rightarrow$ larger a_μ
- Larger $-\Pi_2 \rightarrow$ smaller a_μ
- HPQCD 16 has slightly smaller Π_1^{ud} and larger $-\Pi_2^{ud}$ than BMWc 16 and RBC/UKQCD 18 $\rightarrow$ combine to give smaller $a_{\mu,ud}^{\text{LO-HVP}}$
- Suggests that HPQCD 16 has smaller $C(t)$ for $t \sim 1\,\text{fm}$ but larger for $t \gtrsim 2\,\text{fm}$
- Difference comes from HPQCD 16's large corrections

Comparison of derivatives of $\Pi(Q^2)$ at $Q^2 = 0$

Add all flavor components and compare to phenomenology

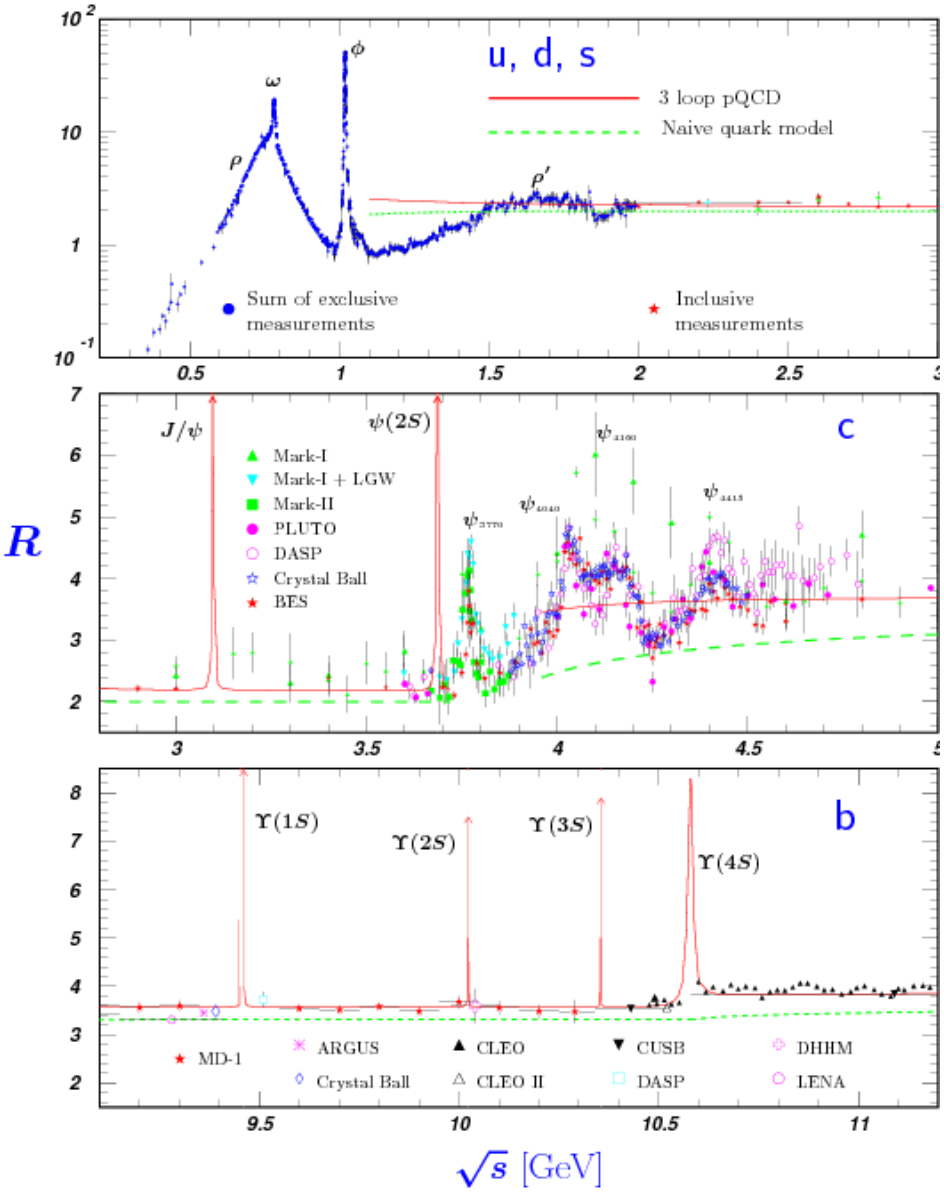




BMWc 16 has Π_1 comparable to phenomenology but smaller $-\Pi_2$

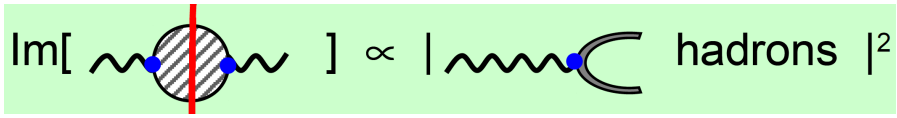
$\rightarrow$ suggests that BMWc (and RBC/UKQCD) has $C(t)$ slightly larger for $t \sim 1\,\mathrm{fm}$ and smaller for $t \gtrsim 2\,\mathrm{fm}$

HVP from $e^+e^- \to$ had (or $\tau \to \nu_\tau +$ had)



(PDG compilation)

Use (Bouchiat et al 61) optical theorem (unitarity)



$$\mathrm{Im}\Pi(s) = -\frac{R(s)}{12\pi}, \qquad R(s) \equiv \frac{\sigma(e^+e^- \to \text{had})}{4\pi\alpha(s)^2/(3s)}$$

and a once subtracted dispersion relation (analyticity)

$$\hat{\Pi}(Q^2) = \int_0^\infty ds \, \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s) = \frac{Q^2}{12\pi^2} \int_0^\infty ds \, \frac{1}{s(s+Q^2)} R(s)$$

$\Rightarrow \hat{\Pi}(Q^2)$ from data: sum of exclusive $\pi^+\pi^-$ etc. channels from CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.

Can also use $I(J^{PC}) = 1(1^{--})$ part of $\tau \to \nu_\tau +$ had and isospin symmetry + corrections

LO-HVP from $e^+e^- \to$ had

- $O(\alpha^2)$ but includes some $O(\alpha^3)$ corrections:
  - final state radiation (FSR): $\sigma(e^+e^- \to \text{had} + \gamma_{\text{FSR}})$, e.g. $\pi^+\pi^-\gamma$ $\to$ required for IR finite cross section at $O(\alpha^3)$
  - radiative modes, e.g. $\pi^0\gamma$ & $\eta\gamma$

- Three recent values:

$$
\begin{aligned}
a_\mu^{\text{LO-HVP}} &= 693.27(2.46) \times 10^{-10} \quad [3.5‰] && \text{(KNT '18)} \\
&= 693.1(3.4) \times 10^{-10} \quad [4.9‰] && \text{(DHMZ '17)} \\
&= 688.1(4.1) \times 10^{-10} \quad [6.0‰] && \text{(Jegerlehner '17)}
\end{aligned}
$$

- Higher orders:

$$
\begin{aligned}
a_\mu^{\text{NLO-HVP}} &= -9.87(0.09) \times 10^{-10} && \text{(Kurz et al '14)} \\
a_\mu^{\text{NNLO-HVP}} &= 1.24(0.01) \times 10^{-10} && \text{(Kurz et al '14)}
\end{aligned}
$$