Hadronic vacuum polarization contribution to the muon magnetic moment from lattice QCD

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<table>
<thead>
<tr>
<th>SM contribution</th>
<th>$a_{\mu}^{\text{contrib.}} \times 10^{11}$</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED [5 loops]</td>
<td>$116584718.841 \pm 0.034$</td>
<td>[Aoyama et al ‘18]</td>
</tr>
<tr>
<td>HVP LO</td>
<td>$6933 \pm 25 [0.32%]$</td>
<td>[KNT ‘18]</td>
</tr>
<tr>
<td></td>
<td>$6939 \pm 40 [0.58%]$</td>
<td>[DHMZ ‘19]</td>
</tr>
<tr>
<td></td>
<td>$6881 \pm 41 [0.60%]$</td>
<td>[Jegerlehner ‘17]</td>
</tr>
<tr>
<td>HVP NLO</td>
<td>$-98.7 \pm 0.9$</td>
<td>[Kurz et al ‘14]</td>
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<tr>
<td></td>
<td></td>
<td>[Kurz et al ‘14, Jegerlehner ‘16]</td>
</tr>
<tr>
<td>HVP NNLO</td>
<td>$12.4 \pm 0.1$</td>
<td>[Jegerlehner ‘16]</td>
</tr>
<tr>
<td>HLbyL</td>
<td>$105 \pm 26 [25%]$</td>
<td>[Prades et al ‘09]</td>
</tr>
<tr>
<td></td>
<td>$54 \pm 14 \pm ?? [??%]$</td>
<td>[RBC ‘16]</td>
</tr>
<tr>
<td>Weak (2 loops)</td>
<td>$153.6 \pm 1.0$</td>
<td>[Gnendiger et al ‘15]</td>
</tr>
<tr>
<td>SM Tot [0.31 ppm]</td>
<td>$116591824 \pm 36$</td>
<td>[w/ KNT ‘18]</td>
</tr>
<tr>
<td></td>
<td>[0.41 ppm]</td>
<td>$116591830 \pm 48$</td>
</tr>
<tr>
<td></td>
<td>[0.42 ppm]</td>
<td>$116591772 \pm 49$</td>
</tr>
<tr>
<td>Exp [0.54 ppm]</td>
<td>$116592091 \pm 63$</td>
<td>[Bennett et al ‘06]</td>
</tr>
<tr>
<td>Exp – SM [3.7\sigma]</td>
<td>$267 \pm 72$</td>
<td>[KNT ‘18]</td>
</tr>
<tr>
<td></td>
<td>[3.3\sigma]</td>
<td>$261 \pm 79$</td>
</tr>
<tr>
<td></td>
<td>[4.0\sigma]</td>
<td>$319 \pm 80$</td>
</tr>
</tbody>
</table>
Today $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} \simeq 3.3 \div 4.0 \sigma$, w/ $\delta a_\mu^{\text{exp}} \sim \delta a_\mu^{\text{SM}}$

SM error dominated by those of the HVP ($\sim 80\%$) and the HLbyL ($\sim 55\%$) contributions

Fermilab E989 began in 2017 and aims for 0.14 ppm vs 0.54 ppm today

J-PARC E34 to begin $\geq$ 2021 and aims for similar precision

⇒ to fully leverage requires $\delta a_\mu^{\text{had}} \leq 1.6 \times 10^{-10}$ vs $4.8 \times 10^{-10}$ today

If exp. & SM central values stay the same:

- E989 alone (exp. error /4) $\rightarrow 5 \div 7 \sigma$
- E989 & equally precise theory $\rightarrow 12 \div 14 \sigma$

But before concluding presence of BSM physics

⇒ need independent, first-principle determinations of these 2, nonperturbative hadronic contributions

⇒ Lattice QCD

Discrepancy is:

| ~ 2× electroweak contribution |
| ~ 3.6× the HLbyL contribution |
| ~ 4% of the LO-HVP contribution |
Consider in Euclidean spacetime (Blum ‘02)

\[
\Pi_{\mu\nu}(Q) = \int d^4x e^{iQ \cdot x} \langle J_\mu(x) J_\nu(0) \rangle = \left( Q_\mu Q_\nu - \delta_{\mu\nu} Q^2 \right) \Pi(Q^2)
\]

\[
w/ J_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s + \frac{2}{3} \bar{c} \gamma_\mu c + \cdots
\]

Then (Lautrup et al ‘69, Blum ‘02)

\[
a^{\text{LO-HVP}}_\ell = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty \frac{dQ^2}{m_\ell^2} w(Q^2/m_\ell^2) \hat{\Pi}(Q^2)
\]

\[
w/ \hat{\Pi}(Q^2) \equiv \left[ \Pi(Q^2) - \Pi(0) \right]
\]

- Integrand peaked for \( Q \sim (m_\ell/2) \sim 50 \text{ MeV} \) for \( \ell = \mu \)
- 80% of \( a^{\text{LO-HVP}}_\mu \) comes from \( Q \leq 300 \text{ MeV} \sim \Lambda_{\text{QCD}} \)

⇒ \( \hat{\Pi}(Q^2) \) must be computed very precisely for \( Q \) in QCD’s nonperturbative domain

(HVP from Jegerlehner, “alphaQEDc17” (2017))
Low-$Q^2$ challenges in finite volume (FV)

A. $\Pi_{\mu\nu}(Q = 0) \neq 0$ in FV $\Rightarrow \Pi_{\mu\nu}(Q)/Q^2 \sim 0 \Pi(Q^2) + \text{cst}/Q^2$

B. On-shell renormalization requires $\Pi(0)$ which is problematic (see above)

C. Need $\hat{\Pi}(Q^2)$ interpolation due to $Q_{\text{min}} = 2\pi/T \sim 135\,\text{MeV} > \frac{m_\mu}{2} \sim 50\,\text{MeV}$ for $T \sim 9\,\text{fm}$

↓

- Compute on $T \times L^3$ lattice
  
  $$C_L(t) = \frac{a^3}{3} \sum_{i=1}^3 \sum_{\vec{x}} \langle J_i(x)J_i(0) \rangle$$

- Decompose ($C_L^{I=1} = \frac{9}{10} C_L^{ud}$)
  
  $$C_L(t) = C_L^{ud}(t) + C_L^{s}(t) + C_L^{c}(t) + C_L^{\text{disc}}(t)$$
  
  $$= C_L^{I=1}(t) + C_L^{I=0}(t)$$

- Define (Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...) (ad A, B, C)

  $$\hat{\Pi}_L'(Q^2) \equiv \Pi_L'(Q^2) - \Pi_L'(0) = \frac{1}{3} \sum_{i=1}^3 \frac{\Pi_{ii,L}'(0) - \Pi_{ii,L}'(Q)}{Q^2} - \Pi_L'(0) = 2a \sum_{t=0}^{T/2} \text{Re} \left[ \frac{e^{iQt}-1}{Q^2} + \frac{t^2}{2} \right] \text{Re} C_L'(t)$$
Combining everything, get $a^{\text{LO-HVP}}_{\ell,f}$ from $C^f_L(t)$:

$$a^{\text{LO-HVP}}_{\ell,f}(Q^2 \leq Q^2_{\text{max}}) = \lim_{a \to 0, L \to \infty, T \to \infty} \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{a}{m^2_{\ell}} \right)^{T/2} \sum_{t=0}^{T/2} W(tm_{\ell}, Q^2_{\text{max}}/m^2_{\ell}) \text{Re} C^f_L(t)$$

where

$$W(\tau, x_{\text{max}}) = \int_0^{x_{\text{max}}} dx \, w(x) \left( \tau^2 - \frac{4}{x} \sin^2 \frac{\tau \sqrt{x}}{2} \right)$$

(BMWc '17, 144 × 96^3, $a \sim 0.064$ fm, $M_{\pi} \sim 135$ MeV)
Simulation challenges

D. $\pi\pi$ contribution very important $\rightarrow$ must have physically light $\pi$

E. Two types of contributions

- quark-connected (qc)
- quark-disconnected (qd)

where qd contributions are SU(3)$_f$ and Zweig suppressed but very challenging

F. $C_L^{ud,\text{disc}}(t)$ have \( N/S \sim \exp\left[(M_\rho - M_\pi)t\right] \) + need high-precision results

$\rightarrow$ need very high statistics + many algorithmic improvements + rigorous bounds on large $t$ behaviour $C_L^{ud,\text{disc}}(t)$
G. Need large volumes to control long-distance, $2\pi$ contributions to $C_{ud, \text{disc}}^L(t)$ (Aubin et al '16)

H. Need controlled continuum limit $\rightarrow$ must have several $a \leq 0.1$ fm

I. Need $\hat{\Pi}(Q^2)$ for $Q^2 \in [0, +\infty[$, but $\frac{\pi}{a} \sim 10$ GeV for $a \sim 0.06$ fm
   $\rightarrow$ match onto perturbation theory

$$a_{\ell, f}^{\text{LO-HVP}} = a_{\ell, f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_{\ell}(Q_{\text{max}}) \hat{\Pi}^f(Q_{\text{max}}^2) + \Delta_{\text{pert}}^{\text{LO-HVP}} a_{\ell, f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

J. Include $c$ quark in sea for higher precision and good matching onto perturbation theory

K. Large gauge ensembles typically generated with $m_u = m_d$ and $\alpha = 0$
   $\Rightarrow$ missing effects that are relevant at % level
   $\rightarrow$ must be included
Some lattice results suggest new physics others not but all compatible with phenomenology.

Differences originate mostly from treatment of noisy, large $t$ behaviour of $C^{ud}_L(t)$ and of its FV effects.

Contributions from $s$, $c$ and disc mainly agree w/ small contributions to total error.

Lattice errors $\gtrsim 2\%$ vs phenomenology errors $\sim 0.4\%$.

Must reduce lattice error to $< 1\%$ to have impact.
15 high-statistics simulations w/ $N_f=2+1+1$ flavors of 4-stout staggered quarks:

- Bracketing physical $m_{ud}$, $m_s$, $m_c$
- 6 a's: $0.134 \rightarrow 0.064$ fm
- $L = 6.1 \div 6.6$ fm, $T = 8.6 \div 11.3$ fm
- Conserved EM current
- Close to 9M / 39M conn./disc. measurements w/ AMA

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a$ [fm]</th>
<th>$T \times L$</th>
<th>#conf-conn</th>
<th>#conf-disc</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.7000</td>
<td>0.134</td>
<td>64 $\times$ 48</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>3.7753</td>
<td>0.111</td>
<td>84 $\times$ 56</td>
<td>1500</td>
<td>1500</td>
</tr>
<tr>
<td>3.8400</td>
<td>0.095</td>
<td>96 $\times$ 64</td>
<td>2500</td>
<td>1500</td>
</tr>
<tr>
<td>3.9200</td>
<td>0.078</td>
<td>128 $\times$ 80</td>
<td>3500</td>
<td>1000</td>
</tr>
<tr>
<td>4.0126</td>
<td>0.064</td>
<td>144 $\times$ 96</td>
<td>450</td>
<td>-</td>
</tr>
</tbody>
</table>

Error on total:

- Stat. = 1.1% (all from $ud$)
- $a \rightarrow 0$ syst. = 1.1% (mostly from $ud$)
- $a$ syst. = 0.8%
- $FV = 1.9%$ (all from $ud$) (NLO $\chi$PT)
- IB = 0.7% (Pheno.)
- Total = 2.7%
Path to HVP Nirvana

To reach sub-percent accuracy on lattice, must:

- reduce statistical error on $ud$ contribution significantly
- correct FV effects reliably (preferably with LQCD)
- control quark-disconnected contributions → already done (BMWc '16 & '17, RBC/UKQCD '16)
- determine $m_u \neq m_d$ and EM corrections on lattice: only leading order in $(m_d - m_u)$ and $\alpha$ is needed
- determine $a$ very precisely: $\frac{\sigma_{a_{LO-HVP}}^\mu}{a_{LO-HVP}} \sim 2 \times \frac{\sigma_a}{a}$
- staggered: must improve continuum extrapolation of $a_{\mu,ud}^{LO-HVP}$
Use all-mode averaging (AMA) \cite{Blum13} as in BMWc ’16,’17 w/ low-mode averaging (LMA) \cite{Giusti03}:

- compute many low modes and evs of Dirac operator on each configuration
- use to explicitly construct largest, long-distance contribution to correlator from all-to-all lattice points (instead of single to all)
- contributions of high modes obtained by inversion in orthogonal complement

→ allows a huge increase in statistics \( \propto T \times L^3 \)
→ implemented in RBC/UKQCD ’18, Aubin et al ’19 and BMWc (in progress)

\( a_{\mu, ud} \) LO-HVP \( \times 10^{10} \)

(old: PRL2018  
new: with LMA)

(BWMc in progress, very numerically expensive part has yet to be done)

\( \Rightarrow \) can reach few permil statistical precision at fixed lattice spacing and volume w/ an additional trick …
In FV, spectrum is discrete, and spectral decomposition of EM current correlator is 

\[ C^{ud}_L(t) = \sum_{n \geq 1} |Z_n|^2 \varphi(E_n, t) \]

\[ \varphi(E_n, t) = \cosh [E_n(T/2 - t)] \]

Upper and lower bounds for \( t > t_c \) (BMWc ’16 & ’17, RBC/UKQCD ’18, Blum et al ’19)

\[ 0 \leq \frac{C^{ud}_L(t_c)}{\varphi(E_c, t_c)} \varphi(E_c, t) \leq C^{ud}_L(t) \leq \frac{C^{ud}_L(t_c)}{\varphi(E_1, t_c)} \varphi(E_1, t) \]

- \( E_1 \) is FV ground state energy \( \sim \) two back-to-back pions w/ one lattice unit of momentum
- \( E_c \) is effective mass of \( C^{ud}_L(t) \) at \( t_c \)

Use lattice \( C^{ud}_L(t) \) for \( t \leq t_c \)

Use average of bounds for \( t > t_c \) above meeting point

Vary \( t_c \) for systematic error

Similar bounds for \( C^{disc}_L(t) \)
Determine $E_n$ & $|Z_n|$ for lowest $N$ states

Requires:

- constructing a matrix of correlation functions composed of $\geq N$ two-pion and rho-meson operators and $j_\mu$

- solving a GEVP to extract the desired $E_n$ & $|Z_n|$
- RBC/UKQCD (Lattice '19) checked possible contributions from four-pion states!

Use to reconstruct large $t$ tail of $C^{ud}_L(t)$

$C^{ud}_L(t) \xrightarrow{t \to \infty} \sum_{n=1}^{N} |Z_n|^2 \varphi(E_n, t)$

Can be used for improving bound method

(Mainz '19, also RBC/UKQCD (Lattice '18))
Use $N$ lowest states to improve bounds (Mainz '19, RBC/UKQCD Lattice '18)

$$\Delta C_{L}^{ud}(t) = C_{L}^{ud}(t) - \sum_{n=1}^{N} |Z_n|^2 \varphi(E_n, t), \quad \varphi(E_n, t) = \cosh \left[ E_n \left( T/2 - t \right) \right]$$

Upper and lower bounds are now

$$0 \leq \Delta C_{L}^{ud}(t) \frac{\varphi(E_c, t)}{\varphi(E_c, t_c)} \leq \Delta C_{L}^{ud}(t) \frac{\varphi(E_{N+1}, t)}{\varphi(E_{N+1}, t_c)}$$

Meet for earlier $t_c$ and resulting $C_{L}^{ud}(t)|_{new} \equiv \Delta C_{L}^{ud}(t)|_{bnd \ avg} + \sum_{n=1}^{N} |Z_n|^2 \varphi(E_n, t)$ has reduced errors
Dominated by long-range, low-lying $2\pi$ states and surprisingly large ($\sim 3\%$ for $L \sim 6$ fm)

According to NLO $\chi$PT (Aubin et al '16), correction is $\sim (13 \pm 13) \times 10^{-10}$ (BMWc '17) for $L \sim 6$ fm

Modeling of low-lying $2\pi$ contributions (Meyer '11) using Lüscher for $E_n$ and Lellouch-Lüscher for $|Z_n|$, using Gounaris-Sakurai phase shift, gives $\sim 20 \times 10^{-10}$

$\chi$PT (Aubin et al '19, Bijnens et al '17) gives $\sim 18 \times 10^{-10}$

$\sim 20 \times 10^{-10}$ confirmed by LQCD calculations in different volumes, w/ large errors (Shintani et al '19, RM123 '18, RBC/UKQCD Lattice '19)

Combination of LQCD calculations of FV corrections and models will help reduce current FV uncertainty to few permil

(Shintani et al '19, $L = 5.4, 10.8$ fm)
Including isospin breaking on the lattice

\[ S_{\text{QCD+QED}} = S_{\text{QCD}}^{\text{iso}} + \frac{1}{2} (m_u - m_d) \int (\bar{u}u - \bar{d}d) + ie \int A_\mu j_\mu, \quad j_\mu = \bar{q} \gamma_\mu q \]

- Separation into isospin limit results and corrections requires an unambiguous definition of this limit (scheme and scale)
- Must be included not only in calculation of \( \langle j_\mu j_\nu \rangle \) correlator BUT ALSO of all quantities used to fix quark masses and scale

(1) operator insertion method (RM123 '12, '13, ...) 

\[ \langle \mathcal{O} \rangle_{\text{QCD+QED}} = \langle \mathcal{O} \rangle_{\text{QCD}}^{\text{iso}} - \frac{1}{2} (m_u - m_d) \langle \mathcal{O} \rangle_{\text{QCD}}^{\text{iso}} \left( \int (\bar{u}u - \bar{d}d) \right)_{\text{QCD}}^{\text{iso}} \]

\[ + \frac{1}{2} e^2 \langle \mathcal{O} \rangle_{xy} \int j_\mu (x) D_{\mu \nu} (x - y) j_\nu (y) \rangle_{\text{QCD}}^{\text{iso}} + \text{hot} \]

(2) direct method (Eichten et al '97, BMWc '14, ...) 

Include \( m_u \neq m_d \) and QED directly in calculation of observables and generation of gauge configurations

(3) combinations of (1) & (2)

For valence effects, (2) is often easier, and for sea effects, (1) is
Strong isospin-breaking corrections

(1) perturbative approach

Sea-quark corrections are \((m_d - m_u)^2 \Rightarrow R = 0\)

For the moment, only \(M\) is calculated

But \(1/N_c\) \(O\) will have to be calculated for < 1% accuracy

\[
\Delta^{\text{SIB-M}} a_{\mu}^{\text{LO-HVP}} = 10.6(4.3)(6.8) \times 10^{-10} \quad (\text{RBC/UKQCD '18})
\]

\[
\Delta^{\text{SIB-M}} a_{\mu}^{\text{LO-HVP}} = 6.0(2.3) \times 10^{-10} \quad (\text{ETMC '19})
\]

(2) direct approach \((\text{HPQCD '18})\)

\[
\Delta^{\text{SIB-M}} a_{\mu}^{\text{LO-HVP}} = 7.7(3.7) \times 10^{-10}, \text{ only valence}
\]

\[
\Delta^{\text{SIB-M}} a_{\mu}^{\text{LO-HVP}} = 9.0(2.3) \times 10^{-10}, \text{ w/ sea}
\]
QED corrections

For the moment, only $V$, $S$ and $F$ are calculated.

Other diagrams are $1/N_c$, but must be calculated for $< 1\%$ accuracy.

Results

\[ \Delta_{\text{QED-V+S}} a_\mu^{\text{LO-HVP}} = 5.9(5.7)(1.7) \times 10^{-10} \quad (\text{RBC/UKQCD '18}) \]

\[ \Delta_{\text{QED-V+S}} a_\mu^{\text{LO-HVP}} = 1.1(1.0) \times 10^{-10} \quad (\text{ETMC '19}) \]

\[ \Delta_{\text{QED-F}} a_\mu^{\text{LO-HVP}} = -6.9(2.1)(1.4) \times 10^{-10} \quad (\text{RBC/UKQCD '18}) \]
\( C(t) \) may be more precise in certain euclidean time ranges on lattice than in phenomenology.

→ combine lattice with phenomenology to reduce error (RBC/UKQCD '18 + Jegerlehner '17)

\[
a_{\mu, \text{LO-HVP}} = a_{\mu, \text{SD, pheno}} + a_{\mu, \text{W, lat}} + a_{\mu, \text{LD, pheno}}
\]
Conclusions

- Current error on lattice $a^{\text{LO-HVP}}_{\mu}$ is $\sim 18 \times 10^{-10}$
- Needs to be reduced by $\sim 4$ to match phenomenology and $\sim 10$, future $a_{\mu}$ measurements
- Error dominated by
  - Large distance noise on $C^{ud}_{L}(t)$
  - Uncontrolled FV corrections
  - Calculation of SID and QED corrections
  - Uncertainty on lattice spacing determination
  - Staggered: continuum extrapolation of $a^{\text{LO-HVP}}_{\mu, ud}$

As shown, all of these issues are being aggressively addressed and progress is being made quickly

- $\lesssim 1\%$ results should be available w/in a year and phenomenology should be matched by time of final FNAL E989 results

→ Combine LQCD and phenomenology to improve overall uncertainty (RBC/UKQCD '18), only if the two agree statistically with comparable errors

- If not, difference should be understood

- The muon $(g - 2)$ Theory Initiative has had a number of meetings and is preparing a first white paper
  → anyone interested is welcome to participate
BACKUP
Matching to perturbation theory: ad I & J

Consider separation ($\ell = e, \mu, \tau$)

\[
a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_{\ell}(Q_{\text{max}}) \hat{\Pi}(Q_{\text{max}}) + \Delta_{\text{pert}}^{\text{LO-HVP}}(Q > Q_{\text{max}})
\]

- Compute $\Delta_{\text{pert}}^{\text{LO-HVP}}(Q > Q_{\text{max}})$ using $R_{\text{pert}}(s)$ to $O(\alpha_s^4)$ from Harlander et al '03
- Not relevant for $\ell = e, \mu$ but important for $\tau$
- Perfect matching of continuum lattice results for $Q_{\text{max}}^2 \geq 2 \text{ GeV}^2$
  - control $\hat{\Pi}(Q^2)$ up to $Q^2 \rightarrow \infty$
- Get matching systematic from considering $Q_{\text{max}}^2 = 2$ and $5 \text{ GeV}^2$
Goldstone has more massive “taste” partners that dilute Goldstone contribution to $a_{\mu,ud}^{\text{LO-HVP}}$.

“Effective” pion mass larger at larger $a$, e.g. $M_{\pi}^{\text{RMS}} \simeq 310 \text{ MeV}$ for $a = 0.134 \text{ fm}$

Effect disappears in $a \to 0$ limit

$a \to 0$ extrapolation includes $M_{\pi}^{\text{RMS}} \to M_{\pi}^{\text{PDG}}$ extrapolation and is quite pronounced

\begin{center}
\begin{tabular}{|c|c|}
\hline
Function & Value \\
\hline
BMWc 17 & 6.50 \times 10^{-10} \\
BMWc 17 taste+FV & 6.60 \times 10^{-10} \\
BMWc 17 final res. & 6.70 \times 10^{-10} \\
HPQCD 16 & 6.65 \times 10^{-10} \\
HPQCD 16 taste+FV & 6.75 \times 10^{-10} \\
FHM (prelim) & 6.80 \times 10^{-10} \\
FHM (prelim) taste+FV & 6.85 \times 10^{-10} \\
\hline
\end{tabular}
\end{center}

FNAL/HPQCD/MILC 16 & prelim already include large $t$ modeling of $C(t)$
FV effects are long-distance effects, determined by lightest states contributing to process

Here \( I = J = 1 \), 2-\( \pi \) states

Determine in \( \chi \)PT, to LO (Aubin et al 15), i.e.

\[
C_{I=1}^{L,LO-\chi PT}(t) = \frac{1}{3L^3} \sum \frac{(\bar{p}_{\text{free}})}{E_p^{\text{free}}}^2 e^{-2E_p^{\text{free}}t}
\]

with \( E_p^{\text{free}} = \sqrt{M_\pi^2 + \bar{p}_{\text{free}}^2} \)

Then \( C_{I=1}^{L,LO-\chi PT}(t) - C_{I=1}^{L,LO-\chi PT}(t) \) can be used to estimate FV effects

Find, for \( M_\pi \sim 135 \text{ MeV} \) and \( L \sim 6 \text{ fm} \) (BMWc 17),

\[
\Delta_{FV} a_{\mu, I=1}^{LO-HVP} \sim 2.3\% \times a_{\mu, I=1}^{LO-HVP}
\]

\( \rightarrow \) probably \( O(50\%) \) too small (Della Morte et al 17, Shintani et al 19, Aubin et al 19 . . . )

Can do better
**LO $S\chi$PT for taste effects**

- **Taste-breaking effects** also mostly come from low-lying $2-\pi$ states

- Determine in $S\chi$PT, to LO (Aubin et al 15, HPQCD 17), i.e.

  $$C_{L,LO-S\chi PT}^{l=1}(t, a^2\Delta^{KS}) = \frac{1}{3L^3} \sum_{j=0}^{4} w_j \left( \frac{\vec{p}_{\text{free}}}{E_{\text{free}}^{p,j}} \right)^2 e^{-2E_{\text{free}}^{p,j} t}$$

  with $E_{\text{free}}^{p,j} = \sqrt{M_{\pi,j}^2 + \vec{p}_{\text{free}}^2}$

- Then $C_{L,LO-S\chi PT}^{l=1}(t, 0) - C_{L,LO-S\chi PT}^{l=1}(t, a^2\Delta^{KS})$ can be used to estimate **taste effects**

- Helps but is it possible to do better?
At LO, the two $\pi$ are free

⇒ omits strong $\rho - \pi \pi$ coupling
⇒ compute at NLO (Bijnens et al 99, Aubin et al 19)

NLO includes LO 2-$\pi$ rescattering and slope of $F_\pi(Q^2)$

However NLO only obtained in continuum (Aubin et al 19)

⇒ helps FV corrections: increase by $O(50\%)$ for $M_\pi \sim 135$ MeV and $L \sim 6$ fm
⇒ does not improve taste corrections
Add point-like $\rho$ to SXPT and model lattice $\rho$

- Construct field theory that couples $\gamma-\rho-\pi\pi$ (Jegerlehner 11)
- Gives much better description of $\Pi(Q^2)$ in continuum
- Add taste breaking to pion loop contributions and work out coupled system to one pion loop (HPQCD 17)

$\rightarrow$ gives FV and taste corrections similar to LO $S\chi$PT

$\Rightarrow$ further correct for taste breaking by measuring $m_{\mu\ell}$ in units of lattice “$M_{\rho}$” from usual correlator fits (ETMC 11, HPQCD 17) or by modelling $C_L(t)$ at large $t$ (FNAL/HPQCD/MILC 17)

Possible issues:

- $\rho$ not treated as a resonance in $M_{\rho}$-rescaling
- Reduction of number of (staggered) states in modelling of $C_L(t)$ is done by fitting, not e.g. by systematically eliminating taste copies
Phenomenology inspired FV corrections

Use (Meyer 11):

\[ C_{L,\text{LO-}\chi\text{PT}}^{l=1}(t) = \frac{1}{3L^3} \sum_{\vec{p}_{\text{free}}} \left( \frac{\vec{p}_{\text{free}}}{E_{\text{free}}} \right)^2 e^{-2E_{\text{free}}^t} \rightarrow C_{L,\text{LLGS}}^{l=1}(t) = \frac{1}{3L^3} \sum_{i} \sum_{p} |\langle 0| J_i |\pi^+ (\vec{p}) \pi^- (-\vec{p}) \rangle|_L^2 e^{-2E_{\text{free}}^t} \]

with

- Lüscher to get \( E_p = \sqrt{M_{\pi}^2 + p^2} \) in FV, where \( p = |\vec{p}| \) is momentum carried by each of the two interacting \( \pi \) in FV

- Lellouch-Lüscher (LL) for interacting \( |\langle 0| J_i |\pi^+ (\vec{p}) \pi^- (p) \rangle|_L \) in FV from free amplitude \( \frac{p_{\text{free}}}{E_p} \)

- \( 2-\pi, \delta_{i=1}^{l=1}(p) \) from phenomenology

Then \( C_{\infty,\text{LLGS}}^{l=1}(t) - C_{L,\text{LLGS}}^{l=1}(t) \) gives estimate of FV

\( \rightarrow \) good for FV corrections: increase by \( O(50\%) \) over LO \( \chi\text{PT} \) for \( M_{\pi} \sim 135 \text{ MeV} \) and \( L \sim 6 \text{ fm} \)

(Della Morte et al 17, Shintani et al 19, Aubin et al 19 . . . )

Implementing simulations at different volumes to test model
Here explore naive staggerization of phenomenological model

\[
C_{L,\text{LLGS}}^I(t) \rightarrow C_{L,\text{SLLGS}}^I(t; a^2\Delta^{KS}) = \frac{1}{3L^3} \sum_i \sum_p \sum_{j=0}^4 w_j \langle 0 | J_i | \pi^+ (\vec{p}) \pi^- (-\vec{p}) \rangle |_{j,L}^2 e^{-2E_{p,j}t}
\]

- Compare model to lattice data using a sliding window

\[
a_{\mu, I=1}^{\text{LO-HVP}} (t_{\text{win}}, \Delta t, \Delta, Q^2 \leq Q_{\text{max}}^2) = \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{a}{m_{\mu}^2} \right)^{T/2} \sum_{t=0}^{T/2} [\Theta(t; t_{\text{win}}, \Delta) \\
- \Theta(t; t_{\text{win}} + \Delta t, \Delta)] W(t_{m_{\mu}}, Q_{\text{max}}^2/m_{\mu}^2) \text{Re} C_{L=1}^I(t)
\]

w/ \( \Theta(t, t_0, \Delta) = [1 + \tanh[(t - t_0)]/\Delta]/2 \) as in RBC/UKQCD 18

- Take \( \Delta t = 0.5 \text{ fm}, \Delta = 0.15 \text{ fm} \) and slide window in steps of 0.1 fm

- Implement using Gounaris-Sakurai (GS) model (Francis et al 13)
\[ \Delta_{\text{taste}}^{\text{lat}}(t_{\text{win}}) = a_{\mu, l=1, \text{lat}}^{\text{LO-HVP}}(t_{\text{win}}, L, a_{\text{fine}}) - a_{\mu, l=1, \text{lat}}^{\text{LO-HVP}}(t_{\text{win}}, L, a_{\text{coarse}}) \]

\[ \Delta_{\text{taste}}^{\text{LO-S} \chi \text{PT}}(t_{\text{win}}) = a_{\mu, l=1, \text{LO-S} \chi \text{PT}}^{\text{LO-HVP}}(t_{\text{win}}, L, (a^2 \Delta_{\text{KS}})_{\text{fine}}) - a_{\mu, l=1, \text{LO-S} \chi \text{PT}}^{\text{LO-HVP}}(t_{\text{win}}, L, (a^2 \Delta_{\text{KS}})_{\text{coarse}}) \]

- **LO SXPT** describes **taste-breaking** corrections well for \( t \gtrsim 2.0 \text{ fm} \)

- Correct **taste breaking** in simulations with **LO SXPT** using either \( t \geq 2.0 \text{ fm} \) or \( t \geq 2.5 \text{ fm} \)

- Use spread in continuum systematic error
\[
\Delta_{\text{lat}}^{\text{taste}}(t_{\text{win}}) = a_{\mu,l=1,\text{lat}}^{\text{LO-HVP}}(t_{\text{win}}, L, a_{\text{fine}}) - a_{\mu,l=1,\text{lat}}^{\text{LO-HVP}}(t_{\text{win}}, L, a_{\text{coarse}})
\]

\[
\Delta_{\text{taste}}^{\text{SLLGS}}(t_{\text{win}}) = a_{\mu,l=1,\text{SLLGS}}^{\text{LO-HVP}}(t_{\text{win}}, L, (a^2 \Delta^{\text{KS}})_{\text{fine}}) - a_{\mu,l=1,\text{SLLGS}}^{\text{LO-HVP}}(t_{\text{win}}, L, (a^2 \Delta^{\text{KS}})_{\text{coarse}})
\]

- **SLLGS** describes taste-breaking corrections well for \( t \gtrsim 1.5 \text{ fm} \)
- Correct taste breaking in simulations with **SLLGS** using either \( t \geq 1.5 \text{ fm} \) or \( t \geq 2.0 \text{ fm} \)
- Use spread in continuum systematic error
Continuum extrapolations of $a_{\mu, ud}^{\text{LO-HVP}}$ - PRELIMINARY

<table>
<thead>
<tr>
<th>error [%] / correction</th>
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<th>LO-S$\chi$PT</th>
<th>SLLGS</th>
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<tr>
<td>$\delta^\text{stat}(a_{\mu, ud}^{\text{LO-HVP}})$</td>
<td>1.5</td>
<td>1.5</td>
<td>1.6</td>
</tr>
<tr>
<td>$\delta^\text{a-extrap}(a_{\mu, ud}^{\text{LO-HVP}})$</td>
<td>2.2</td>
<td>1.0</td>
<td>0.6</td>
</tr>
</tbody>
</table>

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HC2NP 19, Tenerife, 23-28 September 2019
More detailed comparison

- BMWc '17 $ud$ contribution is larger than $N_f = 2 + 1 + 1$
- HPQCD '16 and $N_f = 2$ Mainz '17 results ($\sim 2.1/1.5\sigma$)
- BMWc '17 is only calculation performed directly at physical quark masses w/ 6 $a$'s to fully control continuum extrapolation
- $a_{\mu,\text{LO-HVP}}$ already known with high enough precision for FNAL E989
Time moments

\[ \Pi_n = \frac{1}{n!} \frac{\partial^n}{\partial Q^{2n}} \Pi(Q^2) \bigg|_{Q^2=0} = a \sum_i (-1)^n \frac{t^{2n+2}}{(2n+2)!} C(t) \]

Larger \( n \) probe larger distances

\[ \Pi_n(x) = \frac{5}{9} \Pi_n^{ud}(x) + \frac{1}{9} \Pi_n^s(x) + \frac{4}{9} \Pi_n^c(x) + \frac{1}{9} \Pi_n^{\text{disc}}(x) \]
Derivatives of $\Pi(Q^2)$ at $Q^2 = 0$: $ud$ contribution

- $\Pi_1^{ud}[\text{GeV}^{-2}]$

- $\Pi_2^{ud}[\text{GeV}^{-4}]$

- In Padé picture (and probably generally) larger $\Pi_1$ → larger $a_\mu$

- Larger $-\Pi_2$ → smaller $a_\mu$

- HPQCD 16 has slightly smaller $\Pi_1^{ud}$ and larger $-\Pi_2^{ud}$ than BMWc 16 and RBC/UKQCD 18 → combine to give smaller $a_{\mu, ud}^{\text{LO-HVP}}$

- Suggests that HPQCD 16 has smaller $C(t)$ for $t \sim 1$ fm but larger for $t \gtrsim 2$ fm

- Difference comes from HPQCD 16’s large corrections
Comparison of derivatives of $\Pi(Q^2)$ at $Q^2 = 0$

Add all flavor components and compare to phenomenology

$\Pi_1$ [GeV$^{-2}$]

- $\Pi_2$ [GeV$^{-4}$]

BMWc 16 has $\Pi_1$ comparable to phenomenology but smaller $-\Pi_2$

→ suggests that BMWc (and RBC/UKQCD) has $C(t)$ slightly larger for $t \sim 1$ fm and smaller for $t \gtrsim 2$ fm
HVP from $e^+e^- \rightarrow \text{had}$ (or $\tau \rightarrow \nu_\tau + \text{had}$)

Use (Bouchiat et al 61) optical theorem (unitarity)

$$\text{Im} \Pi(s) = -\frac{R(s)}{12\pi}, \quad R(s) \equiv \frac{\sigma(e^+e^- \rightarrow \text{had})}{4\pi\alpha(s)^2/(3s)}$$

and a once subtracted dispersion relation (analyticity)

$$\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s + Q^2)} \frac{1}{\pi} \text{Im} \Pi(s)$$

$$= \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{1}{s(s + Q^2)} R(s)$$

$\Rightarrow \hat{\Pi}(Q^2)$ from data: sum of exclusive $\pi^+\pi^-$ etc. channels from CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.

Can also use $I(J^{PC}) = 1(1^-^-)$ part of $\tau \rightarrow \nu_\tau + \text{had}$ and isospin symmetry + corrections
LO-HVP from $e^+ e^- \rightarrow \text{had}$

- $O(\alpha^2)$ but includes some $O(\alpha^3)$ corrections:
  - final state radiation (FSR): $\sigma(e^+ e^- \rightarrow \text{had} + \gamma_{\text{FSR}})$, e.g. $\pi^+ \pi^- \gamma \rightarrow$ required for IR finite cross section at $O(\alpha^3)$
  - radiative modes, e.g. $\pi^0 \gamma$ & $\eta \gamma$

- Three recent values:
  \[
  a^{\text{LO-HVP}}_\mu = 693.27(2.46) \times 10^{-10} \quad [3.5\%_0] \quad (\text{KNT '18})
  
  = 693.1(3.4) \times 10^{-10} \quad [4.9\%_0] \quad (\text{DHMZ '17})
  
  = 688.1(4.1) \times 10^{-10} \quad [6.0\%_0] \quad (\text{Jegerlehner '17})
  \]

- Higher orders:
  \[
  a^{\text{NLO-HVP}}_\mu = -9.87(0.09) \times 10^{-10} \quad (\text{Kurz et al '14})
  
  a^{\text{NNLO-HVP}}_\mu = 1.24(0.01) \times 10^{-10} \quad (\text{Kurz et al '14})
  \]