Short-distance constraints and resonance exchange in HLbL

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(in collaboration with L. Cappiello, G. D’Ambrosio, D. Greynat and A. Iyer)
Status of the muon \((g - 2)_\mu\)

- While awaiting for the Fermilab number:
  \[
a^\text{exp}_\mu = 116592091(54)(33) \times 10^{-11}
\]

- Long-standing discrepancy\(^a\) with the SM estimate (3 to 4\(\sigma\)):
  \[
a^\text{SM}_\mu = 116591823(1)(34)(26) \times 10^{-11}
\]

- Excellent control over the dominant EW and EM corrections. Hadronic contributions small but dominate the uncertainty.

\(^a\)Only one experiment, not yet challenged...
Hadronic contributions

- HVP leading effect ($\sim 700 \times 10^{-10}$). Uncertainties can be reduced with $e^+e^-$-based and/or $\tau$-based analyses.
  
  Lattice QCD at a really advanced stage.  

  [Davier et al, Teubner et al]

  [Mainz, BMWc, RBC/UKQCD]

- HLbL much harder to estimate. Connection to experiment more convoluted, albeit dispersion analyses promising.
  
  Lattice QCD catching up fast.  

  [Bern, Mainz]

  [Mainz, RBC/UKQCD]

- Experimentally, we have a much improved projected uncertainty, $16 \times 10^{-11}$.

- Hadronic contributions cannot account for the present discrepancy, but we need better control of theoretical uncertainties to claim NP interpretations, when(if) the time comes.
HLbL estimates

- Three main routes: form factor ansatz, lattice QCD, dispersion relations.

- Main contributions from form factor analyses:

<table>
<thead>
<tr>
<th>Contribution</th>
<th>BPP</th>
<th>HKS,HK</th>
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<th>MV</th>
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<th>N,JN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0, \eta, \eta'$</td>
<td>85(13)</td>
<td>82.7(6.4)</td>
<td>83(12)</td>
<td>114(10)</td>
<td>114(13)</td>
<td>99(16)</td>
</tr>
<tr>
<td>axial vectors</td>
<td>2.5(1.0)</td>
<td>1.7(1.7)</td>
<td>-</td>
<td>22(5)</td>
<td>15(10)</td>
<td>22(5)</td>
</tr>
<tr>
<td>scalars</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-7(7)</td>
<td>-7(2)</td>
</tr>
<tr>
<td>$\pi, K$ loops</td>
<td>-19(13)</td>
<td>-4.5(8.1)</td>
<td>-</td>
<td>-</td>
<td>-19(19)</td>
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<td>$\pi,K$ loops</td>
<td></td>
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<td>-</td>
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<td>0(10)</td>
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<td>+subl. $N_C$</td>
<td></td>
<td></td>
<td>-</td>
<td>-</td>
<td></td>
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</tr>
<tr>
<td>quark loops</td>
<td>21(3)</td>
<td>9.7(11.1)</td>
<td>-</td>
<td>-</td>
<td>2.3</td>
<td>21(3)</td>
</tr>
<tr>
<td>Total</td>
<td>83(32)</td>
<td>89.6(15.4)</td>
<td>80(40)</td>
<td>136(25)</td>
<td>105(26)</td>
<td>116(39)</td>
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- Overall agreement with the 'pion-pole' contribution, main discrepancies in other contributions.

- Used PDG average not really coming from a calculation.

- A number of theoretical issues still open.
Form factor analysis

Pion-pole contribution:

- Vertices given by the $\pi \gamma \gamma$ form factor,

$$\int d^4x \ e^{iq_1 \cdot x} \langle 0 | T \{ J_{EM}^\mu (x) \ J_{EM}^\nu (0) \} | \pi^0(p) \rangle = \epsilon^{\mu \nu \alpha \beta} q_1^\alpha q_2^\beta F_{\gamma^* \gamma^* \pi^0} (Q_1^2, Q_2^2)$$

- $F_{\pi^0 \gamma \gamma}$ not known from first principles. Information only on certain kinematical limits:
  
  (a) $F_{\gamma^* \gamma^* \pi^0} (0, 0) = -\frac{N_C}{12 \pi^2 f_\pi} K(0, 0), \ K(0, 0) = 1$ (Anomaly)

  (b) $\lim_{Q^2 \to \infty} K(Q^2, Q^2) = \frac{8 \pi^2 f_\pi^2}{N_c} \frac{1}{Q^2}$ (OPE)

  (c) $\lim_{Q^2 \to \infty} K(0, Q^2) \sim \frac{1}{Q^2}$ (Brodsky-Lepage)
Form factor analysis

- Ansätze with different short and long-distance constraints:

\[ K(q_1^2, q_2^2) = 1; \]
\[ K(q_1^2, q_2^2) = \frac{m_V^2}{m_V^2 - q_1^2 - q_2^2}; \]
\[ K(q_1^2, q_2^2) = \frac{m_V^4}{(q_1^2 - m_V^2)(q_2^2 - m_V^2)} \]
\[ K(q_1^2, q_2^2) = \frac{m_V^4 - \frac{4\pi^2 f_\pi^2}{N_c}(q_1^2 + q_2^2)}{(q_1^2 - m_V^2)(q_2^2 - m_V^2)} \]

- In principle, the more constraints the better (closer to QCD). However, interesting to play with them to test which ones are numerically important.

- The same strategy can be repeated for the other contributions.

Main Hurdles:

- Hard to pin down the discrepancies: different interpolators for different channels, subject to different constraints.

- Not always clear how the short distances can be incorporated into form factors.

- Related to discussions on 'on-shellness' vs 'off-shellness' of the pion.
Correlators vs form factors

- In general,

\[
\text{Correlator} \neq \sum \text{(particle exchange)}
\]

- The presence of contact terms is important to fulfill general properties e.g. gauge invariance or anomaly matching.

- Simple example: AA correlator at low energies.

\[
\Pi_{AA}^{\mu\nu}(q) = \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) f_\pi^2
\]

- Pion propagation is not enough. Contact terms are fundamental.

- How can one implement the correct contact terms? External sources in Lagrangian formulations, e.g. ChPT.

- Problem: Lagrangians only at specific kinematical regimes (pQCD, ChPT).

Q1: Is there a way to make progress (apart from lattice QCD)?

Q2: Is this an issue for \( a_\mu \)?
**The Melnikov-Vainshtein limit**

**•** In the limit $Q_2^2 \simeq Q_3^2 \gg Q_1^2 \gtrsim \Lambda_{\text{QCD}}$ an OPE links VVVV to the (anomalous) VVA.

**•** The resulting short-distance constraint leads to a (sizeable) increase

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**•** Attempts to implement it with form factors not entirely successful. E.g., [Melnikov, Vainshtein]

$$A_{\pi^0} = F_{\pi\gamma\gamma}(q_2, q_3) \frac{1}{q_2^2 - m_{\pi}^2} F_{\pi\gamma\gamma}(q_1, 0)$$

consistent only if $F_{\pi\gamma\gamma}(q_1, 0) = 1$. Hard to argue phenomenologically...
A toy model

• 5-dimensional model:

\[ S_5 = \int d^4x \int_0^{z_0} dz \left\{ -\lambda \sqrt{-g} \text{tr} \left[ F_{(L)}^{MN} F_{(L)MN} + F_{(R)}^{MN} F_{(R)MN} \right] + c \text{tr} \left[ \omega_5(L_M) - \omega_5(R_M) \right] \right\} \]

with \( \omega_5(L) = \text{tr} \left[ LF_{(L)}^2 + \frac{i}{2} L^3 F_{(L)} - \frac{1}{10} L^5 \right] \)

• Choose AdS\(_5\) space, \( ds^2 = g_{MN} dx^M dx^N = \frac{1}{z^2} (-dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu) \), with \( \eta_{\mu\nu} \) mostly negative.
A toy model

- UV boundary conditions (AdS/CFT prescription): fields on the boundary are sources of the 4d theory, i.e.,

\[ L_\mu(x, 0) = l_\mu(x) \quad R_\mu(x, 0) = r_\mu(x) \]

coupled to \( J^a_{L\mu} = \bar{q}_L \gamma^\mu t^a q_L \) and \( J^a_{R\mu} = \bar{q}_R \gamma^\mu t^a q_R \).

- IR boundary conditions:

\[ L_\mu(x, z_0) - R_\mu(x, z_0) = 0 \quad F_{L\mu}^{z_0}(x, z_0) + F_{R\mu}^{z_0}(x, z_0) = 0 \]

such that chiral symmetry is (spontaneously) broken to \( SU(3)_L \times SU(3)_R \rightarrow SU(3)_V \).

- Pion multiplet related to \( A_5(x, z) \). Defining the Wilson line

\[ \xi_L(x, z) = P \exp \left\{ -i \int_{z}^{z_0} dz' L_z(x, z') \right\} \]

the change of variables

\[ L^\xi_M(x, z) = \xi_L^\dagger(x, z) \left[ L_M(x, z) + i \partial_M \right] \xi_L(x, z), \]

replaces \( A_5 \) by

\[ U(x) \equiv \xi_L(x) \xi_R^\dagger(x) = \exp \left[ \frac{2i\pi^a(x)t^a}{f_\pi} \right] \]
A toy model

• Important: the change of variables does not leave the CS term invariant, but induces a shift

\[ \omega_5(L^\xi) = \omega_5(L) + \omega_5(\Sigma_L) + d\alpha_4(L, \Sigma_L) \]

where

\[ \alpha_4(L, \Sigma_L) = \frac{1}{2} \text{tr} \left[ \Sigma_L (L F(L) + F(L) L) + i \Sigma_L L^3 - \frac{1}{2} \Sigma_L L \Sigma_L L - i \Sigma_L^3 L \right] \quad , \quad \Sigma_L = d\xi L^\xi \]

• The 3 free parameters of the model can be fixed to

\[ \lambda = \frac{N_c}{48\pi^2} ; \quad z_0^2 = \frac{N_c}{6\pi^2 f_\pi^2} ; \quad c = \frac{N_c}{24\pi^2} \]

from the pQCD quark loop in \( \Pi_{AA} \), \( f_\pi \) and the chiral anomaly.

**Holographic recipe:** Given an action \( S_5(A_M) \),

1. Split the fields as \( A_\mu(x, z) = a(x, z) \hat{a}_\mu^\perp(x) + \bar{a}(x, z) \hat{a}_\mu^\parallel(x) + \frac{\alpha(z)}{f_\pi} \partial_\mu \pi(x) \)

2. Solve the EoM and plug them back in. This defines \( S_{\text{eff}}(\hat{a}_\mu(x)) \).

3. Correlators fully analytical:

\[ \Pi_{AA}^{\mu\nu} = \frac{\delta^2 S_{\text{eff}}}{\delta \hat{a}_\mu \delta \hat{a}_\nu} \]
How far can we go with the toy model?

Not QCD but interesting features:

- Via Kaluza-Klein reduction, the 4d theory is a full-fledged realization of large-$N_c$ QCD.
- Lagrangian approach: guarantees not just unitarity, but allows to compute correlators.
- The topological term ensures that the chiral anomaly is correctly implemented.
- With the AdS metric, one reproduces all the short-distance constraints we tested so far.
- The spectrum of V and A is not accurately reproduced. However, on the Euclidean this has a tiny impact. Consider e.g. $\Pi_{LR}$,

Excellent laboratory to explore QFT issues in HLbL.
The HLbL tensor

Using the effective action, one finds a close expression for it:

$$
\Pi_{\mu\nu\lambda\rho} = \varepsilon_{\mu\nu\alpha\beta} \varepsilon_{\rho\alpha'\beta'} \left[ \frac{c^2}{\lambda} \int dz \int dz' T_{12}^\beta(z) G_A^{\alpha\alpha'}(z, z'; p) T_{34}^{\beta'}(z') + F^{(12)}_{\pi\gamma\gamma} \frac{q_1^\alpha q_2^\beta q_3^{\alpha'} q_4^{\beta'}}{p^2 - m_\pi^2} F^{(34)}_{\pi\gamma\gamma} \right]
$$

where

$$
F_{\pi\gamma\gamma}(Q_1^2, Q_2^2) = \frac{N_c}{12\pi^2 f_\pi} \int_0^{z_0} dz \alpha'(z) v(z, Q_1^2) v(z, Q_2^2)
$$

- This represents the full contribution of the Goldstone modes and axial excitations.
- The (inclusive) calculation of the HLbL in the large-$N_c$ limit is straightforward.
• The longitudinal piece of the HLbL tensor can be projected via
\[ G_{A}^{\mu\nu} = \frac{p^{\mu}p^{\nu}}{p^2} G_{A}^{\parallel} \]

The result is simplified to
\[ W_{L} = \frac{N_{c}^{2}}{144\pi^{4} f_{\pi}^{2} p^2} \left[ \int_{0}^{z_{0}} dz \alpha'(z)v_{1}(z)v_{2}(z)v_{3}(z)v_{4}(z) \right] \]
\[ + \frac{N_{c}^{2}}{144\pi^{4} f_{\pi}^{2} p^2} \int_{0}^{z_{0}} dz' \alpha'(z)v_{1}(z)v_{2}(z) \int_{0}^{z_{0}} dz' \alpha'(z')v_{3}(z')v_{4}(z') - F^{(12)}_{\pi\gamma\gamma} \frac{1}{p^2 - m_{\pi}^{2}} F^{(34)}_{\pi\gamma\gamma} \]
\[ = \frac{N_{c}^{2}}{144\pi^{4} f_{\pi}^{2} p^2} \left[ \int_{0}^{z_{0}} dz \alpha'(z)v_{1}(z)v_{2}(z)v_{3}(z)v_{4}(z) \right] + \mathcal{O}(m_{\pi}^{2}) \]

i.e., there is a cancellation and one is left (in the chiral limit) with a contact piece only!

• This piece is responsible also for the correct implementation of the MV short-distance constraint, i.e., the MV constraint is not saturated by propagating terms.

• The origin of this contact term is closely related to the chiral anomaly of the VVA triangle.
Consider the correlator
\[ \Gamma_{\mu\nu\lambda}(q_3) = i \int d^4x d^4y \ e^{iq_3 \cdot (x-y)} \langle 0 | T \{ j^\text{em}_\mu(x) j^\text{em}_\nu(y) j^5_\lambda(0) \} | 0 \rangle \]
\[ = \frac{1}{24\pi^2} \left[ \omega_L(q_3^2) t^\parallel_{\mu\nu\lambda} + \omega_T(q_3^2) t^\perp_{\mu\nu\lambda} \right] \]

It is known that
\[ \omega_L(q) = -\frac{2N_c}{q^2} \]
to all orders in pQCD. Corrections are \( \mathcal{O}(m_\pi^2) \).

In the model VVA can be computed from the effective action:
\[ (S^{(3)}_{\text{CS}})^\perp = \frac{2c}{3} \varepsilon^{\mu\nu\lambda\rho} \int d^4x \left[ 1 + 3 \int_0^{z_0} dz a(z, k_1) v(z, k_2) v'(z, k_3) \right] \hat{a}^\perp_{\mu}(k_1) \partial_\nu \hat{v}_\lambda(k_2) \hat{v}_\rho(k_3) \]
\[ (S^{(3)}_{\text{CS}})^\parallel = \frac{c}{3} \varepsilon^{\mu\nu\lambda\rho} \int d^4x \left[ 1 + 3 \int_0^{z_0} dz a'(z) v(z, k_2) v(z, k_3) \right] \frac{\partial^\alpha \hat{a}^\parallel_{\alpha}(k_1)}{\Box} \partial_\nu \hat{v}_\lambda(k_2) \partial_\mu \hat{v}_\rho(k_3) \]
together with the pion propagation.
The VVA triangle

- Longitudinal component: the pion cancels against the energy-dependent part of the 3-point vertex.

- Only the boundary piece survives (in the chiral limit) and $\omega_L(q)$ is recovered. In general,

$$\omega_L(q) = -\frac{2N_c}{q^2} \left[ 1 - \frac{12\pi^2 f_\pi}{N_c} \frac{m^2_\pi}{q^2 - m^2_\pi} F_{\pi\gamma\gamma}(0, q) \right]$$

- Compare with the continuation proposed in [Melnikov, Vainshtein]

$$\omega_L(q) = -\frac{2N_c}{q^2 - m^2_\pi}$$

Different chiral continuations, which could be numerically important.

- If one takes only the pion piece at low energies one finds

$$\lim_{q \to 0} \omega^{(\pi)}_L(q) = -\frac{2N_c}{q^2}$$

Pion dominance cannot be such a bad approximation after all.

- Subtlety: the action is in the LR prescription, so one should actually shift to the Adler-Bardeen one with

$$\Gamma_{\mu\nu\lambda} = \hat{\Gamma}_{\mu\nu\lambda} + \frac{N_c}{12\pi^2} \varepsilon_{\mu\nu\lambda\alpha}(q_1 - q_2)^\alpha$$
Conclusions

• A Lagrangian approach to HLbL provides an inclusive analysis of the leading large-$N_c$ effects. One has access to full correlators and can clarify unresolved issues of form factor analyses. Nice perks: unitarity, anomaly and short-distance constraints correctly implemented.

• Pion dominance is not compatible with the correct implementation of the chiral anomaly, but numerically it gives an excellent estimate.

• The anomaly is saturated by a contact term. This is the main contribution to HLbL. The uncertainty on HLbL must be reassessed: presumably smaller than previously estimated.

• The contact term is crucial to fulfill the MV short-distance constraint, i.e., it is not saturated with a form factor. The infamous structureless form factor is just a manifestation of this.

• The previous results are generic QFT consequences for the leading large-$N_c$ contributions to HLbL. Specific numbers will of course depend on the model, but even part of the result should be model-independent. Preliminary numbers seem to indicate so.

• Lattice simulations should be able to (hopefully) confirm these features.