The Lattice (RBC-UKQCD) Kaon Physics Programme

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Outline of Talk

1. $K \to \pi\pi$ decays
2. Long-distance contributions
   - 2a) Δm_K
   - 2b) $K^+ \to \pi^+ \nu \bar{\nu}$ rare decays

- All unpublished results presented here are preliminary.

1. Directly computing $\mathbf{K} \to \pi\pi$ decay amplitudes

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- $K \to \pi\pi$ decays are a very important class of processes for standard model phenomenology with a long and noble history.
  - It is in these decays that both indirect and direct CP-violation was discovered.
- Bose Symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2.
- Among the very interesting issues are the origin of the $\Delta I = 1/2$ rule (Re A_0/Re $A_2 \simeq 22.5$) and an understanding of the experimental value of ε'/ε, the parameter which was the first experimental evidence of direct CP-violation.

- In 2015 RBC-UKQCD published our first result for ε'/ε computed at physical quark masses and kinematics, albeit still with large relative errors:

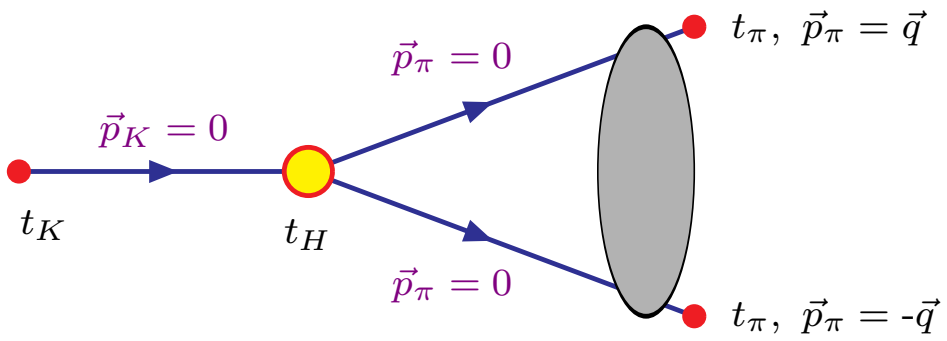
  Z.Bai et al. (RBC-UKQCD), arXiv:1505.07863

$$\left.\frac{\varepsilon'}{\varepsilon}\right|_{\text{RBC-UKQCD}} = (1.38 \pm 5.15 \pm 4.59) \times 10^{-4}$$

  to be compared with

$$\left.\frac{\varepsilon'}{\varepsilon}\right|_{\text{Exp}} = (16.6 \pm 2.3) \times 10^{-4}\,.$$

  - Is this 2.1σ deviation real? $\Rightarrow$ must reduce the uncertainties.
- This is by far the most complicated project that I have ever been involved with.
- Puzzle: For the $I = 0$ s-wave $\pi\pi$ phase shift we obtained $\delta_0 = (23.8 \pm 4.9 \pm 2.2)^\circ$, to be compared with the dispersive results of 34°. G.Colangelo et al.



- $K \to \pi\pi$ correlation function is dominated by lightest state, i.e. the state with two-pions at rest. Maiani and Testa, PL B245 (1990) 585

$$C(t_\pi) = A + B_1 e^{-2m_\pi t_\pi} + B_2 e^{-2E_\pi t_\pi} + \cdots$$

  - Solution 1: Study an excited state. Lellouch and Lüscher, hep-lat/0003023
  - Solution 2: Introduce suitable boundary conditions such that the $\pi\pi$ ground state is $|\pi(\vec{q})\pi(-\vec{q})\rangle$. RBC-UKQCD, C.h.Kim hep-lat/0311003
    N.Christ, C.Kelly, D.Zhang, arXiv:1908.08640

For B-decays, with so many intermediate states below threshold, this is the main obstacle to producing reliable calculations.

Results for A_2

- The amplitude A_2 is considerably simpler to evaluate that A_0.
- Our first results for A_2 at physical kinematics were obtained at a single, rather coarse, value of the lattice spacing ($a \simeq 0.14$ fm). Estimated discretization errors at 15%. arXiv:1111.1699, arXiv:1206.5142
- Our latest results were obtained on two new ensembles, 48^3 with $a \simeq 0.11$ fm and 64^3 with $a \simeq 0.084$ fm so that we can make a continuum extrapolation:

$$\begin{aligned} \mathrm{Re}(A_2) &= 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\ \mathrm{GeV}. \\ \mathrm{Im}(A_2) &= -6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\ \mathrm{GeV}. \end{aligned}$$

arXiv:1502.00263

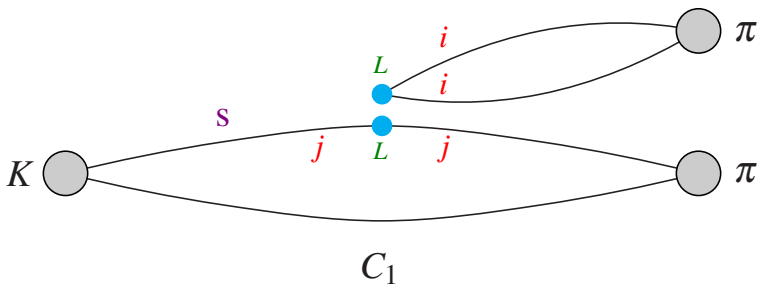
- The experimentally measured value is $\mathrm{Re}(A_2) = 1.479(4) \times 10^{-8}\ \mathrm{GeV}$.
- Although the precision can still be significantly improved (partly by perturbative calculations), the calculation of A_2 at physical kinematics can now be considered as standard.
- We are not currently working towards improving this result.

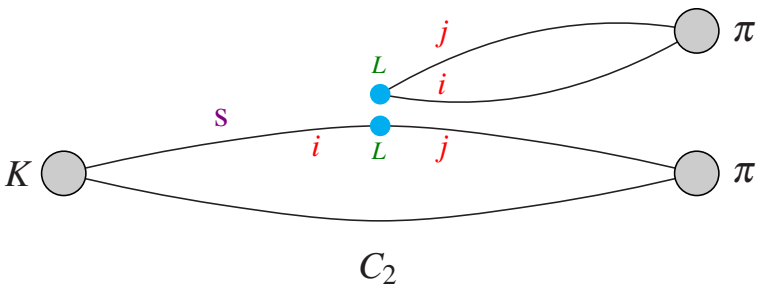
RBC-UKQCD Collaboration, arXiv:1212.1474

- Re A_2 is dominated by a simple operator:

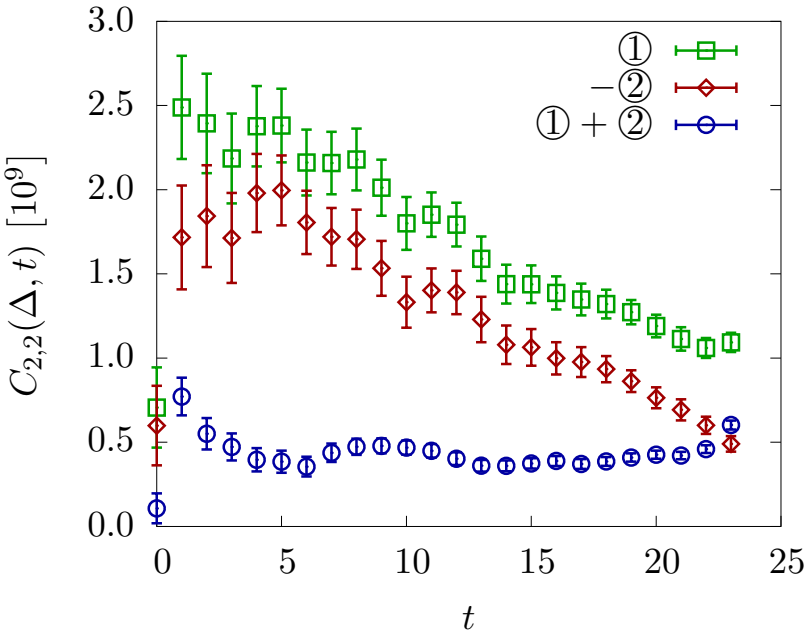
$$O^{3/2}_{(27,1)} = (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L \right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_L$$

and two diagrams:





- Re A_2 is proportional to $C_1 + C_2$.
- The contribution to Re A_0 from Q_2 is proportional to $2C_1 - C_2$ and that from Q_1 is proportional to $C_1 - 2C_2$ with the same overall sign.
- Colour counting might suggest that $C_2 \simeq \frac{1}{3} C_1$.
- We find instead that $C_2 \approx -C_1$ so that A_2 is significantly suppressed!
- **The strong suppression of Re A_2 is a major element in the $\Delta I = 1/2$ rule.**



Physical Kinematics

- Notation ⓘ $\equiv C_i,\ i = 1, 2.$
- Of course before claiming a quantitative understanding of the $\Delta I = 1/2$ rule we needed to compute Re A_0 at physical kinematics and found a results of $\simeq 31 \pm 12$ to be compared to the experimental value of 22.5.
- Much early phenomenology was based on the vacuum insertion approach. although the qualitative picture we find had been suggested by Bardeen, Buras and Gerard in 1987.

Extension and Improvement of the Calculation of A_0

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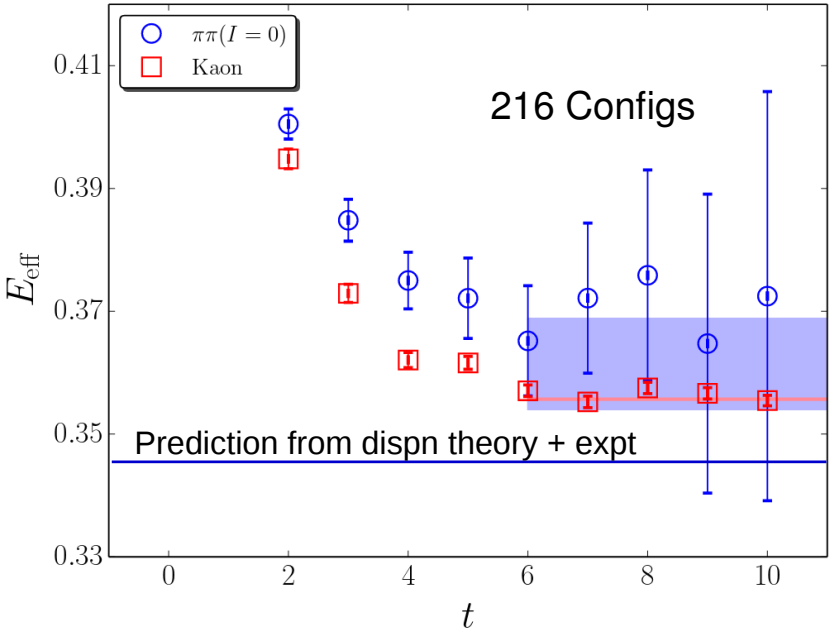
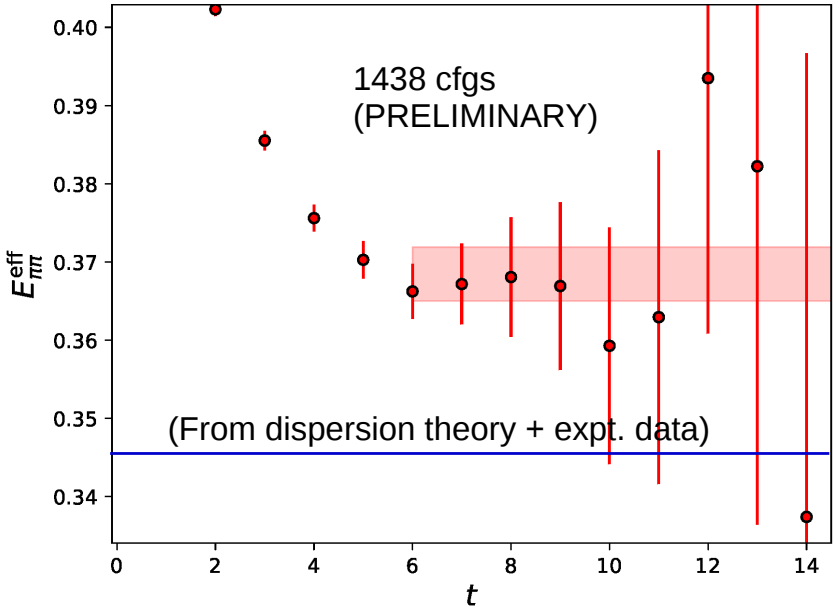
- Increase the statistics: 216 $\rightarrow$ 1438 configurations.
  - Reduce the statistical error;
  - Improved statistics allows for an in-depth study of the systematics.
- Use an expanded set of operators to create the $\pi\pi$ state.
- Improve the non-perturbative renormalisation, including step-scaling to match at a higher energy.
- Significantly improve the analysis techniques.





- Increasing the statistics from 216 to 1438 configurations, the $\pi\pi$ correlation function is still well described by a single $\pi\pi$ state.
  - It does not solve the δ_0 puzzle however:

$$\delta_0 = (23.8 \pm 4.9 \pm 2.2)^\circ \quad \rightarrow \quad \delta_0 = (19.1 \pm 2.5 \pm 1.2)^\circ \qquad (\chi^2/\mathrm{dof} = 1.6)$$

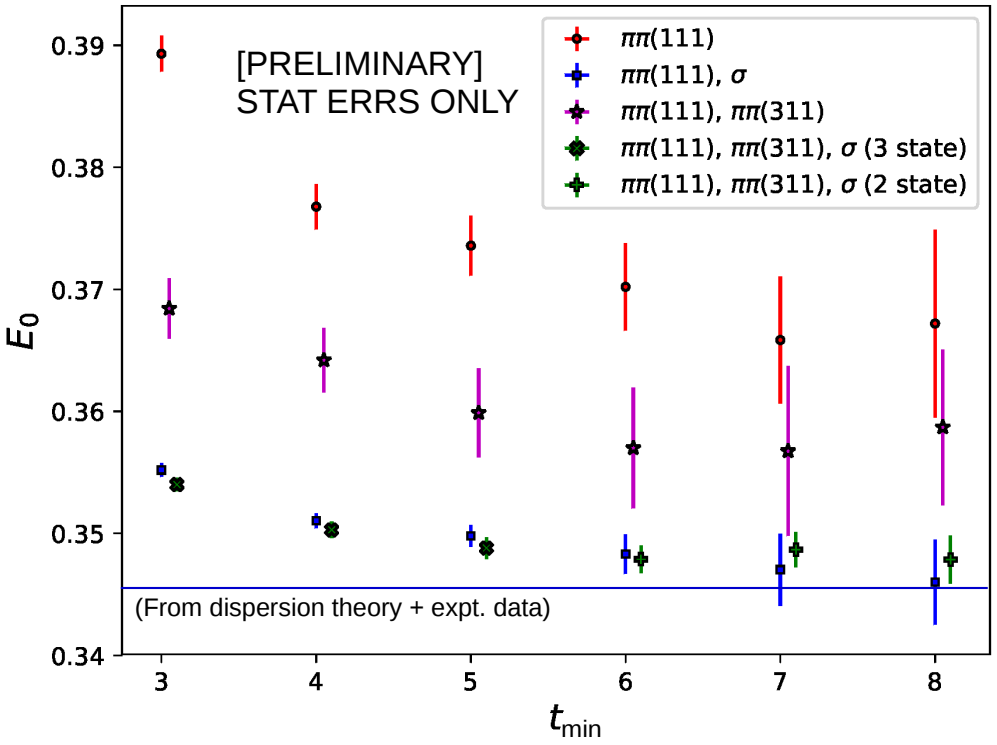
Adding more $\pi\pi$ interpolating operators

- The δ_0-puzzle has been resolved by adding more interpolating operators for the $\pi\pi$ states.
- In particular the inclusion of a σ-like two-quark operator ($\bar{u}u + \bar{d}d$) has exposed a second state, e.g. for $t_f - t_i = 5$

$$\det \begin{pmatrix} \langle \pi\pi(t_f)\pi\pi(t_i)\rangle & \langle \pi\pi(t_f)\sigma(t_i)\rangle \\ \langle \sigma(t_f)\pi\pi(t_i)\rangle & \langle \sigma(t_f)\sigma(t_i)\rangle \end{pmatrix} = 0.439(50) \neq 0$$

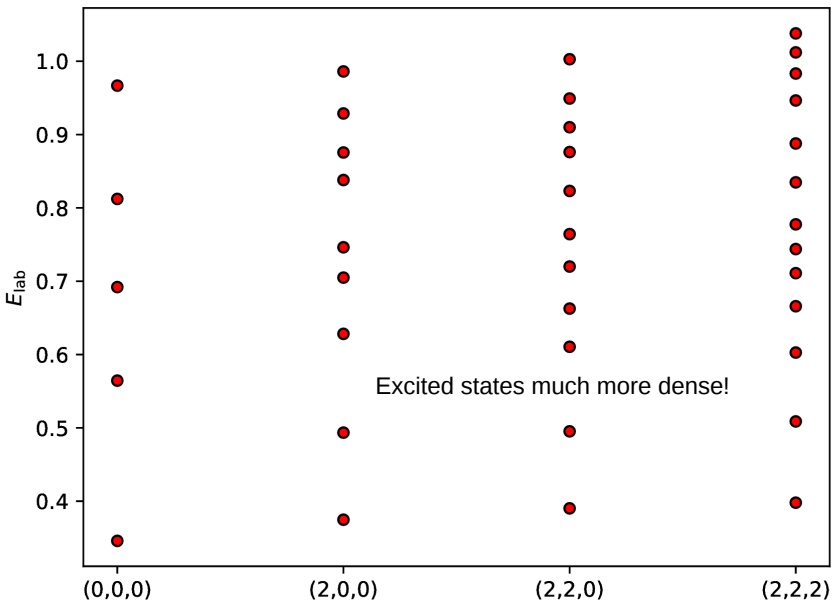
- We have also included a third operator giving each pion a larger momentum $\pm(3,1,1)\pi/L$.
- We have only 741 configurations with the additional operators.

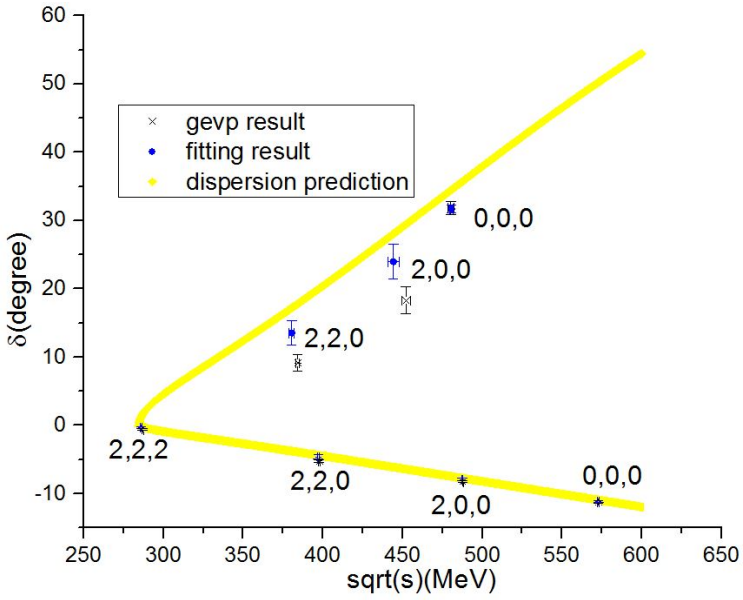
Adding more $\pi\pi$ interpolating operators (cont.)



- $\delta_0 = (31.7 \pm 0.6)^\circ$ from a fit in the range $t = 5$ - 15 (statistical error only).
  - Recall that the fit from dispersion theory is about 34°.
- The $\pi\pi(3,1,1)$ operator turns out not to be very important.

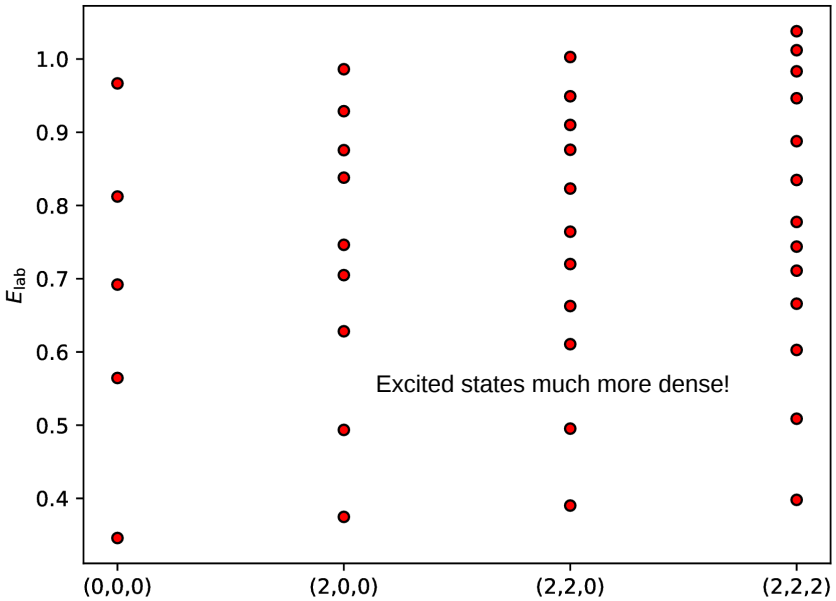
Attempting to determine the phase-shifts with $\vec{p}_{\pi\pi} \neq 0$

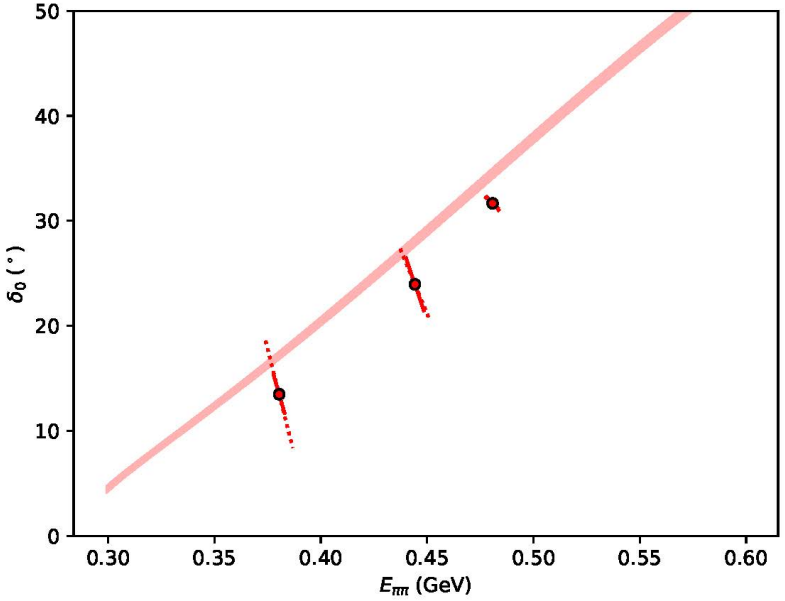




- We also evaluate the phase-shifts from the correlation functions at non-zero CoM momenta. (Components of momenta given in units of $2\pi/L$.)
- The increasing density of excited states makes it difficult to separate the states $\Rightarrow$ poor plateaus.
- In the right-hand plot, only statistical errors are included and the curve comes from G.Colangelo, J.Gasser and H.Leutwyler, Nucl. Phys. B **603** (2001) 125

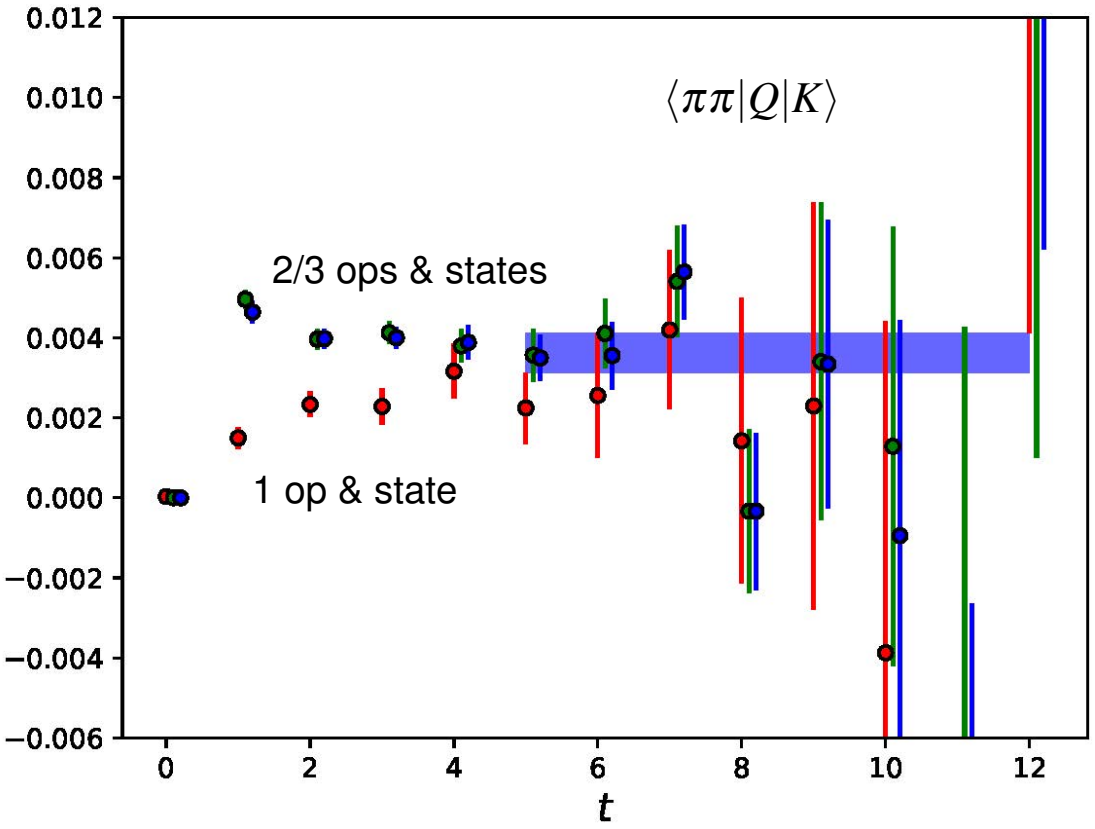
Attempting to determine the phase-shifts with $\vec{p}_{\pi\pi} \neq 0$





- We also evaluate the phase-shifts from the correlation functions at non-zero CoM momenta. (Components of momenta given in units of $2\pi/L$.)
- The increasing density of excited states makes it difficult to separate the states $\Rightarrow$ poor plateaus.
- In the right-hand plot, the three points from right-to-left correspond to (0,0,0), (2,0,0) and (2,2,0) and the curve comes from

G.Colangelo, J.Gasser and H.Leutwyler, Nucl. Phys. B **603** (2001) 125



- We are currently completing the analysis of the $\langle\pi\pi|Q_i|K\rangle$ matrix elements, the amplitude A_0 and ε'/ε.
- The above is a sample plot for the matrix element of an unspecified (here) operator Q. $(t = t_{\pi\pi} - t_{\mathrm{op}})$

- The calculation of $A_0(K \to \pi\pi)$ and ε'/ε will be very substantially improved over our 2015 result.
  - Statistical improvement: 216 $\to$ 741 configurations.
  - 3 $\pi\pi$ interpolating operators used to separate the ground and excited states.
  - Significantly improved analysis techniques to quantify the effects of autocorrelations and to obtain correct p-values (blocked jacknife errors, inclusion of fluctuations in covariance matrix etc..
- The δ_0 puzzle now appears to be solved:

$$\delta_0(m_K) = 31.7(6)^\circ\,.$$

- Draft of paper is in preparation.

Results will be published “soon".

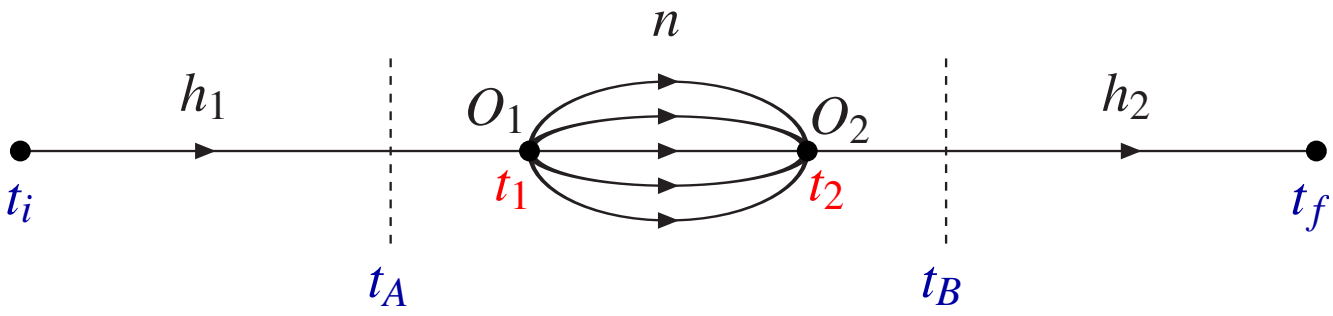
- See the talk of Maria Cerdà-Sevilla for an improved calculation of the Wilson Coefficients relevant for $\mathrm{Im}A_0/\mathrm{Re}A_0$.
  - Matching of the matrix elements renormalised in the RI-SMoM scheme to $\overline{\mathrm{MS}}$ still only known at one-loop. C.Sturm & C.Lehner, arXiv:1104.4948

- RBC-UKQCD Collaborations are developing and exploiting techniques to evaluate long-distance contributions to kaon physics, i.e. evaluating matrix elements of the form

$$\int d^4x \, \langle f \,|\, T[O_1(x)\, O_2(0)] \,|\, i \rangle \,.$$

  - Long-distance here means scales $\gtrsim \frac{1}{m_c}$.
  - As well as computing the non-perturbative long-distance contributions from scales of $O(\Lambda_{\rm QCD})$, we aim to avoid the necessity of performing perturbation theory at the scale of m_c. For Δm_K this has proved particularly slowly convergent. J.Brod & M.Gorbahn, arXiv:1108.2036

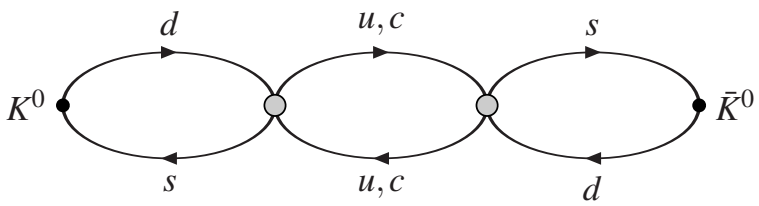
- The techniques are being applied to
  1. $\Delta m_K = m_{K_L} - m_{K_S}$ and ε_K;
  2. Rare kaon decays $K^+ \to \pi^+ \ell^+ \ell^-$ and $K_S \to \pi^0 \ell^+ \ell^-$;
  3. The rare kaon decay $K^+ \to \pi^+ \nu \bar{\nu}$.
- I will discuss some recent work on Δm_K and $K^+ \to \pi^+ \nu \bar{\nu}$ decays.



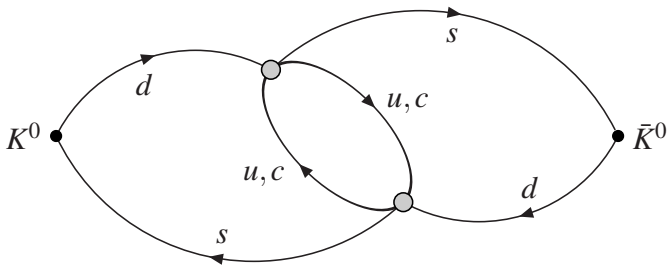
1. Fiducial volume: the integration over $t_{1,2}$ is performed in a large, but finite, interval ($t_A \leq t_{1,2} \leq t_B$). This is required to allow sufficiently large intervals $t_A - t_i$ and $t_f - t_B$ to ensure that it is indeed the hadrons $h_{1,2}$ in the initial and final states.
2. Growing exponentials: If there are intermediate states n with lower energies that those of the external states, then unphysical terms of relative size $e^{(E_{i,f}-E_n)T}$ (where $T = t_B - t_A$) are generated.
   • For kaon physics the number of such terms is small and can be handled. For heavy mesons this is much more challenging.
3. Renormalisation: New UV divergences may be generated as $x_1 \to x_2$.
   • For Δm_K and $K \to \pi\ell^+\ell^-$ decays this doesn't happen with $N_f = 4$.
   • The additional renormalisation, necessary for ε_K and $K \to \pi\nu\bar{\nu}$ decays, has been developed and implemented. N.H.Christ, X.Feng, CTS, A.Portelli, arXiv:1605.04442
4. Finite-volume effects: Power-like FV effects can be calculated by developing an extension of the Lüscher formalism. N.H.Christ, X.Feng, G.Martinelli, CTS, arXiv:1504.01170

RBC-UKQCD

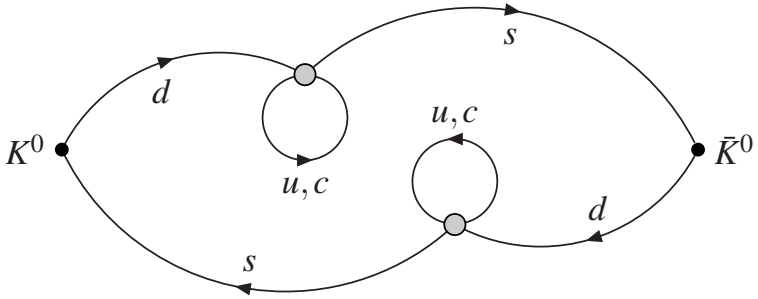
- There are four types of diagram to be evaluated:



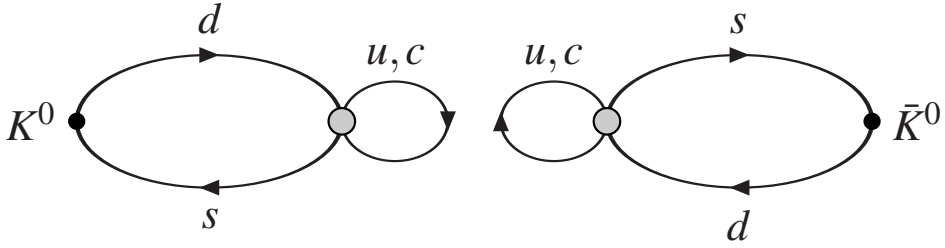
Type 1



Type 2



Type 3



Type 4

- Following the development of the theoretical background and exploratory numerical studies, we presented the first numerical results at physical masses at Lattice 2017 and updated them at Lattice 2018 and Lattice 2019.

Bigeng Wang, Lattice 2019; results are still preliminary

$$C^D = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0|H_W|n\rangle \langle n|H_W|K^0\rangle}{E_n - m_K} \left\{ T + \frac{e^{-(E_n-m_K)T} - 1}{E_n - m_K} \right\}$$

$$C^S = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0|H_W|n\rangle \langle n|H_W|K^0\rangle}{E_n - m_K} \left\{ 1 - e^{-(E_n-m_K)T} \right\}$$

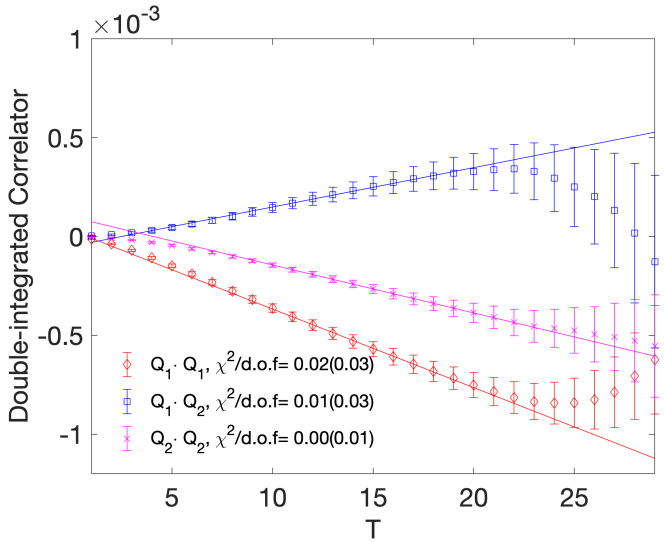
T is the range of integration.

- The calculation is performed on a $64^3 \times 128 \times 12$ lattice with Möbius DWF and the Iwasaki gauge action. a^{-1}=2.359(7) GeV, $m_\pi = 135.5(2)$ MeV and $m_K = 496.5(2)$ MeV.

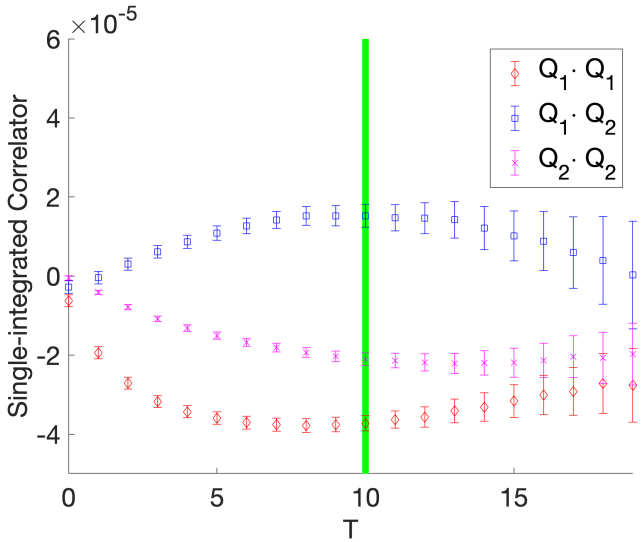
T.Blum et al., RBC-UKQCD Collabs., arXiv:1411.7017

- Charm-physics studies $\Rightarrow am_c \simeq 0.32 - 0.33$. We have used $am_c \simeq 0.31$ and studied the dependence on $m_c \Rightarrow$ largest source of systematic error.
- After completion of the present analysis, the priority is to reduce these artefacts. To this end, a project is beginning on finer ($96^3 \times 128, a^{-1} \simeq 2.8$ GeV) lattices at SUMMIT.

Status of RBC-UKQCD Calculations of Δm_K (cont.)



Double Integrated Correlator



Single Integrated Correlator

- Current preliminary results are

$$\Delta m_K = 7.9(1.2)_{\rm stat}(2.0)_{\rm sys} \times 10^{-12}\,{\rm MeV}\,, \qquad \text{Double Integration}$$

$$\Delta m_K = 6.7(0.6)_{\rm stat}(2.0)_{\rm sys} \times 10^{-12}\,{\rm MeV}\,, \qquad \text{Single Integration}$$

to be compared to the physical value $\Delta m_K^{\rm phys} = 3.483(6) \times 10^{-12}\,{\rm MeV}$.

- The dominant systematic error is due to discretisation effects because $am_c \simeq 0.31$. We have estimated these to be about 25%.
- Finite-volume effects are small ($-0.22(7) \times 10^{-12}$ MeV); are included in the above.

2b) Status of RBC-UKQCD Calculations of $K \to \pi \nu \bar{\nu}$ Decays

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- NA62 ($K^+ \to \pi^+ \nu \bar{\nu}$) and KOTO ($K_L \to \pi^0 \nu \bar{\nu}$) are beginning their experimental programme to study these decays. These FCNC processes provide ideal probes for the observation of new physics effects.
- The dominant contributions from the top quark $\Rightarrow$ they are also very sensitive to V_{ts} and V_{td}.
- Experimental results and bounds:

$$\mathrm{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\mathrm{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10}$$

A.Artamonov et al. (E949), arXiv:0808.2459

$$\mathrm{Br}(K_L \to \pi^0 \nu \bar{\nu}) \leq 2.6 \times 10^{-8} \text{ at 90\% confidence level},$$

J.Ahn et al. (E291a), arXiv:0911.4789

- Sample recent theoretical predictions:

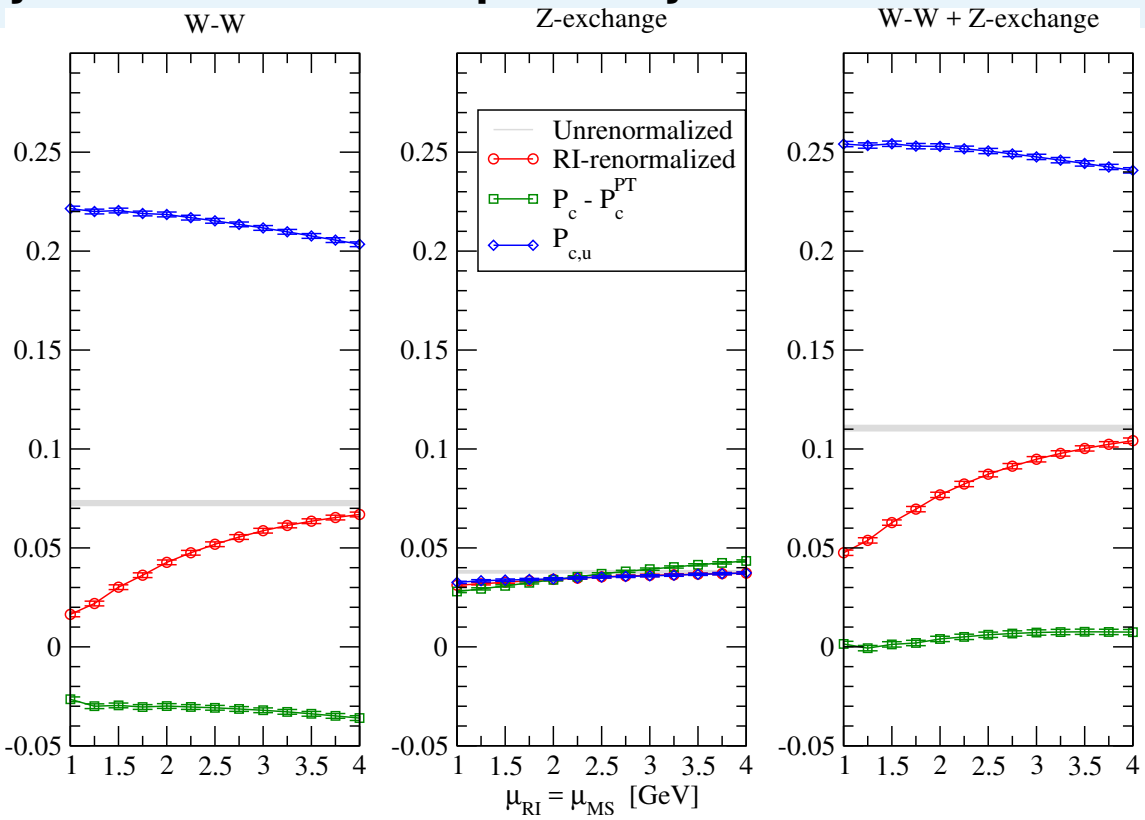
$$\mathrm{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\mathrm{SM}} = (9.11 \pm 0.72) \times 10^{-11}$$
$$\mathrm{Br}(K_L \to \pi^0 \nu \bar{\nu})_{\mathrm{SM}} = (3.00 \pm 0.30) \times 10^{-11},$$

A.Buras, D.Buttazzo, J.Girrbach-Noe, R.Knejgens, arXiv:1503.02693

- To what extent can lattice calculations reduce the theoretical uncertainty?

- To what extent can lattice calculations reduce the theoretical uncertainty?
- $K \to \pi\nu\bar{\nu}$ decays are SD dominated and the hadronic effects can be determined from CC semileptonic decays such as $K^+ \to \pi^0 e^+ \nu$.
- LD contributions, i.e. contributions from distances greater than $1/m_c$ are negligible for K_L decays and are expected to be $O(5\%)$ for K^+ decays.
  - K_L decays are therefore one of the cleanest places to search for the effects of new physics.
  - The aim of our lattice study is to compute the LD effects in K^+ decays. (These provide a significant, if probably still subdominant, contribution to the theoretical uncertainty, which is dominated by the uncertainties in CKM matrix elements.)
- Lattice QCD can provide a first-principles determination of the LD contribution with controlled errors.
- The theoretical framework has been developed and implemented in an exploratory calculation. N.Christ, X.Feng, A.Portelli & CTS, arXiv:1605.04442 Z.Bai, N.Christ, X.Feng, A,Lawson, A.Portelli & CTS, arXiv:1701.02858 & 1806.11520
- Ongoing work, led by X.Feng, includes a study of the momentum dependence on a 32^3 lattice at a^{-1}=1.37 GeV with $m_\pi \simeq 170$ MeV but lighter m_c as well as preparatory work for a physical simulation.

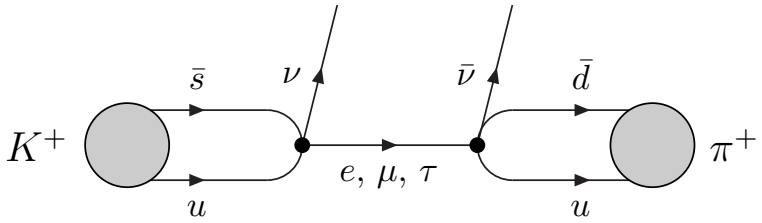
$K^+ \to \pi^+ \nu \bar{\nu}$ decays - Results from exploratory calculation

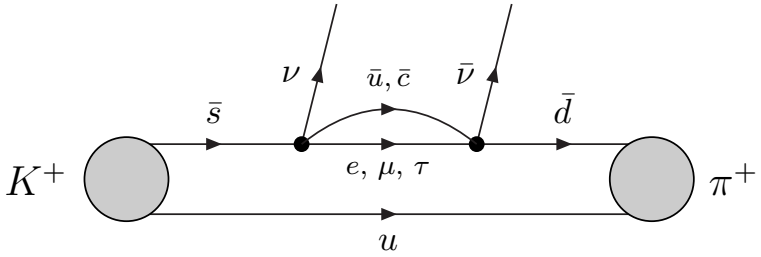


- Details of simulation: 800 configs on a $16^3 \times 32$ lattice with $N_f = 2+1$ DWF, $a^{-1} \simeq 1.73\,\text{GeV}$, $m_\pi \simeq 420\,\text{MeV}$, $m_K \simeq 563\,\text{MeV}$ and $m_c^{\overline{\text{MS}}}(2\,\text{GeV}) \simeq 863\,\text{MeV}$.
- For this unphysical kinematics, we find

$$P_c = 0.2529(\pm 13)(\pm 32)(-45) \quad \text{and} \quad \Delta P_c = 0.0040(\pm 13)(\pm 32)(-45)\,.$$

- Large cancellation between WW and Z-exchange contributions.

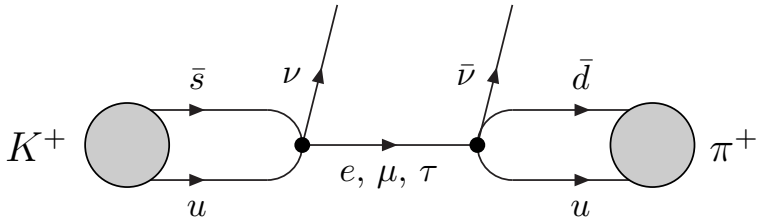


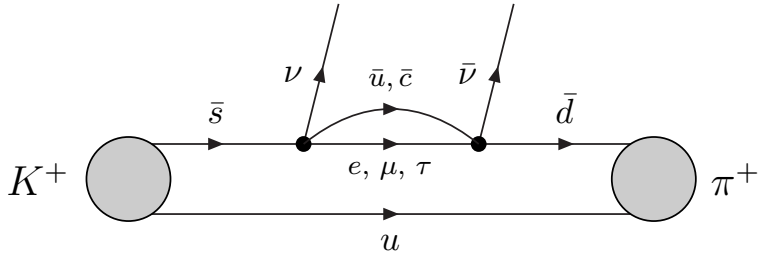


- The Infinite-Volume Reconstruction method has a number of important applications, particularly in QED corrections to hadronic processes.

  X.Feng, L.Jin, arXiv:1812.09817

- The motivation is to eliminate the power-like FV effects (i.e. effects of $O(1/L^n)$) by convoluting hadronic physics computed in lattice simulations with leptonic/photonic physics calculated “analytically" in infinite volume.
- Here we apply it to the rare kaon decay $K^+ \to \pi^+ \nu \bar{\nu}$, which includes contributions from the W-W diagrams shown above (Z-exchange diagrams also contribute but are not relevant for the present discussion). N.H.Christ, X.Feng, L.Jin & CTS, in preparation
- The presence of the almost massless electron $\Rightarrow$ large FV-effects.





$$
\begin{aligned}
A^M_{q\ell} &= i\int d^4x\, \langle \pi^+ \nu\bar{\nu}\,|\,T\left\{O^{\Delta S=1}_{q\ell}(x)\,O^{\Delta S=0}_{q\ell}(0)\right\}|\,K^+\rangle \quad (q=u,c) \\
&\equiv \int d^4x\, H_{\alpha\beta}(x)\,L^{\alpha\beta}(x) \qquad \text{(schematic)}
\end{aligned}
$$

where

$$
\begin{aligned}
H_{\alpha\beta}(x) &= \langle \pi^+\,|\,T\left\{O_{s,\alpha}(x)\,O_{d,\beta}(0)\right\}|\,K^+\rangle \quad \text{where}, \\
O_{s,\alpha} &= \bar{s}\gamma_\alpha(1-\gamma_5)q\,, \quad O_{d,\beta} = \bar{q}\gamma_\beta(1-\gamma_5)d \\
L^{\alpha\beta} &= \bar{u}(p_\nu)\gamma^\alpha(1-\gamma_5)S_\ell(x,0)\gamma^\beta(1-\gamma_5)v(p_{\bar{\nu}})e^{ip_\nu\cdot x}\,.
\end{aligned}
$$

- The challenge is to organise the calculation so that H can be computed on a lattice and L be determined "analytically" in a way which reproduces the physical amplitude, up to exponentially small FV effects.

- Inserting complete sets of eigenstates the physical amplitude $A^M_{q\ell}$ is

$$\int d\phi_n \frac{\langle\pi^+|O_{d,\beta}(0)|n\rangle\langle n|O_{s,\alpha}(0)|K^+\rangle}{E_n+E_{\ell^+}+E_\nu-E_K-i\varepsilon}\,\hat{L}_1^{\alpha\beta}(\vec{p}_n)+\int d\phi_{n_s}\frac{\langle\pi^+|O_{s,\alpha}(0)|n_s\rangle\langle n_s|O_{d,\beta}(0)|K^+\rangle}{E_{n_s}+E_{\ell^-}+E_{\bar{\nu}}-E_K-i\varepsilon}\,\hat{L}_2^{\alpha\beta}(\vec{p}_{n_s})$$

- $|n_s\rangle$ is a charge-2 hadronic state $\Rightarrow$ denominator in second term does not vanish $\Rightarrow$ can drop the $i\varepsilon$ and the Minkowski $\leftrightarrow$ Euclidean connection "straightforward".
- $|n\rangle = |0\rangle$ term given by $f_{\pi,K}$.
- For hadronic $|n\rangle$, assume that for $|t| > |t_s|$, $H^{\rm had}(t,\vec{x})$ is dominated by $|\pi^0\rangle$ and divide the integral into $(t_s,0)$ and the remainder, giving $I^{(s)}$ and $\tilde{I}^{(l)}$ respectively.
- $I^{(s)}$ can be calculated on a lattice with only exponentially small FV effects:

$$I^{(s)} = \int d\phi_n \frac{\langle\pi^+|O_{d,\beta}(0)|n\rangle\langle n|O_{s,\alpha}(0)|K^+\rangle}{E_n+E_{\ell^+}+E_\nu-E_K}\,\hat{L}_1^{\alpha\beta}(\vec{p}_n)\left(1-e^{-(E_n+E_{l^+}+E_\nu-E_K)|t_s|}\right)$$

- Let $\tilde{I}^{(l)}$ be the remainder; it can be written as $\tilde{I}^{(l)} = \int d^3x\, H_{\alpha\beta}(t_s,\vec{x})\tilde{L}_1^{\alpha\beta}(t_s,\vec{x})$ where

$$\tilde{L}_1^{\alpha\beta}(t_s,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\pi,p}}e^{i(\vec{p}-\vec{p}_K)\cdot\vec{x}}\,\hat{L}_1^{\alpha\beta}(\vec{p})\,\frac{e^{-(E_{\ell^+}+E_\nu)|t_s|}}{E_{\pi^0}+E_{\ell^+}+E_\nu-E_K-i\varepsilon}$$

- Thus all components can be computed with only exponentially small FV effects, including the imaginary part.

- We have developed the theoretical techniques necessary to compute long-distance effects in kaon physics, focussing on Δm_K, ε_K and $K \to \pi \ell^+ \ell^-$ and $K^+ \to \pi^+ \nu \bar{\nu}$ rare decays.
  - In all these cases exploratory calculations have been successfully performed, demonstrating the practicability of the methods.
  - For Δm_K a calculation at physical quark masses is well advanced, and the principal systematic error is due to discretisation effects ($am_c \simeq 0.31$). Preparations are being made for a simulation on a finer lattice at Summit.
  - For all these quantities preparations are under way for calculations at physical quark masses.
- The infinite-volume reconstruction method has been primarily introduced to eliminate Finite-Volume power corrections of the form $1/L^n$ when adding QED to QCD computations. Here I have illustrated its use in LD contributions to rare kaon decays where the electron is almost massless.
- Other applications include long-distance contributions to $K \to \nu \bar{\nu}$ decays as described in this talk and $K_L \to \mu^+ \mu^-$ ($\pi^- \to e^+ e^-$) decays.

N.H.Christ & Y.Zhao, Lattice 2019