Far-from-equilibrium hydrodynamics

Michał Spaliński
University of Białystok & National Centre for Nuclear Research

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Abstract

In a number of settings (including models of kinetic theory and strongly coupled supersymmetric Yang-Mills theory), the pressure anisotropy of boost-invariant flow is known to exhibit attractor behaviour well before local equilibration is attained. I will describe some aspects of this phenomenon and its possible implications for relativistic hydrodynamics.

Mueller-Israel Stewart hydrodynamics

$$T^{\mu\nu} = \mathscr{E} u^\mu u^\nu + \mathscr{P}(\mathscr{E})(g^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}$$

$$\nabla_\alpha T^{\alpha\beta} = 0$$

$$\left(\tau_\pi \mathscr{D} + 1\right) \Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \ldots$$

Perturbations around equilibrium reveal both **hydro** and **non-hydrodynamic modes**

$$\omega_{\mathrm{H}}^{(\pm)} = \pm \frac{k}{\sqrt{3}} - \frac{2i}{3\,T}\frac{\eta}{s}k^2 + \ldots \qquad \omega_{\mathrm{NH}} = -i\left(\frac{1}{\tau_\pi} - \frac{4}{3\,T}\frac{\eta}{s}k^2\right) + \ldots$$

The latter act as a **regulator** to ensure causal propagation

$$v = \frac{1}{\sqrt{3}}\sqrt{1 + 4\frac{\eta/s}{T\tau_\pi}} < 1 \iff T\tau_\pi > 2\eta/s$$

Bjorken flow in MIS hydrodynamics

The symmetries of Bjorken flow imply (assuming conformal symmetry)

$$T^\mu_\nu = \mathrm{diag}(-\mathcal{E}, \underbrace{\mathcal{P} - 2\alpha}_{\mathcal{P}_L}, \underbrace{\mathcal{P} + \alpha}_{\mathcal{P}_T}, \underbrace{\mathcal{P} + \alpha}_{\mathcal{P}_T})$$

with $\mathcal{E} = \mathcal{E}(\tau),\ \alpha = \alpha(\tau)$ and $\mathcal{E} = 3\mathcal{P} \sim T^4$ (effective temperature).

Dimensionless **pressure anisotropy**

$$\mathcal{A} \equiv \frac{\mathcal{P}_T - \mathcal{P}_L}{\mathcal{P}} \sim \frac{\alpha}{\epsilon}$$

The equations of MIS hydro imply

- a second order ODE which determines $T(\tau)$
- a first order ODE which determines $\mathcal{A}(\tau T(\tau)) \equiv \mathcal{A}(w)$

Evolution equation for the pressure anisotropy

$$C_{\tau_\pi}\left(1+\frac{\mathcal{A}}{12}\right)\mathcal{A}'+\left(\frac{C_{\tau_\pi}}{3w}+\frac{C_{\lambda_1}}{8C_\eta}\right)\mathcal{A}^2=\frac{3}{2}\left(\frac{8C_\eta}{w}-\mathcal{A}\right)$$

in terms of dimensionless transport coefficients

$$C_{\tau_\pi}=T\tau_\pi,\quad C_\eta=\eta/s,\quad C_{\lambda_1}=T\lambda_1/\eta$$

Asymptotic late-time solution:

$$\mathcal{A}=\underbrace{\frac{8C_\eta}{w}}_{\text{Navier–Stokes}}+\underbrace{\frac{16C_\eta(C_{\tau_\pi}-C_{\lambda_1})}{3w^2}}_{\text{2nd order}}+\ldots$$

Universal - no dependence on initial conditions.

Exponential corrections to the asymptotic gradient solution imply a **transseries** structure

$$\mathscr{A} = \underbrace{\sum_{n>0} \frac{a_n^{(0)}}{w^n}}_{\Phi_0(w)} + \sigma\, e^{-\frac{3}{2C_{\tau_\pi}} w} \underbrace{\left(w^{\frac{C_\eta - 2C_{\lambda_1}}{C_{\tau_\pi}}} \sum_{n\geq 0} \frac{a_n^{(1)}}{w^n} \right)}_{\Phi_1(w)} + \ldots$$

- The form of the transseries is determined by the **non-hydrodynamic** sector

$$\mathscr{A} = \sum_{n=0}^{\infty} \sigma^n\, e^{in\,\Omega w}\, \Phi_n(w), \qquad \Omega = i\frac{3}{2C_{\tau_\pi}} = \frac{3}{2}\mathrm{Im}(\omega)$$

- The hydro sector is universal: **no memory of initial conditions**
- The **transseries parameter** σ contains the integration constant (initial data)
- The transseries describes the **dissipation of initial state information**
- **Resurgence:** all universal coefficients can be recovered from the hydro ones

Similar asymptotic solutions have been found

- Other hydro models (HJSW, aHYDRO)
- At the **microscopic level**
  A. Kinetic theory
  B. N=4 supersymmetric Yang-Mills theory

$$\mathscr{E}(\tau, \boldsymbol{\sigma}) = \sum_{\boldsymbol{n} \in \mathbb{N}_0^\infty} \boldsymbol{\sigma}^{\boldsymbol{n}} \, \mathrm{e}^{-\boldsymbol{n} \cdot \boldsymbol{\Omega} \, \tau^{2/3}} \, \Phi_{\boldsymbol{n}}\left(\tau^{2/3}\right)$$

where $\boldsymbol{\Omega}$ is the vector of black-brane quasinormal modes.

These transseries solutions imply emergence of universal behaviour at late time, which can be approximated by low orders of the gradient expansion.

Attractor solution in MIS hydro

Evolution equation for the pressure anisotropy

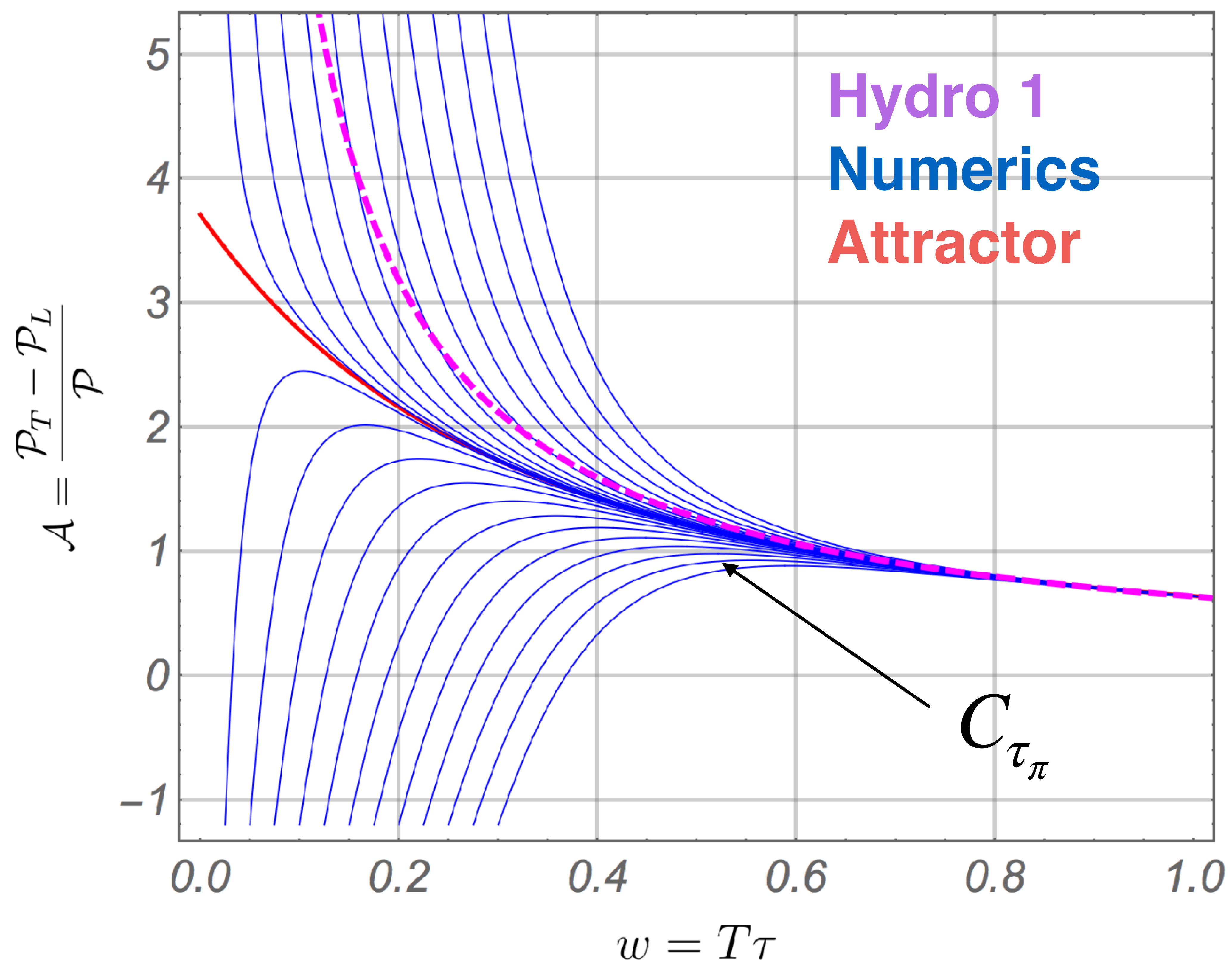
$$C_{\tau_\pi}\left(1+\frac{\mathscr{A}}{12}\right)\mathscr{A}' + \left(\frac{C_{\tau_\pi}}{3w}+\frac{C_{\lambda_1}}{8C_\eta}\right)\mathscr{A}^2 = \frac{3}{2}\left(\frac{8C_\eta}{w}-\mathscr{A}\right)$$

Numerical solutions tend to the asymptotic late-time solution

$$\mathscr{A} = \boxed{\frac{8C_\eta}{w}} + \frac{16C_\eta(C_{\tau_\pi}-C_{\lambda_1})}{3w^2} + \dots$$

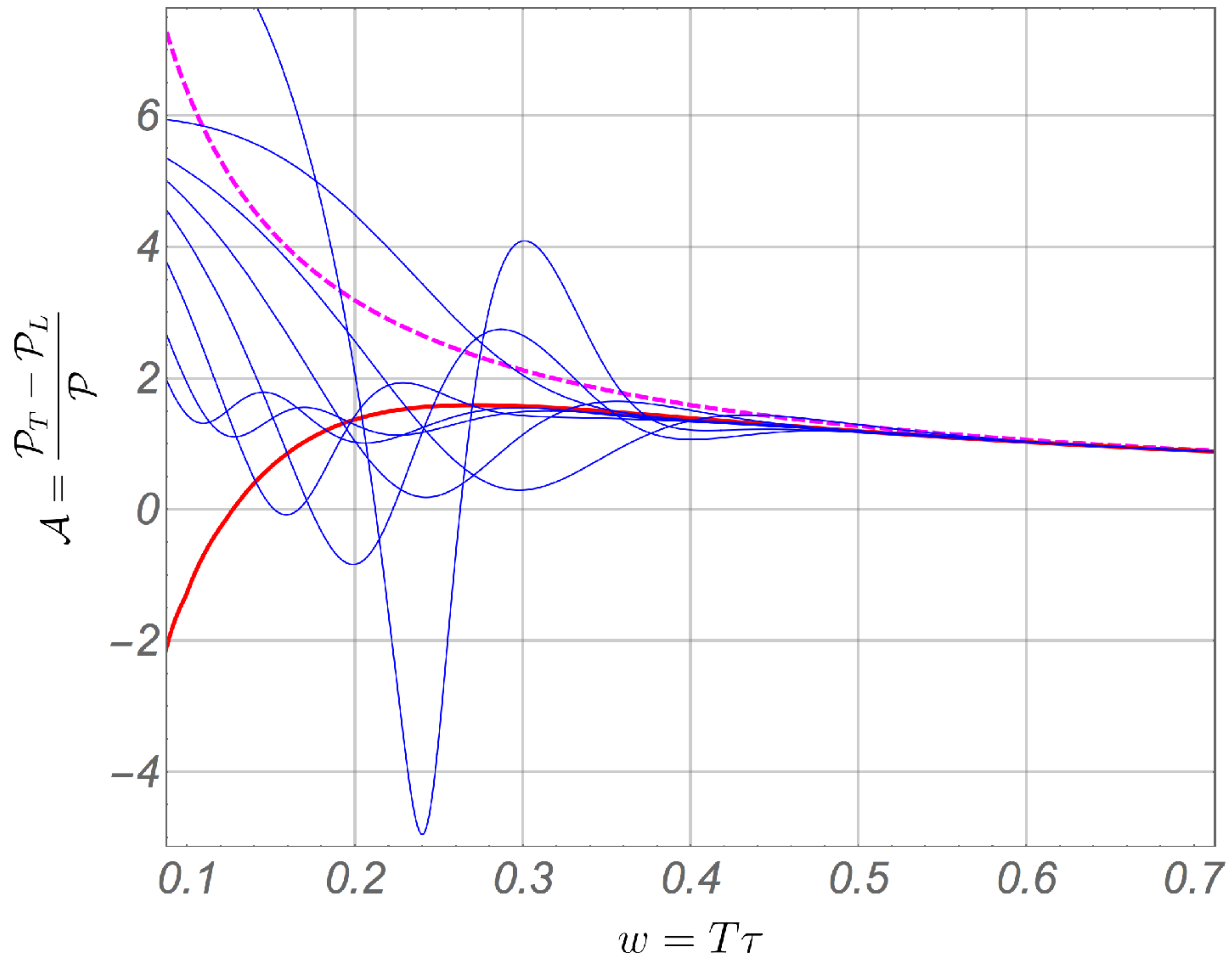
but already at very early times they show characteristic **attractor behaviour.**

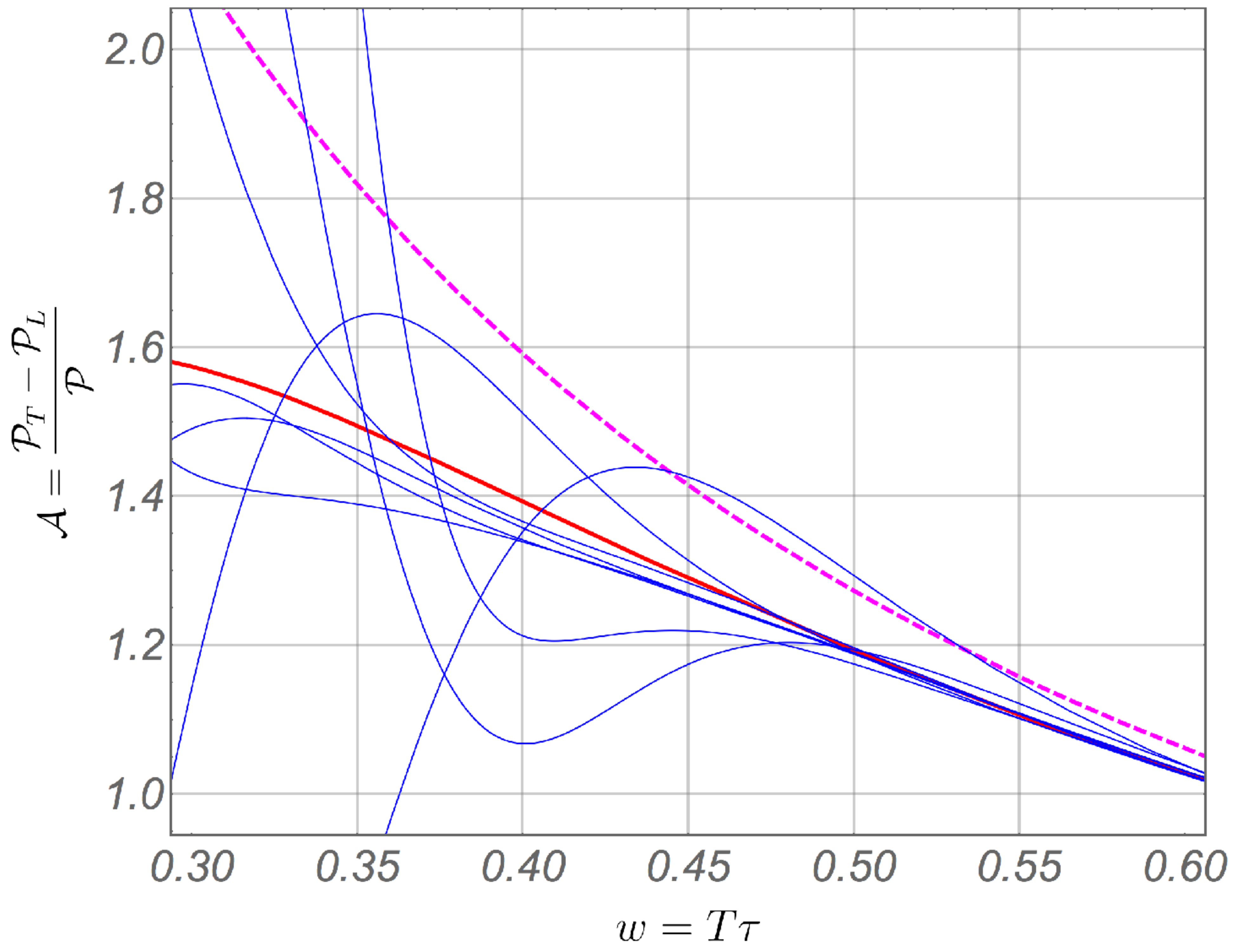
The decay exponential on a scale set by C_{τ_π}.



Similar attractor behaviour has been found in other hydrodynamic models (HJSW, aHYDRO), as well as **at the microscopic level**

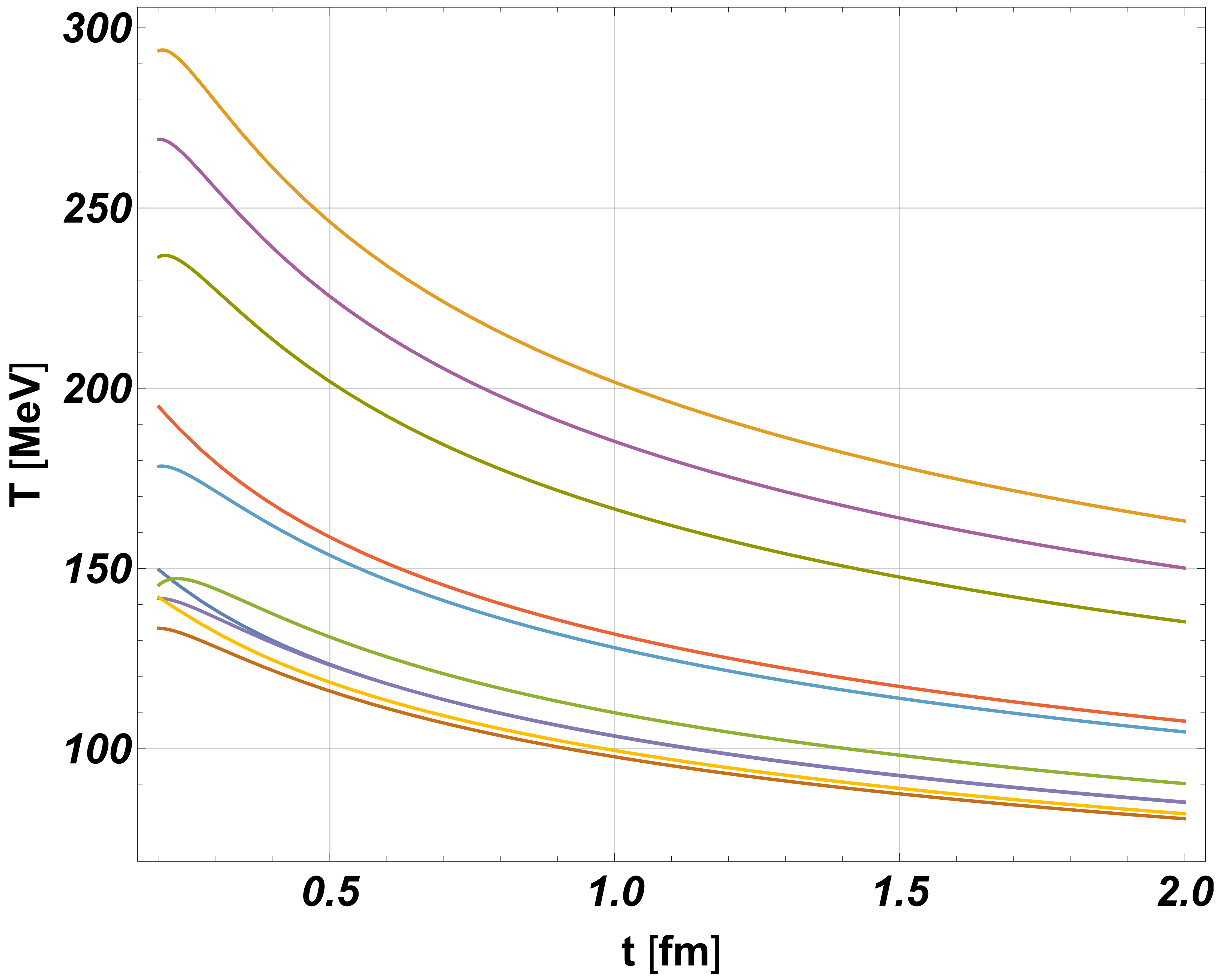
- Kinetic theory
- N=4 supersymmetric Yang-Mills theory



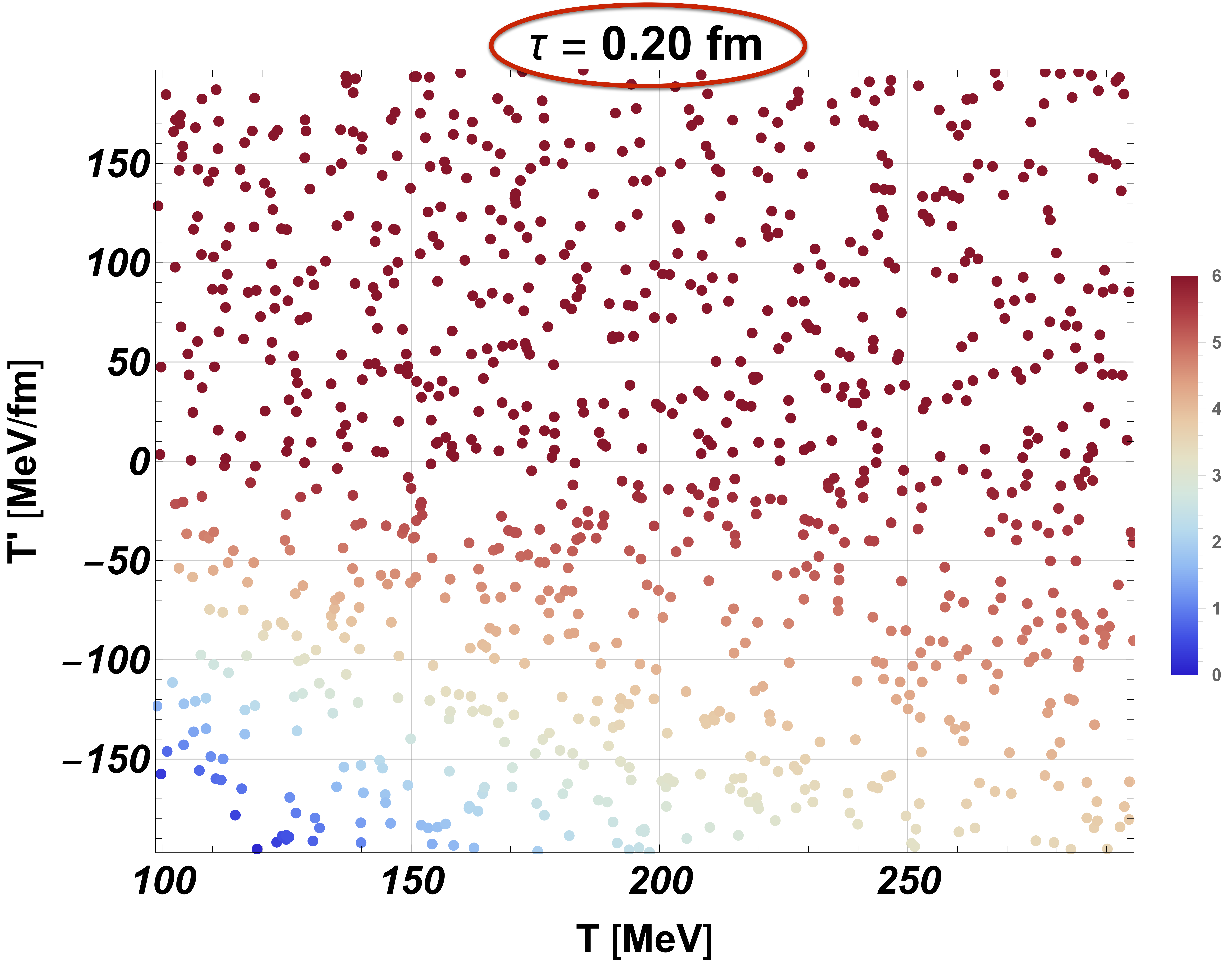


Attractor behaviour of the temperature

Time evolution of the temperature: **hardly any distinctive pattern** at early times.



There is however a **very clear pattern on constant time slices in phase space**.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T' [MeV/fm]
T [MeV]
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100
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−50
−100
−150
100
150
200
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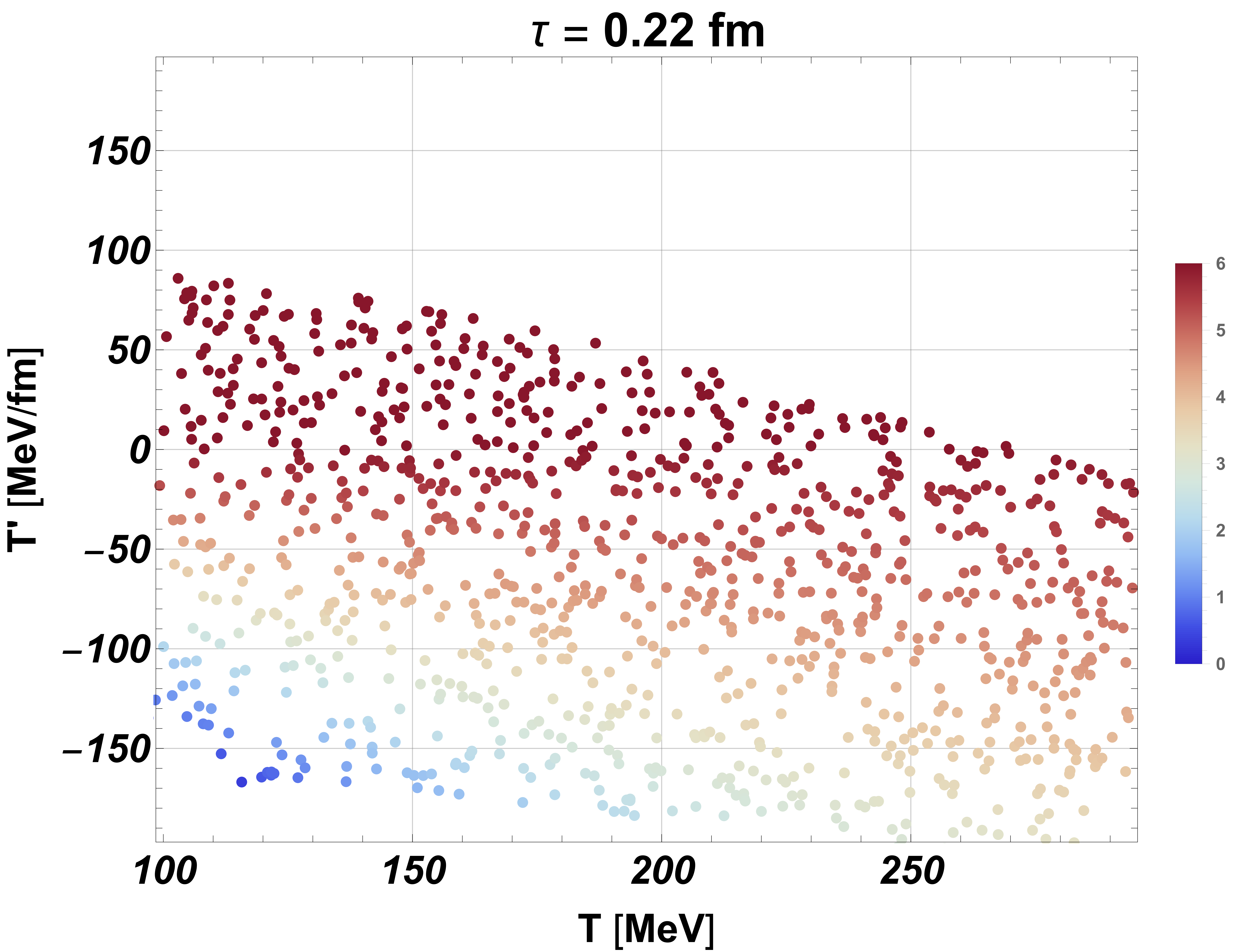
4
3
2
1
0



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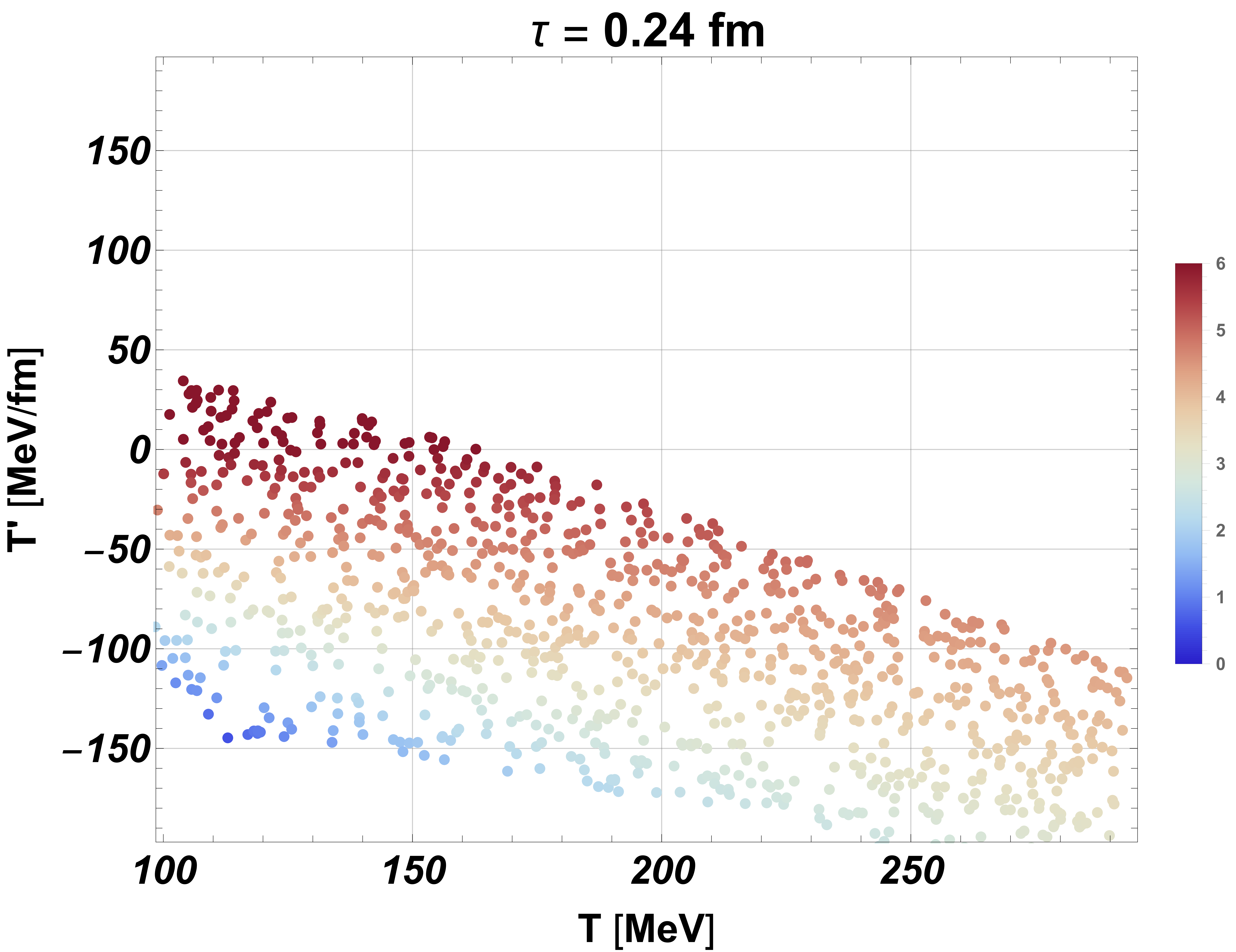

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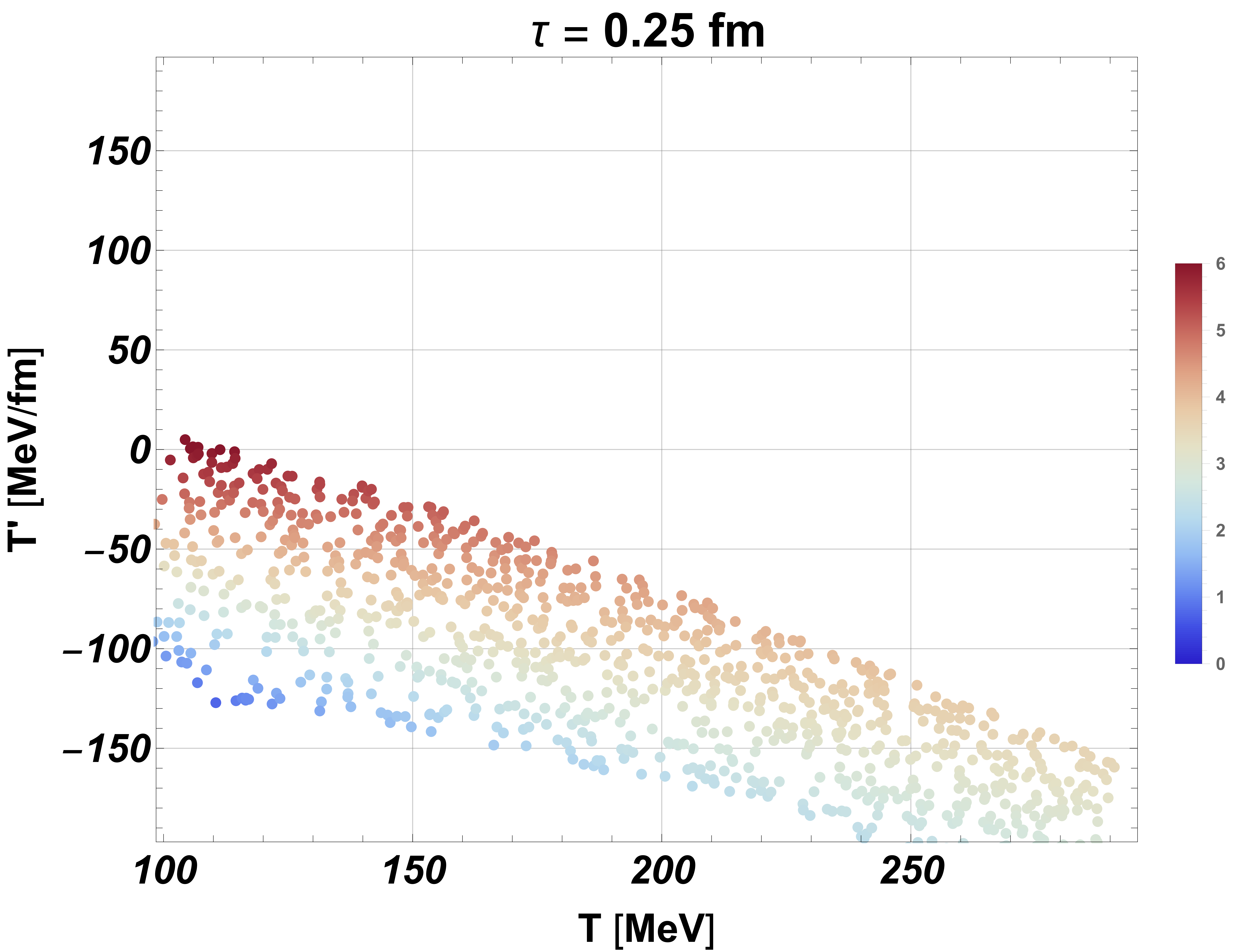

τ = 0.25 fm
T' [MeV/fm]
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T [MeV]
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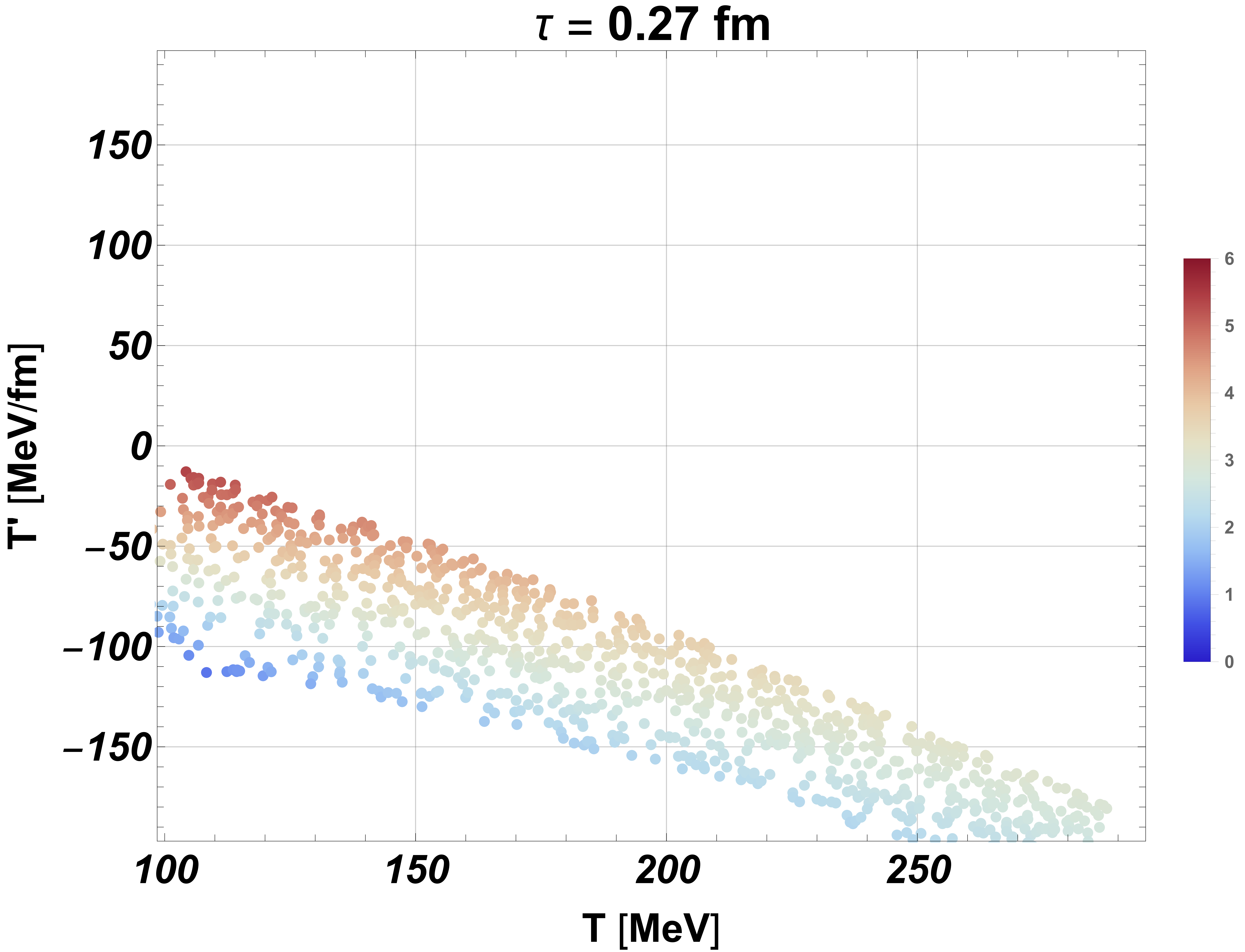

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−50
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−150
T' [MeV/fm]
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150
200
250
T [MeV]
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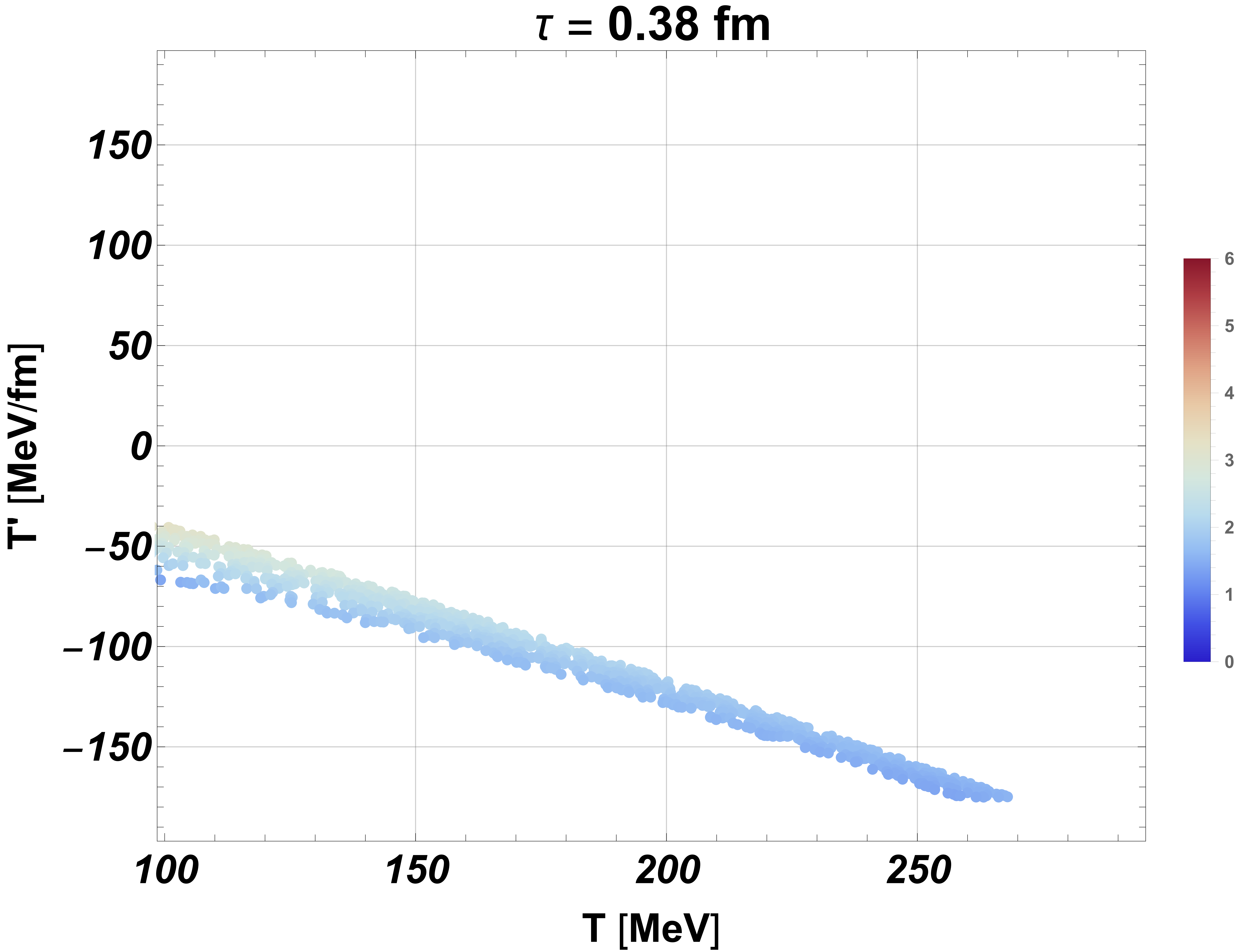
2
1
0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−50
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−150
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150
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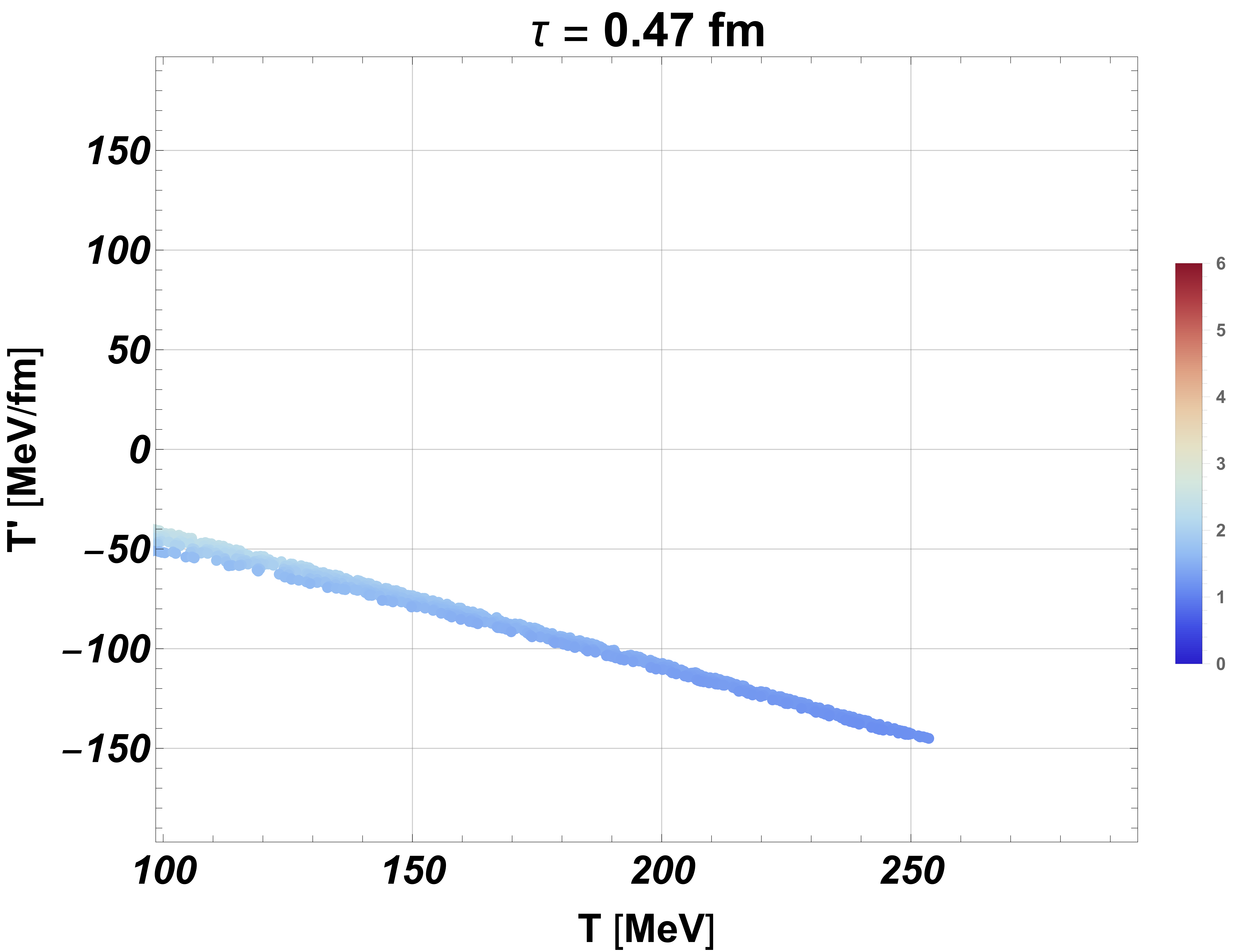

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−150
100
150
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250
T [MeV]
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4
3
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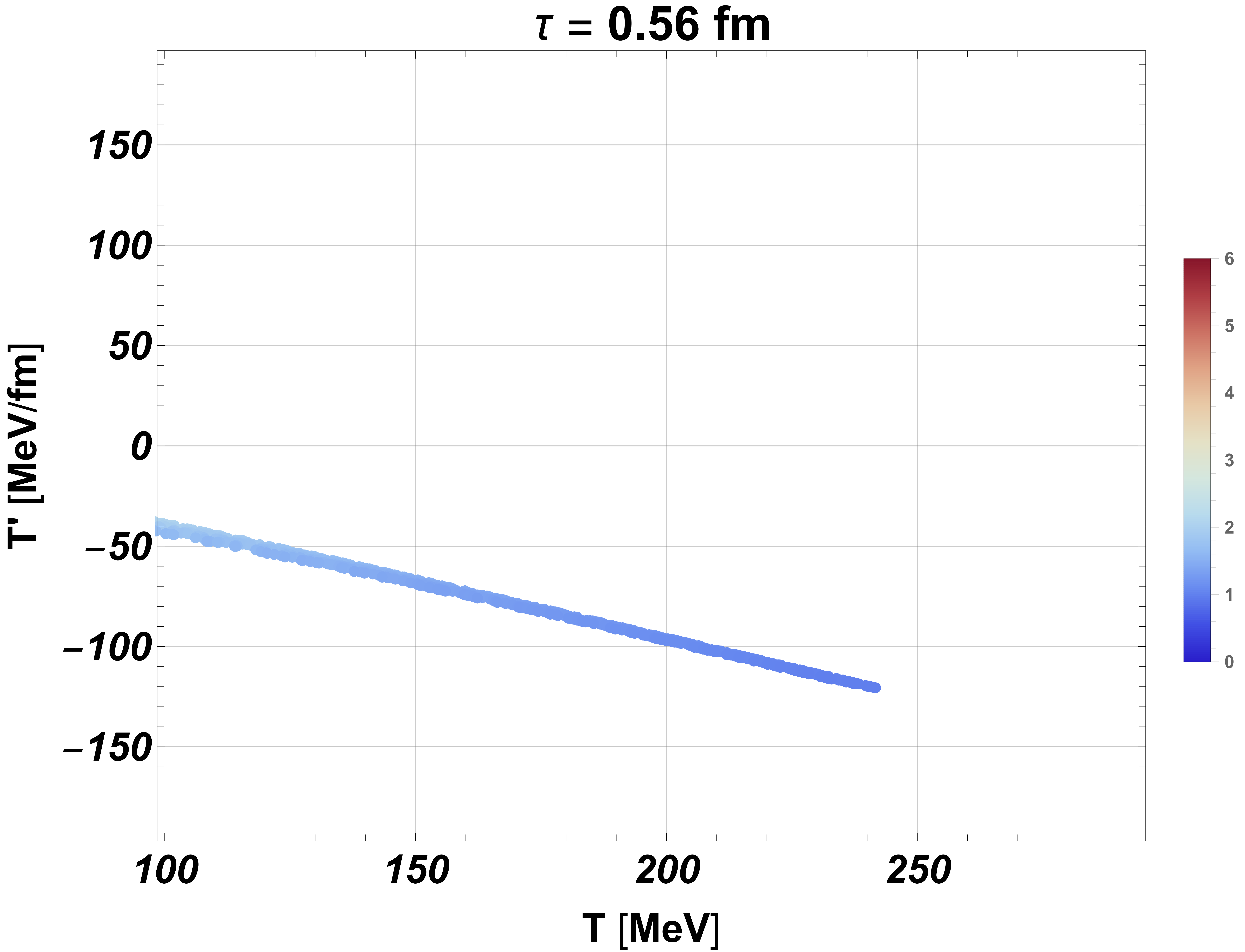
0



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T' [MeV/fm]
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150
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T [MeV]
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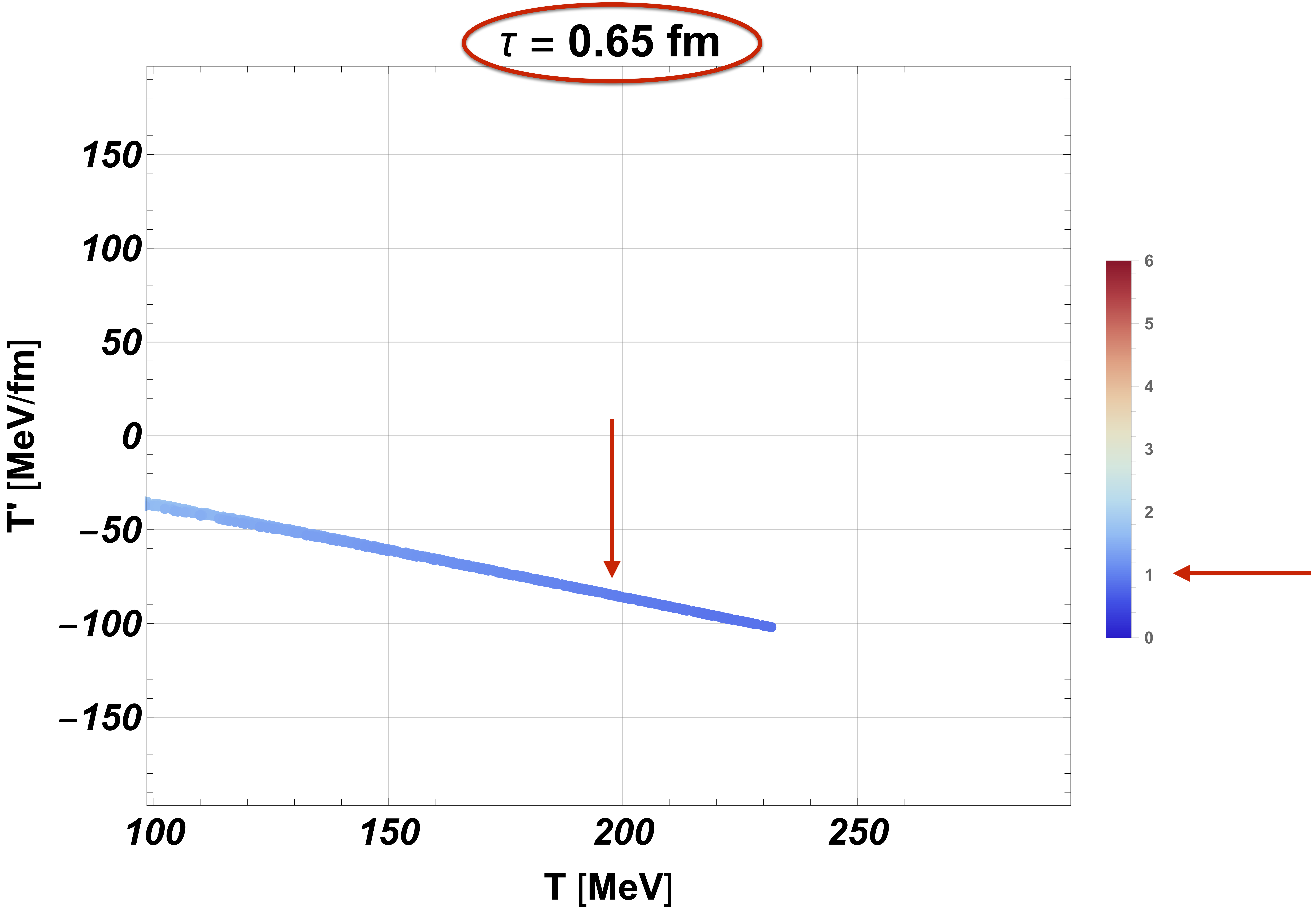

τ = 0.65 fm
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100
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0
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T' [MeV/fm]
100
150
200
250
T [MeV]
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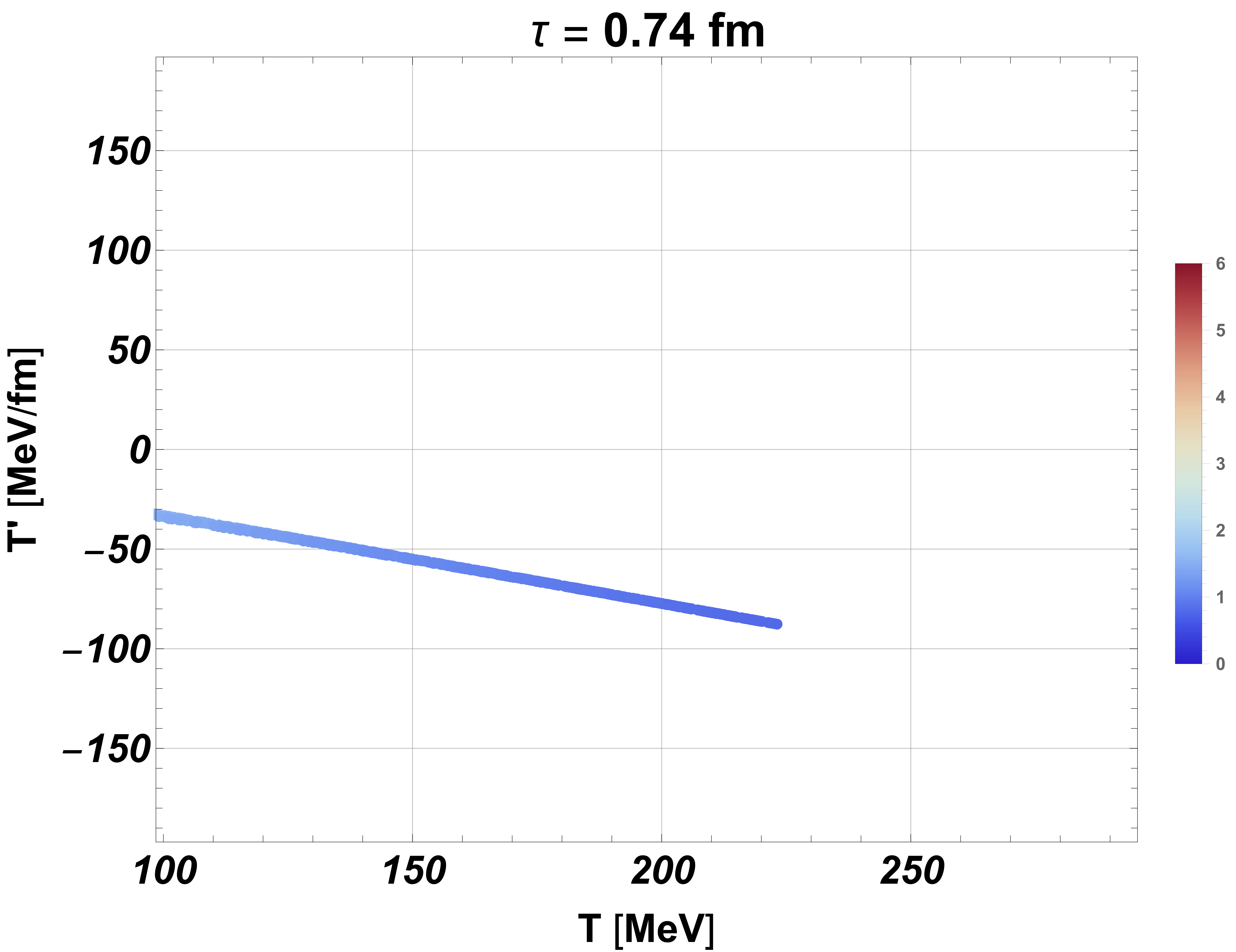
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T' [MeV/fm]
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150
200
250
T [MeV]
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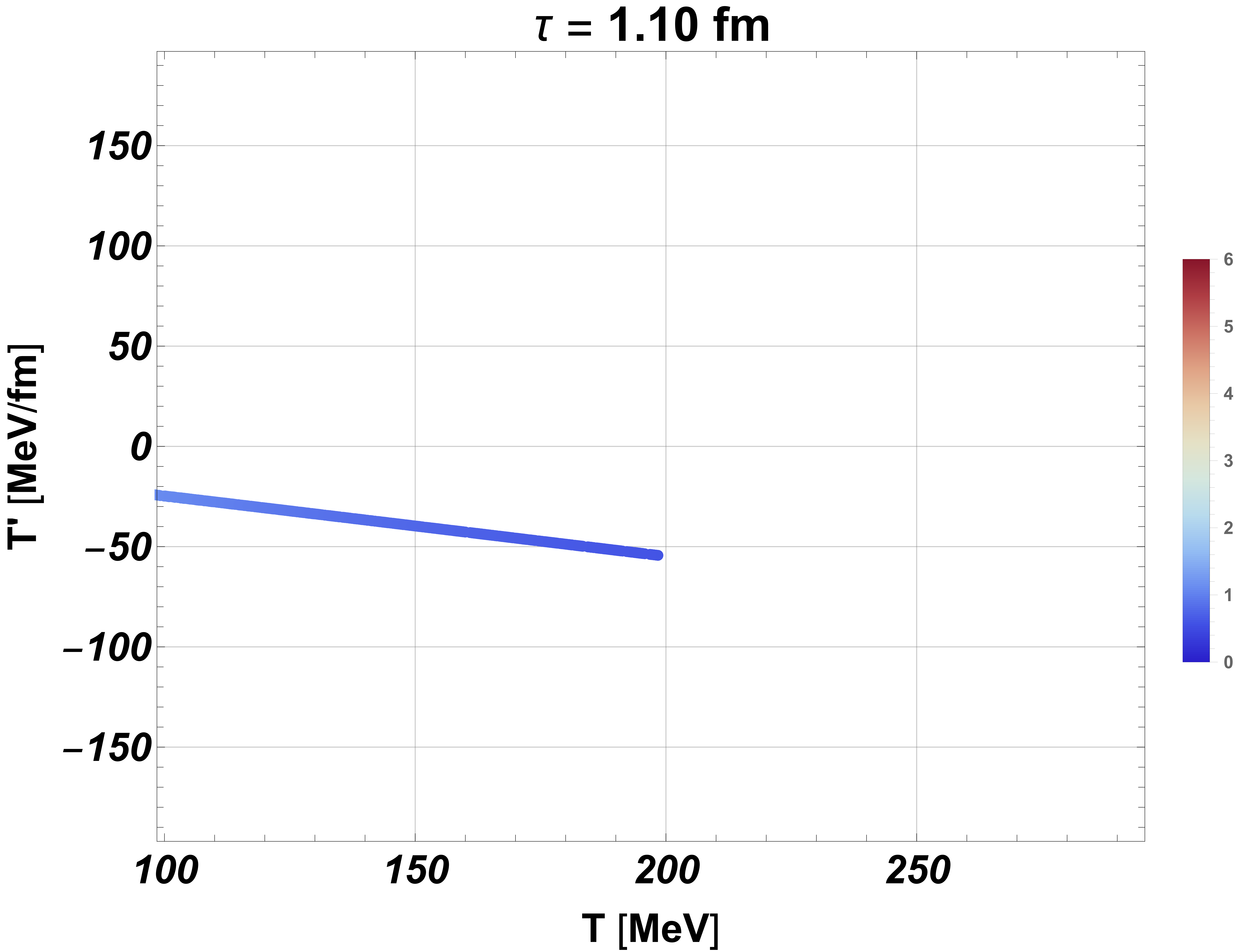
4
3
2
1
0



τ = 1.10 fm
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−100
−150
T' [MeV/fm]
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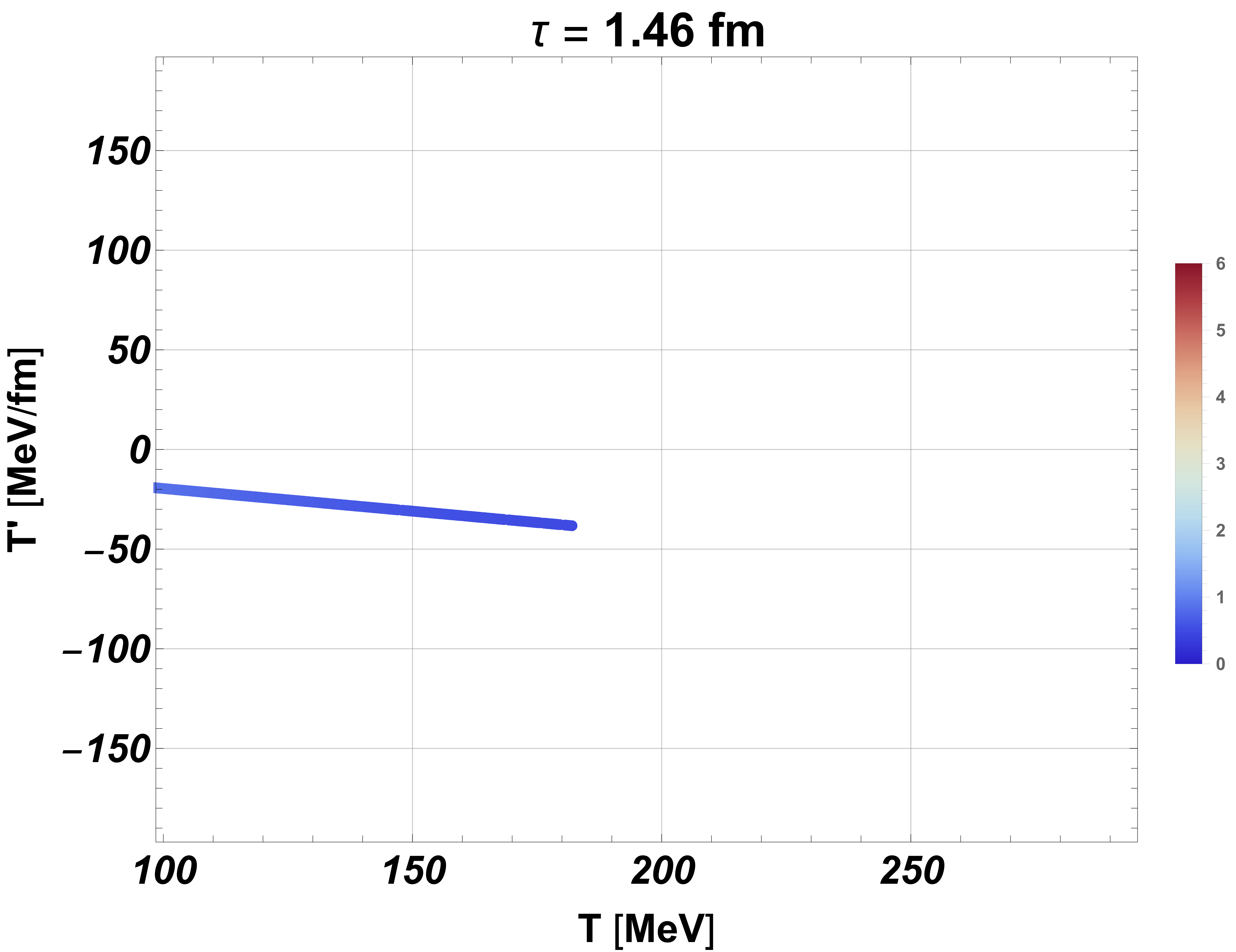
150
200
250
T [MeV]
6
5
4
3
2
1
0



τ = 1.46 fm
T' [MeV/fm]
T [MeV]
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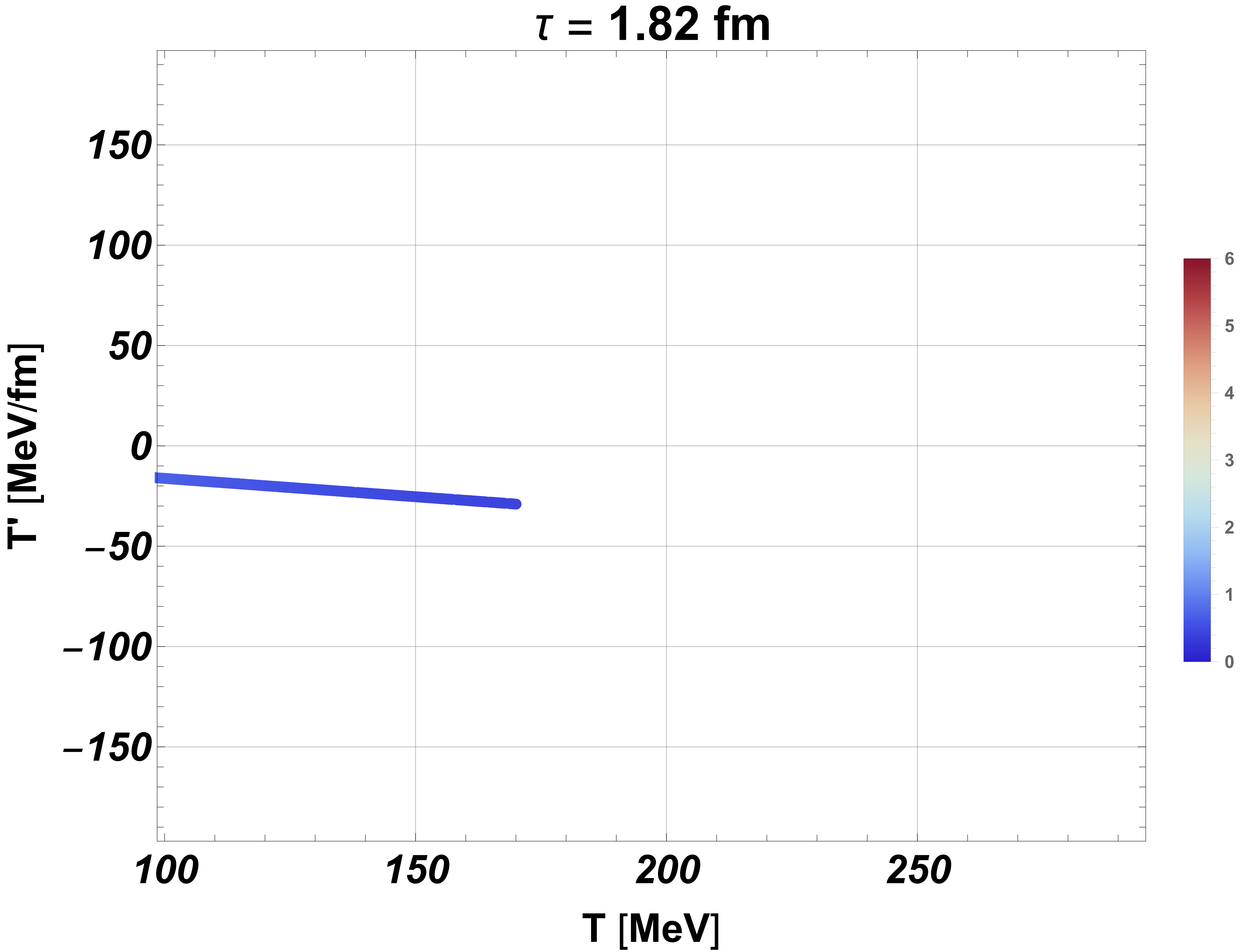

τ = 1.82 fm
T' [MeV/fm]
T [MeV]
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150
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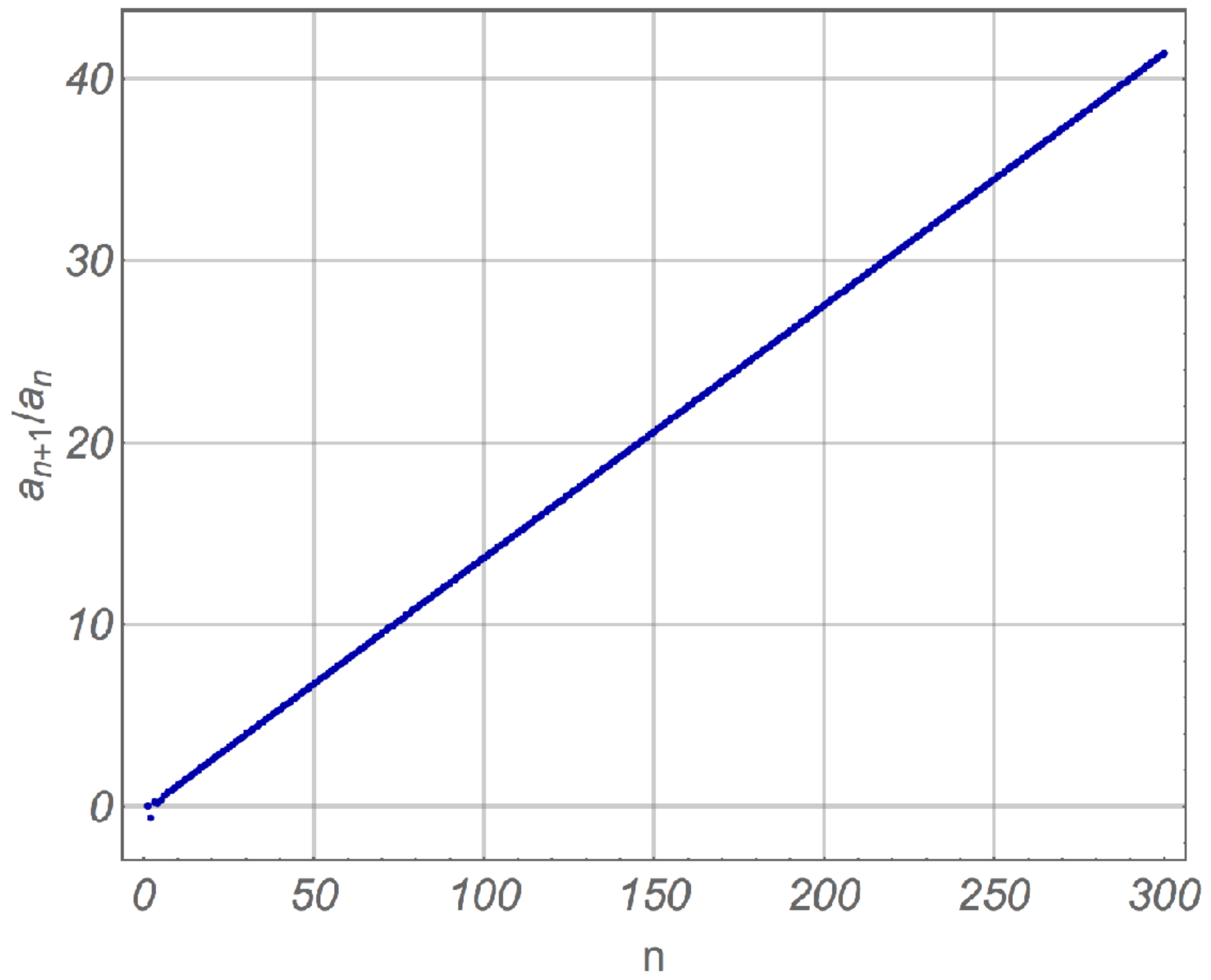
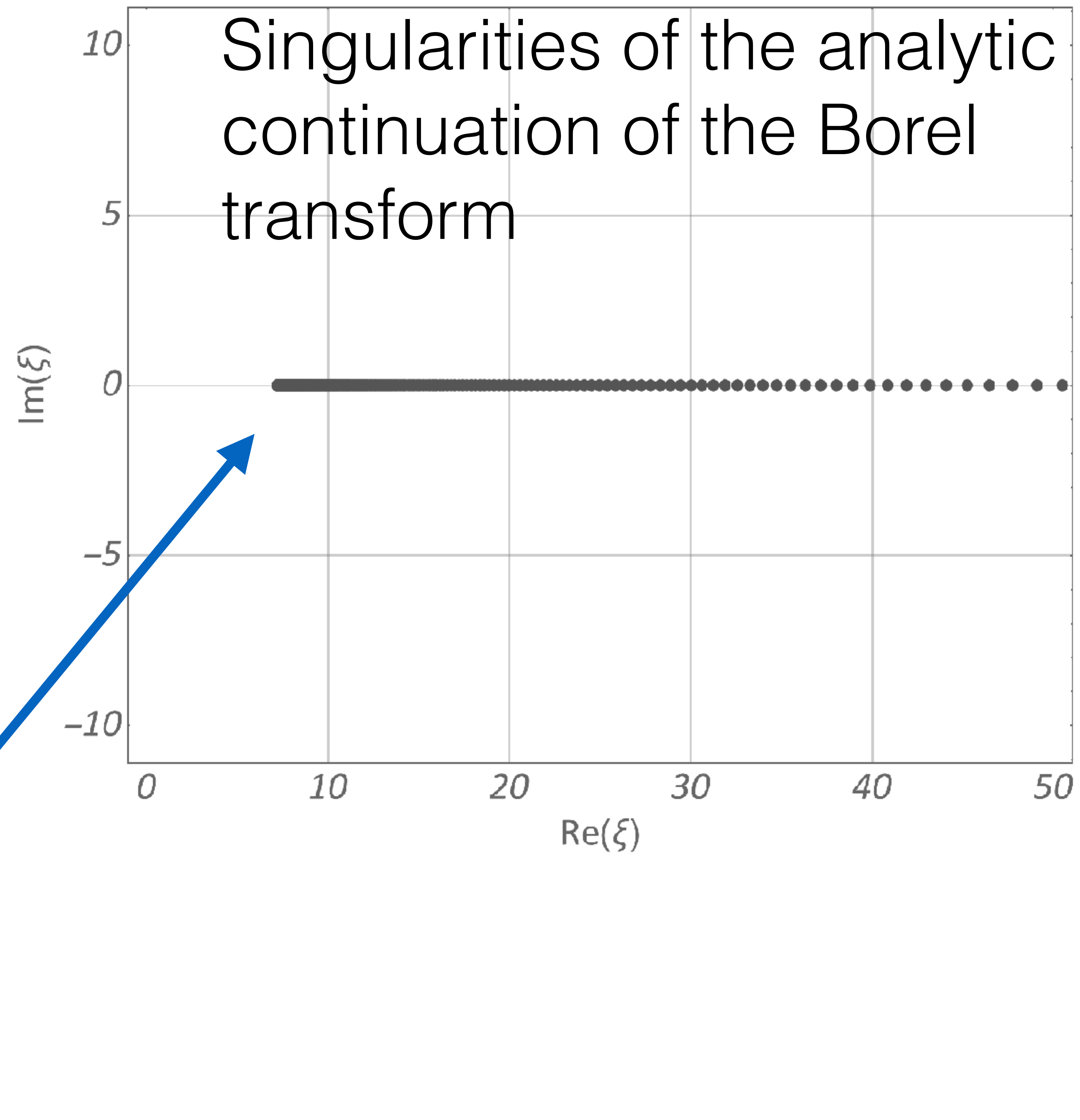
Summary

- The emergence of hydrodynamic behaviour is governed by the decay of non-hydrodynamic transients rather than local equilibration
- Late-time asymptotic solutions have the form of transseries and the expresses the "dissipation" of initial state information
- The transseries solutions suggests the existence of attractors which can be approximated by low orders of the gradient expansion
- These features are seen both in hydrodynamic models and at the microscopic level
- Attractor behaviour can be studied in a phase space picture of the dynamics

Backup material

Asymptotic behaviour in BRSSS





- **Branch point location determined by τ_π**
- Cannot integrate over the real line
- Complex ambiguity

Borel summation

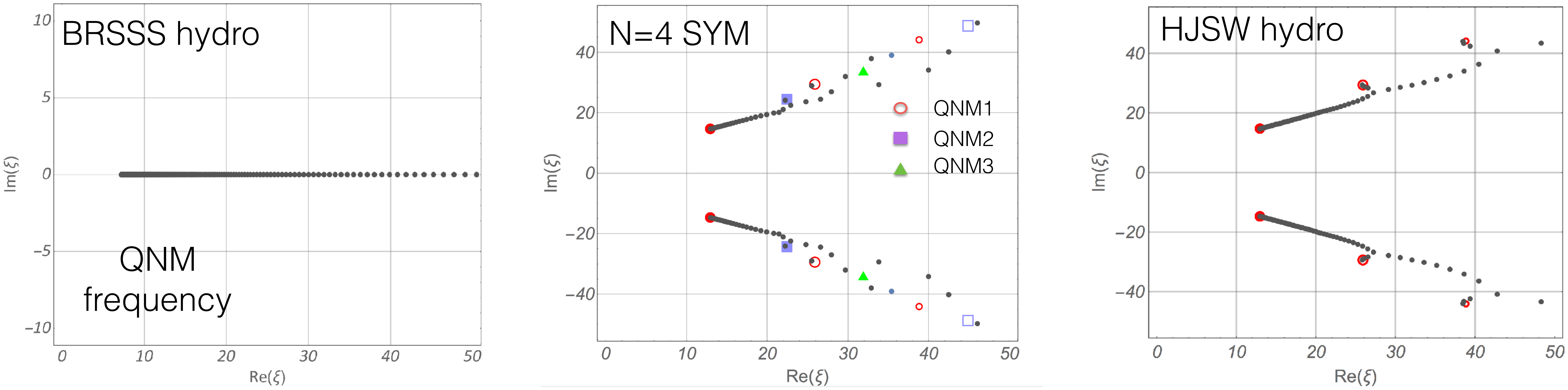
- Borel transform

$$\mathscr{B}[\Phi_0](\xi) = \sum_{n\geq 0} \frac{a_{n+1}^{(0)}}{\Gamma[n+1]}\xi^n$$

- Borel sum

Analytic continuation (Pade)

$$\mathcal{S}\Phi_0(w) = \int_C d\xi\, e^{-w\xi}\, \tilde{\mathscr{B}}[\Phi_0](\xi)$$

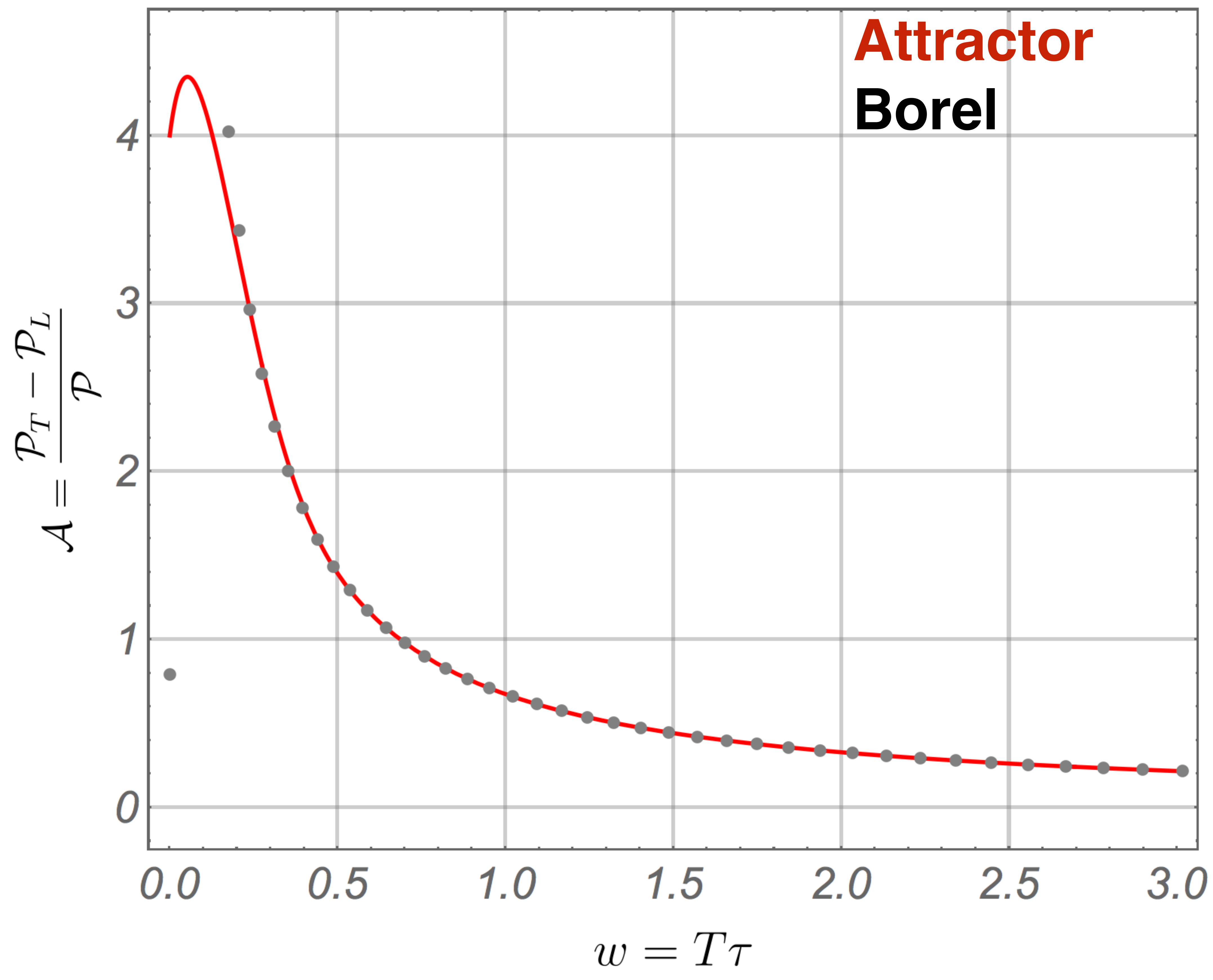


Borel sum in HJSW

We adopt HJSW as a testing ground:

- Use 240 terms of the series
- The result **can be compared** to the numerically determined attractor
- The summation breaks down for w < 0.3, but **gets better for larger values of w**
- Could be improved by including trans-series sectors (this would require determining appropriate values of trans series parameters)

Next: proceed in the same way to sum the series for N=4 SYM.



Seeing the transients in SYM plasma

We can look for the leading transseries correction in AdS/CFT numerics at late times.

The leading transseries correction from our hydro model is of the form

$$\delta\mathscr{A}(w) = e^{-\frac{3}{2}\Omega_I w} w^{\beta_R} \left[\Phi_+(w) \cos\left(\frac{3}{2}\Omega_R w - \beta_I \log(w)\right) + \Phi_-(w) \sin\left(\frac{3}{2}\Omega_R w - \beta_I \log(w)\right) \right]$$

where

$$\Phi_\pm(w) = C_\pm \left(1 + \sum_{n>0} \frac{a_n^{(\pm)}}{w^n}\right) \approx C_\pm$$

The approximate solution

$$\mathscr{A}(w) \sim \mathscr{A}_H(w) + \delta\mathscr{A}(w)$$

can be **compared to numerical solutions** of time evolution obtained at the microscopic level using AdS/CFT.

To see that the transient, damped oscillations can be resolved with the existing numerical methods we can consider pairs of solutions. Because we are looking at a universal observable, **the hydrodynamic part will cancel:**

$$\mathscr{A}_1(w) - \mathscr{A}_2(w) \sim e^{-\frac{3}{2}\Omega_I w} w^{\beta_R} \left[C_{12}^{(+)} \cos\left(\frac{3}{2}\Omega_R w - \beta_I \log(w)\right) + C_{12}^{(-)} \sin\left(\frac{3}{2}\Omega_R w - \beta_I \log(w)\right)\right]$$

Here **all the parameters are fixed apart from the two amplitudes**, which reflect the initial conditions and differ from one pair of solution to another.

The two amplitudes appearing in the formula above can then be fitted to the numerical solution.