Quantum Aspects of Non-Abelian T-Duality

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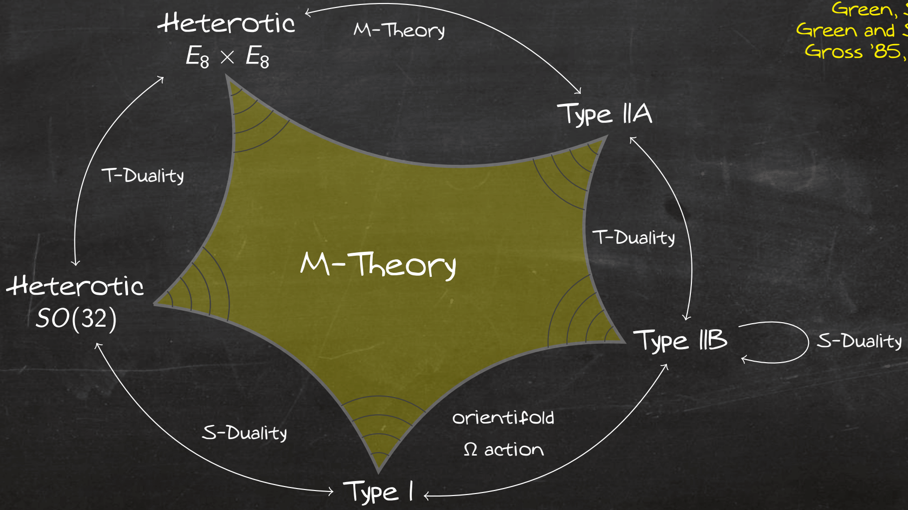
Based on arXiv: 1805.03657 (NPB) with B. Fraser & K. Sfetsos.

Dualities in String Theory



Green, Schwarz '82
Green and Schwarz '84
Gross '85, Witten '95

Example: Compactified Boson

- Compactify: Restrict the domain of variation of a free boson to a circle of radius R.
- Torus modular invariant partition function:

$$Z(R) = \frac{1}{|\eta(\tau)|^2} \sum_{e,m\in\mathbb{Z}} q^{\frac{1}{2}\left(\frac{e}{R}+\frac{mR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{e}{R}-\frac{mR}{2}\right)^2} = \sum_{e,m\in\mathbb{Z}} |\chi^{\text{Vir}}_{e,m}|^2$$

- Electric-magnetic duality ($e \leftrightarrow m$) resulting from the invariance of the partition function under the duality transformation $R \leftrightarrow 2/R$

$$Z(R) = Z(2/R)$$

CFTs at radius R and $R' = 2/R$ are isomorphic; the isomorphism inverts the sign of $J, \bar{J}$.

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Virasoro character:

$$\chi_i(\tau) = \mathrm{Tr}_{R_i}\, q^{L_0 - \frac{c}{24}}$$

$$(q \equiv e^{2\pi i \tau})$$

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Modular invariance:

$\tau \mapsto -\frac{1}{\tau}$

$\tau \mapsto \tau + 1$

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Abelian vs Non-Abelian Duality

PROPERTY	ATD	NATD
Invertibility		
Exactness		
$Z_{\text{original}} \stackrel{?}{=} Z_{\text{dual}}$	✓	
Modular invariance	✓	

Abelian vs Non-Abelian Duality

- Duality with respect to Abelian isometry group G (ATD)

Buscher '87, '88
Rocek & E. Verlinde '92

How: $S[g] \xrightarrow[\text{add Lagrange mult.} v]{\text{gauge isometries}} S_{\text{gauge}}[g, v, A_\pm] \xrightarrow{\text{integrate over } A_\pm} S^*[g, v, A_\pm]$

What: $(g, b, \phi, G) \leftrightarrow (g^*, b^*, \phi^*, G^*)$; G^* is the rep. ring, also a group.

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} Fraser, DM & Sfetsos '18

Reminder: The Coset Construction

Goddard, Kent & Olive '85

- Consider the $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{h}}_\ell$ WZW models, with $\mathfrak{h} \subset \mathfrak{g}$. Their Virasoro modes $L_n^{\mathfrak{g}}$ and $L_n^{\mathfrak{h}}$ satisfy the Virasoro algebra with central charges c_k and c_ℓ, where

$$c_k = \frac{k \dim \mathfrak{g}}{k + \mathrm{g}}, \qquad \mathrm{g} = \text{ dual Coxeter number of } \mathfrak{g}.$$

- Then their difference

$$L_n^{\mathfrak{g}/\mathfrak{h}} := L_n^{\mathfrak{g}} - L_n^{\mathfrak{h}},$$

also satisfies the Virasoro algebra with central charge

$$c = c_k - c_\ell.$$

- We are interested in diagonal cosets:

$$\frac{\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_\ell}{\hat{\mathfrak{g}}_{k+\ell}},$$

with central charge

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Example:

$\widehat{\mathfrak{su}}(2)_k / \widehat{\mathfrak{u}}(1)$

central charge

$$c = \frac{2(k-1)}{k+2}$$

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Example:

$k = 1 : c = 0$

$k = 2 : c = \frac{1}{2}$

$k = 3 : c = \frac{4}{5}$

$k = 4 : c = 1$

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Example:

$\mathfrak{g} = \mathfrak{su}(2)$, $\ell = 1$,

central charge

$$c = 1 - \frac{6}{p(p+1)}, \qquad p = k + 2 \geq 3$$

Unitary Minimal Models

NATD as a Large Level Limit

Sfetsos '94

- For the WZW model consider the global symmetry $g \mapsto hgh^{-1}$, for constant $h \in H$
- NATD can be realized as the limit

$$\lim_{\ell \to \infty} \frac{\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{h}}_\ell}{\hat{\mathfrak{h}}_{k+\ell}} = \hat{\mathfrak{g}}_k \big|_{\text{NATD wrt } \mathfrak{h}}$$

- What about at the quantum level?
- We consider the case where $\mathfrak{g} = \mathfrak{h} = \mathfrak{su}(2)$

$$\frac{\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{su}}(2)_\ell}{\widehat{\mathfrak{su}}(2)_{k+\ell}}$$

Coset Space Geometry of $(SU(2) \times SU(2))/SU(2)$

Polychronakos & Sfetsos '10,'11

- The coset space is parameterized by a pair of spheres: (S^3_k, S^3_ℓ), modded out by the diagonal $SU(2)_{k+\ell}$ action

$$(g_1, g_2) \sim (hg_1h^{-1}, hg_2h^{-1}), \qquad (g_1, g_2) \in (SU(2)_k, SU(2)_\ell) \quad h \in SU(2)_{k+\ell}$$

- Embed both spheres into $\mathbb{R}^4$ as $\alpha_0^2 + |\boldsymbol{\alpha}|^2 = 1$ and $\beta_0^2 + |\boldsymbol{\beta}|^2 = 1$
- Coordinates on the coset space: $(\alpha_0, \beta_0, \gamma \equiv \pm\sqrt{|\boldsymbol{\alpha} \cdot \boldsymbol{\beta}|})$
- Dilaton: $e^{-2\Phi} = (1 - \alpha^2)(1 - \beta^2) - \gamma^2$

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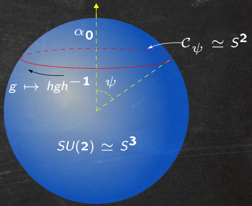

Dual of $SU(2)$ WZW

Polychronakos & Sfetsos '10,'11

We want to take $\ell \to \infty$ so that the radius of the S^3_ℓ becomes large

- Limit 1: Consider focussing around $\beta_0 = 1$ and $\gamma = 0$, i.e. at the North pole of S^3_ℓ

$$\mathrm{d}s^2 = k\left(\mathrm{d}\psi^2 + \frac{\cos^2\psi}{y^2}\mathrm{d}x^2 + \frac{[y\mathrm{d}y + (\sin\psi\cos\psi + x + \psi)\mathrm{d}x]^2}{y^2\cos^2\psi}\right), \qquad e^{-2\Phi} = y^2\cos^2\psi$$

This is the NATD background found e.g. in Giveon & Rocek '94.

- Limit 2: Consider focussing around $\beta_0 = 0$, i.e. at the center of S^3_ℓ

$$\mathrm{d}s^2 = k\left(\mathrm{d}z^2 + \mathrm{d}\psi^2 + \tan^2\psi\,\mathrm{d}\phi^2\right), \qquad e^{-2\Phi} = \cos^2\psi$$

This is the CFT background $\frac{SU(2)_k}{U(1)} \times \mathbb{R}$.

High Spin Limit

Polychronakos & Sfetsos '10,'11

- A coset primary field is labelled by three weights, namely $\{\lambda, \mu; \nu\}$ with conformal weight

$$h_{\{\lambda,\mu;\nu\}} = \frac{\lambda(\lambda+2)}{4(k+2)} + \frac{\mu(\mu+2)}{4(\ell+2)} - \frac{\nu(\nu+2)}{4(k+\ell+2)}$$

- Take $\ell \to \infty$ in such a way so that $\lim_{\ell\to\infty} h_{\{\lambda,\mu;\nu\}} < \infty$
- Specifically, let

$$0 < \lambda < k, \qquad \nu - \mu \in \mathbb{Z}, \qquad \mu = \frac{\delta}{k}\ell \sim \ell$$

then

$$\lim_{\ell\to\infty} h_{\{\lambda,\mu;\nu\}} = \frac{\lambda(\lambda+2)}{4k} + \frac{\delta - 2n}{4k}\delta.$$

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A weight is just twice the usual spin: $\lambda = 2j$.

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Representation Theory

Fraser, DM & Sfetsos '18

- Hilbert space: Charge conjugation CFT

$$\mathcal{H}_{\text{coset}} = \bigoplus_{\substack{\lambda+\mu-\nu\in 2\mathbb{Z} \\ \{\lambda,\mu;\nu\}\sim\{k-\lambda,\ell-\mu;k+\ell-\nu\}}} \mathcal{H}_{\{\lambda,\mu;\nu\}} \otimes \overline{\mathcal{H}}_{\{\lambda,\mu;\nu\}}$$

- Task 1: Extract Coset characters via branching rules

$$R_\lambda \otimes R_\mu = \bigoplus_{\nu=0}^{k+\ell} \mathcal{H}_{\{\lambda,\mu;\nu\}} \otimes R_\nu \quad \Longrightarrow \quad \chi_\lambda^{(k)}(z,\tau)\chi_\mu^{(\ell)}(z,\tau) = \sum_{\nu=0}^{k+\ell} \Xi_{\{\lambda,\mu;\nu\}}(\tau)\chi_\nu^{(k+\ell)}(z,\tau)$$

- Task 2: Construct modular invariant partition function from coset characters:

$$\boxed{\mathcal{Z}_{\text{coset}}(\tau) = \frac{1}{2}\sum_{\lambda+\mu-\nu\in 2\mathbb{Z}} \left|\Xi_{\{\lambda,\mu;\nu\}}(\tau)\right|^2}$$

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Limit of the Partition Function

Fraser, DM & Sfetsos '18

- Result:

$$\lim_{\ell\to\infty} \mathcal{Z}_{\text{coset}}(q) = \#\frac{1}{|\eta(q)|^2\sqrt{\text{Im}\,\tau}}\sum_{\lambda=0}^{k}\sum_{\beta=-k+1}^{k}\left|\eta(q)c_\beta^\lambda\right|^2$$

$$= \#\mathcal{Z}_{\text{Bos}(\mathbb{R})}\cdot\mathcal{Z}_{k\text{-Paraf.}}$$

- Parafermions coupled to an uncompactified boson: $\frac{SU(2)_k}{U(1)}\times\mathbb{R}$

- Modular invariant

Summary & Future Directions

In conclusion

- We found the torus modular invariant partition function for the NATD of the $\widehat{\mathfrak{su}}(2)_k$ WZW model.
- The partition function of the original and dual theories are not the same.
- Interestingly, both limiting backgrounds seem to share the same partition function.

Open questions

- Implement the high spin limit to more general cosets.
- What is the relation with higher spin theories and holography? ($\widehat{\mathfrak{su}}(N)$ diagonal coset)
- It will be interesting to see if our findings may have relevance in the presence of Ramond-Ramond backgrounds.
- We first quantized strings on the coset background and then took the limit in the quantum theory: Does quantization and the large level limit commute?

Questions

Thank you!

Dimitris Manolopoulos

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