Color charge

- QCD is based on the gauge group SU(3) : three colors (red, green, blue)
- **Quarks** carry a SU(3) color

- **Anti-quarks** also carry SU(3) (anti)-colors

- **Gluons** carry a color and a anticolor

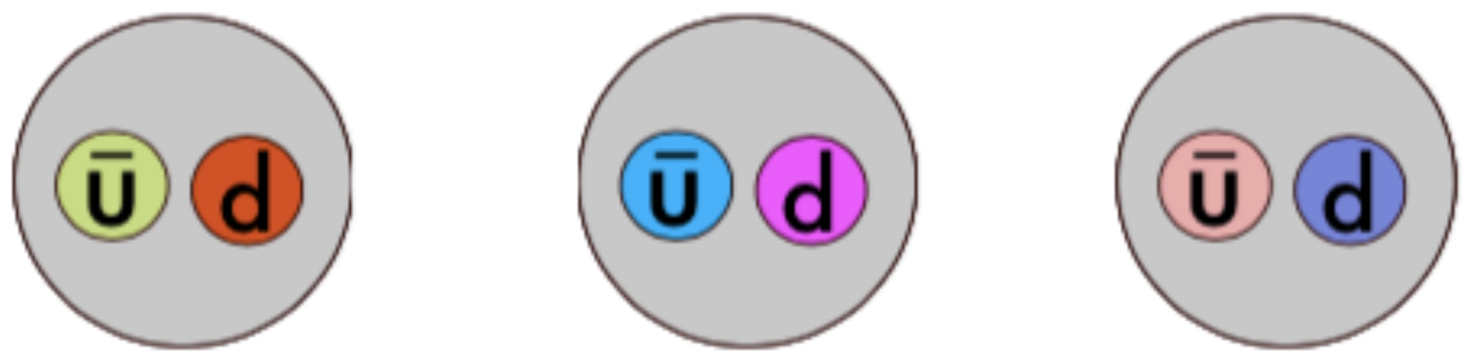
→ Gluons carry a color charge : different from QED (photon electrically neutral)

→ Gluons interact with themself via QCD !

Mesons and baryons

- Quarks and gluons are not directly observed in detectors
- We observe only hadrons (bound states, colorless particles)
  - Mesons (quark + anti-quark)

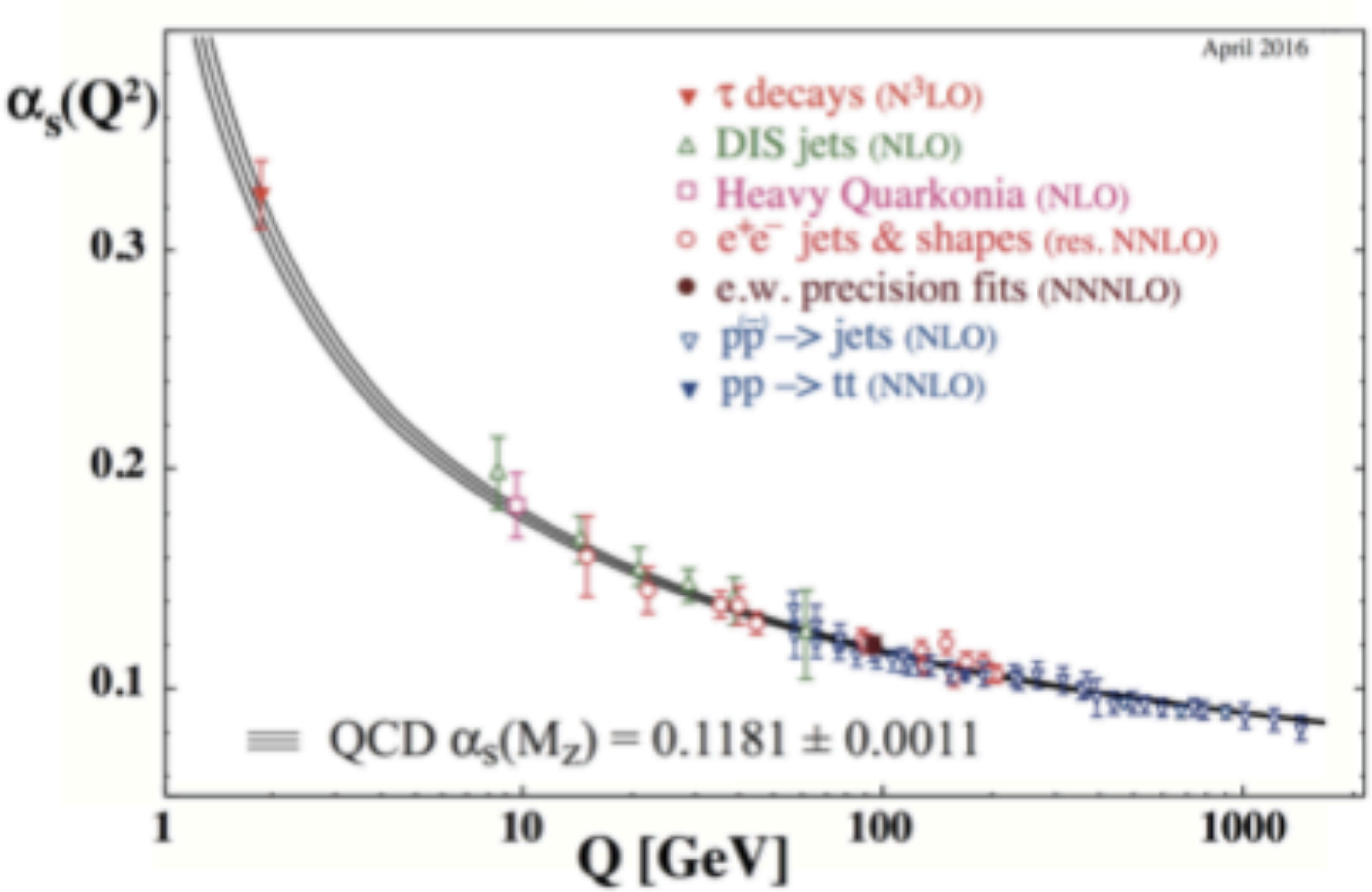
green+anti-green blue+anti-blue red+anti-red



  - Baryons (three quarks or three anti-quarks)

red + blue + green

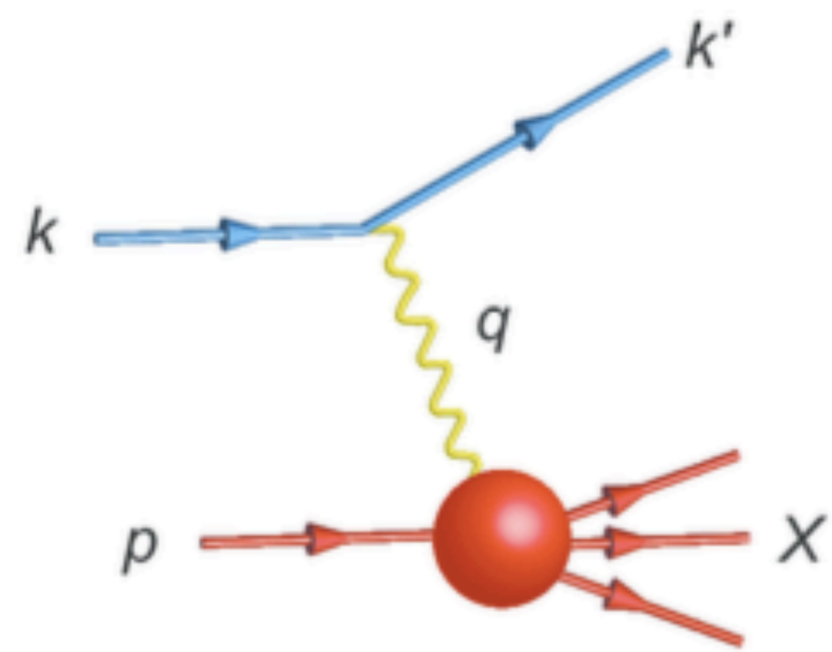
- **Running of the strong coupling**



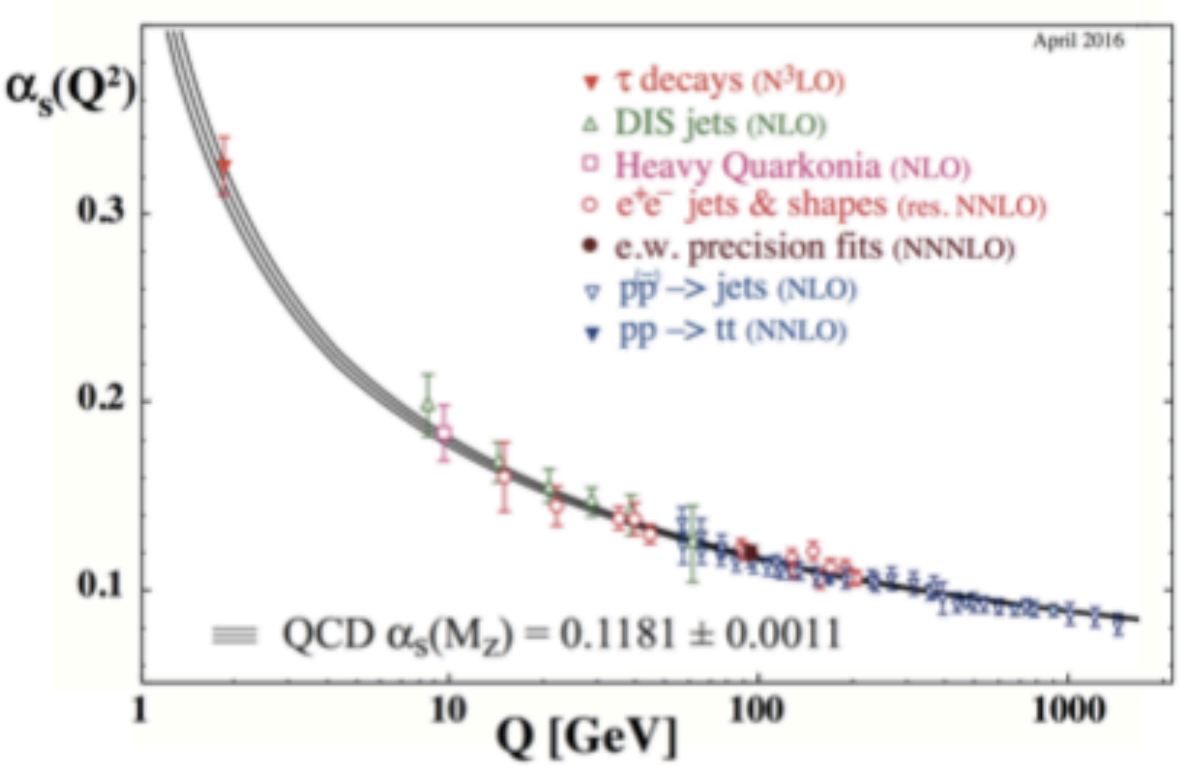
strong force gets weaker at short distances

▶ High-energy regime

- quarks weakly coupled
- "seen" as individual entities by sufficiently energetic probes
- perturbation theory applicable !



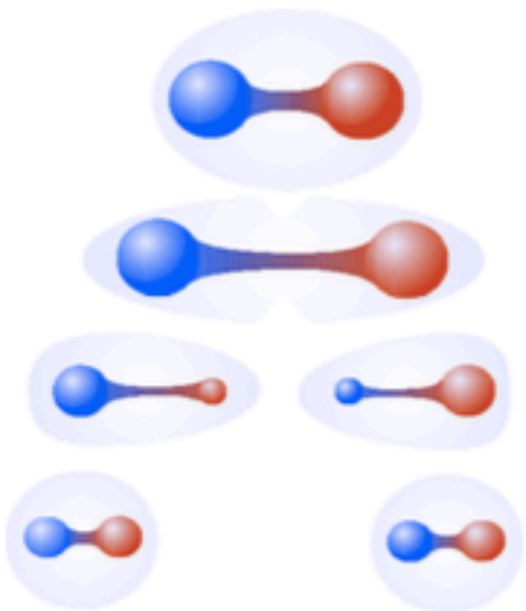
- **Running of the strong coupling**



- Strong force get stronger with the distance
- Only bound states are observed (color singlets)
- As soon as there is enough energy, a new quark/anti-quark pair is created

▶ Low-energy regime

- quarks strongly coupled : mesons and hadrons
- relevant degrees of freedom are hadrons
- perturbation theory breaks down → need different techniques



Why lattice QCD ?

Strong interaction is omnipresent

- Hadrons structure and masses
- The « form factors » of hadrons
- The products of high energy collisions : $pp \to X$ (hadrons)
- No general exact solution to QCD

Goals of Lattice QCD

- validate QCD as fundamental theory of strong interaction
- understand confinement
- compute basic hadron properties
- compute electroweak amplitudes involving hadrons
- study exotic states of matter (quark-gluon plasma, ...)
- ...

Advantages

✓ Non-perturbative tool
✓ Rigorous calculation : it is not a model
✓ The precision can be systematically improved

Disadvantages

✗ Need supercomputers (expensive calculations)
✗ Gives a numbers (not an analytic expression)

The path integral formalism

The vacuum expectation value of an observable $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U] \int \mathcal{D}[\overline{\psi}]\, \mathcal{D}[\psi]\ \mathcal{O}[\overline{\psi}, \psi, U]\ e^{iS[U,\overline{\psi},\psi]}$$

- In the previous formula we have to sum over all « paths » (field configurations)

  $\longrightarrow$ The weight is given by the action

  $\longrightarrow$ factor « i » : quantum interferences between paths

- In QED, on can use perturbation theory to compute $\langle \mathcal{O} \rangle$ order by order in the small coupling α_{QED}

$$S[U, \overline{\psi}, \psi] = S_0[U, \overline{\psi}, \psi] + \alpha_{\mathrm{QED}}\, S_{\mathrm{int}}[U, \overline{\psi}, \psi] + \cdots$$

  $\longrightarrow$ Leads to the diagrammatic expansion (Feynman diagrams)

- The strong coupling is not small (at small energies)

  $\longrightarrow$ perturbation theory does not work

Idea : evaluate the path integral numerically

The vacuum expectation value of an observable $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U] \int \mathcal{D}[\overline{\psi}]\, \mathcal{D}[\psi]\; \mathcal{O}[\overline{\psi}, \psi, U]\; e^{iS[U,\overline{\psi},\psi]}$$

Problems :

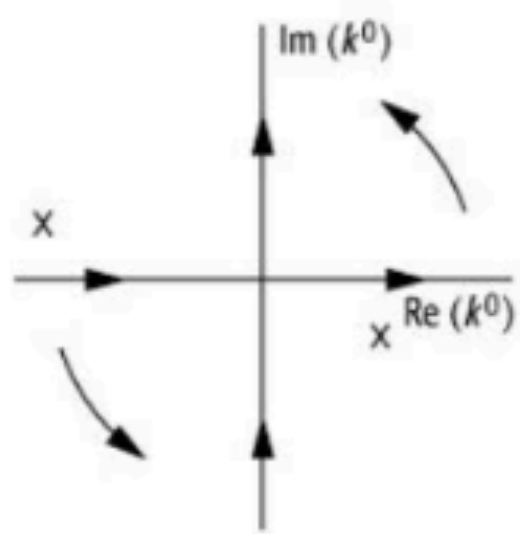
- ill-defined object
- factor « i » : large oscillations, difficult to integrate numerically

Solution :

- discretization of the theory on a hypercubic lattice
  - $\longrightarrow$ this is just a choice of regularization of the theory (like dimensional regularization)
  - $\longrightarrow$ finite number of degrees of freedom : path integral well defined
  - $\longrightarrow$ regularization adapted to numerical calculations
- rotate to imaginary time (Wick rotation)

$$e^{iS[U,\overline{\psi},\psi]} \to e^{-S_E[U,\overline{\psi},\psi]}$$

  - $\longrightarrow$ no oscillation anymore
  - $\longrightarrow$ based on analytic properties of QFT



Lattice QCD

- QCD (euclidean) Lagrangien :

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \sum_{i=1}^{N_f} \overline{\psi}_i(x)\left(\not{D} + m_i\right)\psi_i(x) \quad , \quad \not{D} = \gamma^\mu\left[\partial_\mu - ig A_\mu(x)\right]$$

- Break-up spacetime into a 4D grid : lattice spacing a, spatial extent L, time extent T
- Lattice spacing : natural UV regulator for the theory

1) Rotational/translational Lorentz symmetries are broken

2) Gauge symmetry is preserved

Quark fields $\psi(x), \overline{\psi}(x)$ on each site

- $\psi_\alpha^a(x)$: α = Dirac index
  a = color index

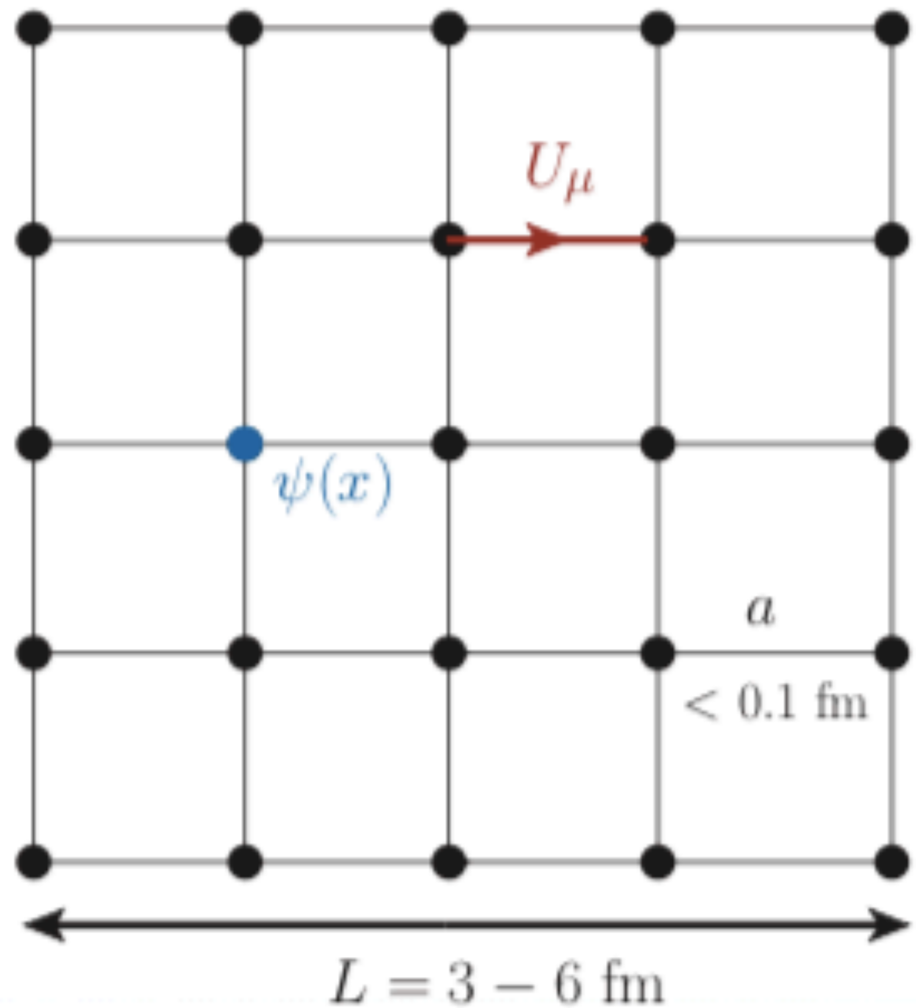
$\Rightarrow 3 \times 4 = 12$ complex numbers per site

Glue field $U_\mu(x)$ on links : parallel transporter

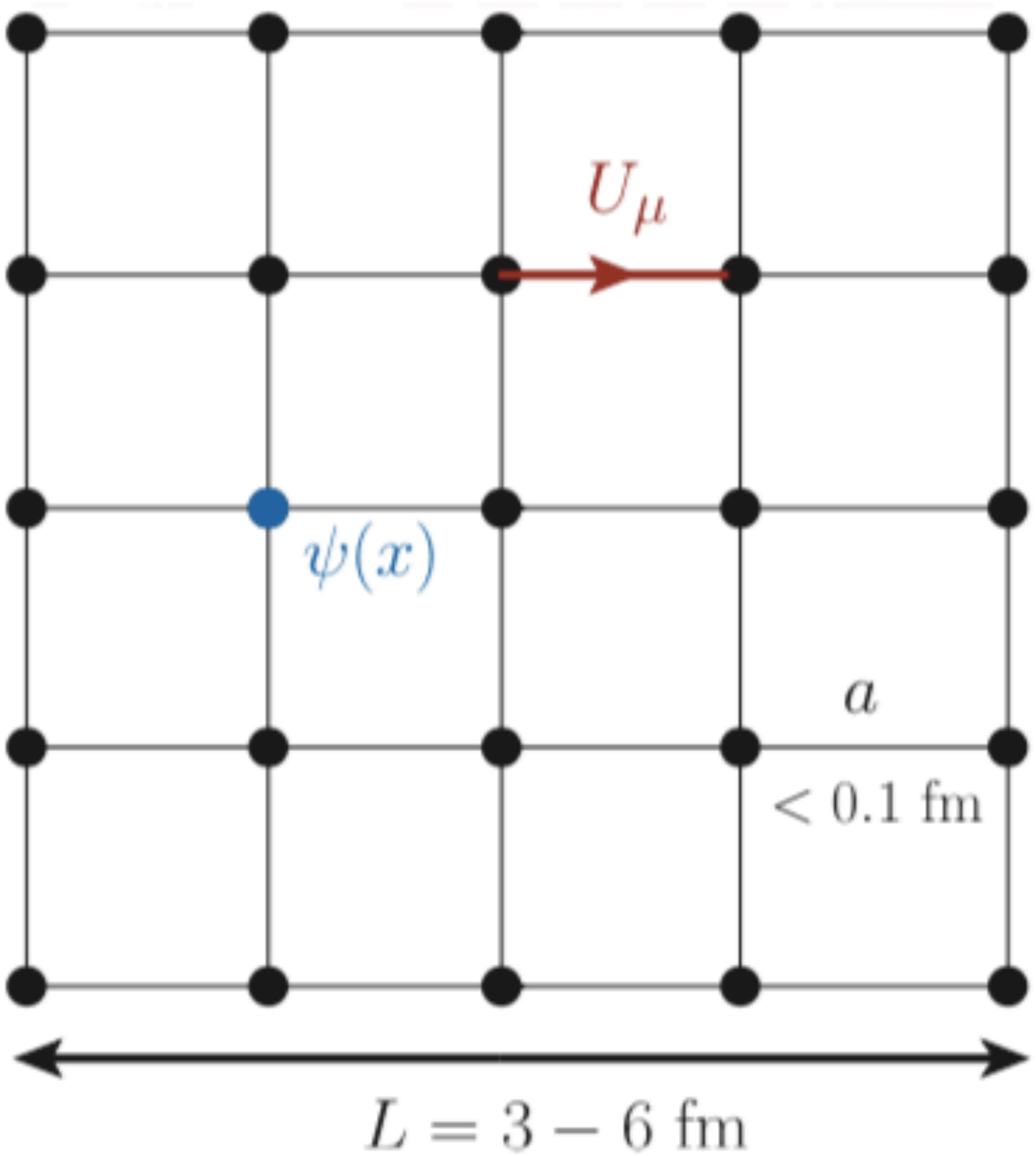
$$U_\mu(x) = \mathcal{P} e^{ig\int_x^{x+a\hat{\mu}} A_\nu(y)\mathrm{d}y^\nu} \in \mathrm{SU}(3)$$

- A field configuration $\{U_\mu\}$ is a set of SU(3) matrices

$\Rightarrow 9 \times 4 = 36$ complex numbers per site



Physical size a lattice



Typical lattice

- $L^3 \times T = 48^3 \times 96$
- $\approx 800 \times 10^6$ degrees of freedom
- $a \in [0.04 : 0.1]$ fm ($L \in 2 - 6$ fm)

Proton radius ≈ 0.9 fm

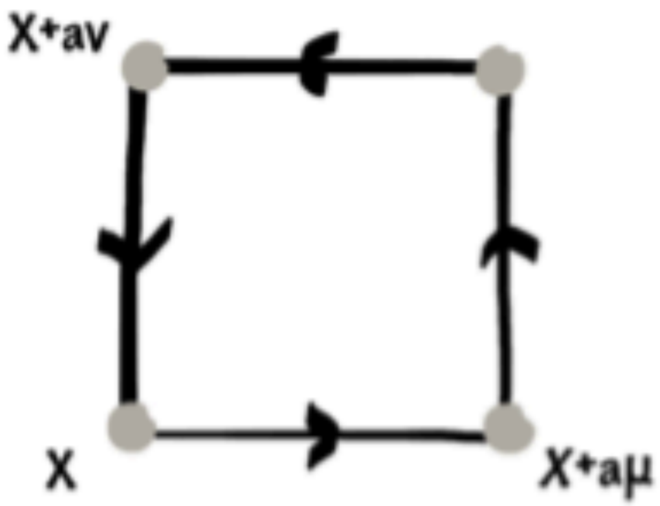
Wilson action for gluons

- In the continuum :

$$S_G^{\mathrm{cont}} = -\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right]$$

- We want to preserve gauge invariance symmetry

$$U_\mu(x) = \exp\left(iaA_\mu^a T^a\right) = 1 + iaA_\mu^a T^a + \ldots \in \mathrm{SU}(3)$$



Plaquette :

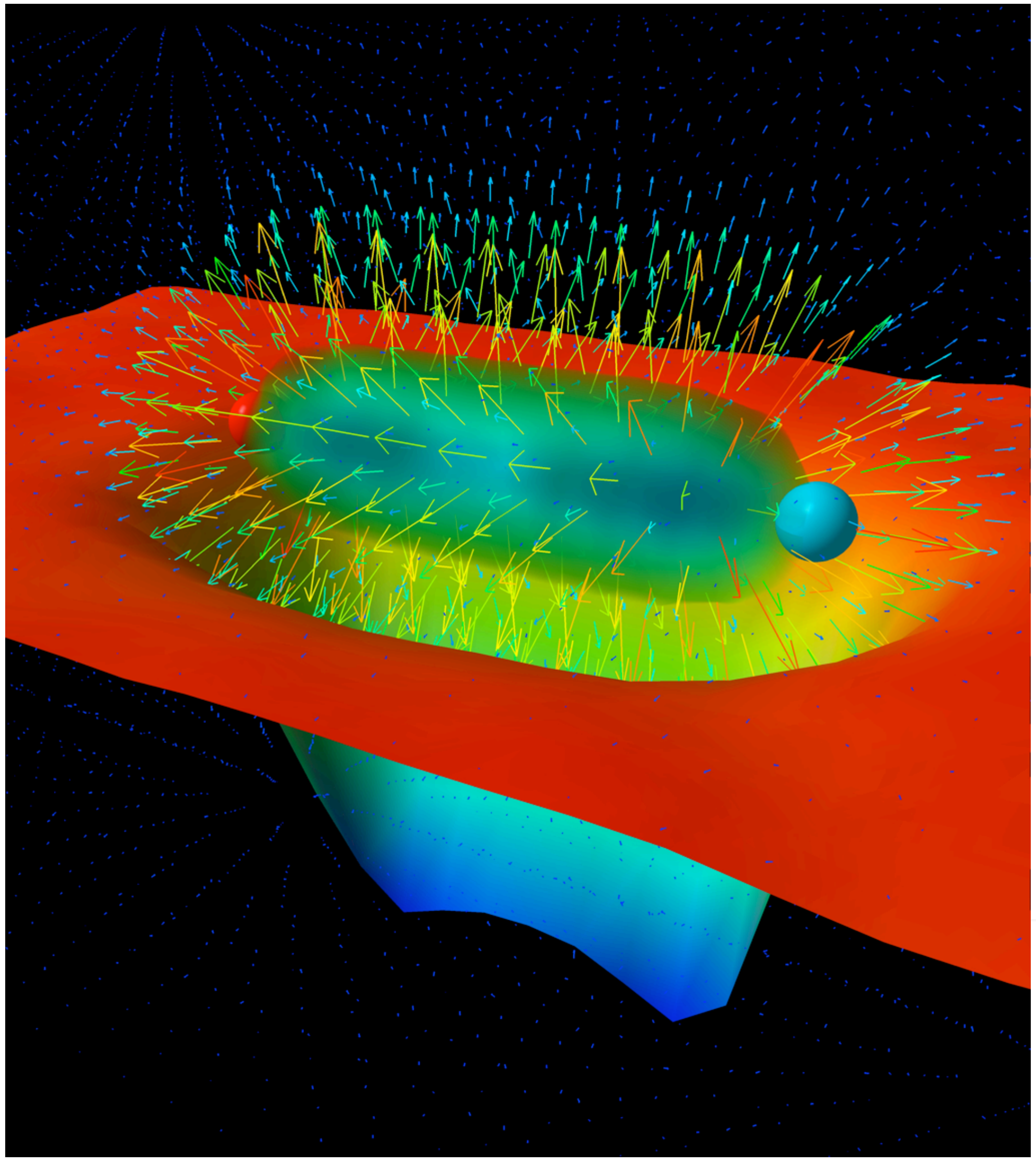
$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x)$$

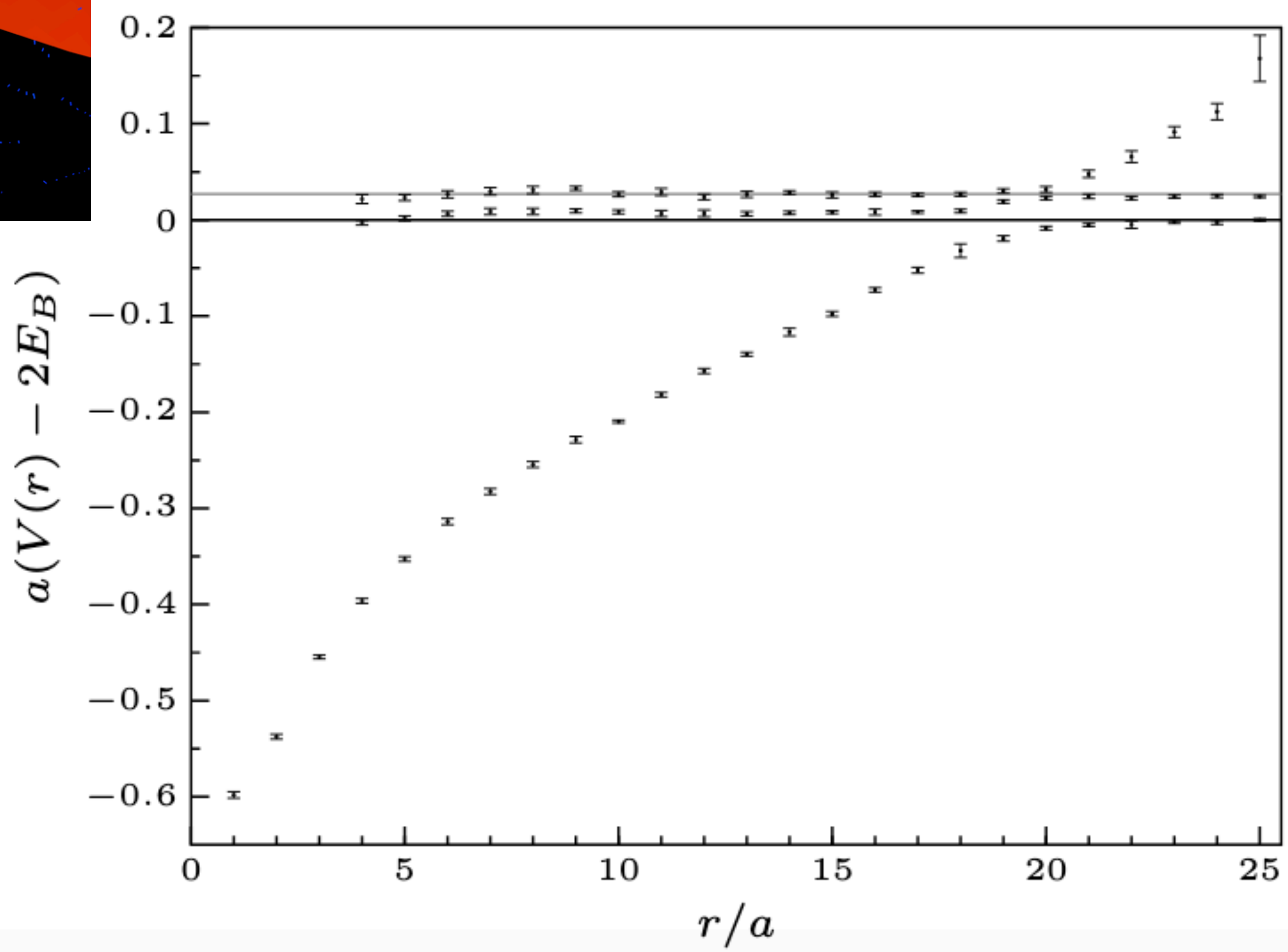
$$P_{\mu\nu}(x) = 1 + ig_0a^2F_{\mu\nu} - \tfrac{1}{2}g_0^2a^4F_{\mu\nu}^2 + \mathcal{O}(a^6)$$

- We define the lattice action

$$S_G[U] = \frac{1}{g_0^2}\sum_{x\in\Lambda}\sum_{\mu,\nu}\mathrm{Re}\,\mathrm{Tr}\left[1 - P_{\mu\nu}(x)\right] = S_G^{\mathrm{cont}}[U] + \mathcal{O}(a^2)$$

- Other choices are possible. They differ by an $\mathcal{O}(a^2)$ ambiguity.
  $\rightarrow$ Can be use to reduce discretization errors





0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
$a(V(r) - 2E_B)$
0
5
10
15
20
25
r/a

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

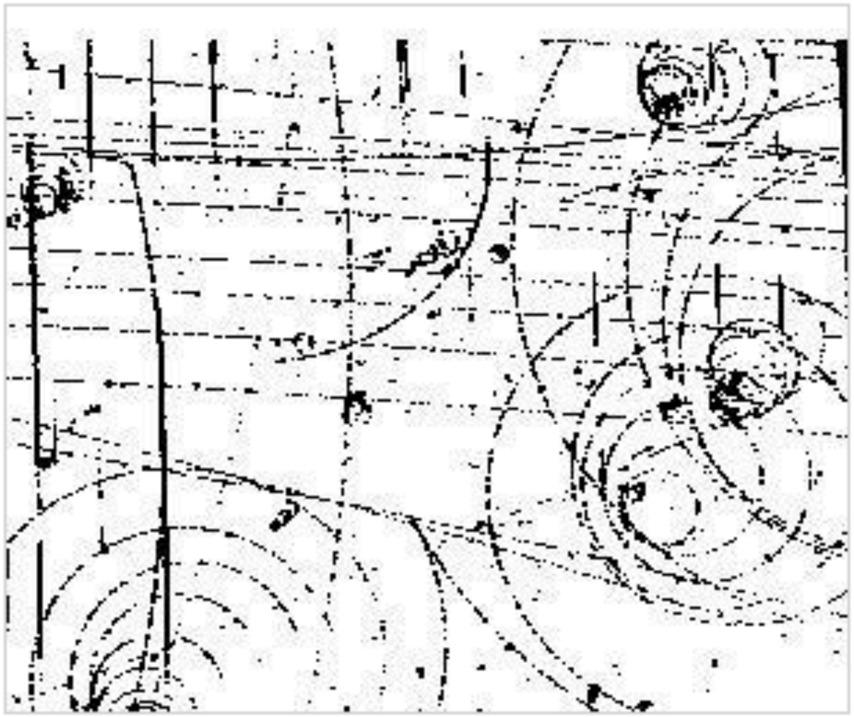
Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

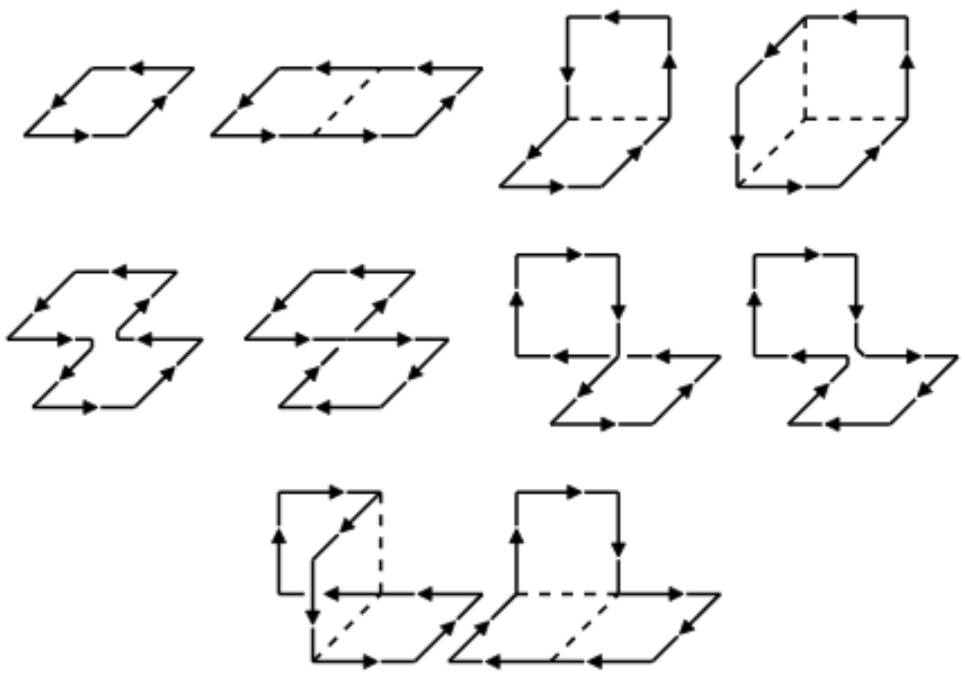
If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

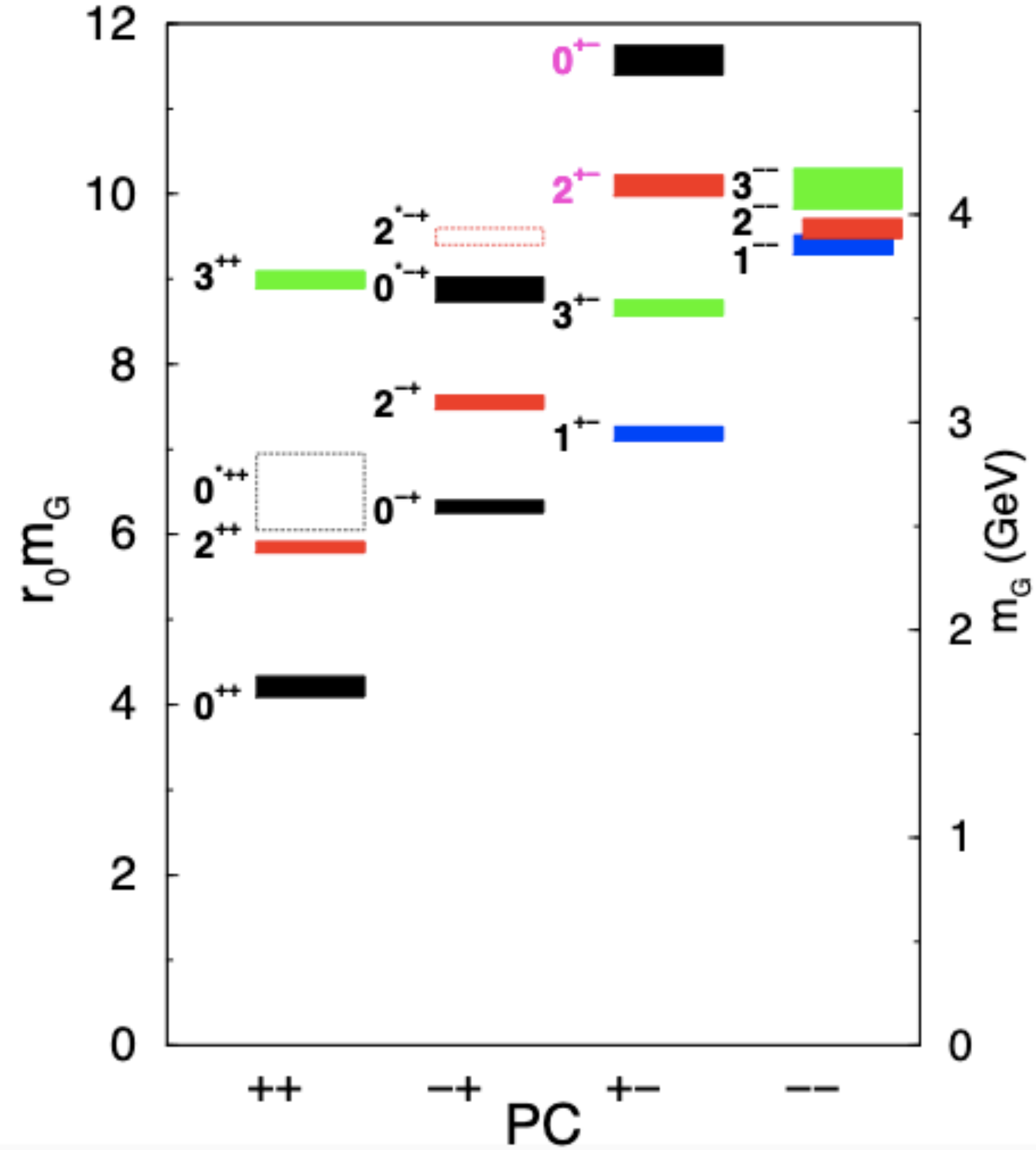
Yang–Mills and Mass Gap



The laws of quantum physics stand to the world of elementary particles in the way that Newton's laws of classical mechanics stand to the macroscopic world. Almost half a century ago, Yang and Mills introduced a remarkable new framework to describe elementary particles using structures that also occur in geometry. Quantum Yang-Mills theory is now the foundation of most of elementary particle theory, and its predictions have been tested at many experimental laboratories, but its mathematical foundation is still unclear. The successful use of Yang-Mills theory to describe the strong interactions of elementary particles depends on a subtle quantum mechanical property called the "mass gap": the quantum particles have positive masses, even though the classical waves travel at the speed of light. This property has been discovered by physicists from experiment and confirmed by computer simulations, but it still has not been understood from a theoretical point of view. Progress in establishing the existence of the Yang-Mills theory and a mass gap will require the introduction of fundamental new ideas both in physics and in mathematics.

This problem is:	Unsolved





12
10
8
6
4
2
0
$r_0 m_G$
4
3
2
1
0
m_G (GeV)
0^{+-}
2^{+-}
3^{--}
2^{--}
1^{--}
2^{*-+}
3^{++}
0^{*-+}
3^{+-}
2^{-+}
1^{+-}
0^{*++}
0^{-+}
2^{++}
0^{++}
++
−+
+−
−−
PC

Discretizations of the fermionic action

- There are many different actions
- They are all equivalent in the continuum limit ($\rightarrow$ QCD !)
- But they have different features at finite value of the lattice spacing

Action	Advantages	Disadvantages
Staggered	✓ computationally very fast	✗ fourth root problem
		✗ complicated Wick contractions
		✗ taste mixing
Wilson-Clover	✓ computationally fast ($\times 10$)	✗ breaks chiral symmetry
		✗ needs operator improvement
Twisted mass fermions	✓ computationally fast ($\times 10$)	✗ breaks chiral symmetry
	✓ automatic O(a)-improvement	✗ violation of isospin
Domain wall	✓ improved chiral symmetry	✗ computationally expensive ($\times 100$)
Overlap fermions	✓ exact chiral symmetry	✗ computationally expensive ($\times 100$)

Wick contractions

- In LQCD, expectation values are given by the (finite) path integral (S_E is the euclidean action) :

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U_\mu]\mathcal{D}[\psi, \overline{\psi}]\ \mathcal{O}[U_\mu, \psi, \overline{\psi}]\ e^{-S_E[U_\mu, \psi, \overline{\psi}]}$$

- $S_E = S_G + S_F$
- Lattice spacing : natural UV regulator for the theory [rigorous definition of the path integral]
- Integration over fermionic variables

$$S_F = a^4 \sum_{x \in \Lambda} \overline{\psi}(x) D_W \psi(x)$$

→ Action quadratic (like in the free theory)

→ can be computed using Wick contractions

$$\int D[\psi]D[\overline{\psi}]\ e^{-S_F} = \det D_W$$

$$\int D[\psi]D[\overline{\psi}]\ \psi_i(y)\ \overline{\psi}_j(x)\ e^{-S_F} = -\left(D_W^{-1}\right)_{ij} \det D_W$$

$$C_2(x) = \sum_{\vec{x}} \langle \left(\overline{\psi}\gamma_5\psi\right)^\dagger (t, \vec{x})\ \left(\overline{\psi}\gamma_5\psi\right)(0) \rangle$$

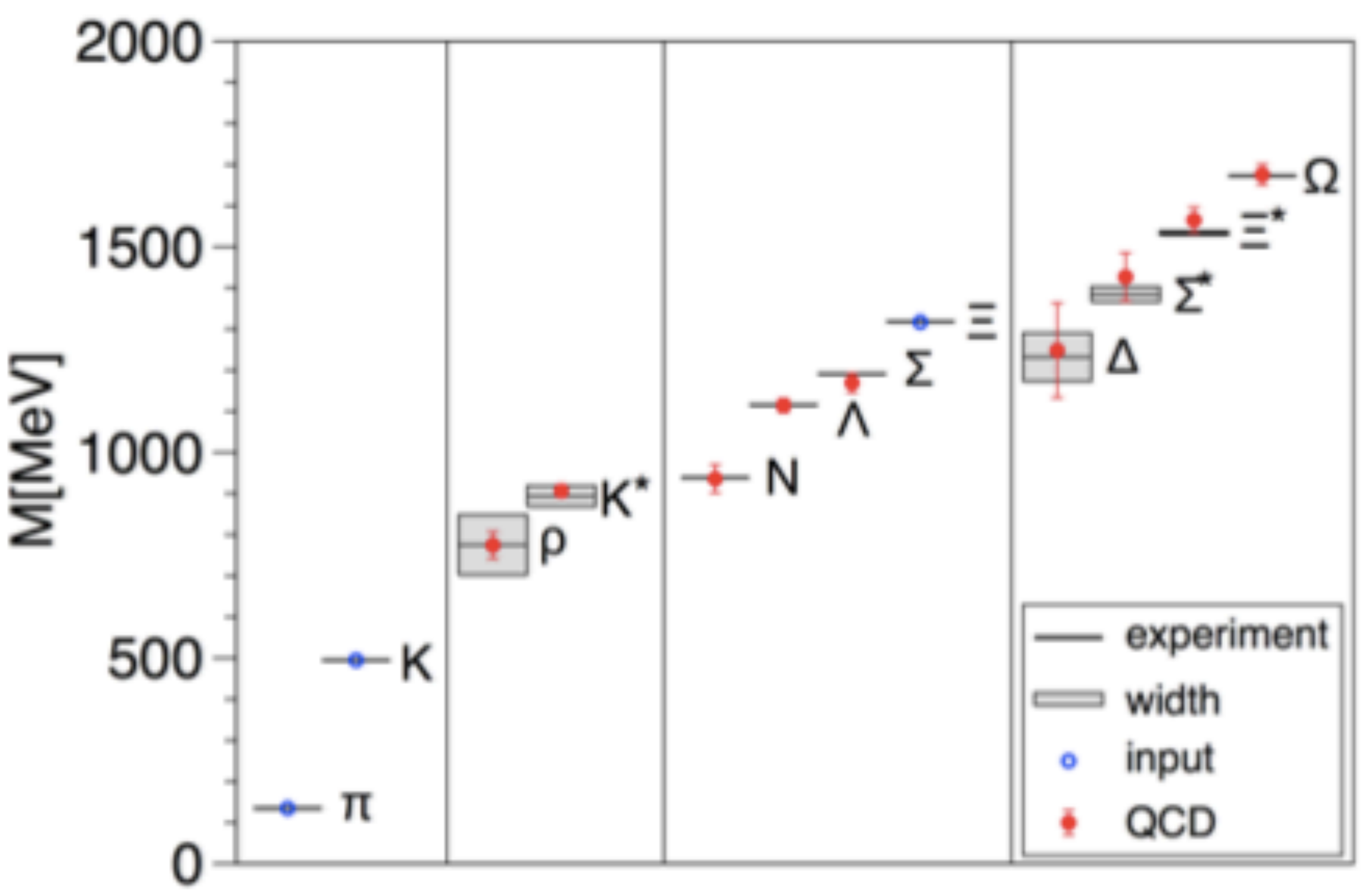
→ The results depends on $D^{-1}[U_\mu]$

→ One needs to compute the inverse of a huge matrix !

$$\boxed{\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U_\mu]\ \langle \mathcal{O} \rangle_F\ [U_\mu]\ \det D\ e^{-S_G}}$$

Meson or baryon masses

- This is one of the simplest quantity to extract on the lattice
- Check if QCD can indeed reproduced the experimental pattern (with correct quantum numbers)



- Predicts new bound states
- Information about the internal structure of hadrons

Weak decays

- La matrice CKM (Cabibbo-Kobayashi-Maskawa) :

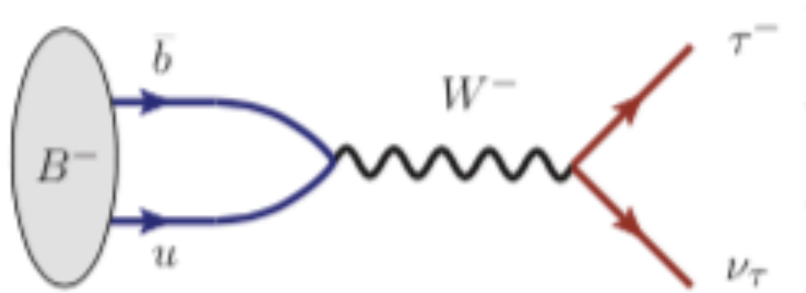
$$\underbrace{\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}}_{\text{weak eig.}} = \underbrace{\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}}_{\text{CKM matrix}} \underbrace{\begin{pmatrix} d \\ s \\ b \end{pmatrix}}_{\text{strong eig.}}$$

→ amplitude de probabilité de changement de saveur des quarks lors d'une désintégration faible

Taux de désintégration = Facteur cinématique × Élément de matrice CKM × Facteur non perturbatif (constante de désintégration, facteur de forme, ...)

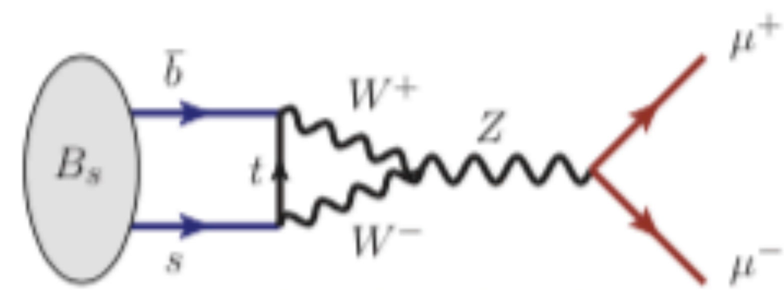
Souvent la source majeure d'incertitude

$$\mathcal{B}\left(B^- \to \tau^- \nu_\tau\right) \propto |V_{ub}|^2 f_B^2$$



- Measured by BELLE and BaBar
- Can extract $|V_{ub}|$
- One can also extract V_{cb} by replacing B by D

$$\mathcal{B}\left(B_s \to \mu^+ \mu^-\right) \propto |V_{tb}^* V_{ts}|^2 f_{B_s}^2$$

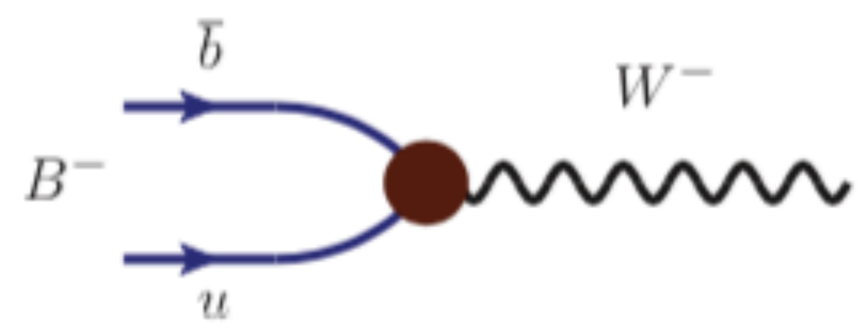


- Observed by CMS and LHCb collaborations
- Rare decay (FCNC process : 1-loop in SM)
- good probe for New Physics

Two-point correlation functions

- The decay constant parametrize the matrix element of the Axial current $\mathcal{A}^\mu = \overline{\psi}\gamma_\mu\gamma_5\psi$:

$$\langle 0|\mathcal{A}^\mu|B(p)\rangle = ip_B^\mu f_B$$



- $\mathcal{O}_B$: interpolating operator for the B meson

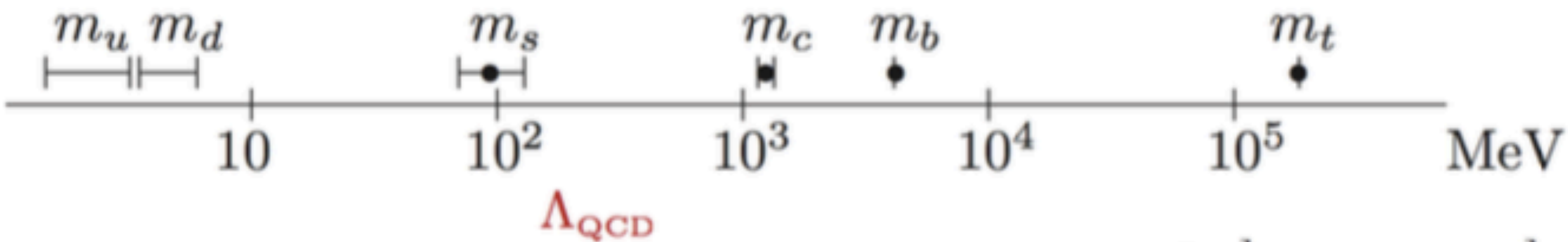
$$\langle 0|\mathcal{O}|B\rangle \neq 0$$

- Two-point function $\langle \mathcal{A}_0^\dagger(t)\mathcal{A}_0(0)\rangle$

$$\begin{aligned}
\langle \mathcal{A}_0^\dagger(t)\mathcal{A}_0(0)\rangle &= \sum_n \langle 0|\mathcal{A}_0^\dagger(t)|B_n\rangle \frac{1}{2E_n}\langle B_n|\mathcal{A}_0(0)|0\rangle \\
&= \sum_n \langle 0|e^{Ht}\mathcal{A}_0^\dagger(0)e^{-Ht}|n\rangle \frac{1}{2E_n}\langle n|\mathcal{A}_0(0)|0\rangle \\
&= \sum_n \langle 0|\hat{\mathcal{A}}_0{}^\dagger|B_n\rangle \frac{1}{2E_n}\langle B_n|\hat{\mathcal{A}}_0|0\rangle\, e^{-E_n t}
\end{aligned}$$

→ One can extract f_B from the two-point correlation function

Heavy quarks on the lattice



$$L^{-1} \ll \mu \ll a^{-1}$$

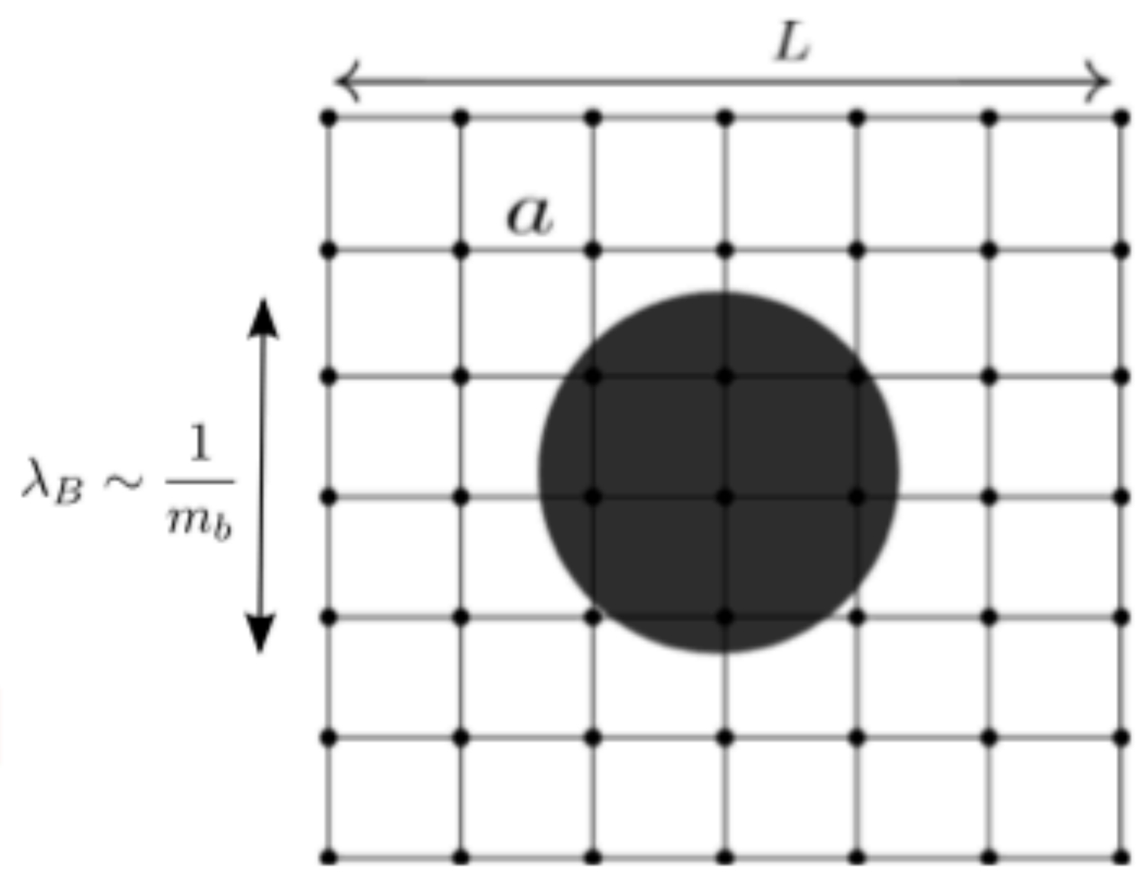
To simulate the B-meson one needs :

$$L^{-1} \ll m_\pi, m_B \ll a^{-1}$$

$\mathcal{O}\left(e^{-m_\pi L}\right)$ — $am_B < 1$

$L \gg 6$ fm — $a \ll 0.05$ fm

volume effects — discretization effects



$\Rightarrow$ $L/a \gg 120$: not possible in present-day lattice simulations

- **Effective field theory for the heavy quark**
  - Heavy Quark Effective Theory (HQET) or Nonrelativistic QCD (NRQCD)

$$\mathcal{L}^{1/m}_{\mathrm{HQET}} = \overline{\psi}_h D_0 \psi_h - \omega_{\mathrm{spin}} \mathcal{O}_{\mathrm{spin}} - \omega_{\mathrm{kin}} \mathcal{O}_{\mathrm{kin}}$$

  where

$$\mathcal{O}_{\mathrm{spin}} = -\overline{\psi}_h(x) \frac{g}{2} \sigma_{\mu\nu} F^{\mu\nu} \psi_h(x) \quad , \quad \mathcal{O}_{\mathrm{kin}} = \overline{\psi}_h(x) (D_\perp)^2 \psi_h(x) \, .$$

  - Can work directly at the b-quark mass ✓
  - Truncation of the expansion in inverse power of m_b : $(\Lambda_{\mathrm{QCD}}/m_b)^2$ ✗
  - New parameters to be tuned non-perturbatively ($\omega_{\mathrm{spin}}, \omega_{\mathrm{kin}}, \ldots$) ✗

FLAG Review 2019

March 5, 2019

Flavour Lattice Averaging Group (FLAG)

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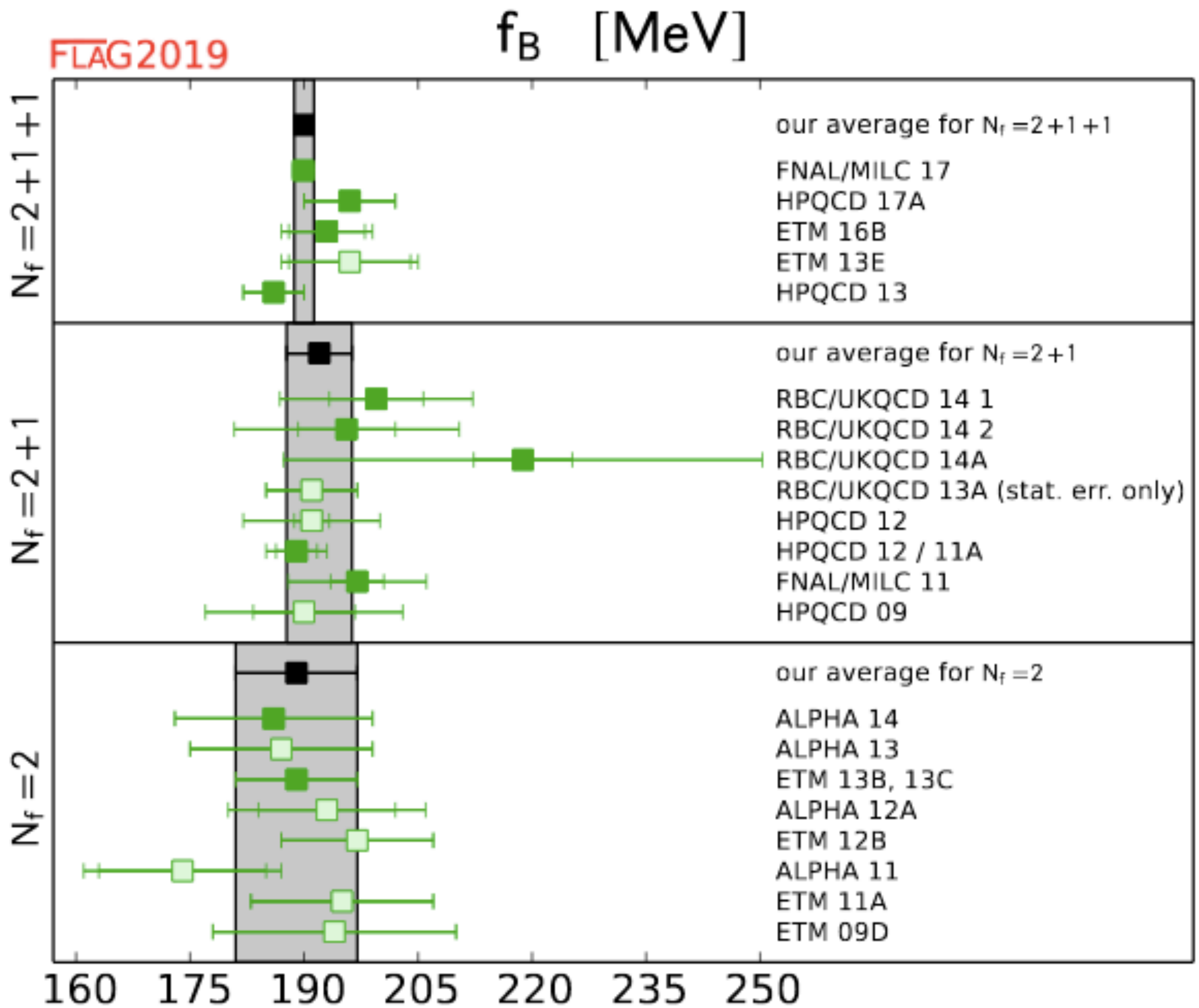
Abstract

We review lattice results related to pion, kaon, D-meson, B-meson, and nucleon physics with the aim of making them easily accessible to the nuclear and particle physics communities. More specifically, we report on the determination of the light-quark masses, the form factor $f_+(0)$ arising in the semileptonic $K \to \pi$ transition at zero momentum transfer, as well as the decay constant ratio f_K/f_π and its consequences for the CKM matrix elements V_{us} and V_{ud}. Furthermore, we describe the results obtained on the lattice for some of the low-energy constants of $SU(2)_L \times SU(2)_R$ and $SU(3)_L \times SU(3)_R$ Chiral Perturbation Theory. We review the determination of the B_K parameter of neutral kaon mixing as well as the additional four B parameters that arise in theories of physics beyond the Standard Model. For the heavy-quark sector, we provide results for m_c and m_b as well as those for D- and B-meson decay constants, form factors, and mixing parameters. These are the heavy-quark quantities most relevant for the determination of CKM matrix elements and the global CKM unitarity-triangle fit. We review the status of lattice determinations of the strong coupling constant α_s. Finally, in this review we have added a new section reviewing results for nucleon matrix elements of the axial, scalar and tensor bilinears, both isovector and flavor diagonal.



f_B [MeV]
FLAG2019
N_f =2+1+1
N_f =2+1
N_f =2
our average for N_f =2+1+1
FNAL/MILC 17
HPQCD 17A
ETM 16B
ETM 13E
HPQCD 13
our average for N_f =2+1
RBC/UKQCD 14 1
RBC/UKQCD 14 2
RBC/UKQCD 14A
RBC/UKQCD 13A (stat. err. only)
HPQCD 12
HPQCD 12 / 11A
FNAL/MILC 11
HPQCD 09
our average for N_f =2
ALPHA 14
ALPHA 13
ETM 13B, 13C
ALPHA 12A
ETM 12B
ALPHA 11
ETM 11A
ETM 09D
160
175
190
205
220
235
250