Introduction to Accelerator Physics
Beam Dynamics for „Summer Students“

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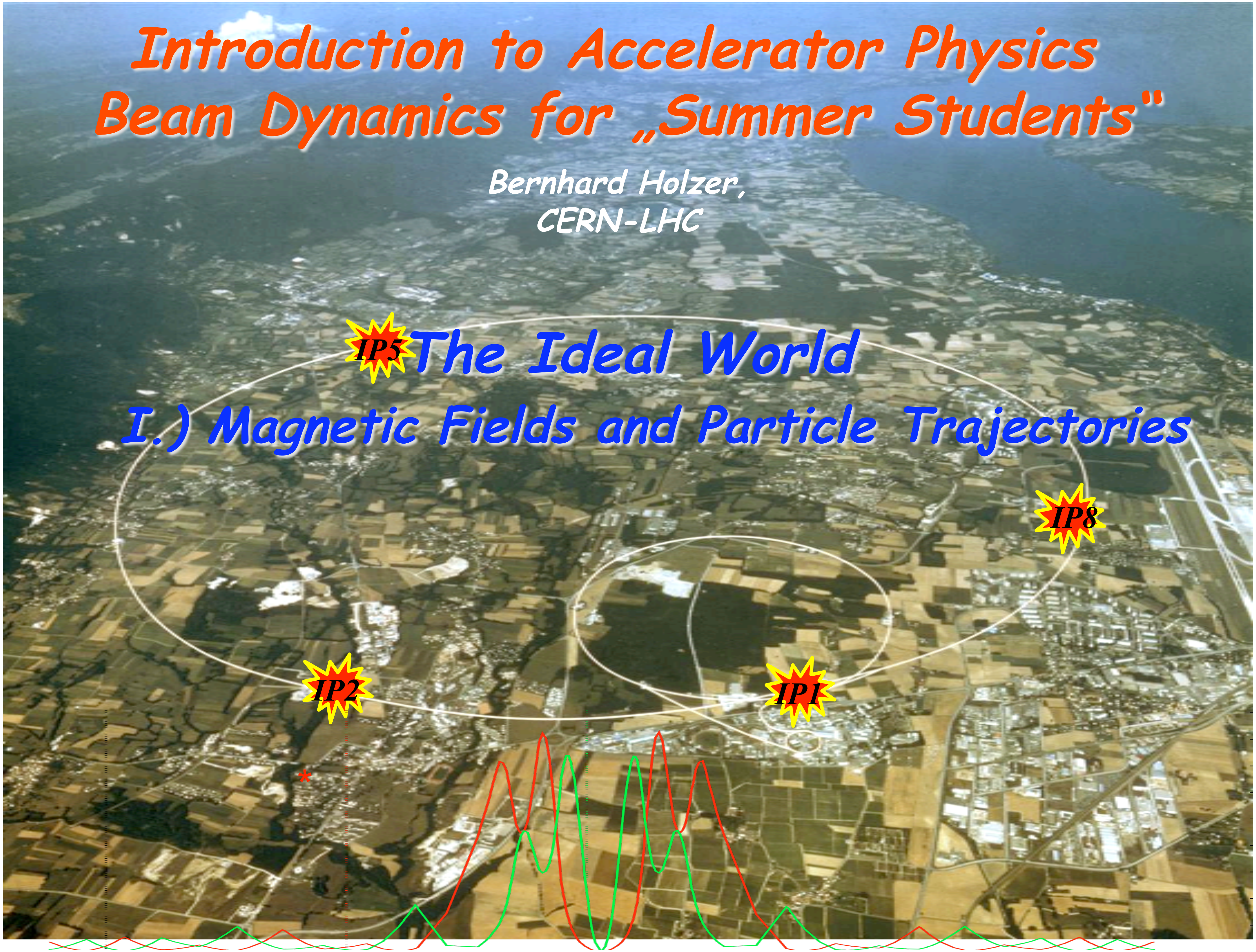


The Ideal World

I.) Magnetic Fields and Particle Trajectories

instead of … :

$$D_n = \beta_C \sin n\phi_C * \delta_{\text{supr}} * \sum_{i=1}^{n} \cos\left(i\phi_C - \frac{1}{2}\phi_C \pm \varphi_m \right) * \sqrt{\frac{\beta_m}{\beta_C}} -$$

$$- \cos n\phi_C * \delta_{\text{supr}} * \sum_{i=1}^{n} \sqrt{\beta_m \beta_C} * \sin\left(i\phi_C - \frac{1}{2}\phi_C \pm \varphi_m \right)$$

$$D_n = \sqrt{\beta_m \beta_C} * \sin n\phi_C * \delta_{\text{supr}} * \sum_{i=1}^{n} \cos\left((2i-1)\frac{\phi_C}{2} \pm \varphi_m \right) -$$

$$- \sqrt{\beta_m \beta_C} * \delta_{\text{supr}} * \cos n\phi_C \sum_{i=1}^{n} \sin\left((2i-1)\frac{\phi_C}{2} \pm \varphi_m \right)$$

Remembering the trigonometric gymnastics shown above we get

$$D_n = \delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \sin n\phi_C * \sum_{i=1}^{n} \cos\left((2i-1)\frac{\phi_C}{2} \right) * 2\cos\varphi_m -$$

$$- \delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos n\phi_C \sum_{i=1}^{n} \sin\left((2i-1)\frac{\phi_C}{2} \right) * 2\cos\varphi_m$$

$$D_n = 2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sum_{i=1}^{n} \cos\left((2i-1)\frac{\phi_C}{2} \right) * \sin(n\phi_C) - \right.$$

$$\left. - \sum_{i=1}^{n} \sin\left((2i-1)\frac{\phi_C}{2} \right) * \cos(n\phi_C) \right\}$$

$$D_n = 2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \sin(n\phi_C) \frac{\sin\frac{n\phi_C}{2} * \cos\frac{n\phi_C}{2}}{\sin\frac{\phi_C}{2}} -$$

$$- 2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos\varphi_m * \cos(n\Phi_C) * \frac{\sin\frac{n\Phi_C}{2} * \sin\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}}$$

$$D_n = \frac{2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\phi_C}{2}} \left\{ 2\sin\frac{n\phi_C}{2} \cos\frac{n\phi_C}{2} * \cos\frac{n\phi_C}{2} \sin\frac{n\phi_C}{2} - \right.$$

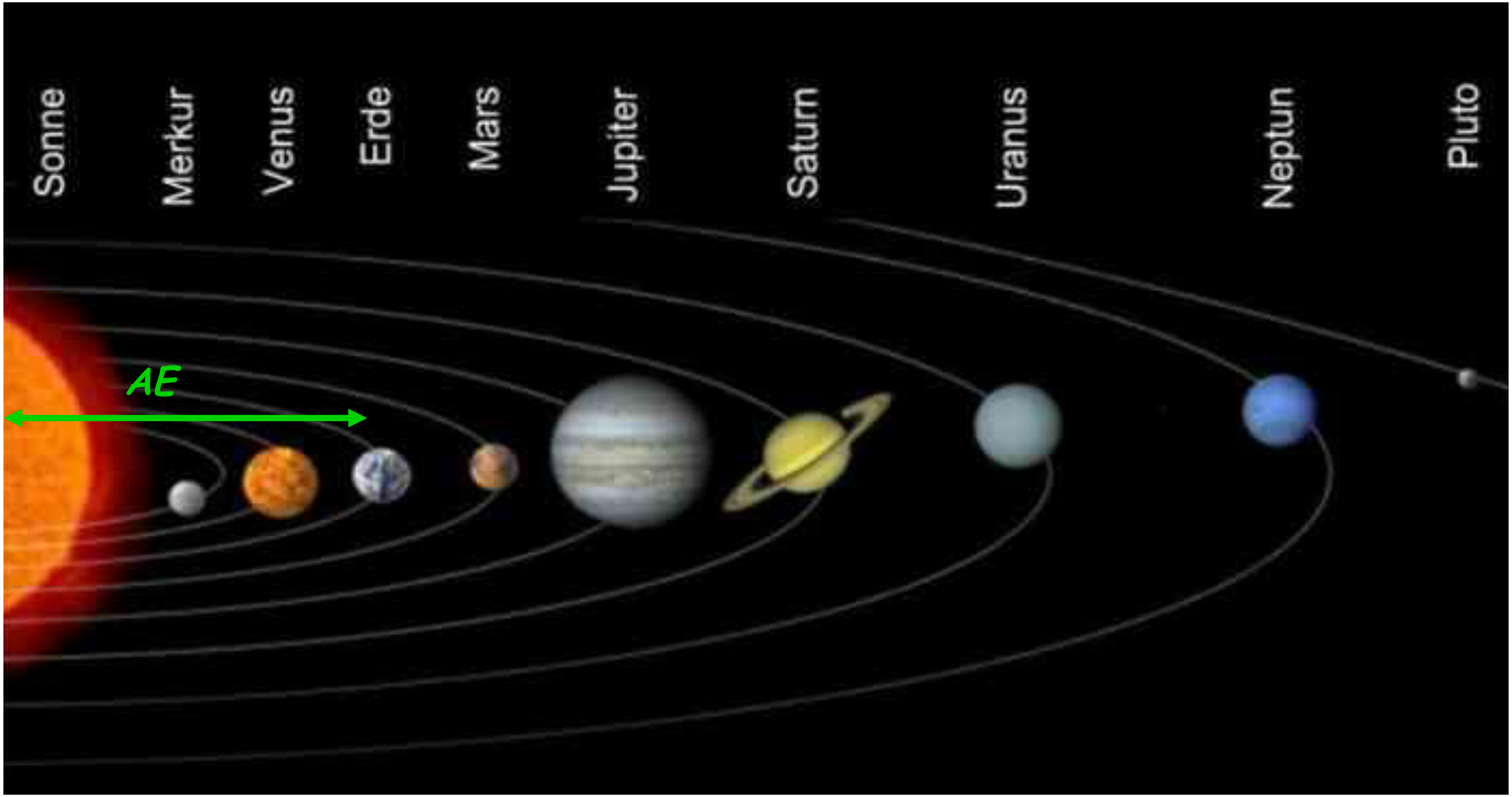
$$\left. - (\cos^2\frac{n\phi_C}{2} - \sin^2\frac{n\phi_C}{2}) \sin^2\frac{n\phi_C}{2} \right\}$$

replace by …

"after some TLC transformations" ***… or …*** ***" after some beer"***

Largest storage ring: The Solar System

astronomical unit: average distance earth-sun
$1AE \approx 150 * 10^6$ km
Distance Pluto-Sun ≈ 40 AE



Luminosity Run of a typical storage ring:

LHC Storage Ring: Protons accelerated and stored for 12 hours

distance of particles travelling at about v ≈ c

L = 10^{10}-10^{11} km

... several times Sun - Pluto and back ♪

- → ***guide the particles on a well defined orbit („design orbit“)***
- → ***focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.***

1.) Introduction and Basic Ideas

„ … in the end and after all it should be a kind of circular machine“

→ need transverse deflecting force

Lorentz force

$$\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$$

typical velocity in high energy machines:

$$v \approx c \approx 3 * 10^8 \, {}^{m}\!/_{s}$$

Example:♪

$$B = 1\,T \quad \rightarrow \quad F = q * 3 * 10^8 \, \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = q * \underbrace{300 \, \frac{MV}{m}}$$

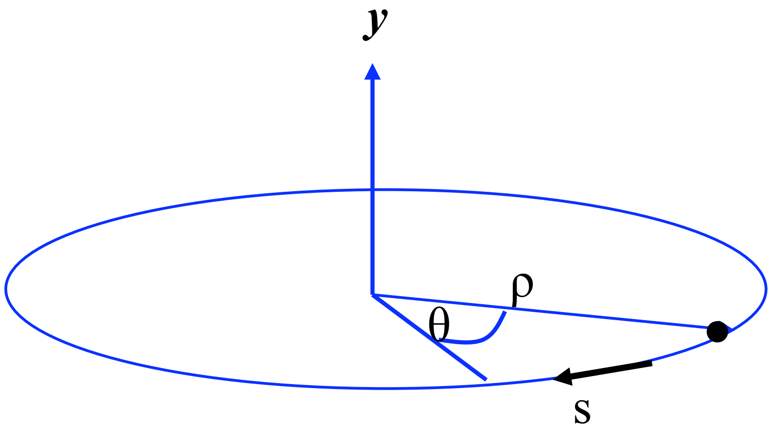
equivalent el. field *…♪* E

technical limit for el. field:♪

$$E \leq 1 \, \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $F_L = e\,v\,B$

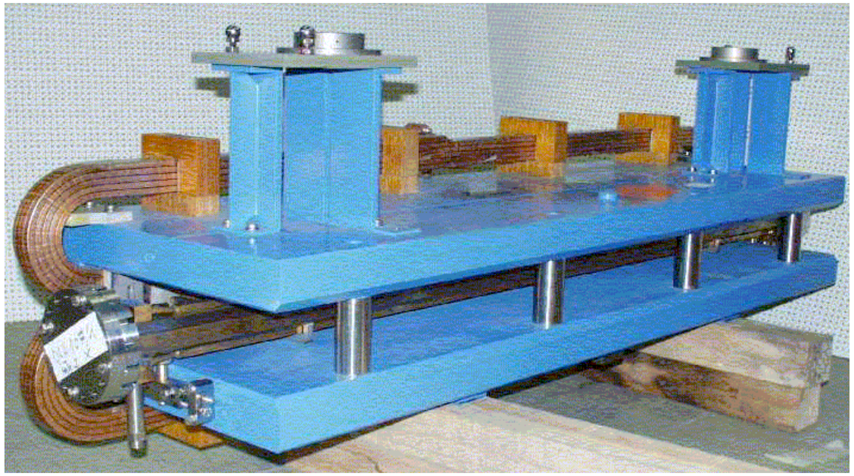
centrifugal force $F_{centr} = \frac{\gamma\, m_0\, v^2}{\rho}$

$$\frac{\gamma\, m_0\, \cancel{v^2}}{\rho} = e\,\cancel{v}\,B$$

$$\boxed{\frac{p}{e} = B\,\rho}$$

B ρ = "beam rigidity"

2.) The Magnetic Guide Field



Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 \, n \, I}{h}$$

Normalise magnetic field to momentum:

$$\frac{p}{e} = B\rho \quad \longrightarrow \quad \frac{1}{\rho} = \frac{eB}{p}$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

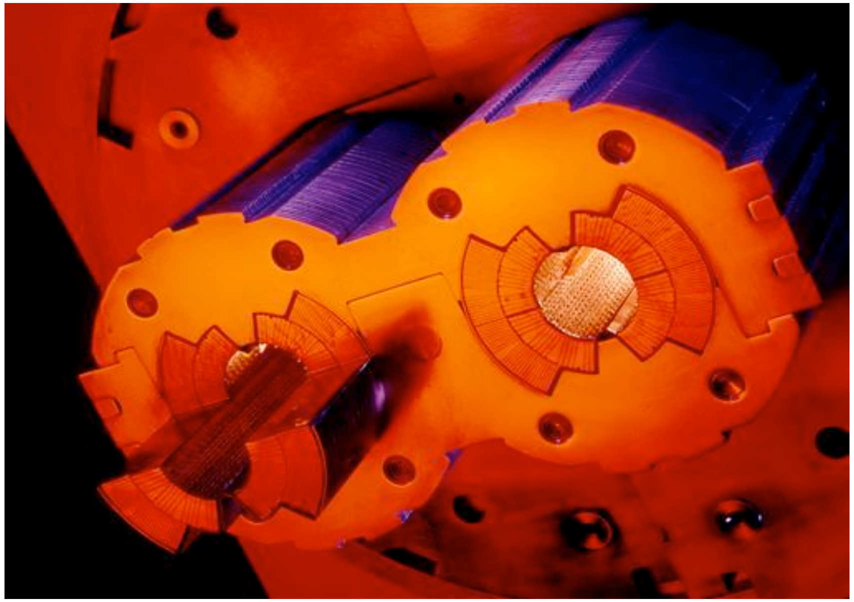
Example LHC:

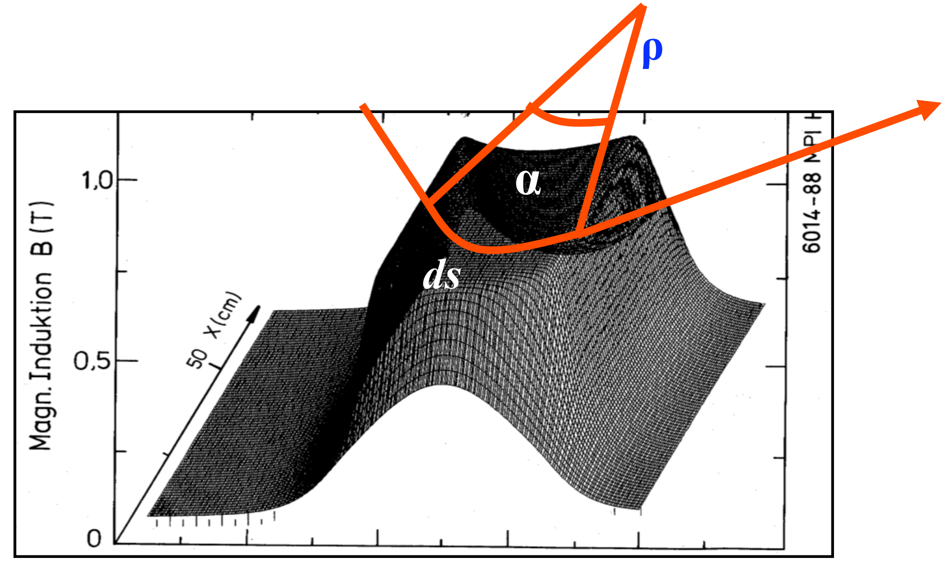
$$\left.\begin{array}{l} B = 8.3\,T \\ \\ p = 7000\,\dfrac{GeV}{c} \end{array}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\,{Vs}/{m^2}}{7000 * 10^9\, {eV}/{c}} = \frac{8.3\,s * 3 * 10^8\, {m}/{s}}{7000 * 10^9\, m^2}$$

$$\frac{1}{\rho} = 0.333\,\frac{8.3}{7000}\,{1}/{m}$$

The Magnetic Guide Field





field map of a storage ring dipole magnet

$\rho = 2.53\ \boldsymbol{km}$ → $2\pi\rho = 17.6\ km$

$\approx 66\%$

$B \approx 1 \ldots 8\ \boldsymbol{T}$

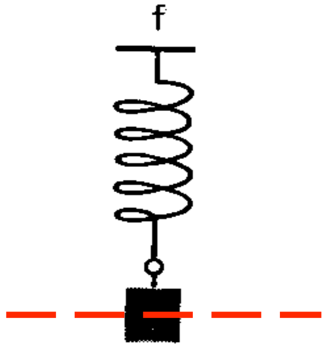
rule of thumb:

$$\frac{1}{\rho} \approx 0.3\,\frac{B[T]}{p[GeV/c]}$$

„normalised bending strength“

3.) Focusing Properties – Transverse Beam Optics

classical mechanics: pendulum



there is a restoring force, proportional to the elongation x:

$$m * \frac{d^2 x}{dt^2} = -c * x$$

general solution: free harmonic oszillation

$$x(t) = A * \cos(\omega t + \varphi)$$

Storage Ring:* we need a *Lorentz force* that rises as a function of the *distance to .... ?

.... the design orbit

$$F(x) = q * v * B(x)$$

Quadrupole Magnets:

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*

linear increasing Lorentz force

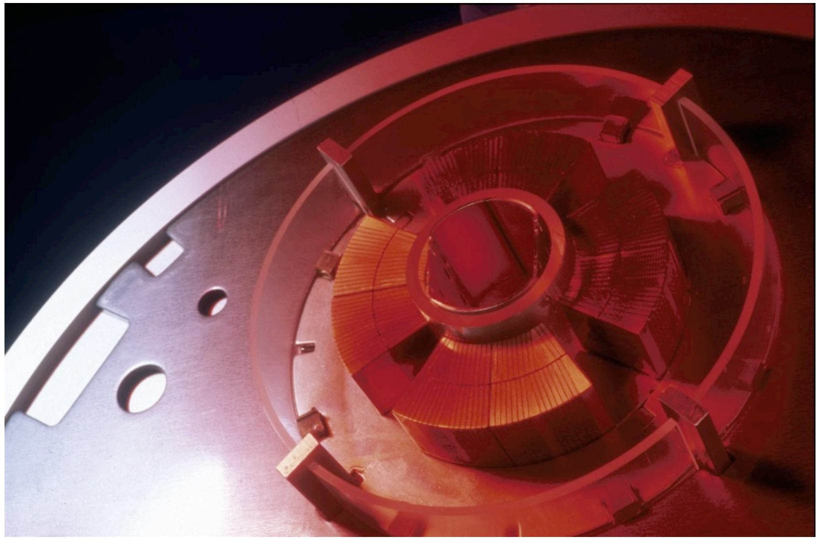
linear increasing magnetic field $B_y = g\,x \qquad B_x = g\,y$

normalised quadrupole field:

$$\longrightarrow \qquad k = \frac{g}{p/e}$$

simple rule:

$$\boxed{k = 0.3\,\frac{g(T/m)}{p(GeV/c)}}$$



LHC main quadrupole magnet

$$g \approx 25 \ldots 220\ \ T/m$$

what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \qquad \Rightarrow \qquad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} = g$$

Focusing forces and particle trajectories:

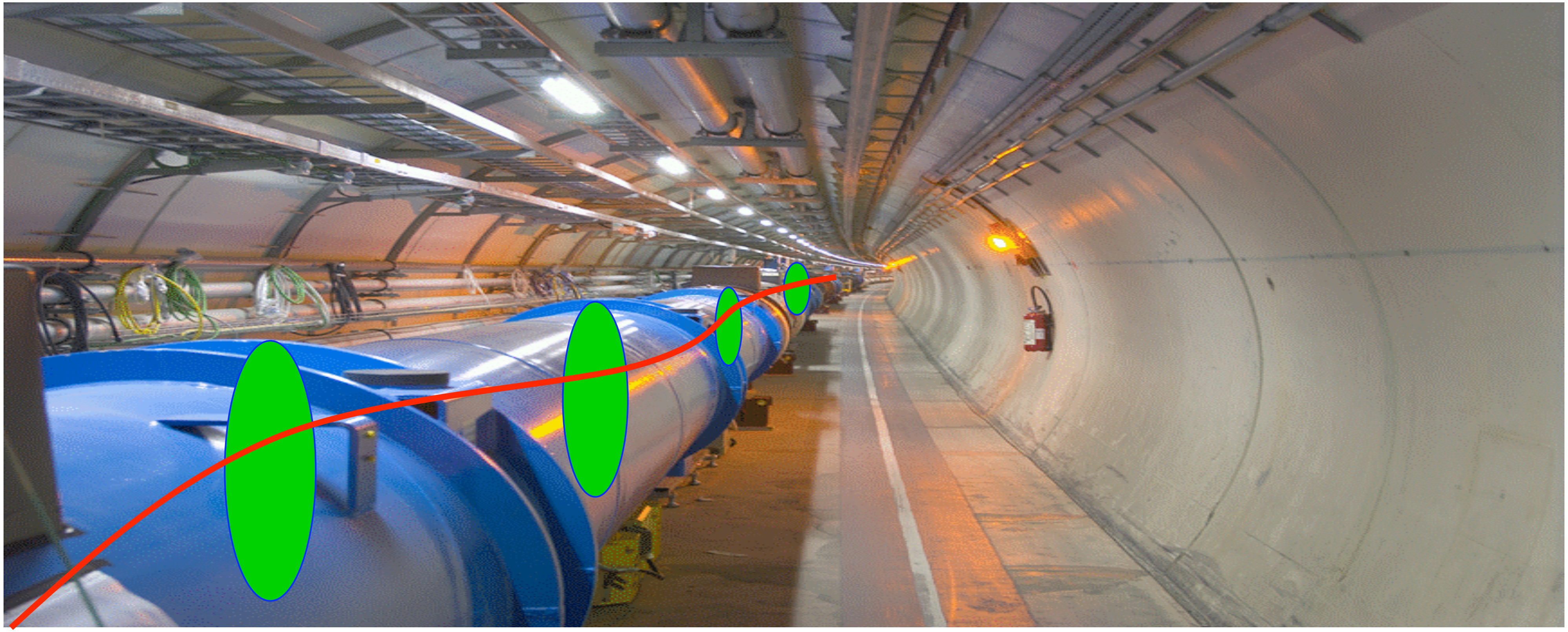
normalise magnet fields to momentum
(remember: $\mathbf{B*\rho = p / q}$ *)*

Dipole Magnet

$$\frac{B}{p/q} = \frac{B}{B\rho} = \frac{1}{\rho}$$

Quadrupole Magnet

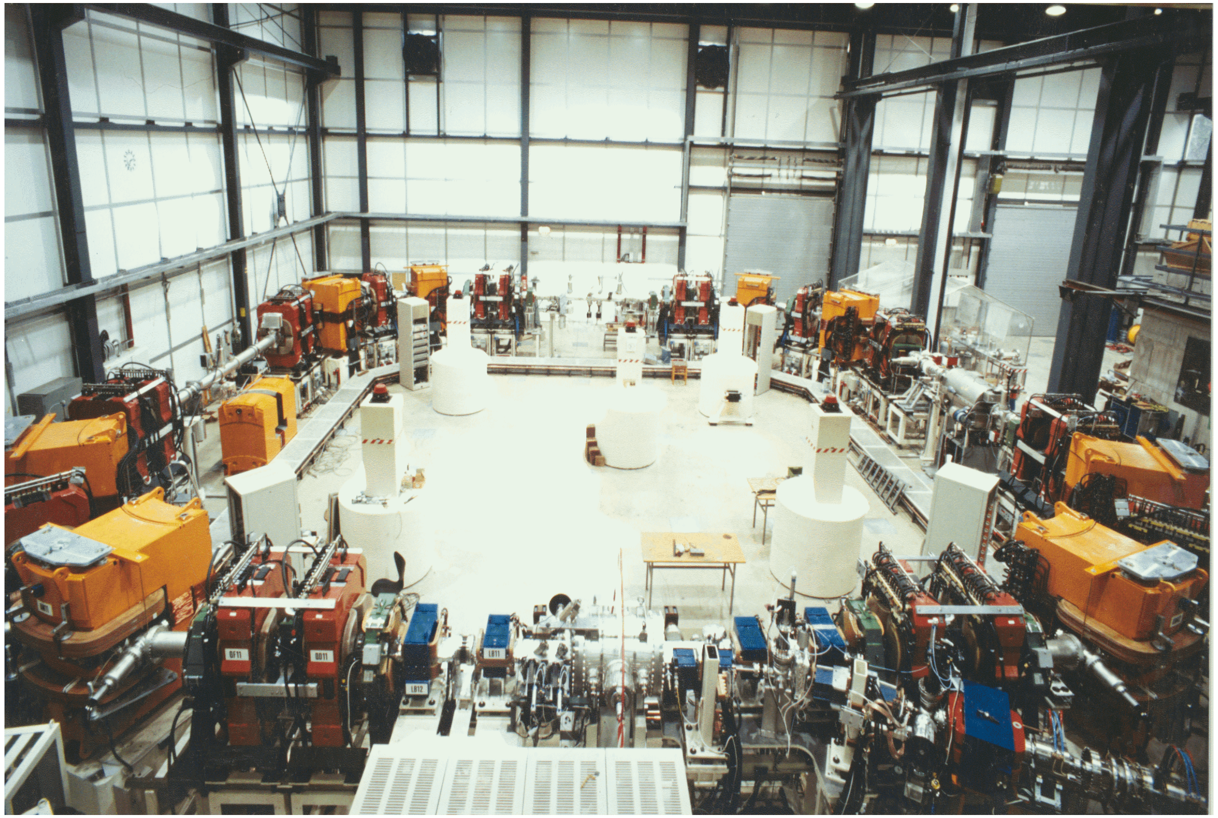
$$k := \frac{g}{p/q}$$



4.) The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \xcancel{\frac{1}{2!}\,m\,x^2} + \xcancel{\frac{1}{3!}\,n\,x^3} + \ldots$$

only terms linear in x, y taken into account *dipole fields*
quadrupole fields



Separate Function Machines:

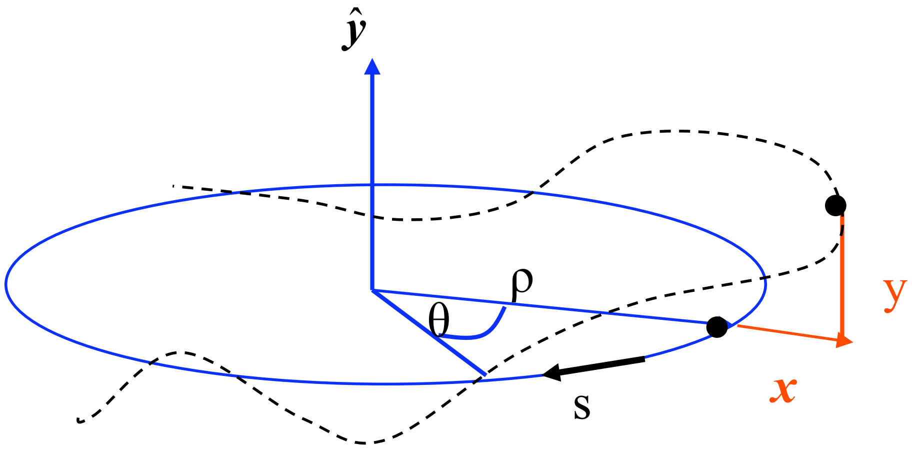
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

** man sieht nur*
dipole und quads → linear

The Equation of Motion:



* *Equation for the horizontal motion:*

$$x'' + x\left(\frac{1}{\rho^2} + k\right) = 0$$

x = particle amplitude

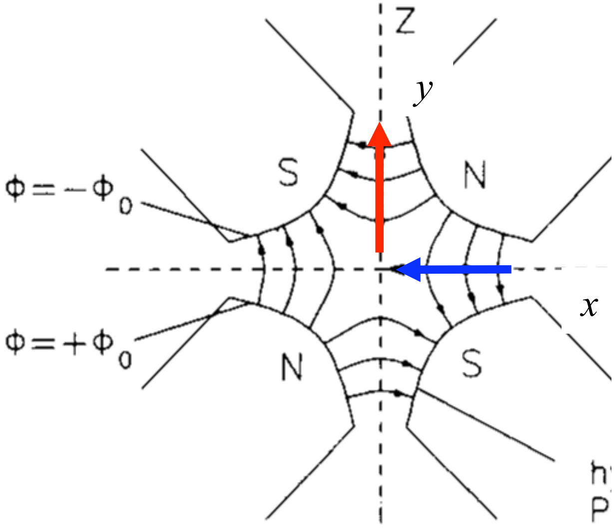
x’ = angle of particle trajectory (wrt ideal path line)

* *Equation for the vertical motion:*

$\frac{1}{\rho^2} = 0$ *no dipoles … in general …*

$k \leftrightarrow -k$ *quadrupole field changes sign*

$$y'' - k\,y = 0$$



5.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 + k$

… vert. Plane: $K = -k$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \qquad \boxed{x'' + K\,x = 0}$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz:

$$x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \longrightarrow \quad \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

determine a_1 , a_2 by boundary conditions:

$$s = 0 \quad \longrightarrow \quad \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

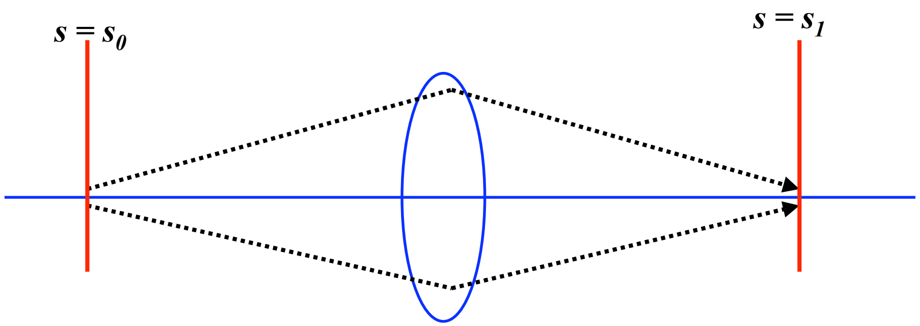
Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

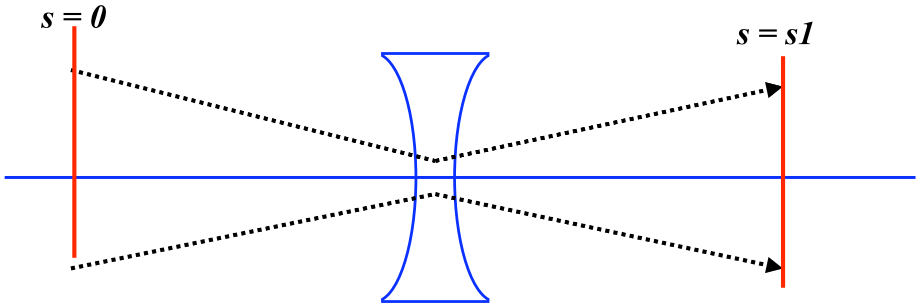
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \dfrac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\ x = 0$$



Remember from school:

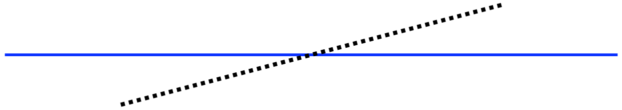
$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

$$x(s) = x_0 \cdot \cosh(\sqrt{|K|}s) + x_0{}' \cdot \sinh(\sqrt{|K|}\ s)$$

$$M_{def\ oc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

drift space:

$$K = 0$$



$$x(s) = x_0 + x_0{}' * s$$

$$M_{drif\ t} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

!** with the assumptions made, the motion in the horizontal and vertical planes are independent* ***„ ... the particle motion in x & y is uncoupled“

Thin Lens Approximation:

matrix of a quadrupole lens

$$M = \begin{pmatrix} \cos\sqrt{|k|}l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}l \\ -\sqrt{|k|}\sin\sqrt{|k|}l & \cos\sqrt{|k|}l \end{pmatrix}$$

in many practical cases we have the situation:

$$f = \frac{1}{kl_q} >> l_q$$

... focal length of the lens is much bigger than the length of the magnet

limes: $l_q \rightarrow 0$ ***while keeping*** $k\, l_q$ ***= const***

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad M_z = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$$

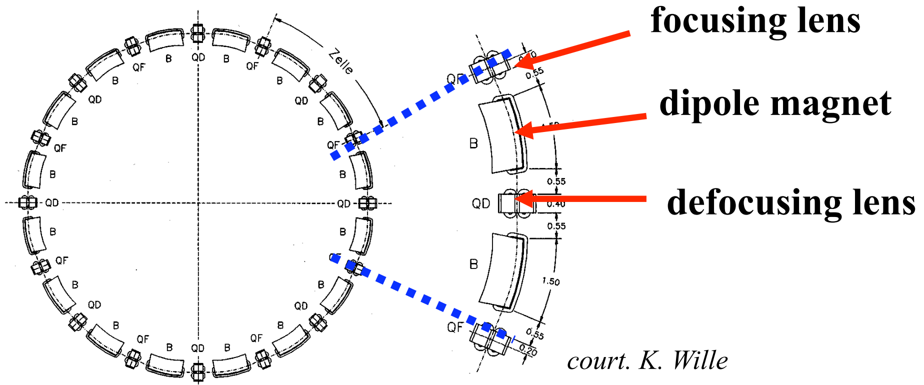
... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

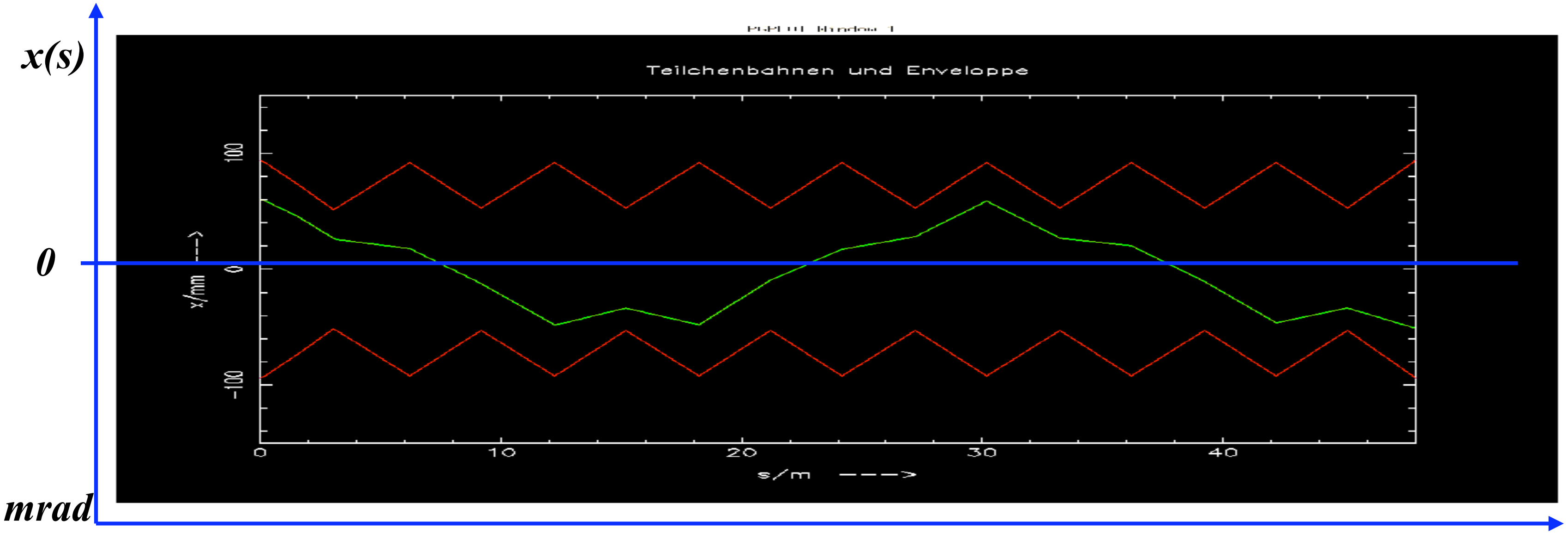
$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D^*}.....$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2,s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



court. K. Wille

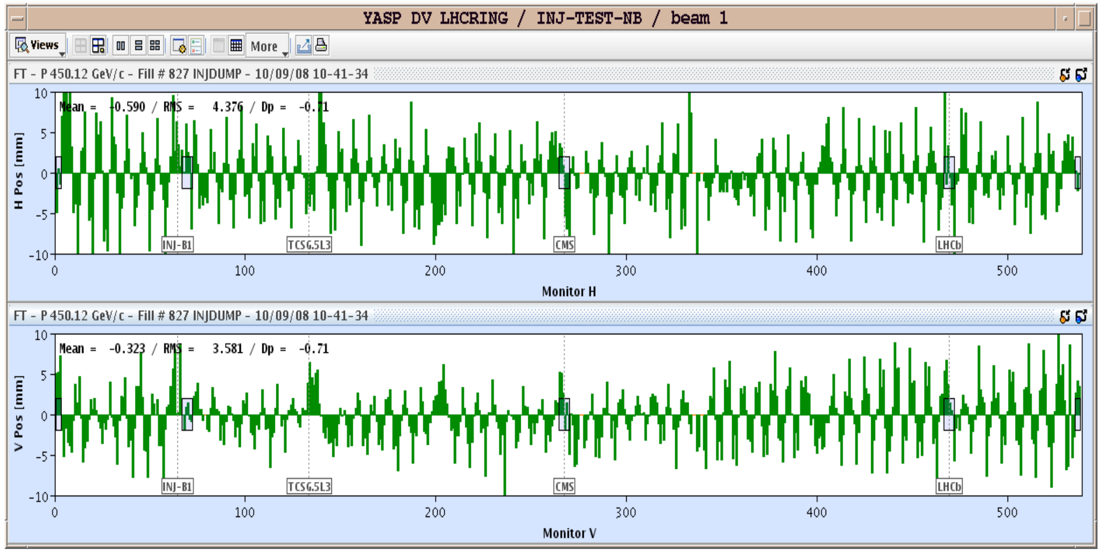
in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values in a strong foc. machine:

$x \approx mm$, $x' \leq mrad$

6.) Orbit & Tune:



YASP DV LHCRING / INJ-TEST-NB / beam 1

FT - P 450.12 GeV/c - Fill # 827 INJDUMP - 10/09/08 10-41-34

Mean = -0.590 / RMS = 4.376 / Dp = -0.71

FT - P 450.12 GeV/c - Fill # 827 INJDUMP - 10/09/08 10-41-34

Mean = -0.323 / RMS = 3.581 / Dp = -0.71

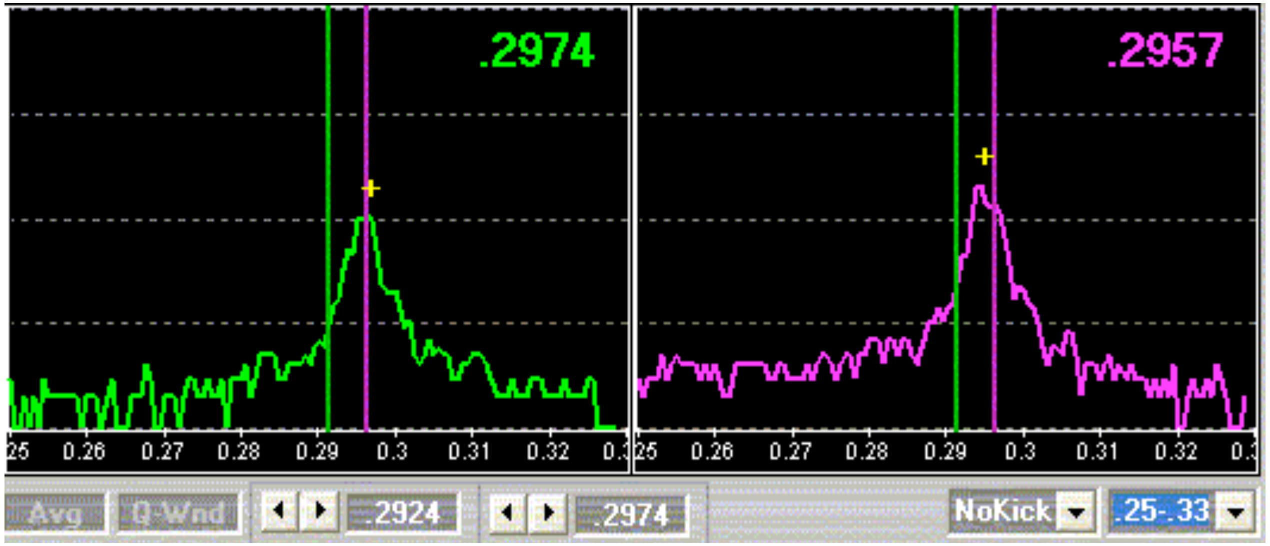
Tune: number of oscillations per turn

64.31
59.32

Relevant for beam stability:
non integer part

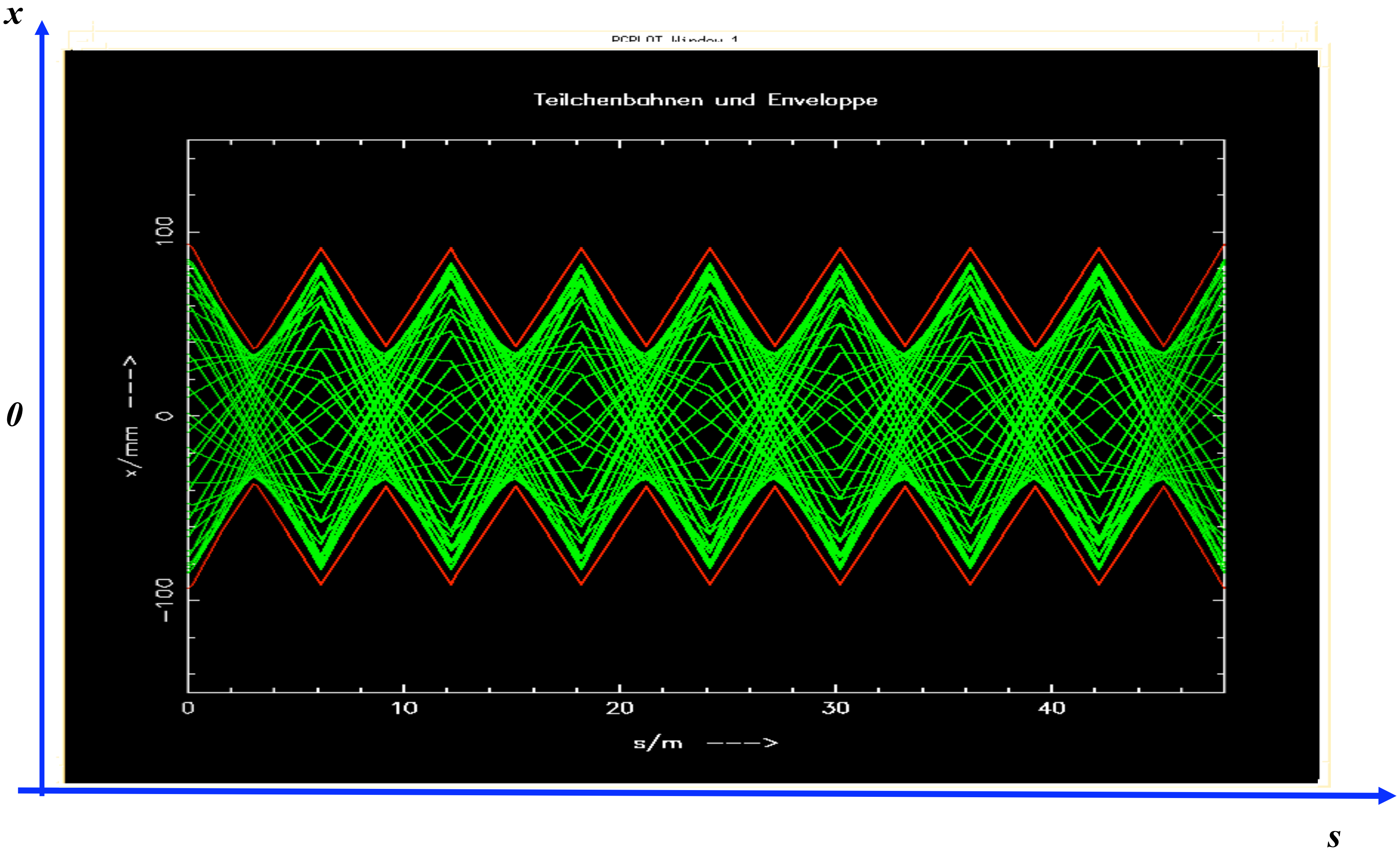
LHC revolution frequency: 11.3 kHz

$$0.31*11.3 = 3.5kHz$$



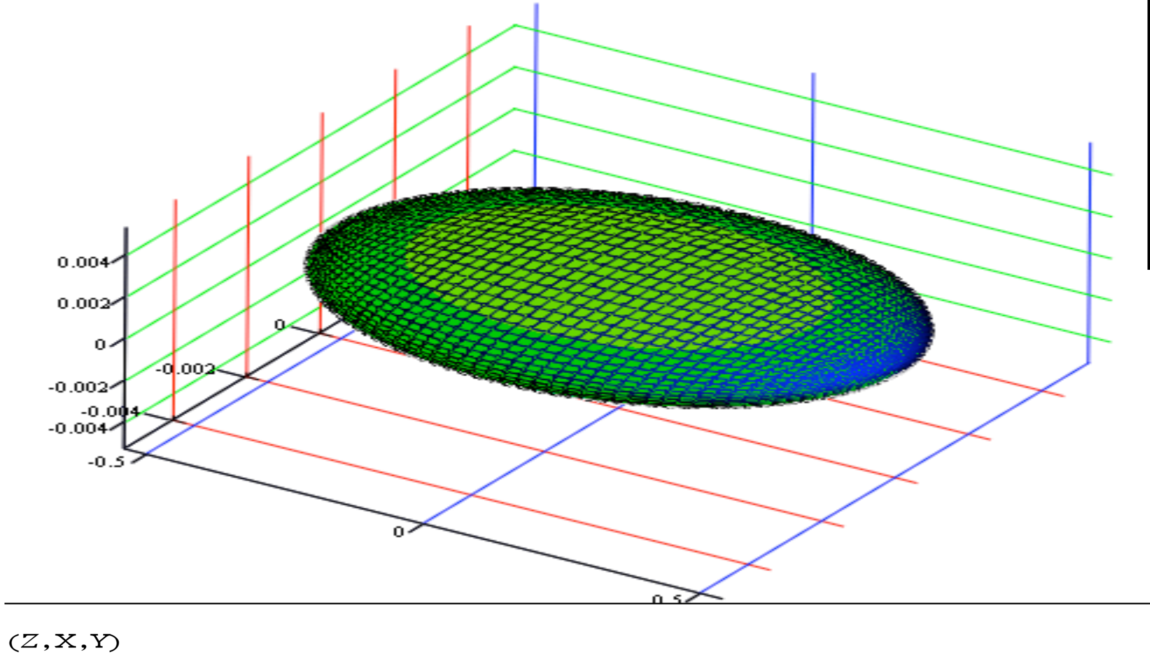
Question: what will happen, if the particle performs a second turn ?

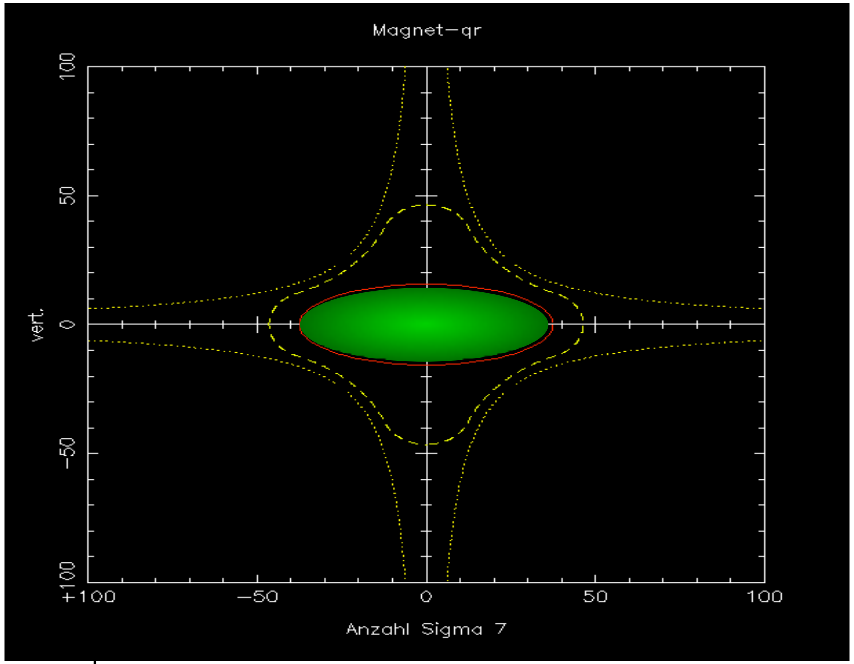
... or a third one or ... 10^{10} turns



II.) The Ideal World:

Particle Trajectories, Beams & Bunches



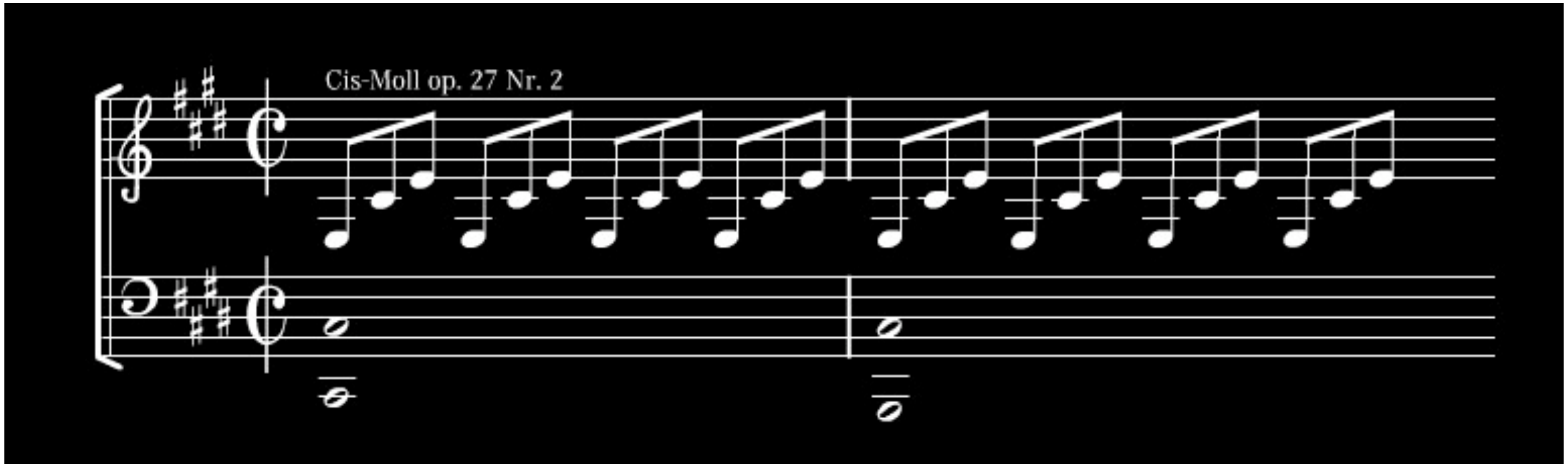


Bunch in a Storage Ring

19th century:

Ludwig van Beethoven: „Mondschein Sonate“

Sonate Nr. 14 in cis-Moll (op. 27/II,   1801)



Astronomer Hill:

differential equation for motions with periodic focusing properties „Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $x''(s) - k(s)x(s) = 0$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

$\}$ ***we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.***

7.) The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s + L) = \beta(s)$$

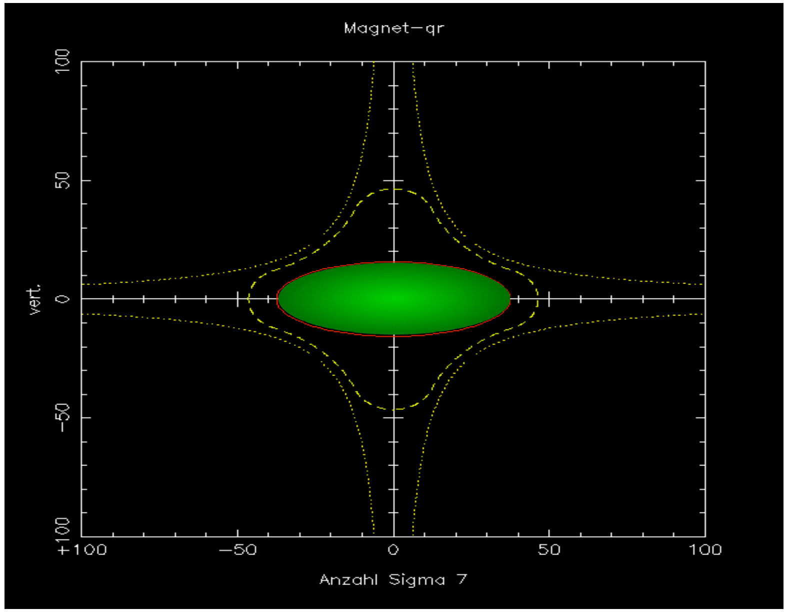
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

***Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“***

$$Q_y = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

8.) The Beta Function

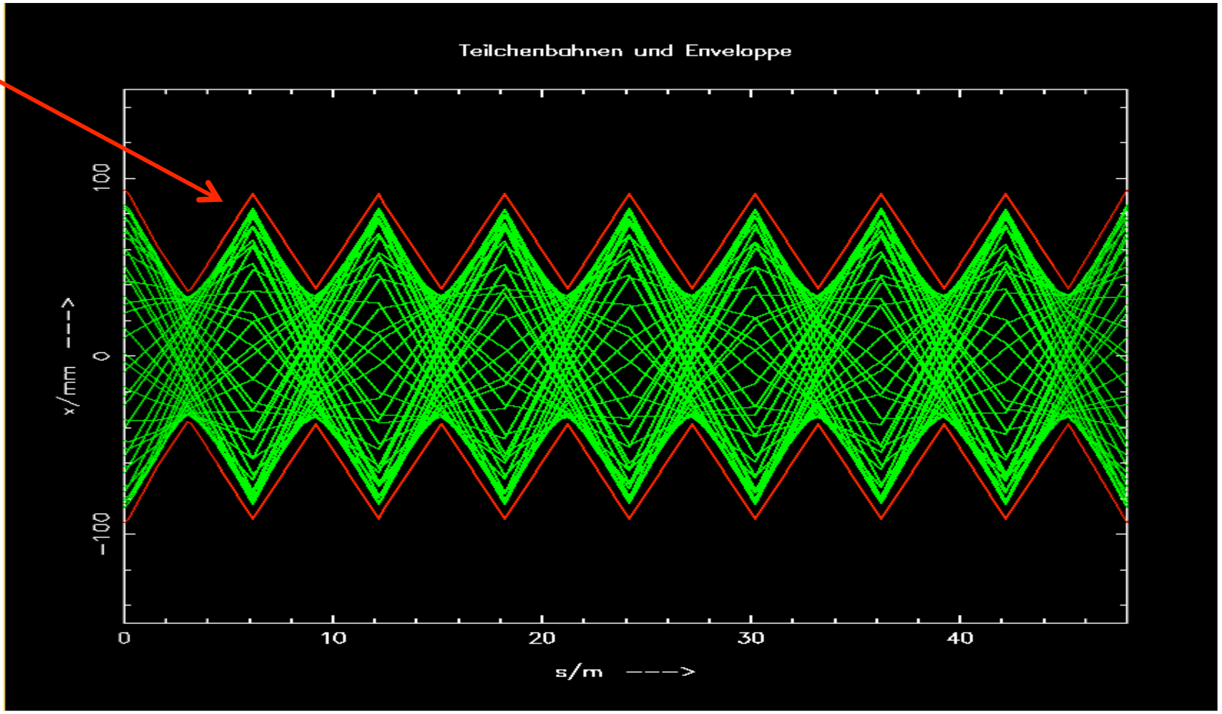


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

9.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) \qquad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) \qquad x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\,\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

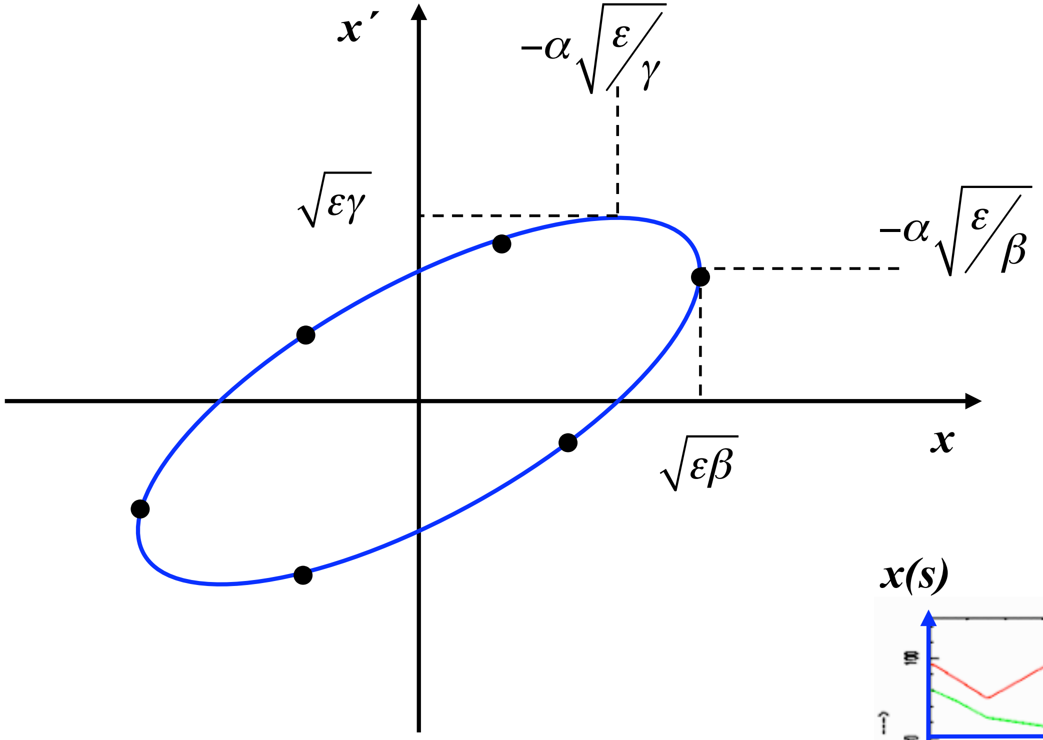
Insert into (2) and solve for ε

$$\varepsilon\ = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

***** *ε is a constant of the motion … it is independent of „s“***
***** *parametric representation of an ellipse in the x x‘ space***
***** *shape and orientation of ellipse are given by α, β, γ***

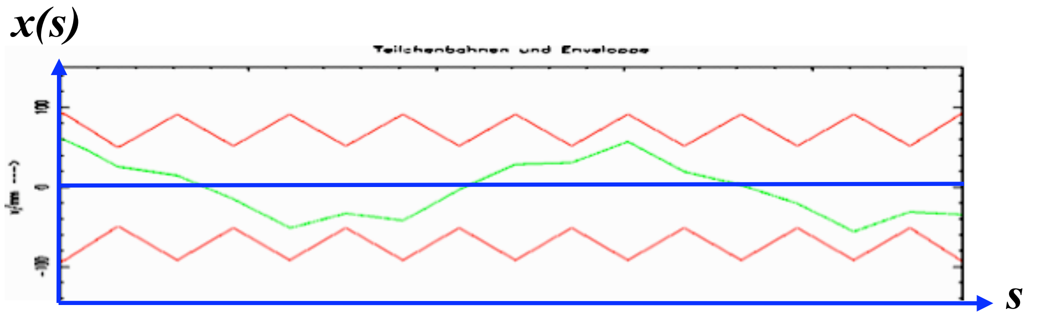
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const



ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely spoken: area covered in transverse x, x´ phase space … and it is constant !!!

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s) + \phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ → ***x´ at that position …?***

… put $\hat{x}(s)$ ***into*** $\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$ ***and solve for x´***

$$\varepsilon = \gamma\cdot\varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\longrightarrow \quad x' = -\alpha\cdot\sqrt{\varepsilon/\beta}$$

✻ ***A high β-function means a large beam size and a small beam divergence.*** ***!***
… et vice versa !!!

$$\left.\begin{array}{r}\text{✻ In the middle of a quadrupole } \beta = \text{maximum,}\\ \alpha = \text{zero}\end{array}\right\}\quad x' = 0$$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

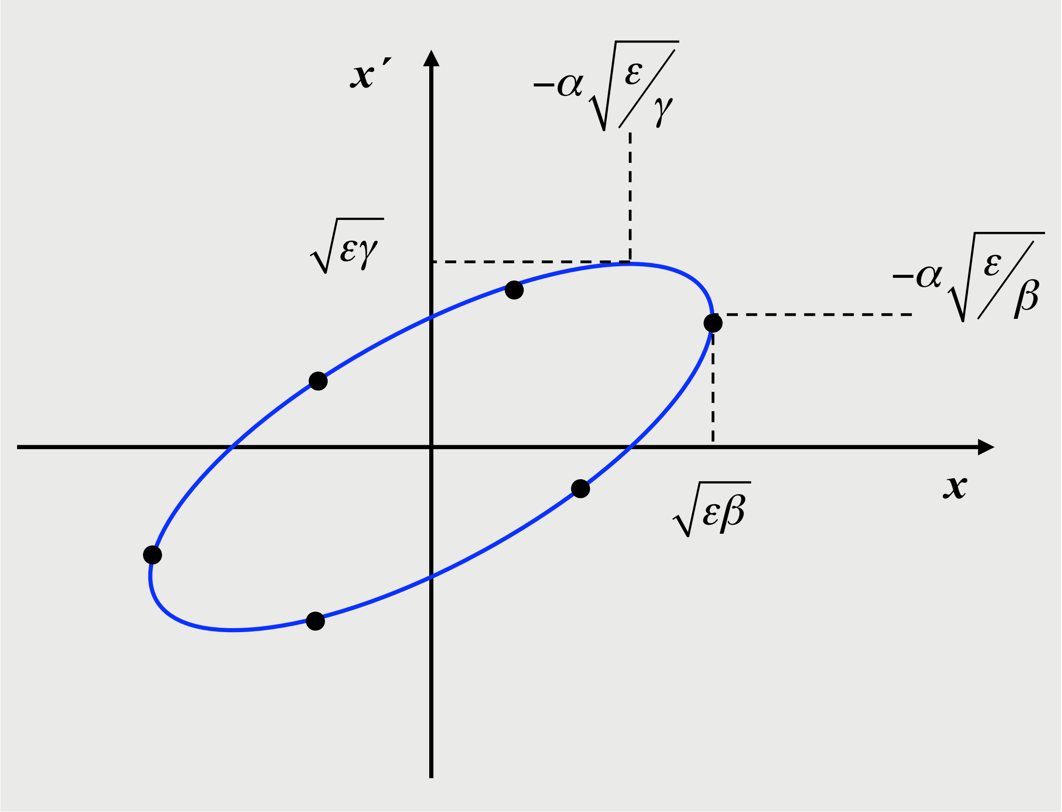
$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2$$

… solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ ***via:*** $\quad \dfrac{dx'}{dx} = 0$

$$\longrightarrow \quad \boxed{\hat{x}' = \sqrt{\varepsilon\gamma}}$$

$$\longrightarrow \quad \boxed{\hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}}$$

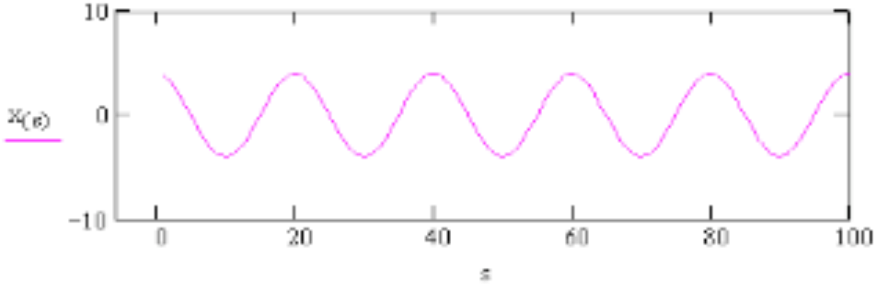


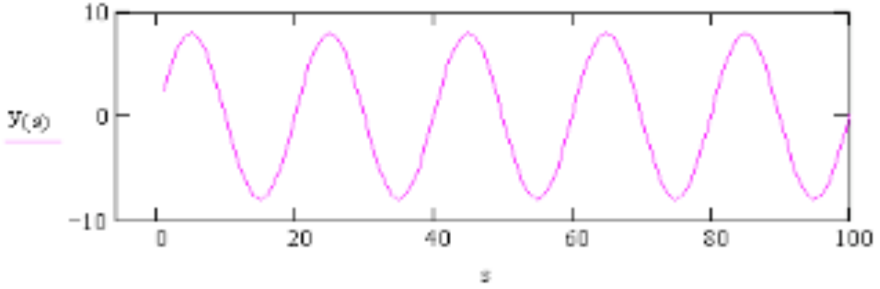
shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

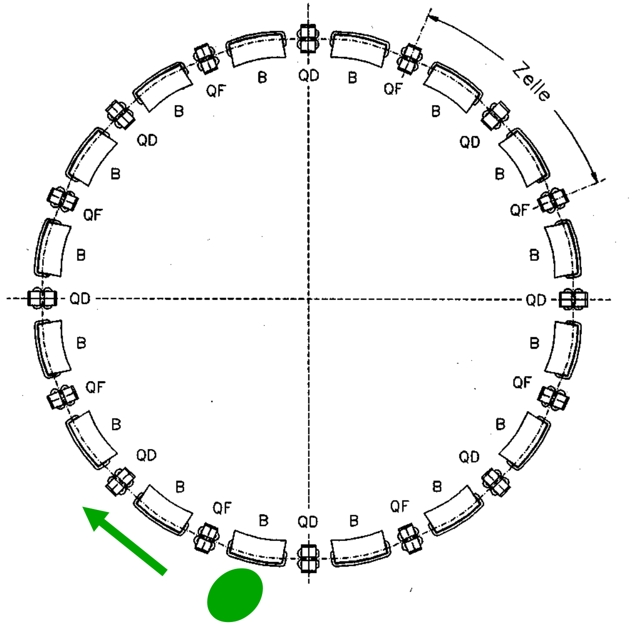
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"

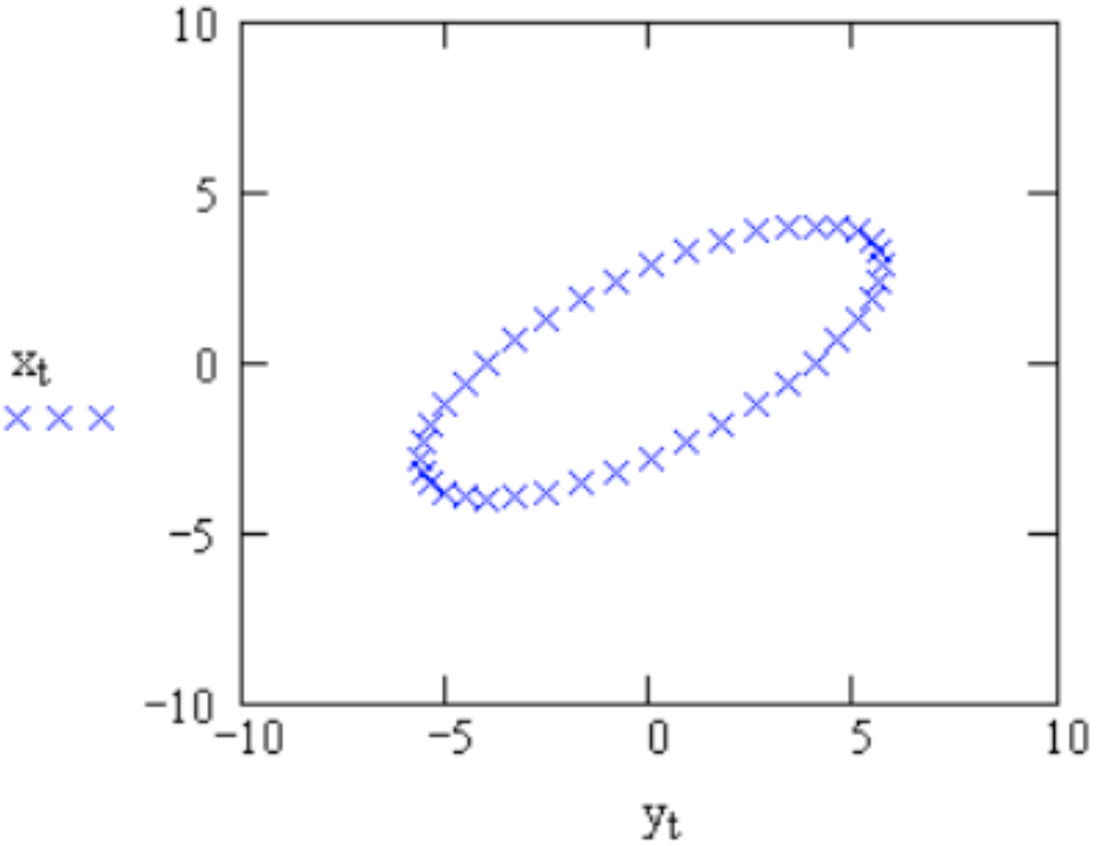




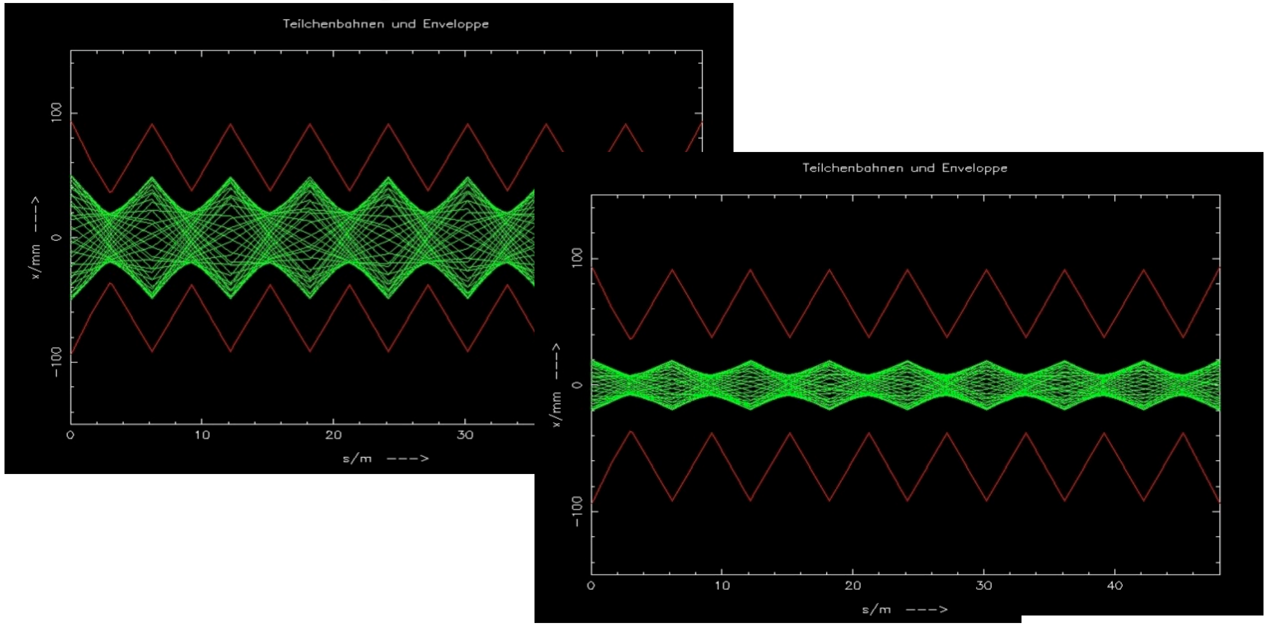


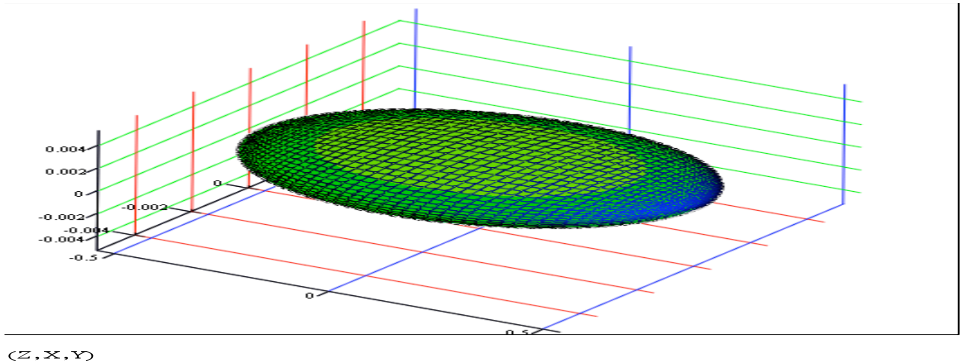
… and now the ellipse:

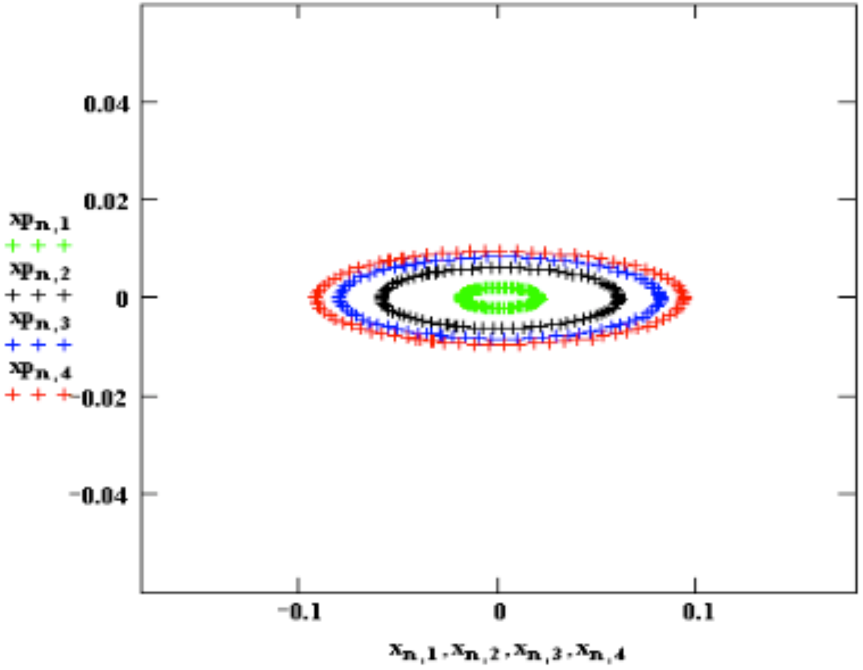
note for each turn x, x´at a given position „s_1" and plot in the phase space diagram



Emittance of the Particle Ensemble:

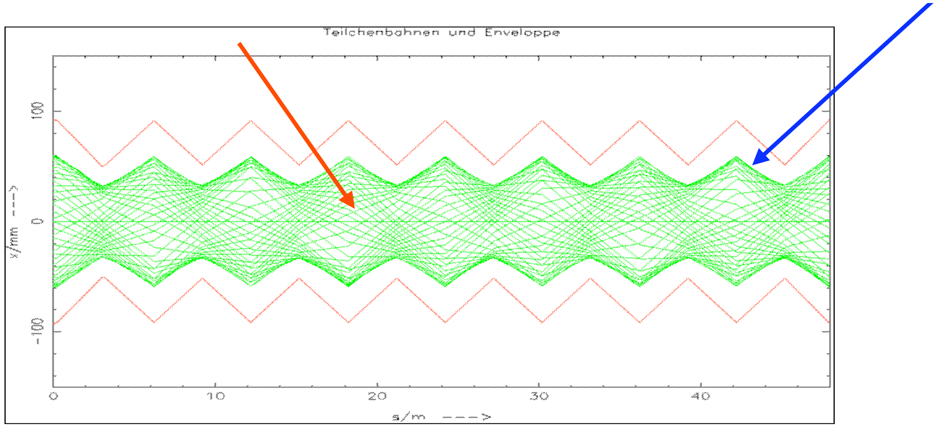






Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}$$



single particle trajectories, $N \approx 10^{11}$ per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\,\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

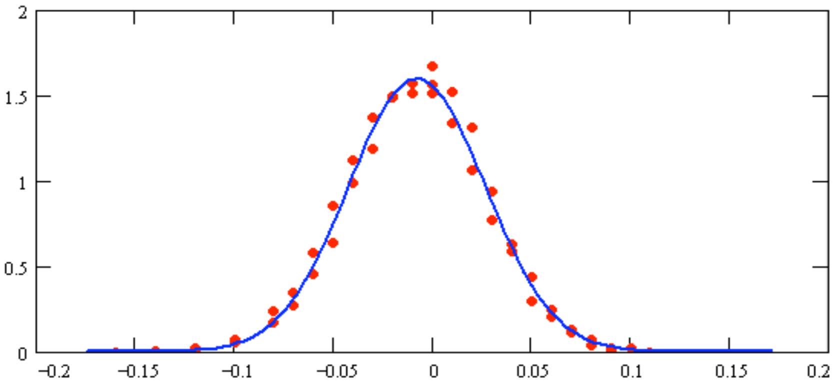
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

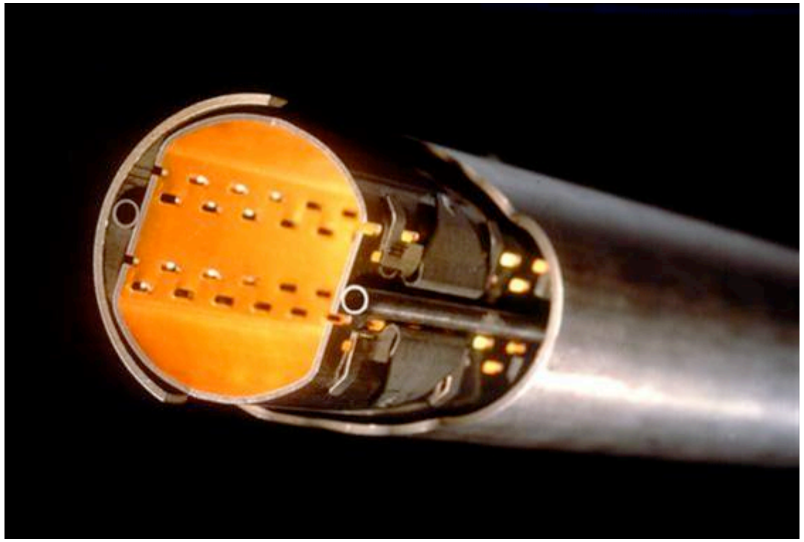
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5 * 10^{-10}\,m\,rad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5 * 10^{-10}\,m * 180\,m} = 0.3\,mm$$





aperture requirements: $r_0 = 12 * \sigma$

Résumé:

beam rigidity: $B \cdot \rho = {}^{p}\!/_{q}$

bending strength of a dipole: $\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$

focusing strength of a quadrupole: $k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$

focal length of a quadrupole: $f = \frac{1}{k \cdot l_q}$

equation of motion: $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

matrix of a foc. quadrupole: $x_{s2} = M \cdot x_{s1}$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix} , \qquad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

10.) Bibliography:

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Radiation Facilicties, Teubner, Stuttgart 1992

3.) **Peter Schmüser:** Basic Course on Accelerator Optics, CERN Acc.
School: 5th general acc. phys. course CERN 94-01

4.) **Bernhard Holzer:** Lattice Design, CERN Acc. School: Interm.Acc.phys course,
http://cas.web.cern.ch/cas/ZEUTHEN/lectures-zeuthen.htm

5.) **Herni Bruck:** Accelerateurs Circulaires des Particules,
presse Universitaires de France, Paris 1966 (english / francais)

6.) **M.S. Livingston, J.P. Blewett:** Particle Accelerators,
Mc Graw-Hill, New York,1962

7.) **Frank Hinterberger:** Physik der Teilchenbeschleuniger, Springer Verlag 1997

8.) **Mathew Sands:** The Physics of e+ e- Storage Rings, SLAC report 121, 1970

9.) **D. Edwards, M. Syphers :** An Introduction to the Physics of Particle
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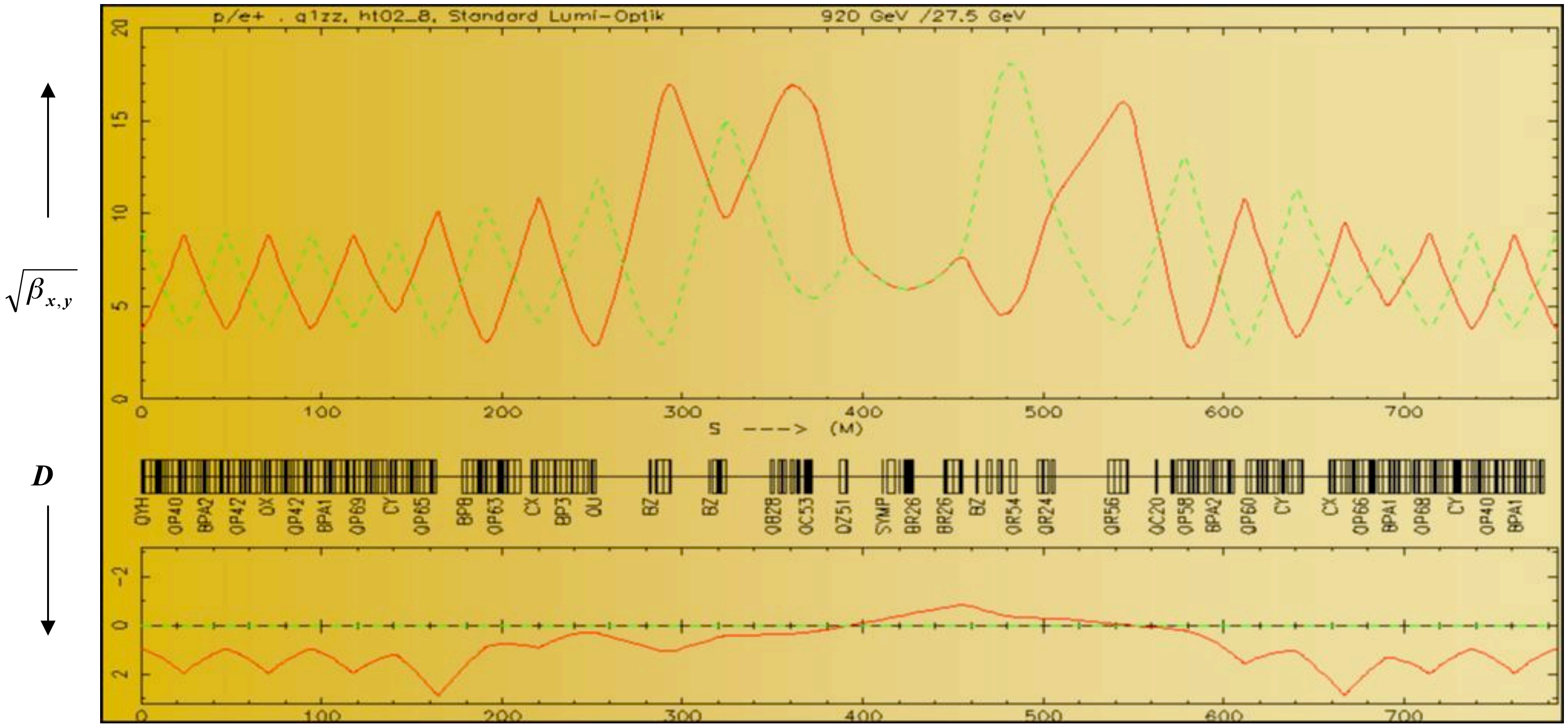
III.) The „not so ideal“ World

Lattice Design in Particle Accelerators



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

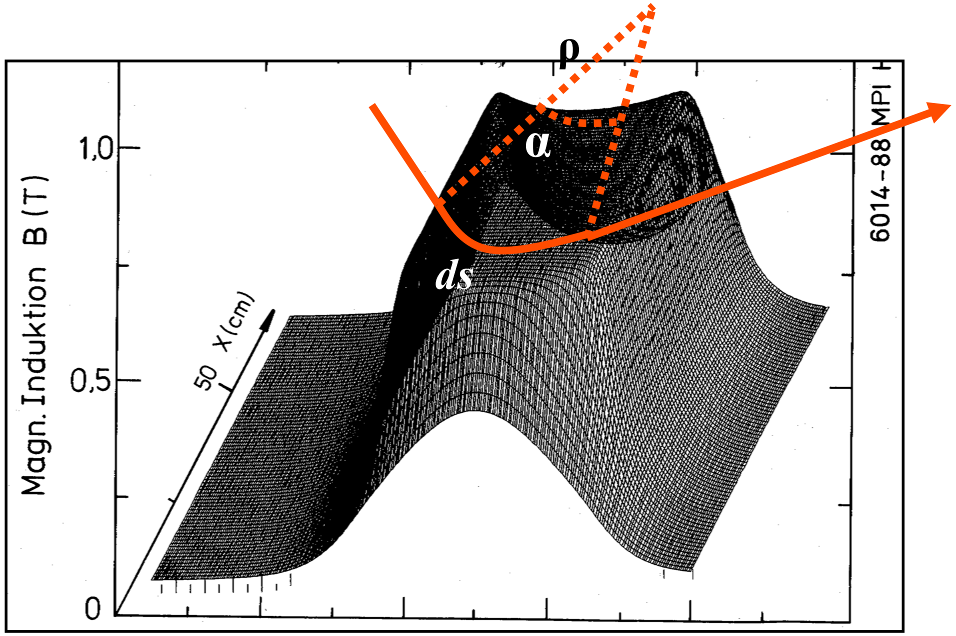
11.) Lattice Design:

„… how to build a storage ring“

$$\boxed{B\,\rho = p\,/\,q}$$

Circular Orbit: dipole magnets to define the geometry

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$



field map of a storage ring dipole magnet

The angle run out in one revolution must be 2π, so

… for a full circle $\alpha = \frac{\int Bdl}{B\,\rho} = 2\pi \;\rightarrow\; \int Bdl = 2\pi\,\frac{p}{q}$ *… defines the integrated dipole field around the machine.*

Nota bene: ♪ $\frac{\Delta B}{B} \approx 10^{-4}$ *is usually required !!*

Example LHC:

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\,dl \approx N\,l\,B = 2\pi\;\; p/e$$

$$B \approx \frac{2\pi\;\;7000\;\;10^{9}\,eV}{1232\;\;15\,m\;\;3\;\;10^{8}\,\frac{m}{s}\;\;\;e} = \underline{\underline{8.3\,Tesla}}$$

Recapitulation: storage ring elements
... the story with the matrices !!!

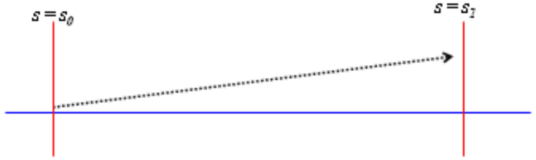
Equation of Motion:

$$x'' + K\,x = 0 \qquad K = 1/\rho^2 - k \;\ldots\; \textit{hor. plane:}$$

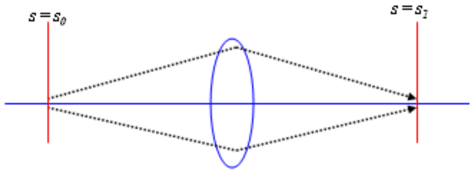
$$K = k \qquad \ldots\; \textit{vert. Plane:}$$

Solution of Trajectory Equations

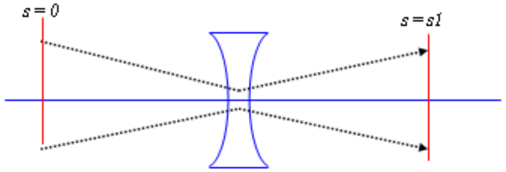
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$



$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}l) & \cosh(\sqrt{|K|}l) \end{pmatrix}$$

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \quad \ldots$$

12.) Transfer Matrix M

… yes we had the topic already

general solution of Hill´s equation

$$\left\{\begin{aligned} x(s) &= \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) &= \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right] \end{aligned}\right.$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put $\Psi(0) = 0$

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$\underline{x(s)} = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0 \sin\psi_s\right\}\underline{x_0} + \left\{\sqrt{\beta_s\beta_0}\,\sin\psi_s\right\}\underline{x_0'}$$

$$\underline{x'(s)} = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\}\underline{x_0} + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s \sin\psi_s\right\}\underline{x_0'}$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***

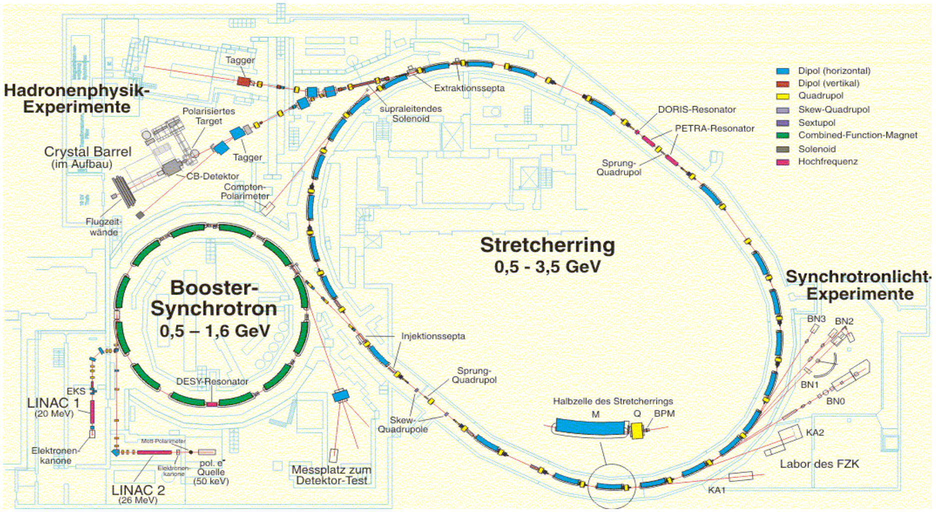
* ***and nothing but the α β γ at these positions.***

* ... ***!***

* *Äquivalenz der Matrizen*

13.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



ELSA Electron Storage Ring

„This** *rather formidable looking matrix* **simplifies considerably if we** *consider one complete revolution* **…“

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

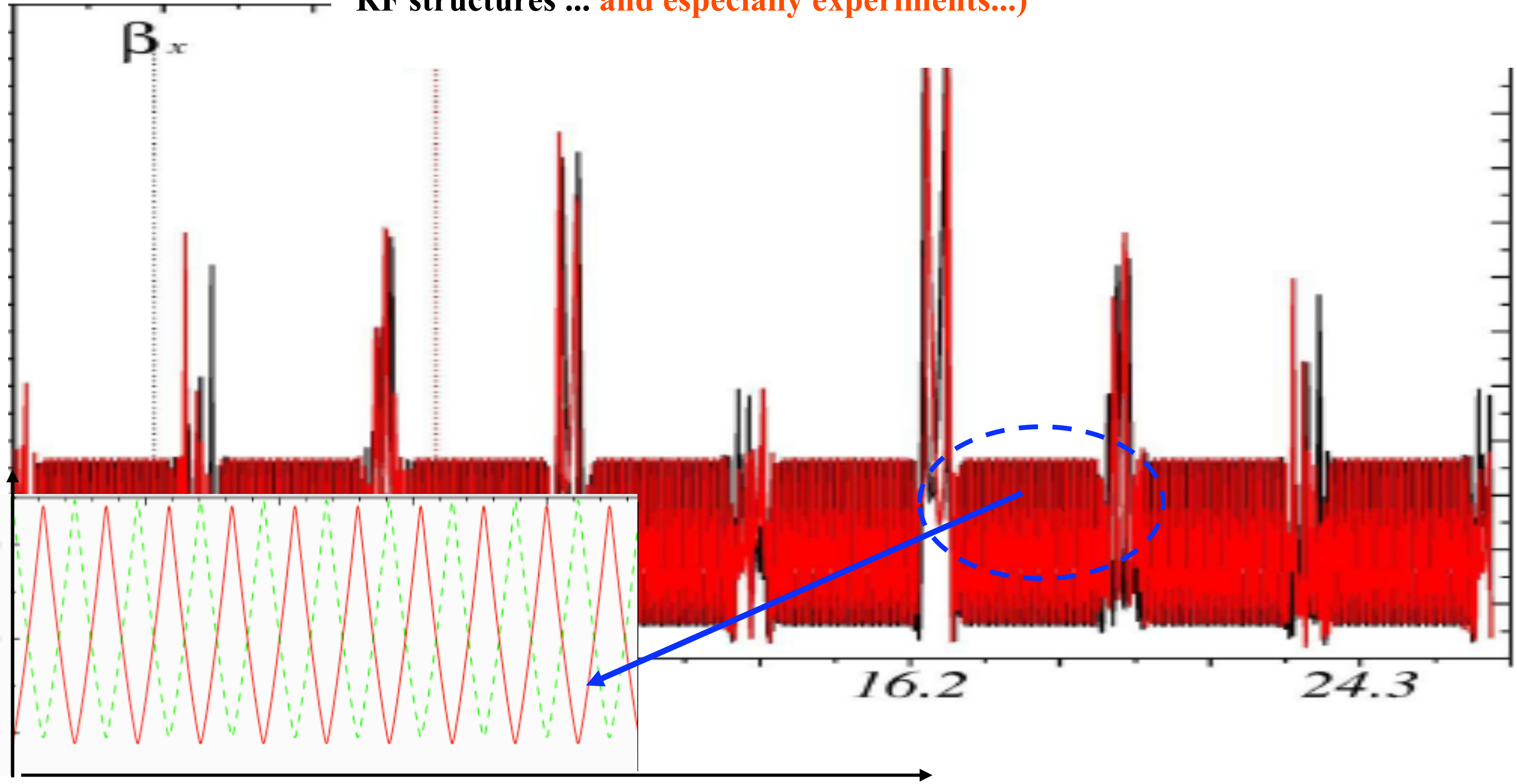
ψ_{turn} = phase advance per period

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

***Tune:* Phase advance per turn in units of 2π**

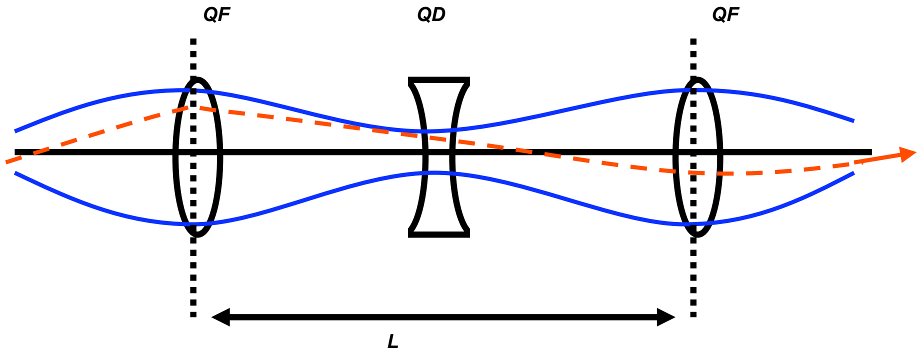
FoDo-Lattice

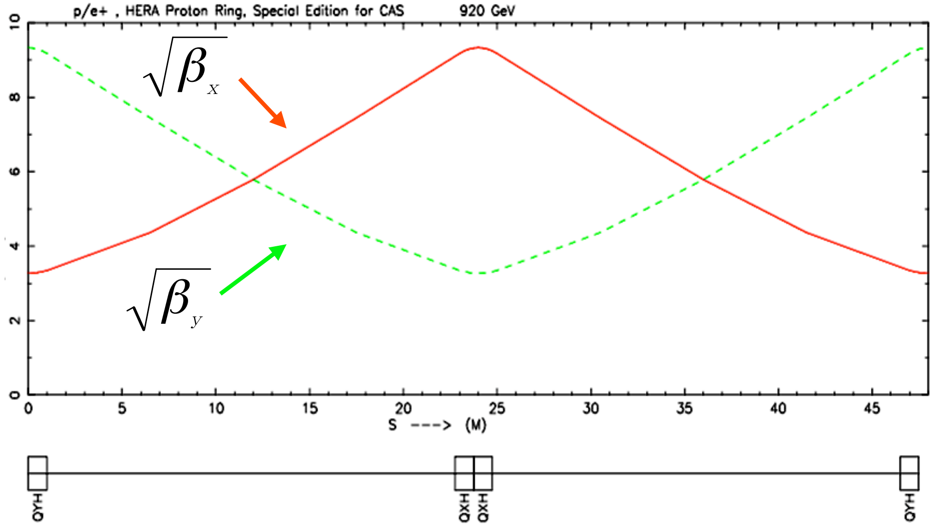
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in .
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell μ = 45°,
→ calculate the twiss parameters for a periodic solution

Periodic solution of a FoDo Cell





Output of the optics program:

Nr	Type	Length	Strength	β_x	α_x	ψ_x	β_y	α_y	ψ_y
		m	1/m2	m		1/2π	m		1/2π
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

Q_X= 0,125 Q_Y= 0,125

$0.125 * 2\pi = 45°$

Can we understand, what the optics code is doing?

matrices

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l_q) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l_q) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l_q) & \cos(\sqrt{|K|}l_q) \end{pmatrix} \qquad M_{drift} = \begin{pmatrix} 1 & l_d \\ 0 & 1 \end{pmatrix}$$

strength and length of the FoDo elements

$K = +/- 0.54102\ m^{-2}$
$lq = 0.5\ m$
$ld = 2.5\ m$

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qf}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for one period gives us all the information that we need !

Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\gamma\sin\psi & \cos\psi - \alpha\sin\psi \end{pmatrix} \rightarrow$$

$$\cos(\psi) = \frac{1}{2}\boldsymbol{Trace}(\boldsymbol{M}) = 0.707$$

$$\psi = \boldsymbol{arc}\cos\left(\frac{1}{2}\boldsymbol{Trace}(\boldsymbol{M})\right) = \underline{\underline{45°}}$$

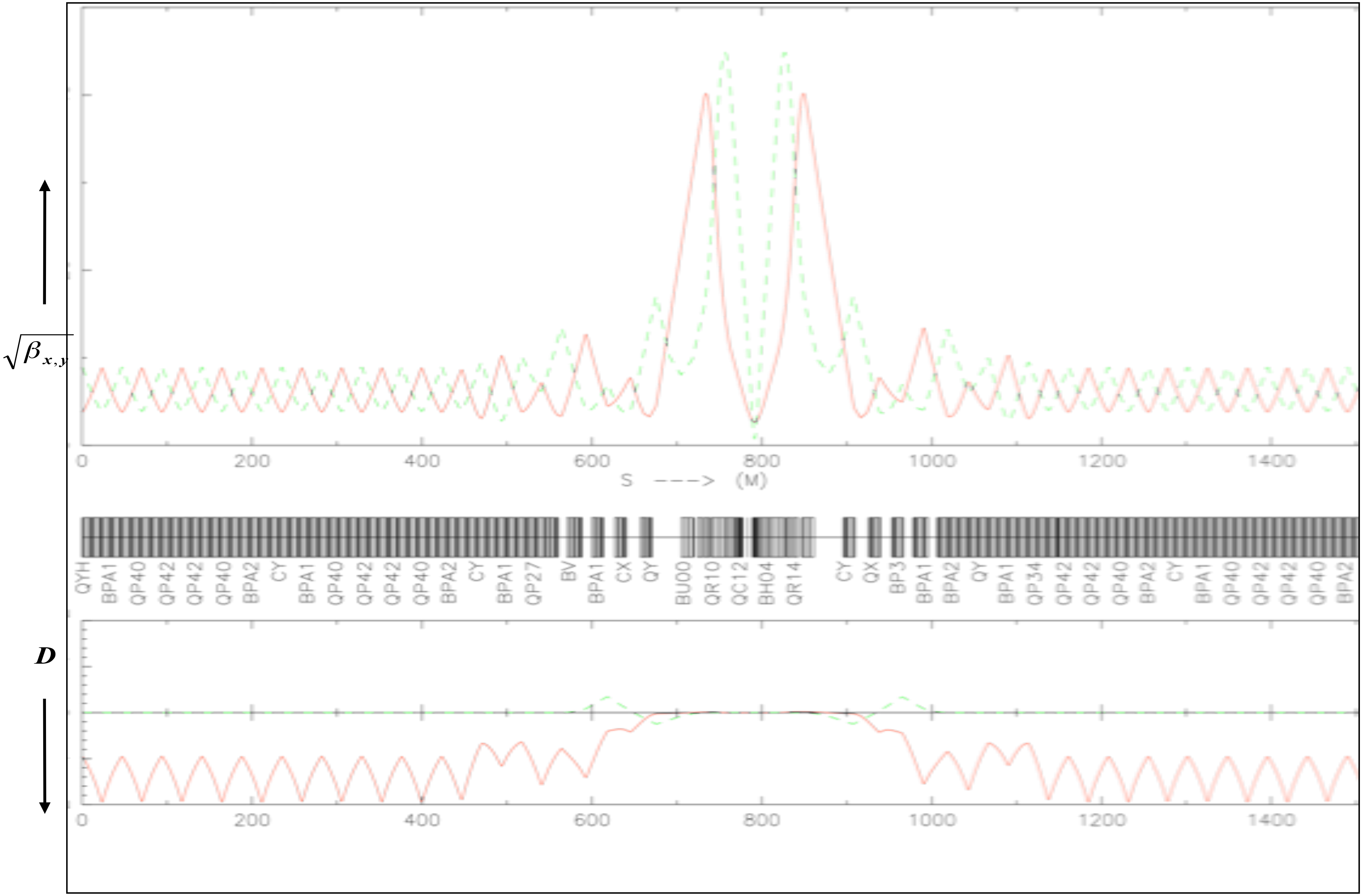
hor β-function

$$\beta = \frac{\boldsymbol{M}_{1,2}}{\sin\psi} = \underline{\underline{11.611\ \boldsymbol{m}}}$$

hor α-function ♪

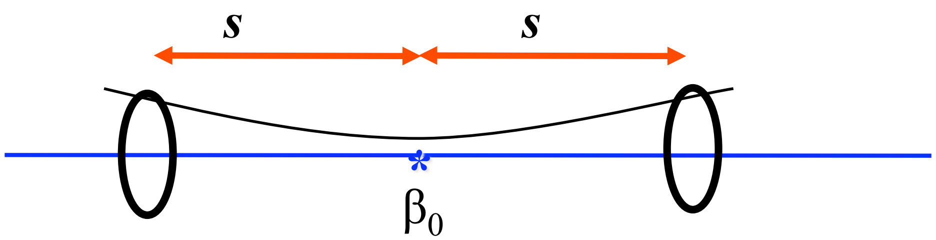
$$\alpha = \frac{\boldsymbol{M}_{1,1} - \cos\psi}{\sin\psi} = \underline{\underline{0}}$$

14.) Insertions



β-Function in a Drift:

let's assume we are at a symmetry point in the center of a drift.

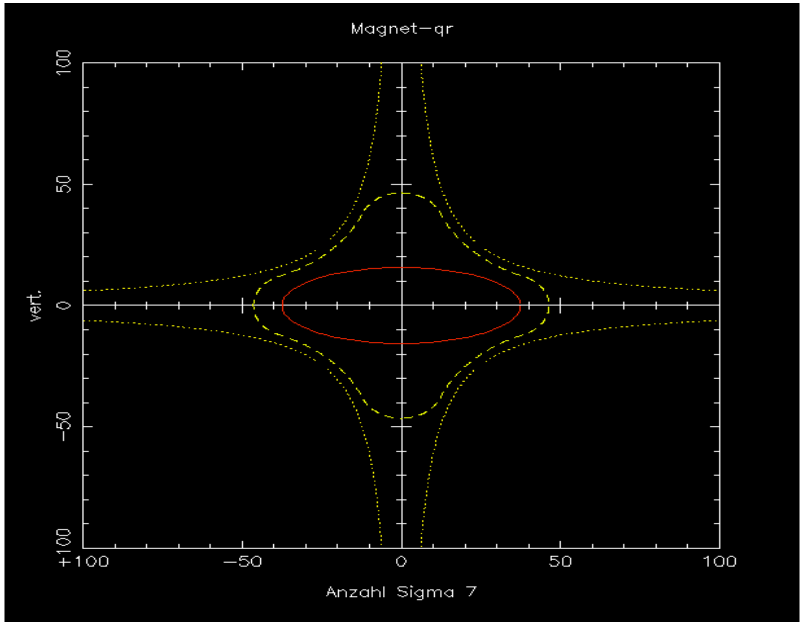


β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

At the end of a long symmetric drift space the beta function reaches its maximum value in the complete lattice.
-> here we get the largest beam dimension.

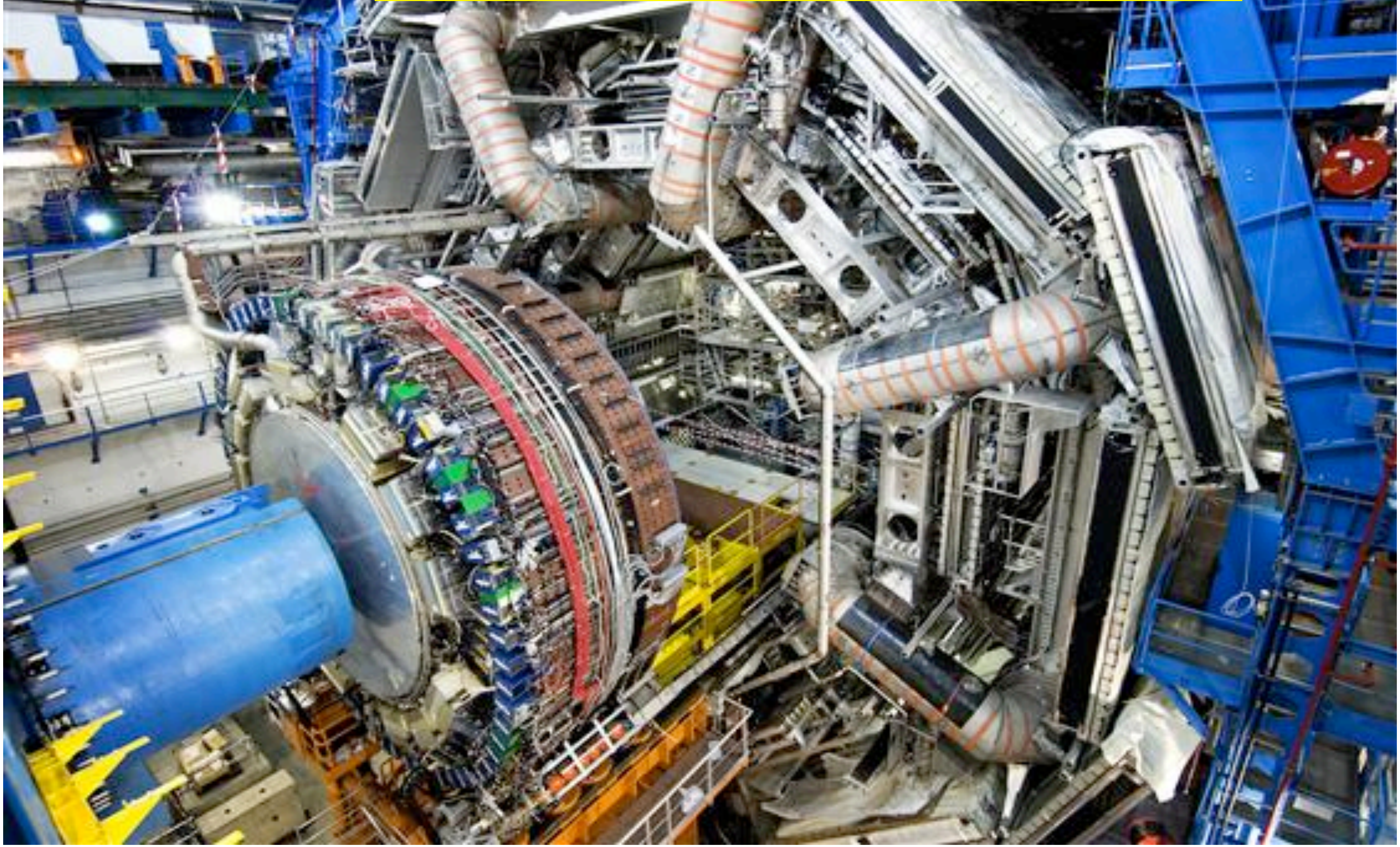
-> keep l as small as possible



7 sigma beam size inside a mini beta quadrupole

... clearly there is an

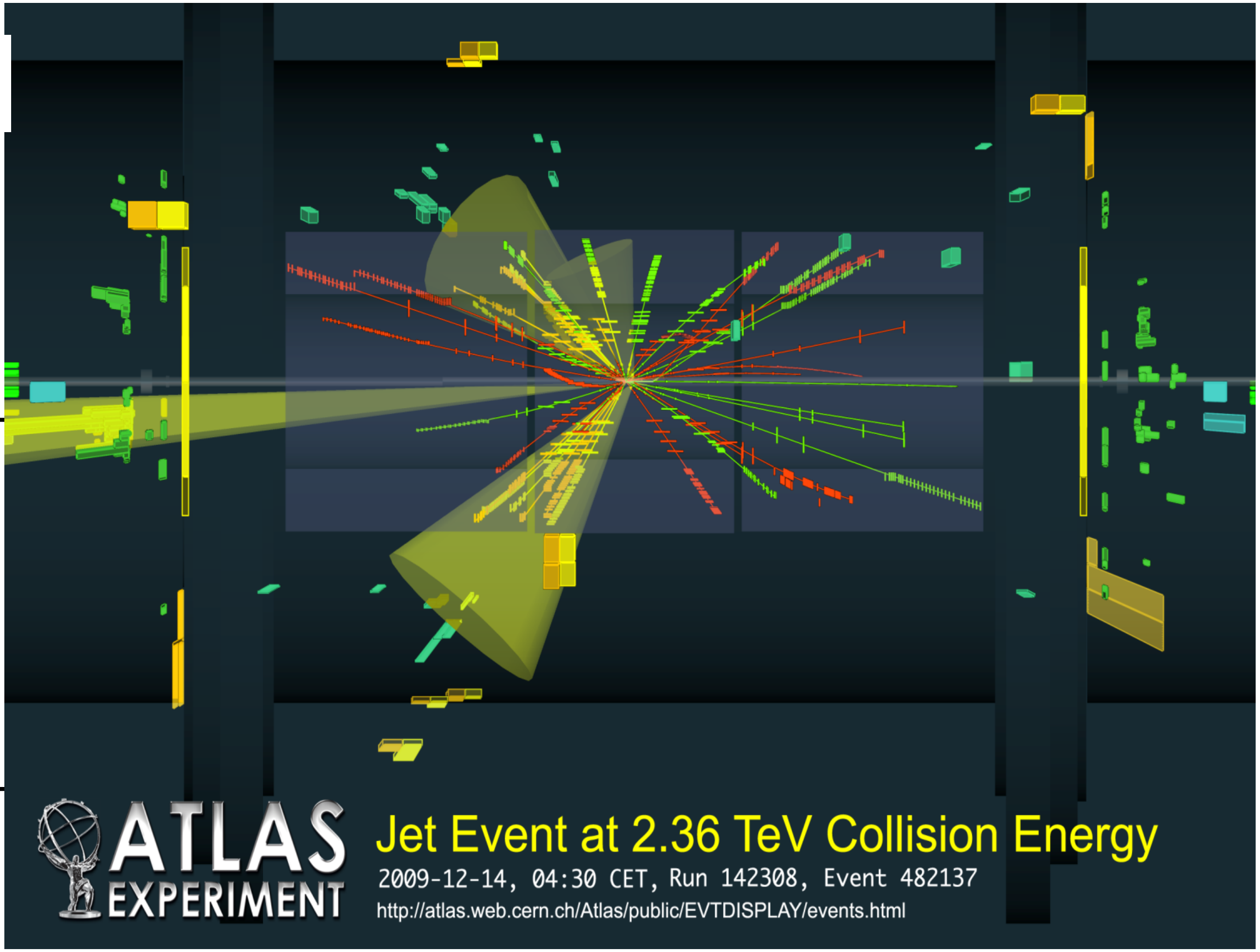
... unfortunately ... in general
high energy detectors that are
installed in that drift spaces
are a little bit bigger than a few centimeters ...



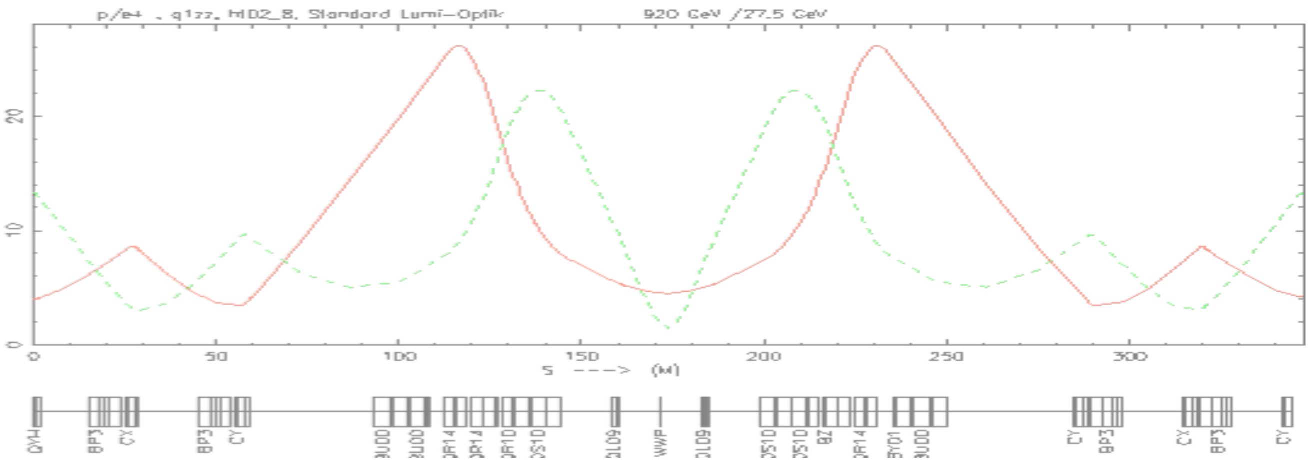
The Mini-β Insertion:

$$R = L * \Sigma_{react}$$

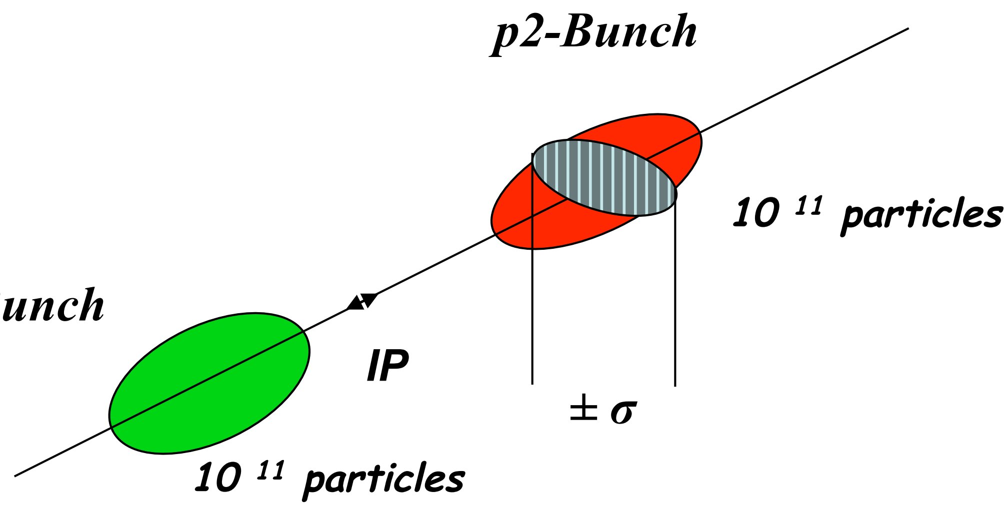
production rate of events
is determined by the
cross section Σ_{react}
and a parameter L that is given
by the design of the accelerator:
… the luminosity



$$L = \frac{1}{4\pi e^2 f_0 \, b} * \frac{I_1 * I_2}{\sigma^*_x * \sigma^*_y}$$



15.) Luminosity



Example: Luminosity run at LHC

$\beta_{x,y} = 0.55\ m$

$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m$

$\sigma_{x,y} = 17\ \mu m$

$f_0 = 11.245\ kHz$

$n_b = 2808$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$I_p = 584\ mA$

$$L = 1.0 * 10^{34}\ {}^{1}\!/_{cm^2 s}$$

beam sizes in the order of my cat’s hair !!

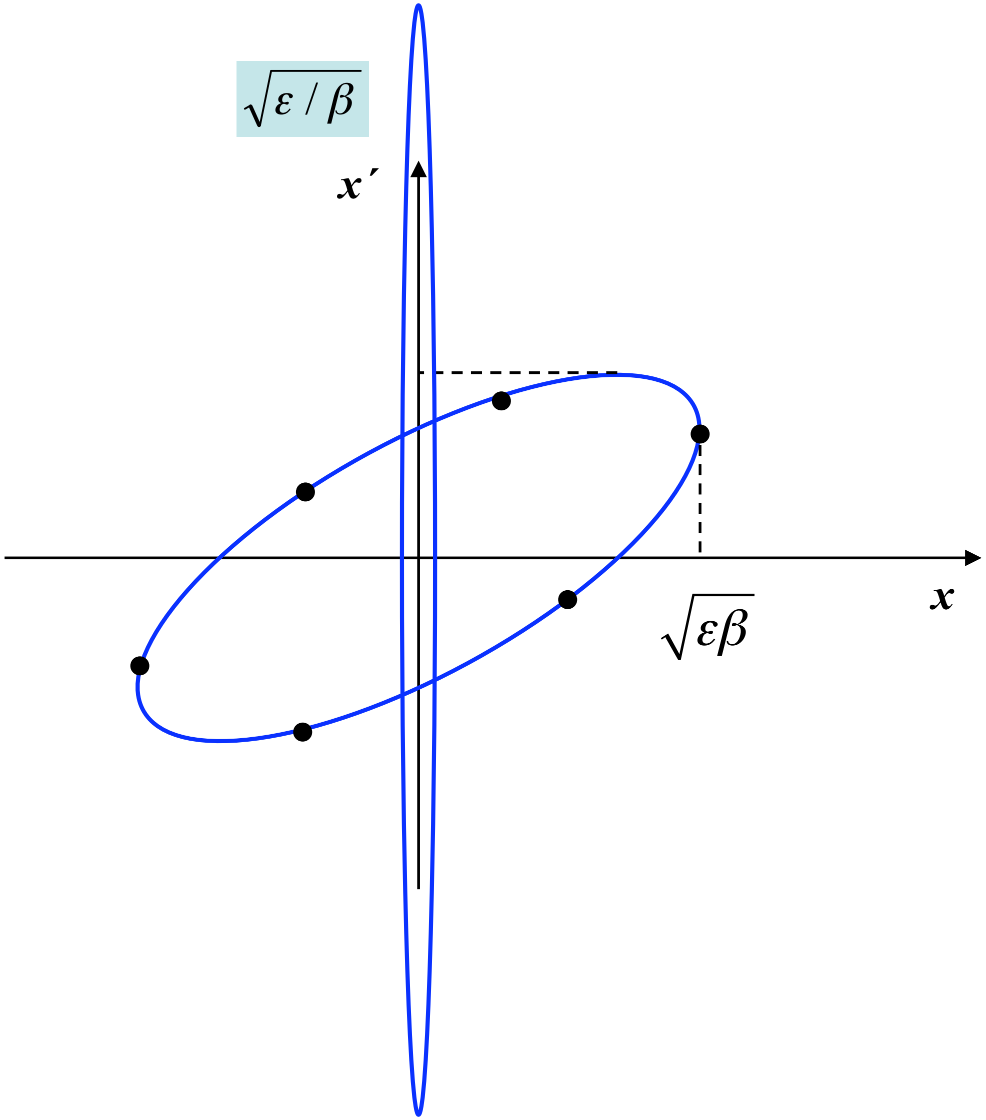
Mini-β Insertions: Betafunctions

A mini-β insertion is always a kind of special symmetric drift space.

→greetings from Liouville

the smaller the beam size

the larger the bam divergence



Mini-β Insertions: some guide lines

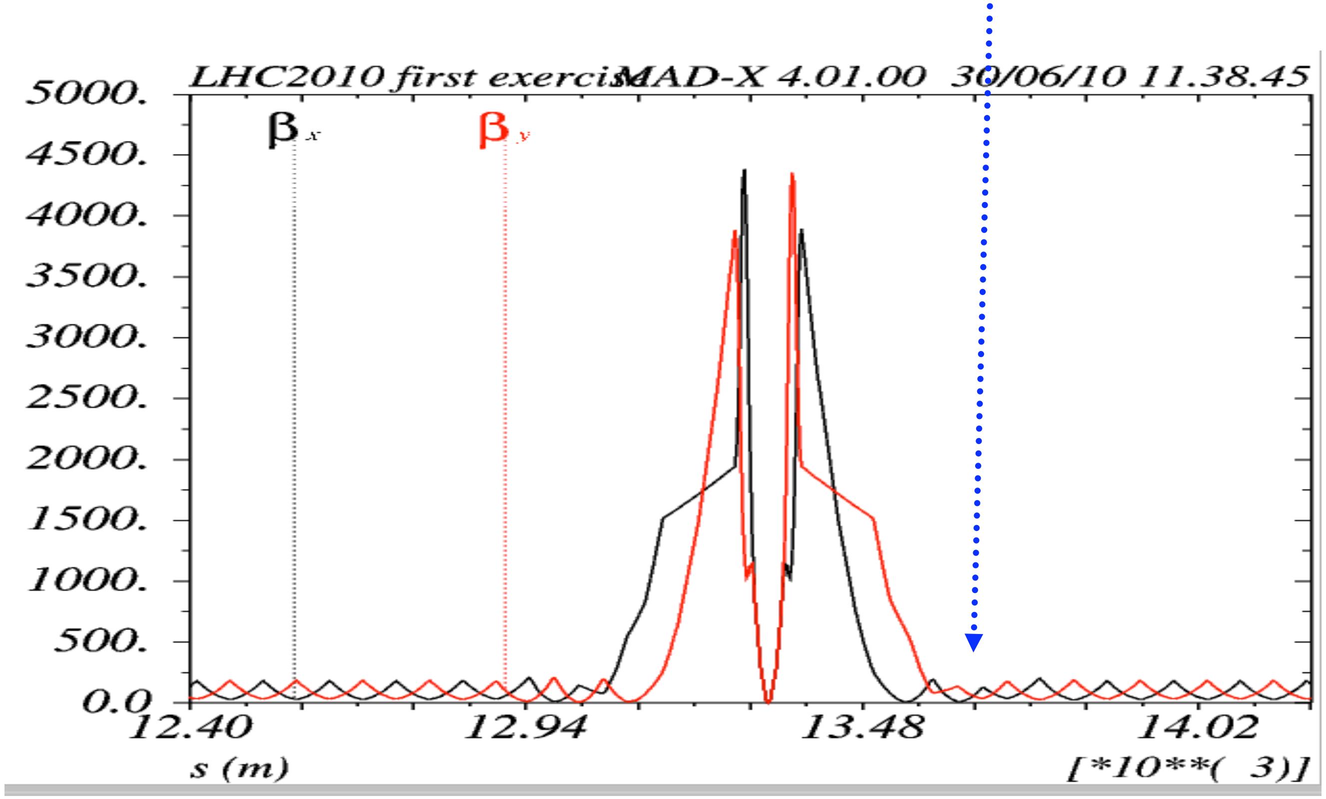
- ***calculate the periodic solution in the arc***
- ***introduce the drift space needed for the insertion device (detector ...)***
- ***put a quadrupole doublet (triplet ?) as close as possible***
- ***introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure***

parameters to be optimised & matched to the periodic solution:

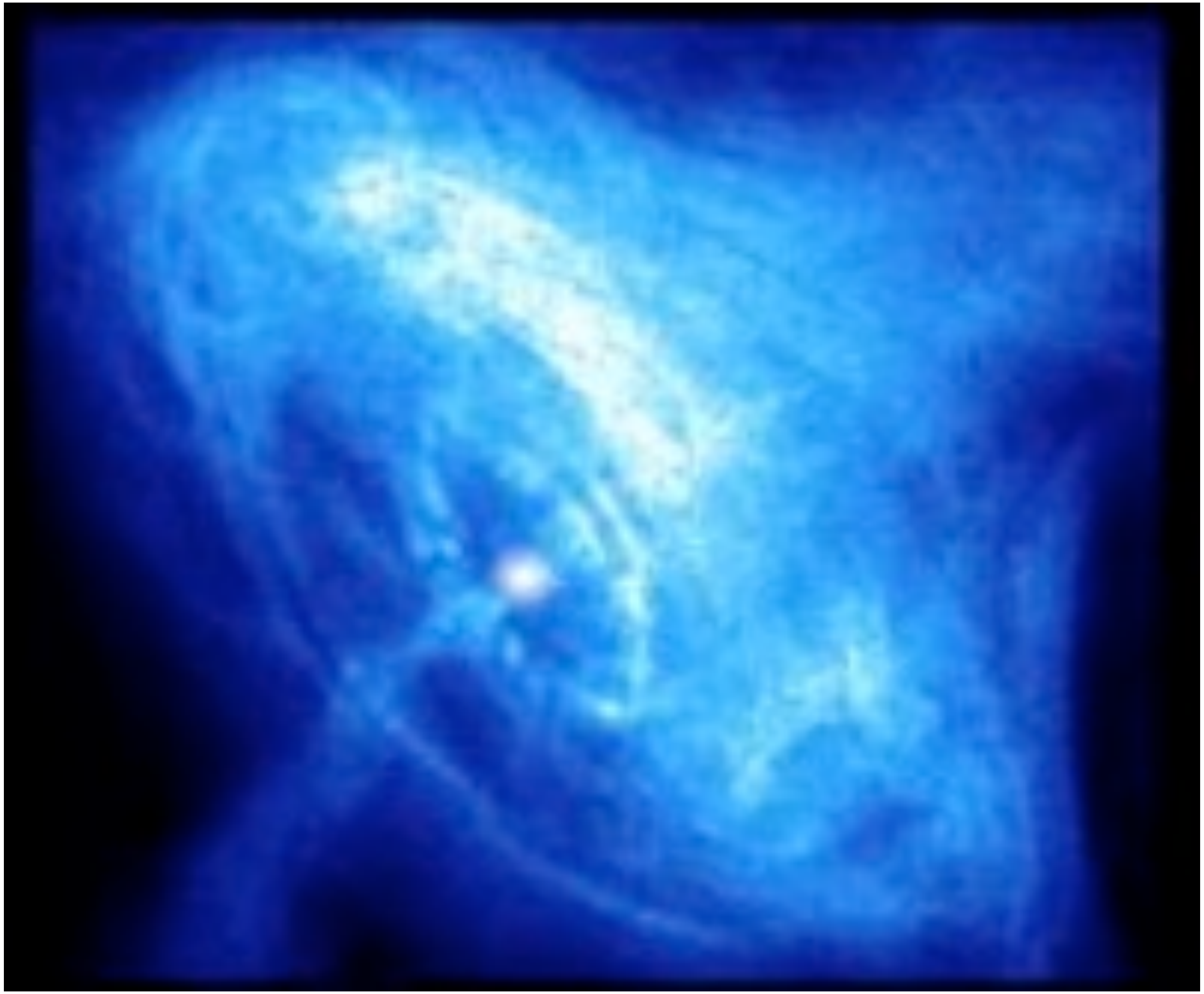
$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$

$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

8 individually powered quad magnets are needed to match the insertion
(... at least)



IV) … let´s talk about acceleration



crab nebula,

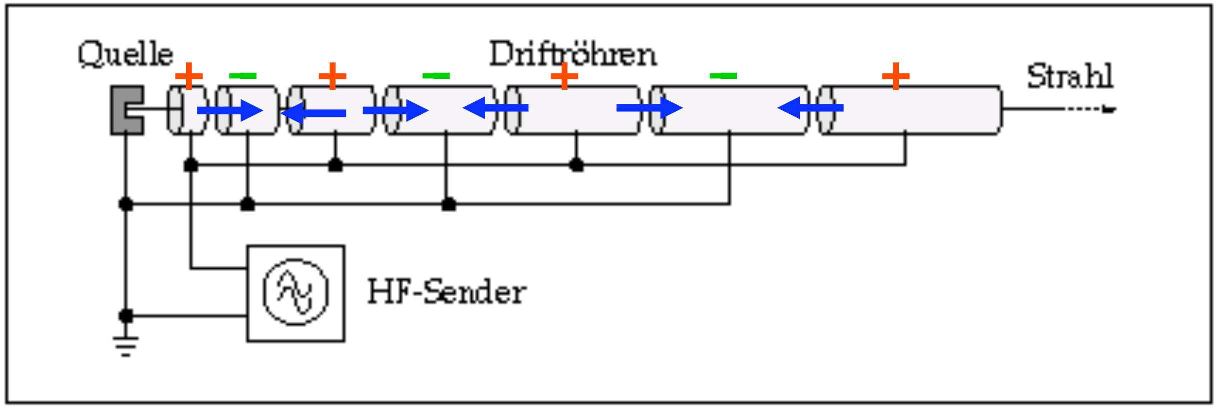
burst of charged particles E = 10 ^{20} eV

16.) Electrostatic Machines

Example for such a „steam engine“: 12 MV-Tandem van de Graaff Accelerator at MPI Heidelberg

17.) RF Acceleration

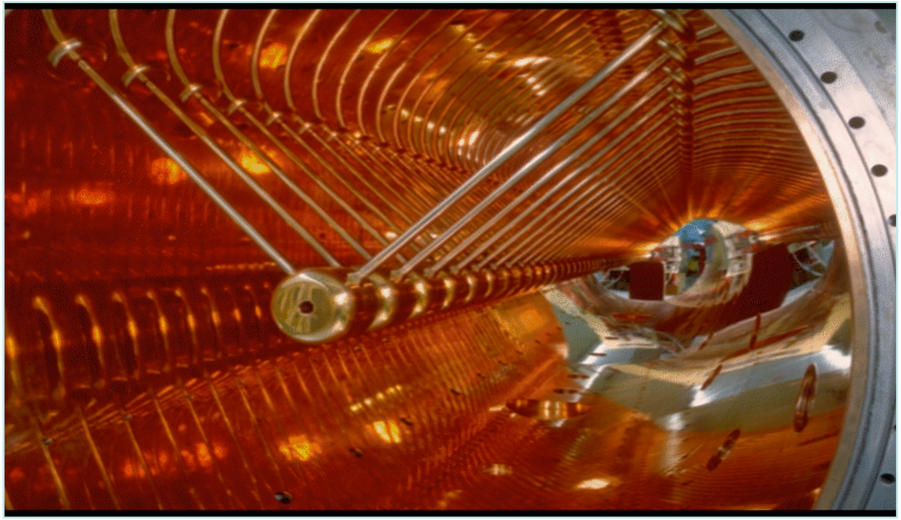
1928, Wideroe



Energy Gain per „Gap“:

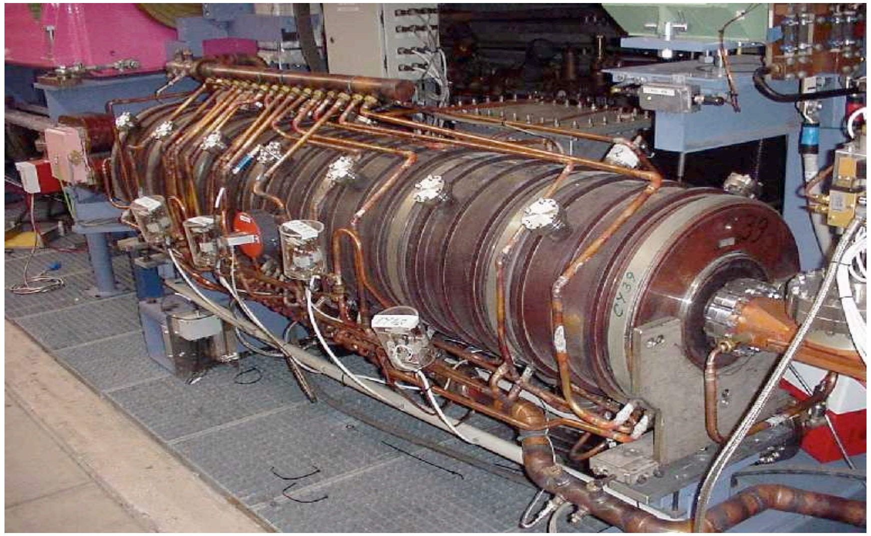
$$W = q\, U_0 \sin \omega_{RF} t$$

drift tube structure at a proton linac (GSI Unilac)



* *RF Acceleration:* ***multiple application of the same acceleration voltage; brillant idea to gain higher energies***

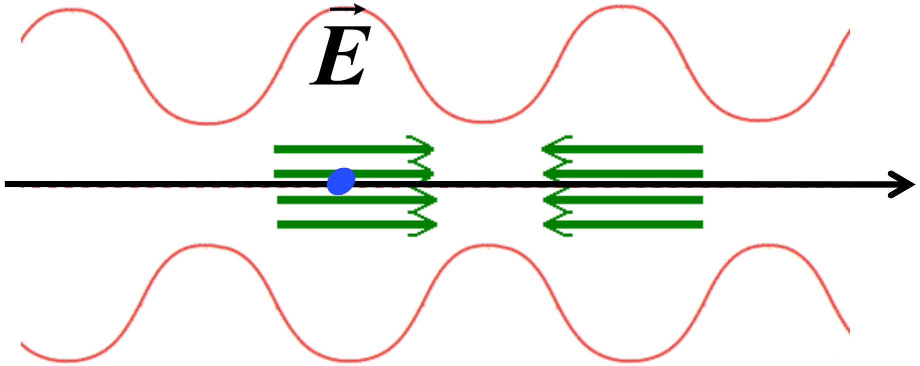
500 MHz cavities in an electron storage ring

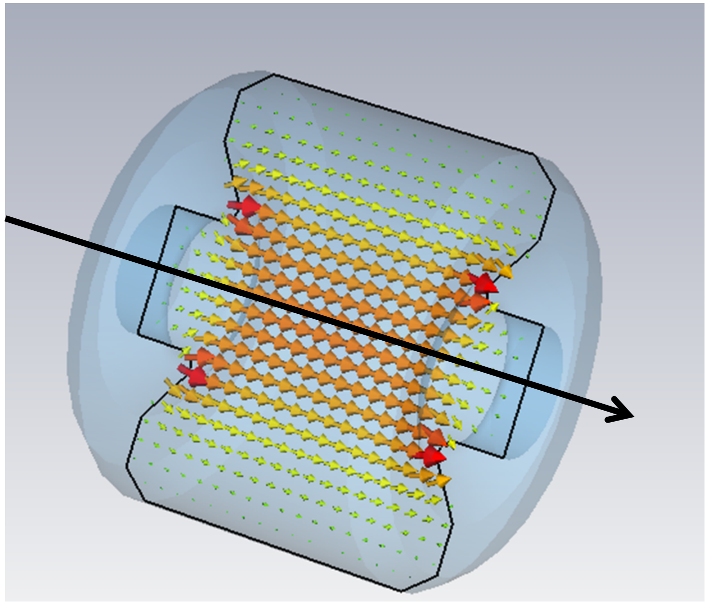


RF Acceleration

Where is the acceleration?

Install an RF accelerating structure in the ring:





B. Salvant
N. Biancacci

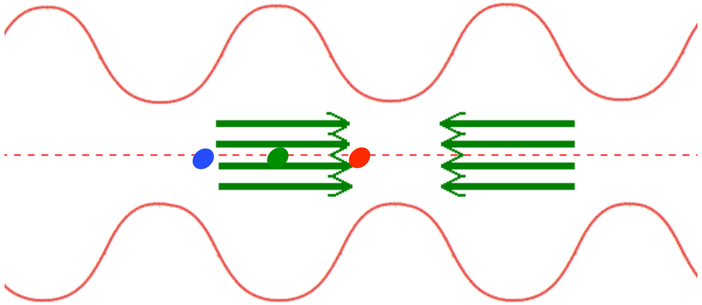
18.) The Acceleration for Δp/p≠0

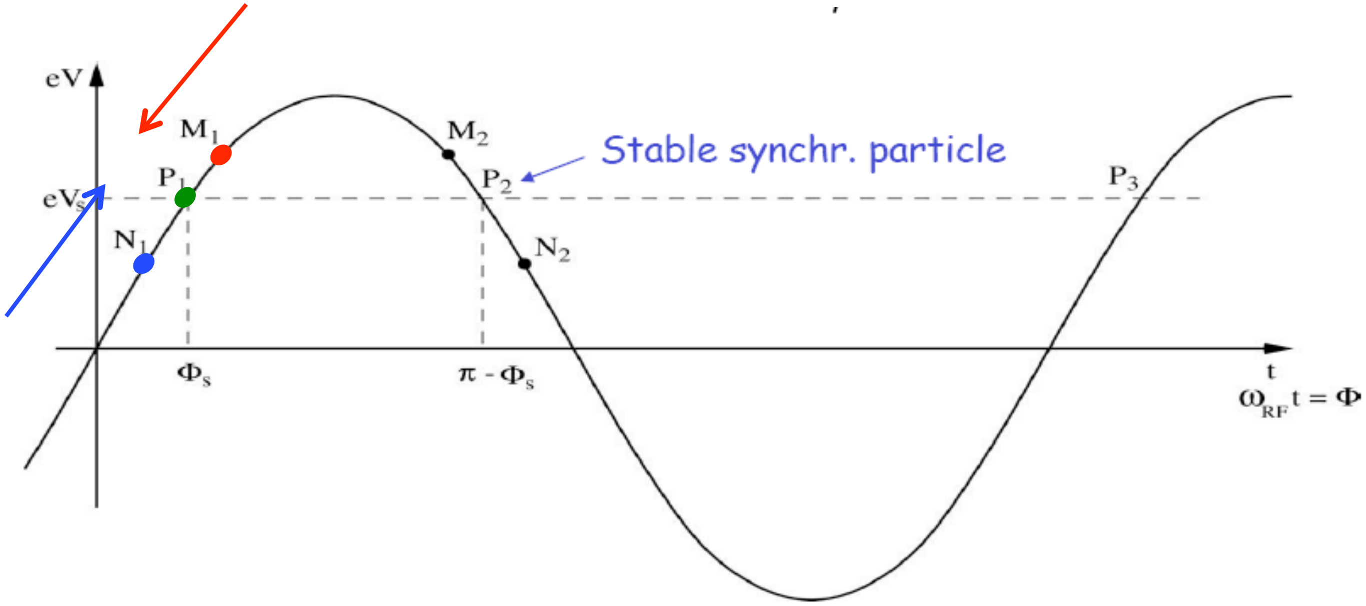
"Phase Focusing" below transition

ideal particle ●

particle with Δp/p > 0 ● *faster*

particle with Δp/p < 0 ● *slower*



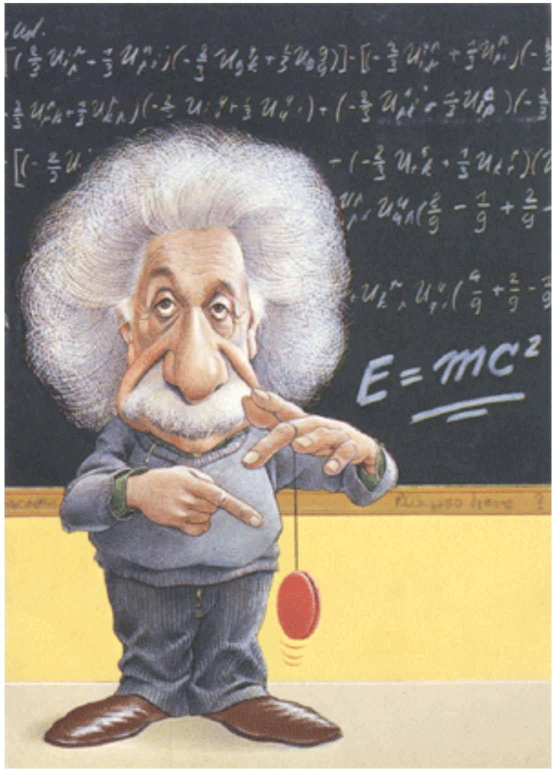


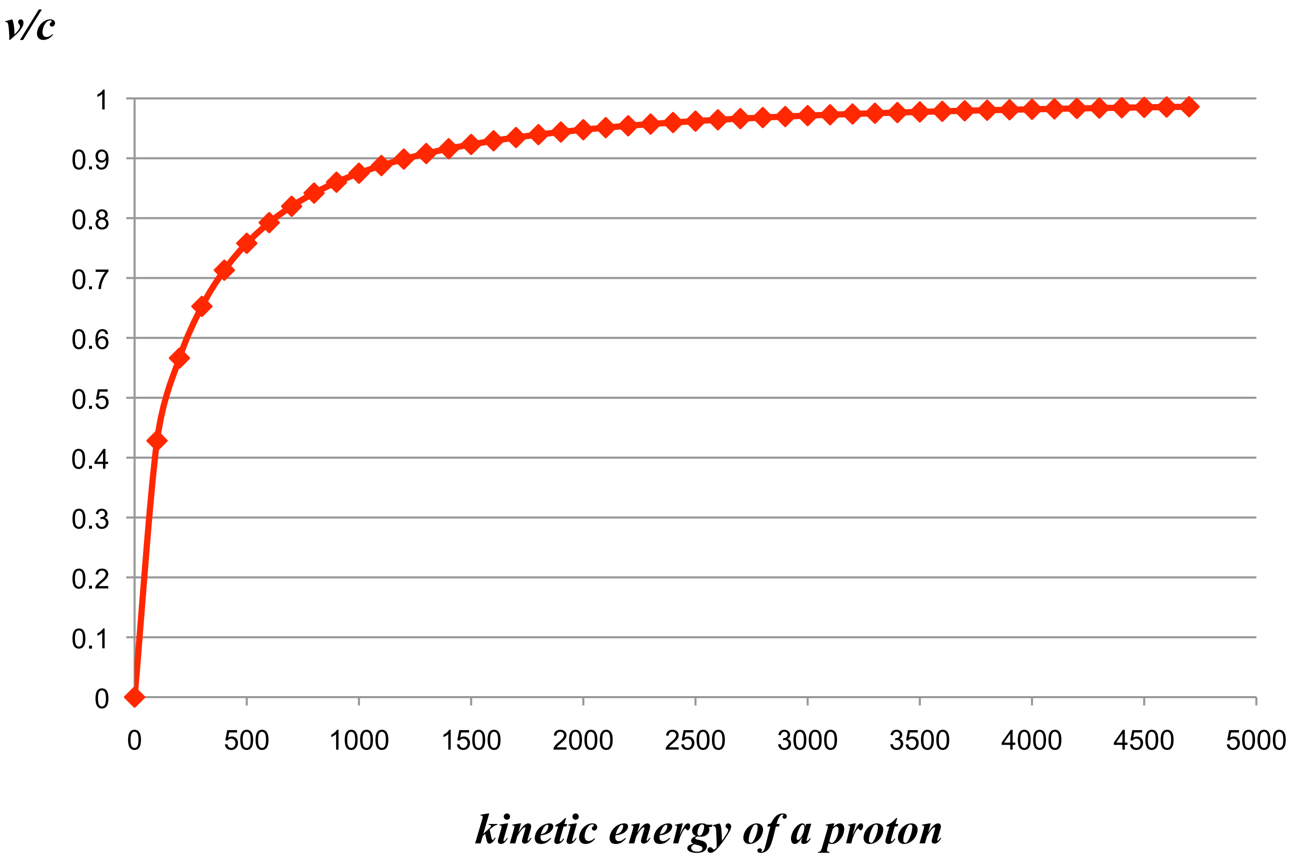
Focussing effect in the longitudinal direction keeping the particles close together … forming a "bunch"

oscillation frequency: $f_s = f_{rev}\sqrt{-\frac{h\alpha_s}{2\pi} * \frac{qU_0\cos\phi_s}{E_s}}$ *≈ some Hz*

... so sorry, here we need help from Albert:

$$\gamma = \frac{E_{total}}{mc^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \longrightarrow \frac{v}{c} = \sqrt{1-\frac{mc^2}{E^2}}$$





... some when the particles do not get faster anymore

.... but heavier !

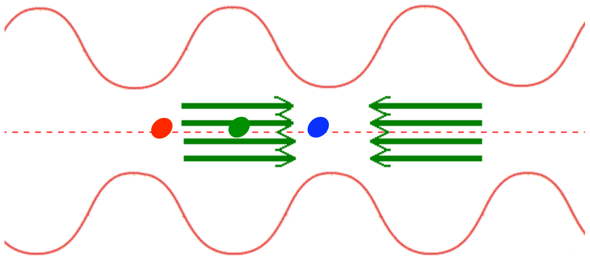
19.) The Acceleration for $\Delta p/p \neq 0$

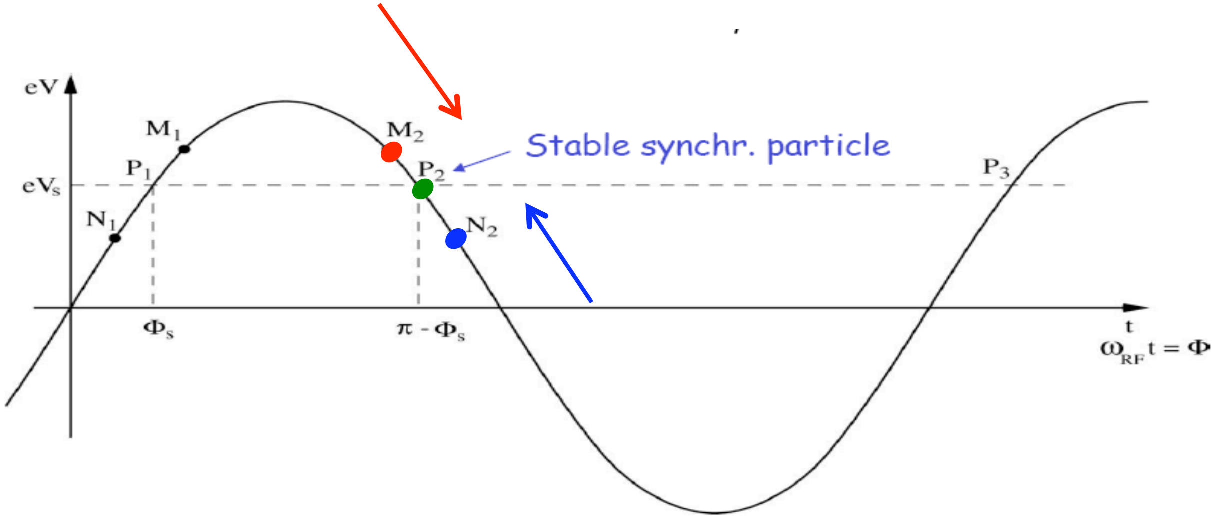
“Phase Focusing” above transition

ideal particle ●

particle with $\Delta p/p > 0$ ● *heavier*

particle with $\Delta p/p < 0$ ● *lighter*



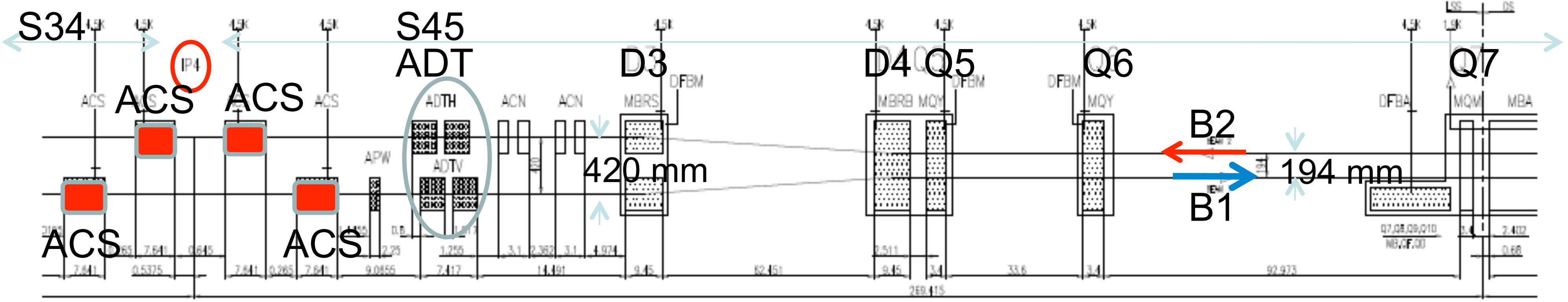


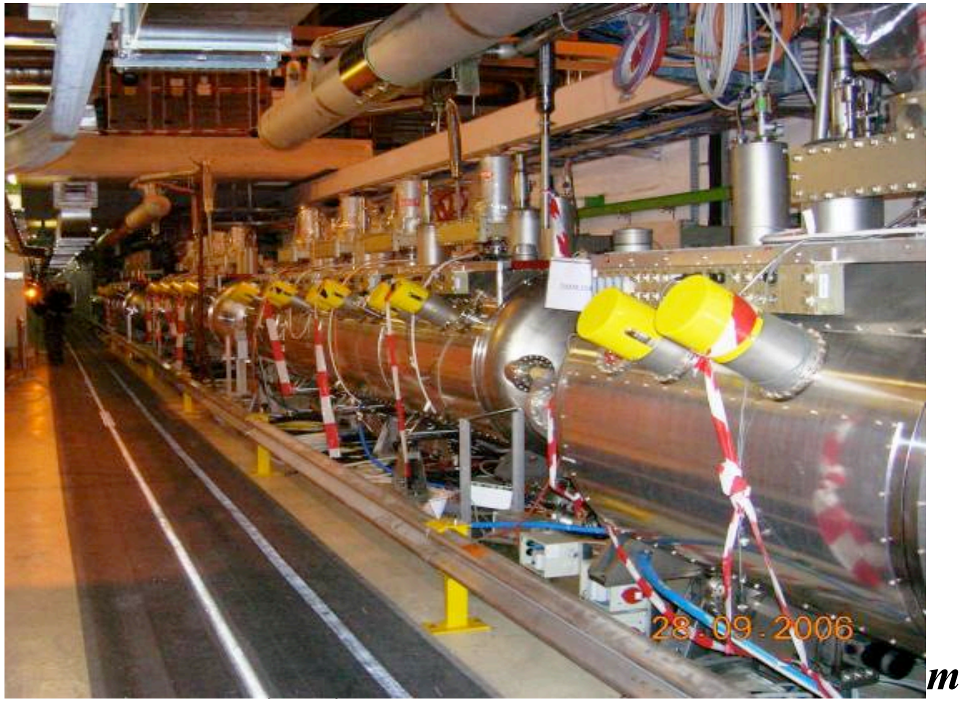
Focussing effect in the longitudinal direction
keeping the particles close together ... forming a “bunch”

... and how do we accelerate now ???

with the dipole magnets !

The RF system: IR4



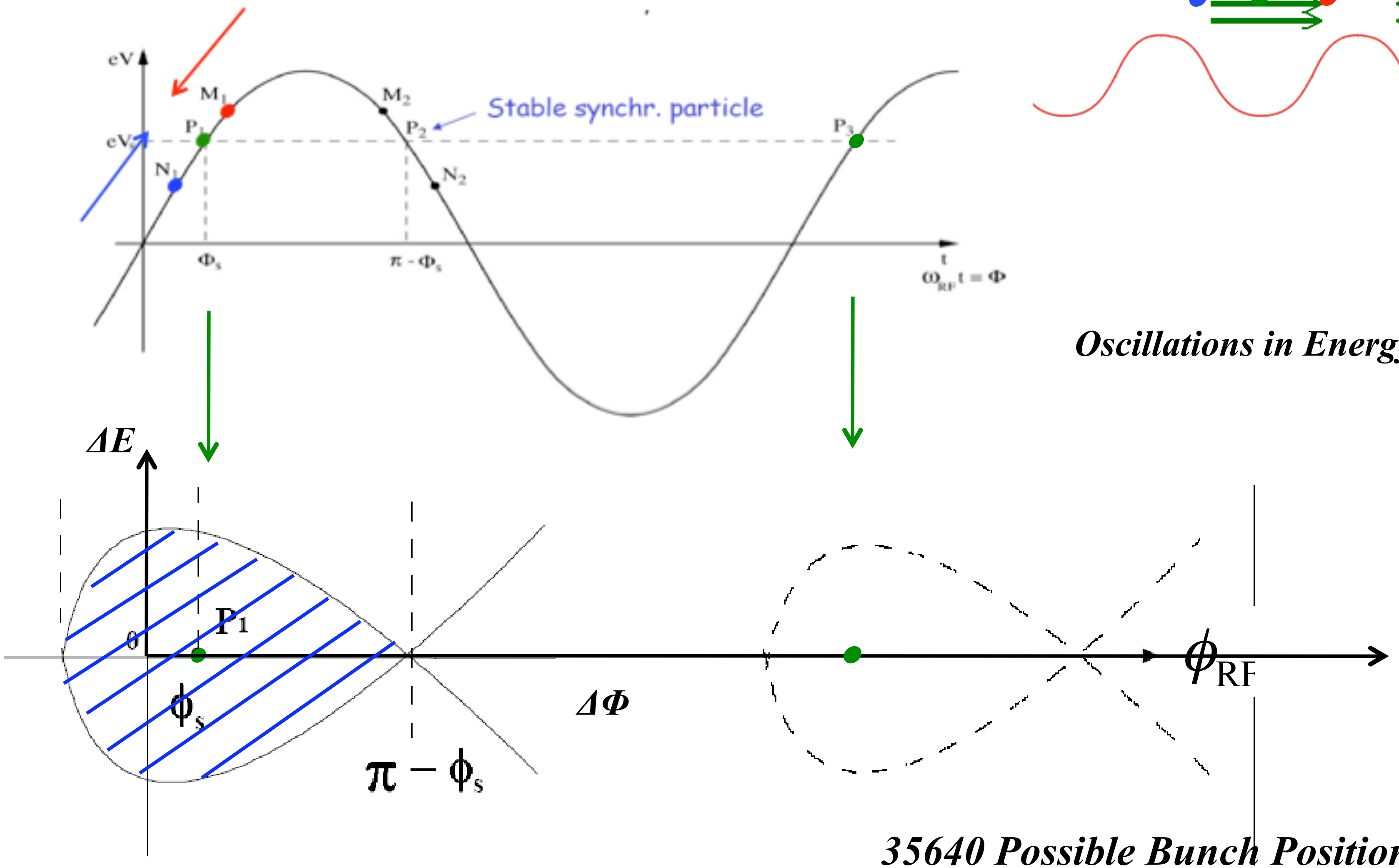


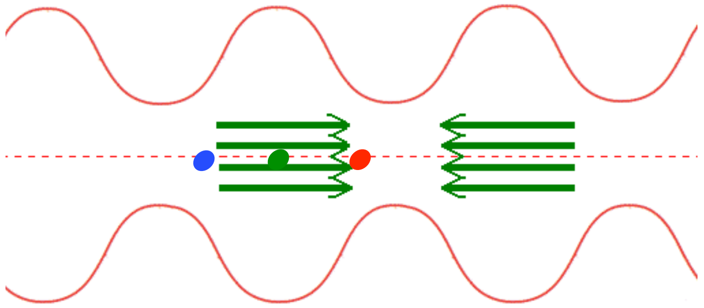
m

Nb on Cu cavities @4.5 K (=LEP2)
Beam pipe diam.=300mm

Bunch length (4σ)	*ns*	*1.06*
Energy spread (2σ)	*10^{-3}*	*0.22*
Synchr. rad. loss/turn	*keV*	*7*
Synchr. rad. power	*kW*	*3.6*
RF frequency	*M Hz*	*400*
Harmonic number		*35640*
RF voltage/beam	*MV*	*16*
Energy gain/turn	*keV*	*485*
Synchrotron frequency	*Hz*	*23.0*

RF Buckets & long. dynamics in phase space

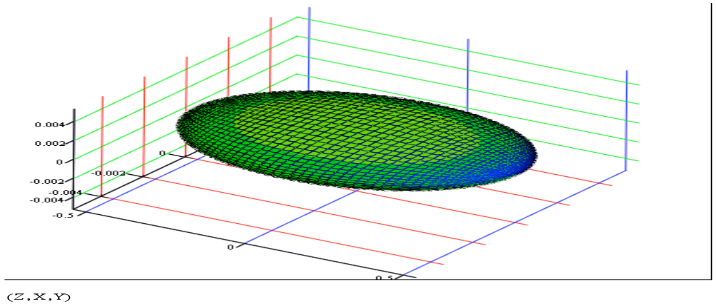




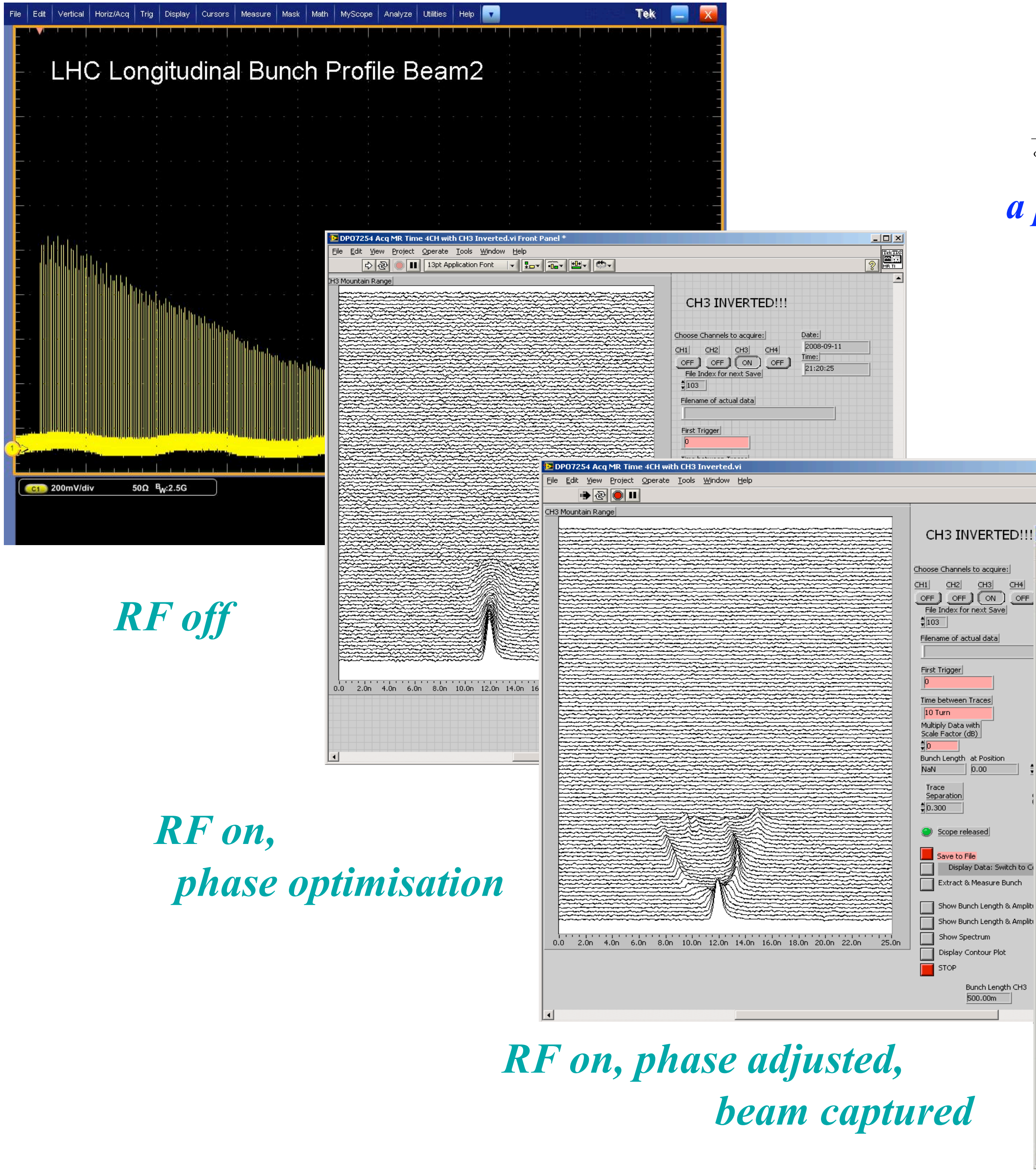
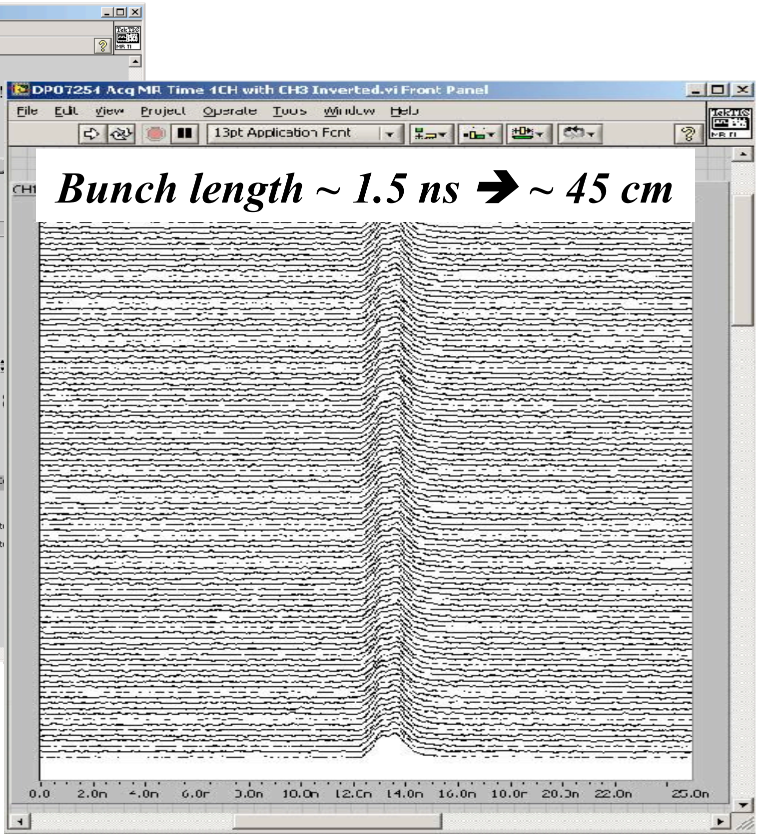
Oscillations in Energy and Phase

35640 Possible Bunch Positions (“buckets”)
2808 Bunches

LHC Commissioning: RF



a proton bunch: focused longitudinal by the RF field

Choose Channels to acquire:
CH1 OFF CH2 OFF CH3 ON CH4 OFF
Date: 2008-09-11
Time: 21:20:25
File Index for next Save: 103
Filename of actual data:
First Trigger: 0
Time between Traces: 10 Turn
Multiply Data with Scale Factor (dB): 0
Bunch Length: NaN at Position: 0.00
Trace Separation: 0.300
Bunch Length CH3: 500.00m

RF off

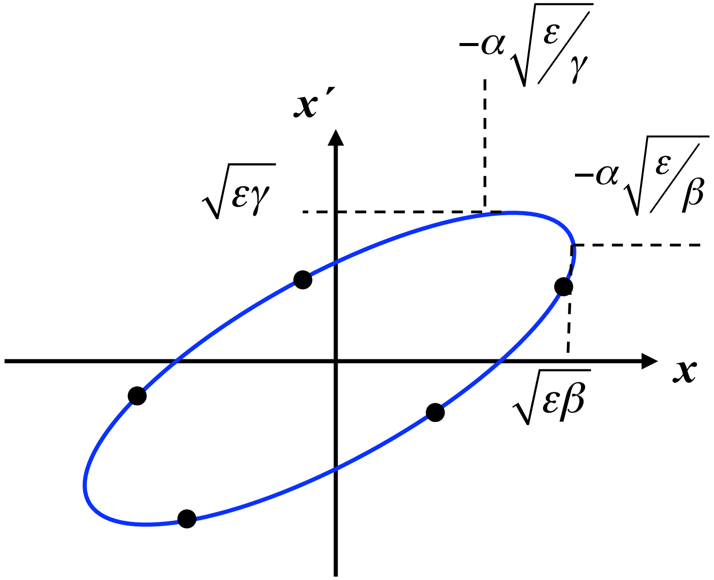
RF on,
phase optimisation

Bunch length ~ 1.5 ns ➔ ~ 45 cm

RF on, phase adjusted,
beam captured

20.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the x, x´ Phase Space Ellipse*

***Liouville:** Area in phase space is constant.*

But so sorry ... $\varepsilon \neq const$!

Classical Mechanics:

phase space = diagram of the two canonical variables
position & momentum
x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

$$\boldsymbol{q = position = x}$$
$$\boldsymbol{p = momentum = \gamma m v = mc\gamma\beta_x}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\boldsymbol{\beta_x = v_x / c}$

$$\int p\,dq = mc\int \gamma\beta_x dx$$

$$\int p\,dq = mc\gamma\beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\boxed{\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}}$$

the beam emittance
shrinks during
acceleration $\varepsilon \sim 1/\gamma$

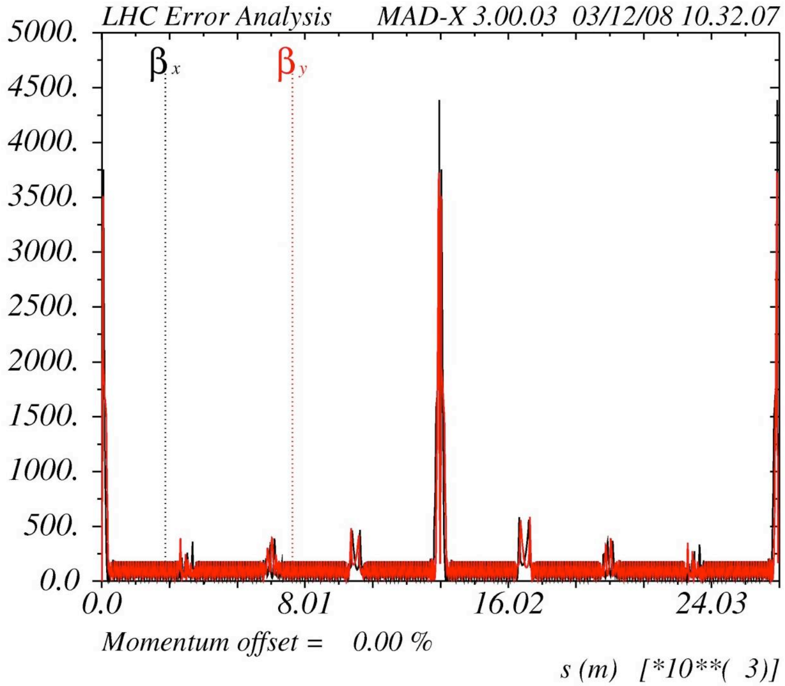
Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

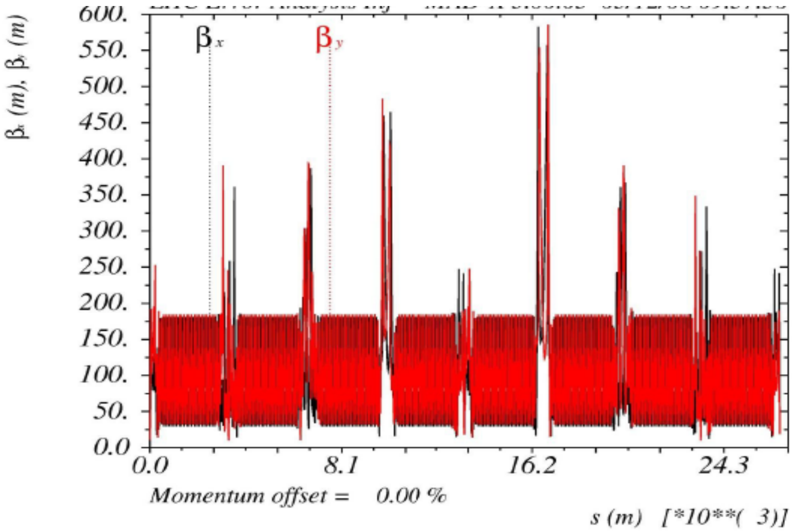
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

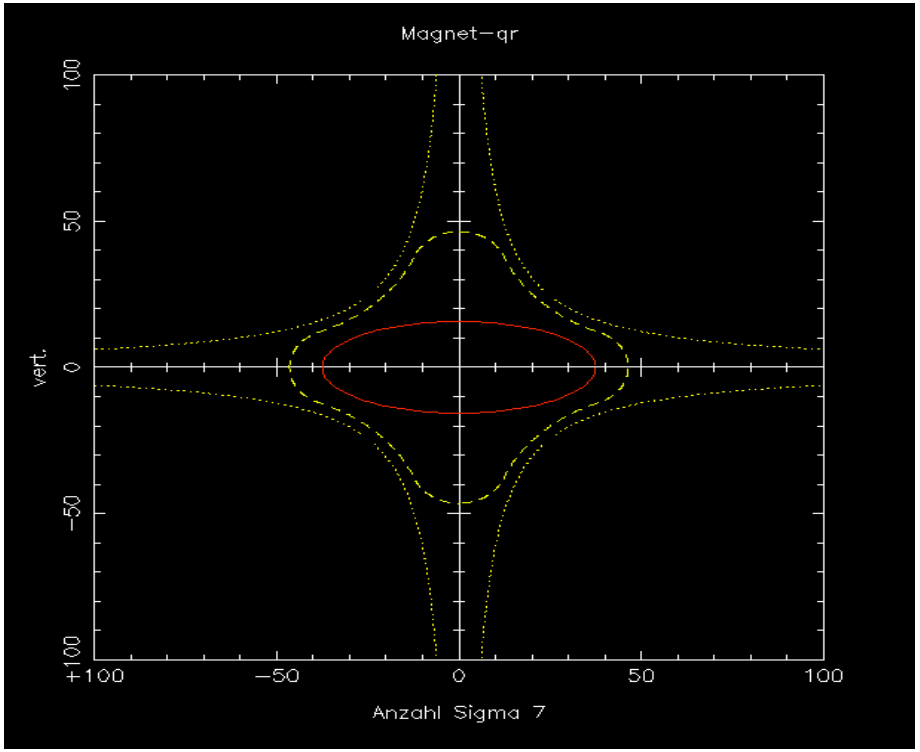


LHC injection optics at 450 GeV

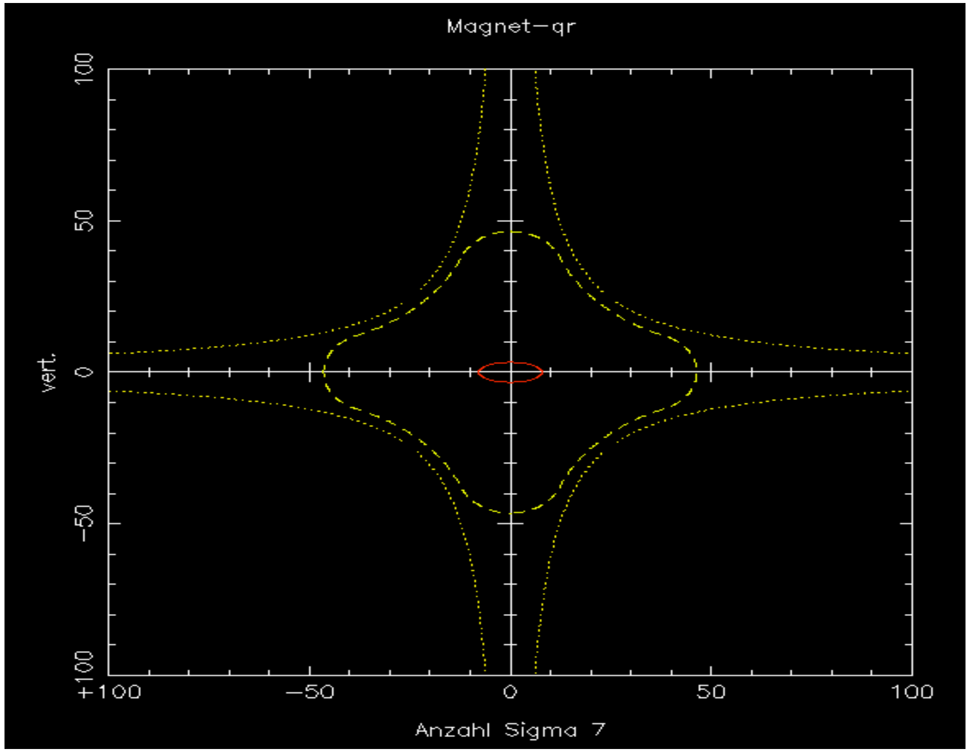
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε *(40GeV)* $= 1.2 * 10^{-7}$
ε *(920GeV)* $= 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

The „ not so ideal world “

21.) The „ Δp / p ≠ 0" Problem

ideal accelerator: *all particles will see the same accelerating voltage.*

→ $\Delta p / p = 0$

„nearly ideal“ accelerator: *Cockroft Walton or van de Graaf*

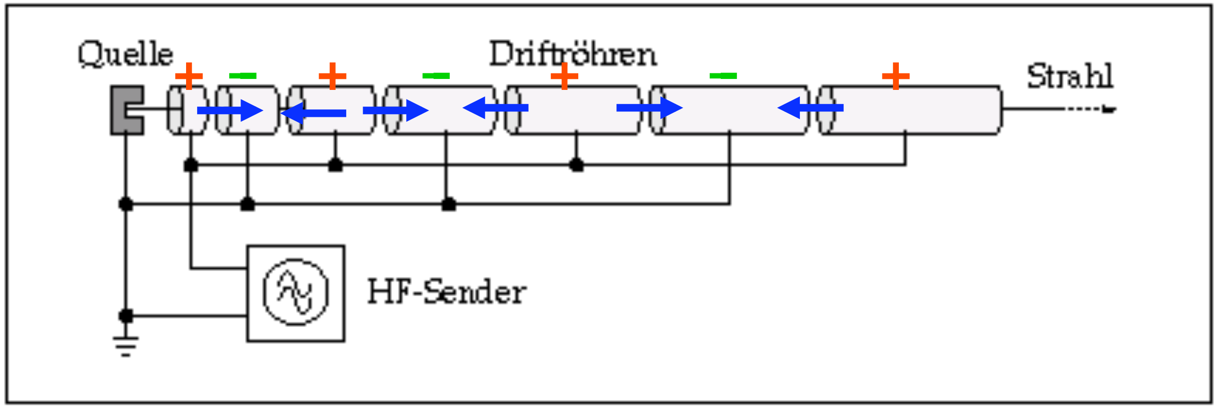
$\Delta p / p \approx 10^{-5}$

Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

Linear Accelerator

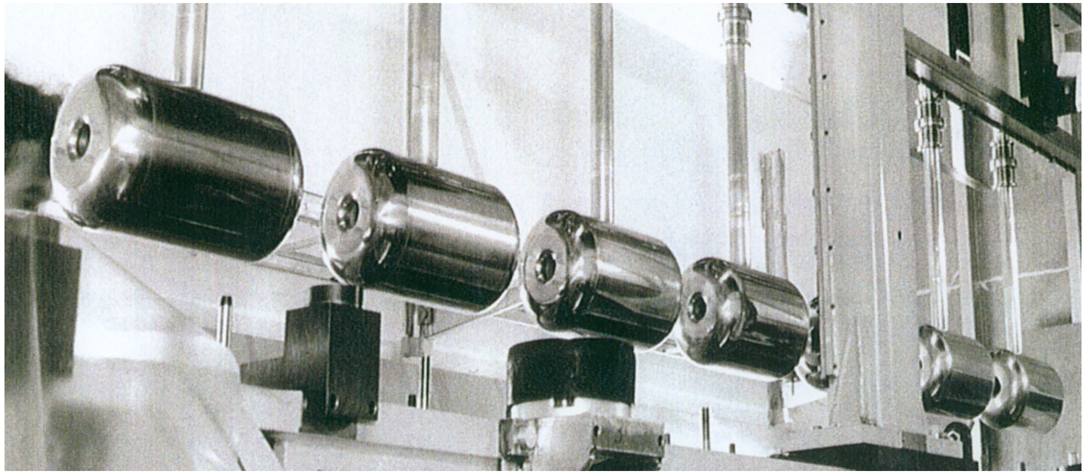
1928, Wideroe schematic Layout:



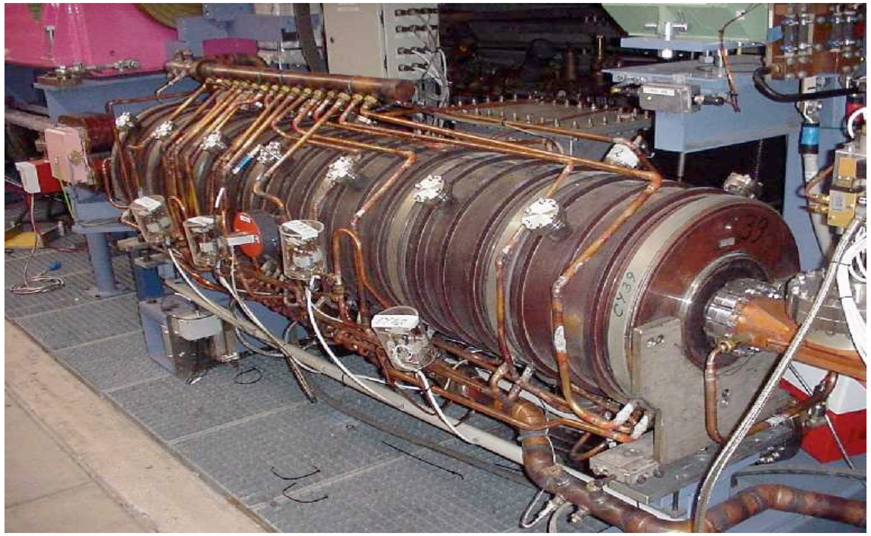
Energy Gain per „Gap":

$$W = q\, U_0 \sin \omega_{RF} t$$

drift tube structure at a proton linac



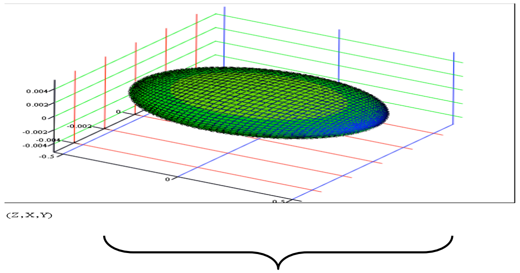
500 MHz cavities in an electron storage ring



* *RF Acceleration:* ***multiple application of the same acceleration voltage; brillant idea to gain higher energies***

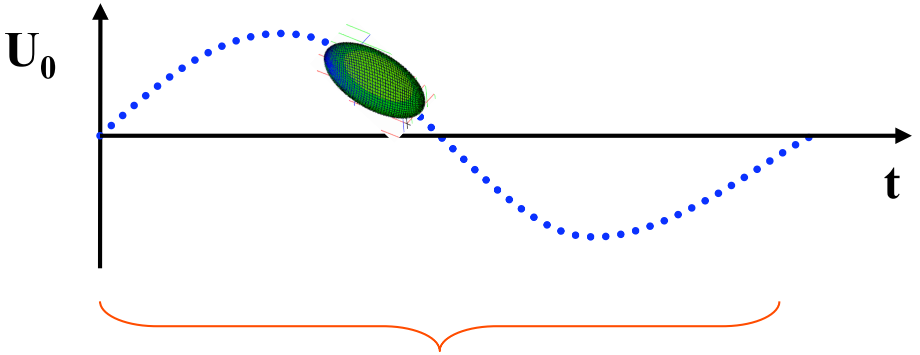
RF Acceleration-Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

just a stupid (and nearly wrong) example)



$\lambda = 75\ cm$

$$\nu = 400\,MHz$$
$$c = \lambda\,\nu$$
$$\lambda = 75\ cm$$

$$\sin(90^o) = 1$$
$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$$

22.) Dispersive and Chromatic Effects: $\Delta p/p \neq 0$

Are there any Problems ???

Sure there are !!!

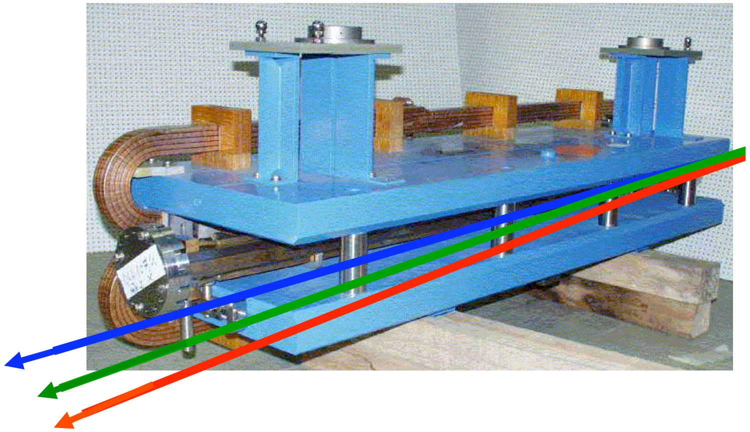
font colors due to pedagogical reasons

Dispersion and Chromaticity:
Magnet Errors for Δp/p ≠ 0

***Influence of external fields on the beam:** prop. to magn. field & prop. zu 1/p*

dipole magnet

$$\alpha = \frac{\int B\,dl}{p/e}$$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens

$$k = \frac{g}{p/e}$$

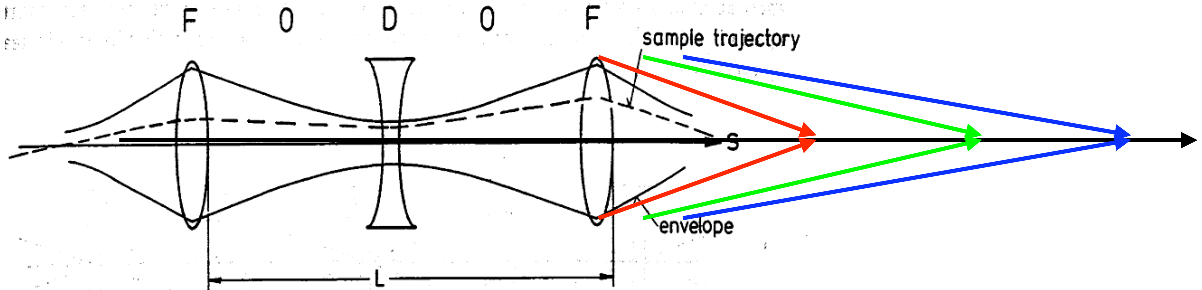
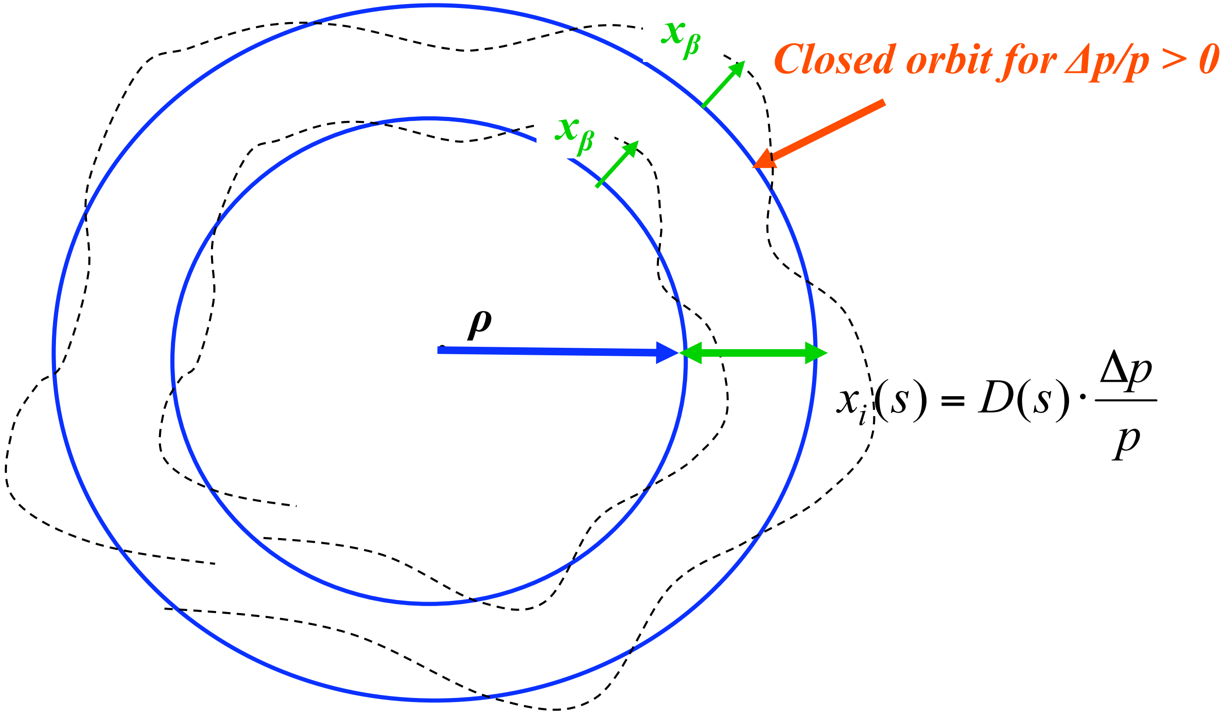


Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

Dispersion

Example: homogeneous dipole field



$$x_i(s) = D(s) \cdot \frac{\Delta p}{p}$$

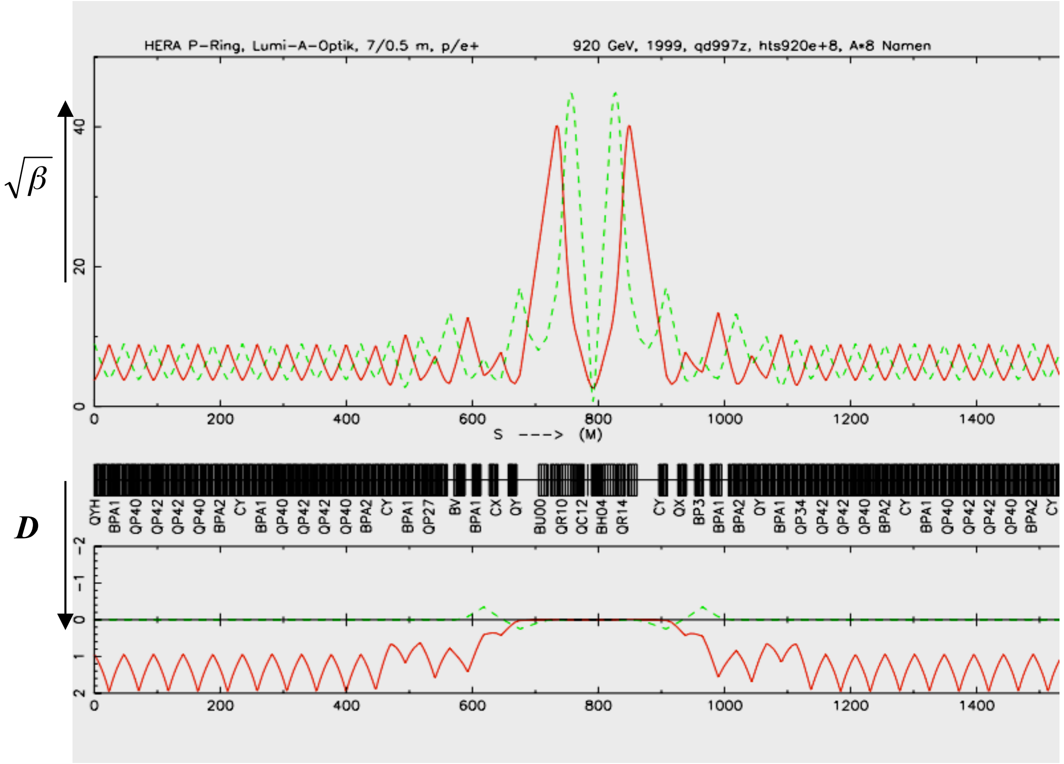
Matrix formalism:

$$\left.\begin{aligned} x(s) &= x_\beta(s) + D(s) \cdot \frac{\Delta p}{p} \\ x(s) &= C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p} \end{aligned}\right\}$$

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x'} \end{pmatrix}_s = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{S} \\ \boldsymbol{C'} & \boldsymbol{S'} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x'} \end{pmatrix}_0 + \frac{\boldsymbol{\Delta p}}{\boldsymbol{p}} \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{D'} \end{pmatrix}_0$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \ldots 2\,mm$$
$$D(s) \approx 1 \ldots 2\,m$$
$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

→ Dispersion must vanish at the collision point

!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}$$

23.) Chromaticity:
A Quadrupole Error for Δp/p ≠ 0

Influence of external fields on the beam: prop. to magn. field & prop. zu 1/p

focusing lens $$k = \frac{g}{p/e}$$

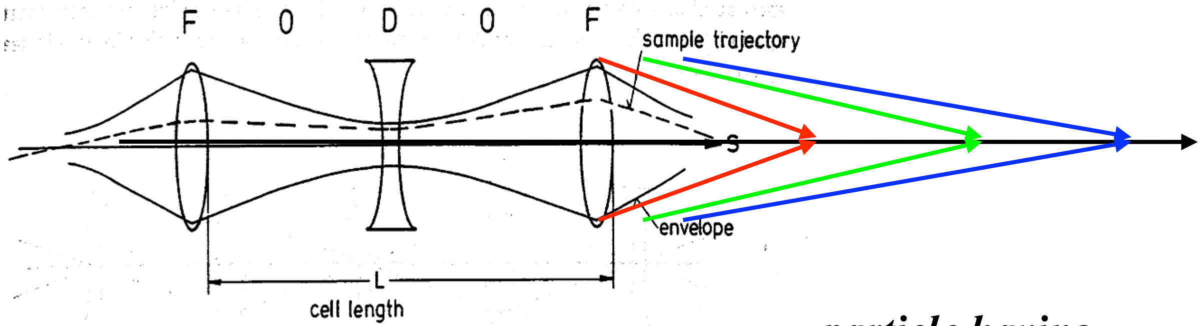


Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

... which acts like a quadrupole error in the machine and leads to a tune spread:

$$\Delta Q = -\frac{1}{4\pi}\frac{\Delta p}{p_0} k_0 \beta(s) ds$$

definition of chromaticity:

$$\boxed{\Delta Q = Q' * \frac{\Delta p}{p}}$$

… what is wrong about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' ***is a number indicating the size of the tune spot in the working diagram,***
Q' ***is always created if the beam is focussed***
→ it is determined by the focusing strength k of all quadrupoles

$$\boxed{Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds}$$

k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

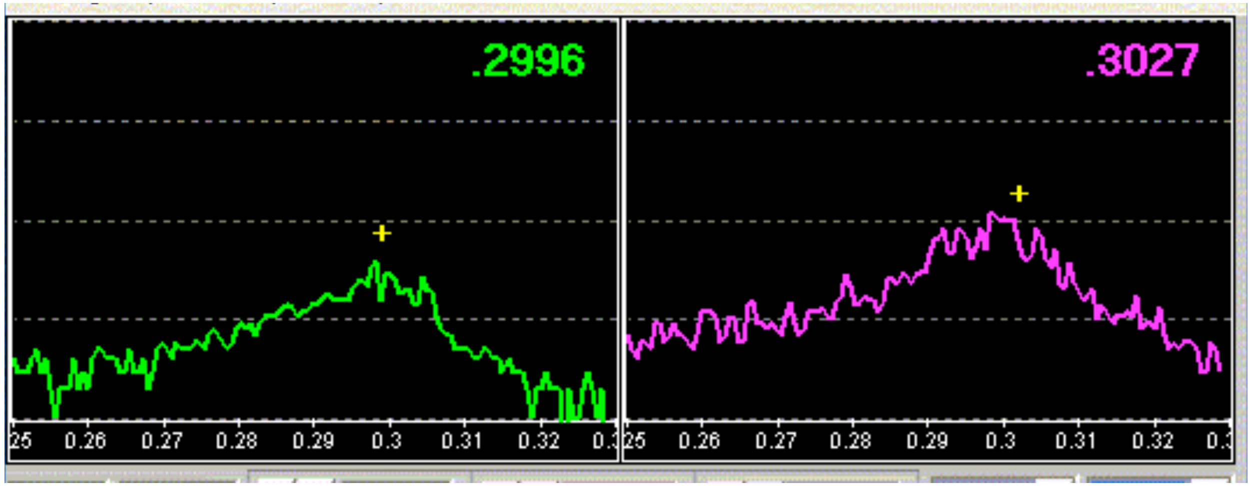
Example: LHC

Q' = 250
Δ p/p = +/- 0.2 *10⁻³

$Q' = 250$
$\Delta p/p = +/- 0.2 *10^{-3}$
$\Delta Q = 0.256 \dots 0.36$

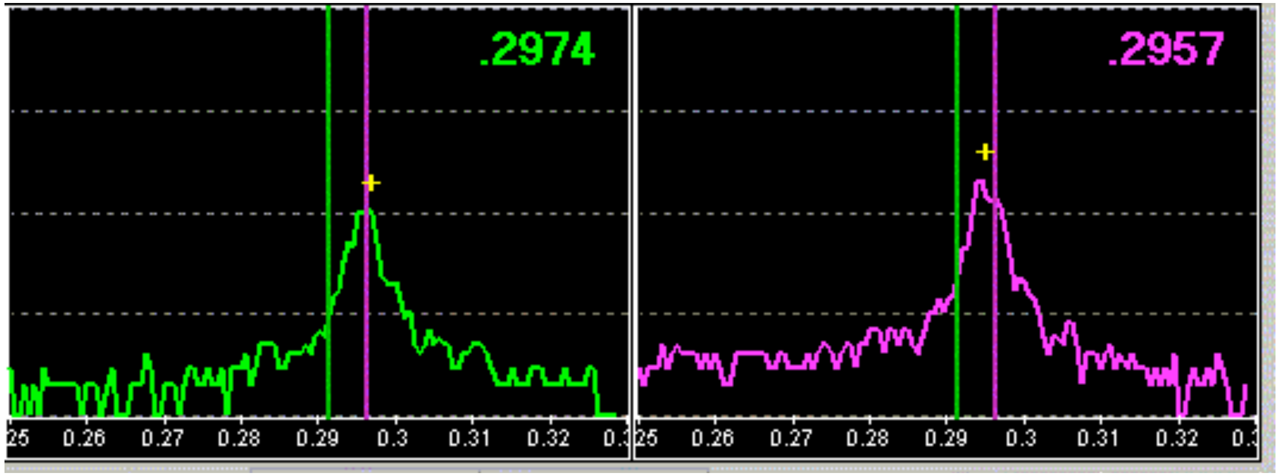
$\Bigg\}$ ***→Some particles get very close to resonances and are lost***

in other words: the tune is not a point
it is a pancake



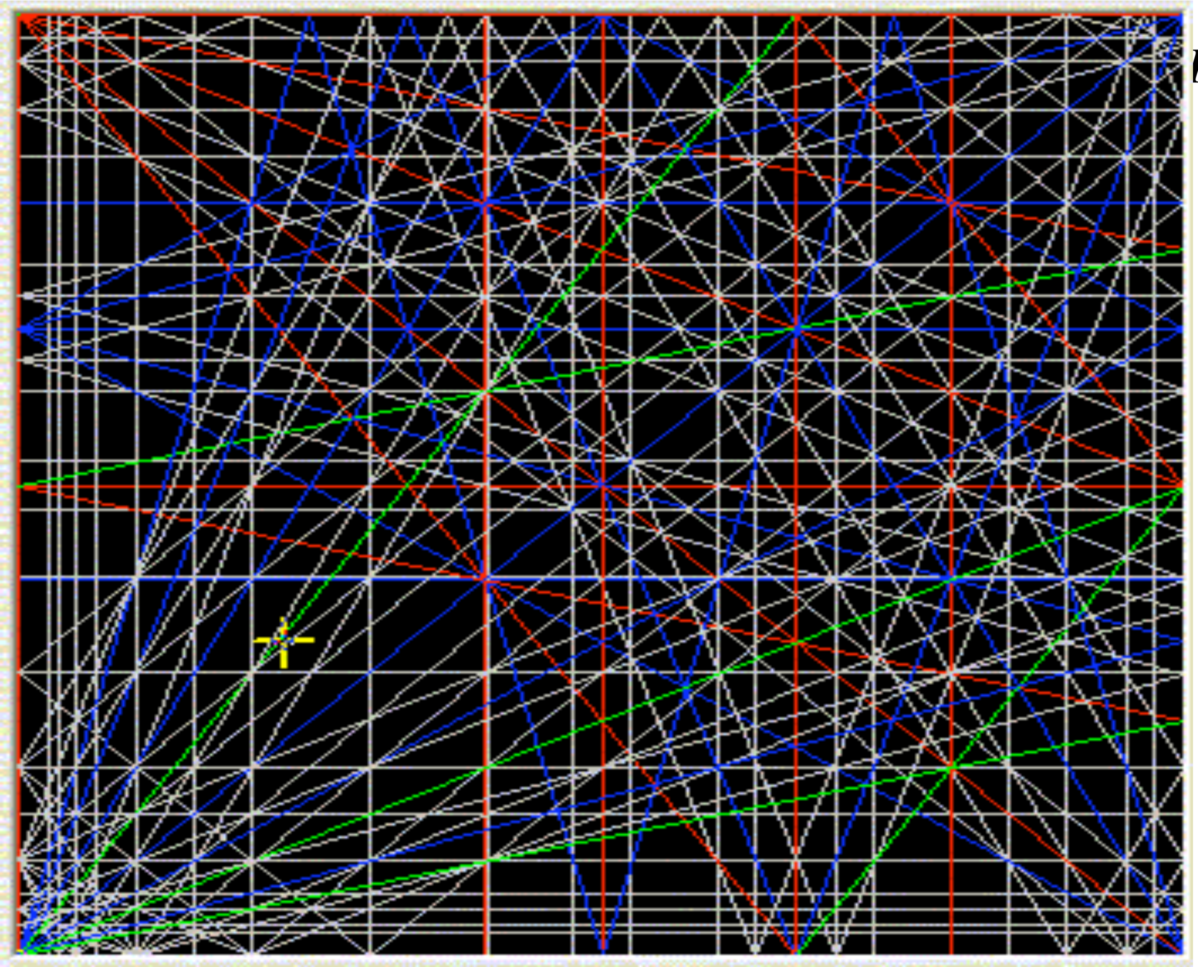
Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)



Tune and Resonances

$$m*Q_x+n*Q_y+l*Q_s = integer$$



RA e Tune diagram up to 3rd order

… and up to 7th order

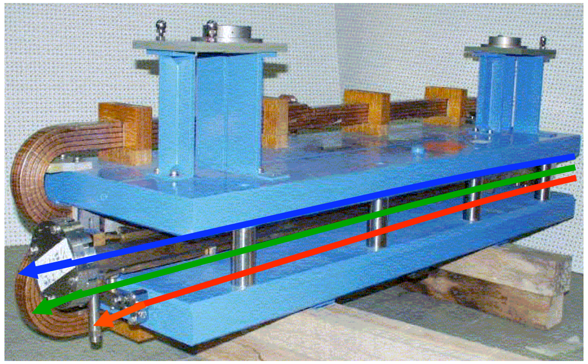
Homework for the operateurs:
find a nice place for the tune
where against all probability
the beam will survive

Correction of Q':

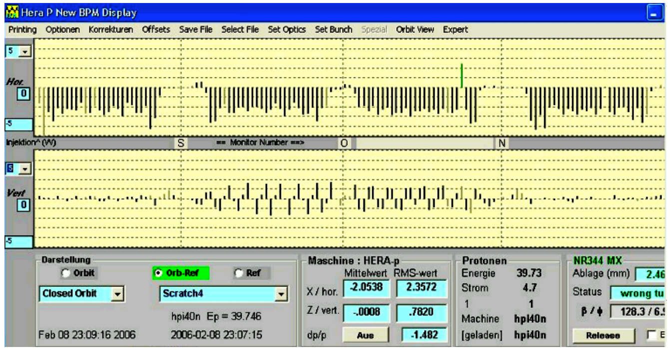
Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum

$$x_D(s) = D(s)\frac{\Delta p}{p}$$



… using the dispersion function



Darstellung

○ Orbit ● Orb-Ref ○ Ref

Closed Orbit

Scratch4

hpi40n Ep = 39.746

Feb 08 23:09:16 2006

2006-02-08 23:07:15

Maschine : HERA-p

	Mittelwert	RMS-wert
X / hor.	-2.0538	2.3572
Z / vert.	-.0008	.7820
dp/p	Aus	-1.482

Protonen

Energie	39.73
Strom	4.7
1	1
Machine	hpi40n
[geladen]	hpi40n

NR344 MX

Ablage (mm) 2.46

Status wrong tu

β / φ 128.3 / 6

2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

linear rising „gradient“:

Correction of Q':

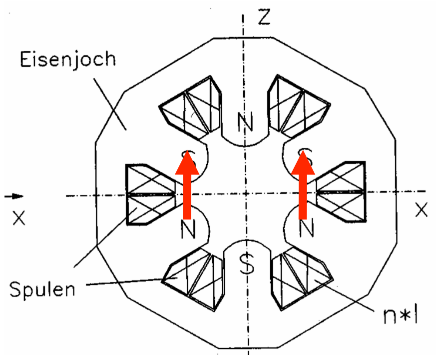
k_1 normalised quadrupole strength

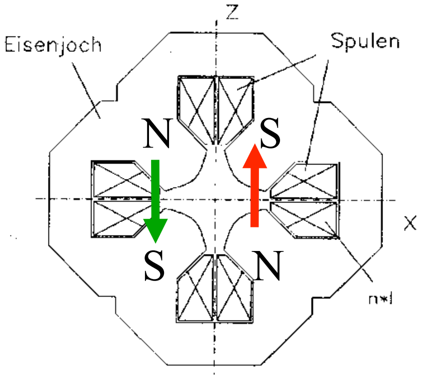
k_2 normalised sextupole strength

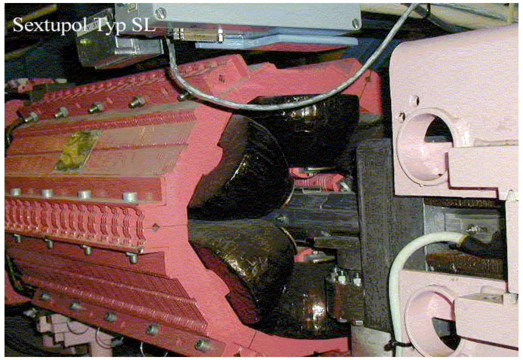
$$k_1(sext) = \frac{\tilde{g}\,x}{p/e} = k_2 * x$$

$$k_1(sext) = k_2 * D * \frac{\Delta p}{p}$$

Sextupole Magnets:







corrected chromaticity

considering a single cell:

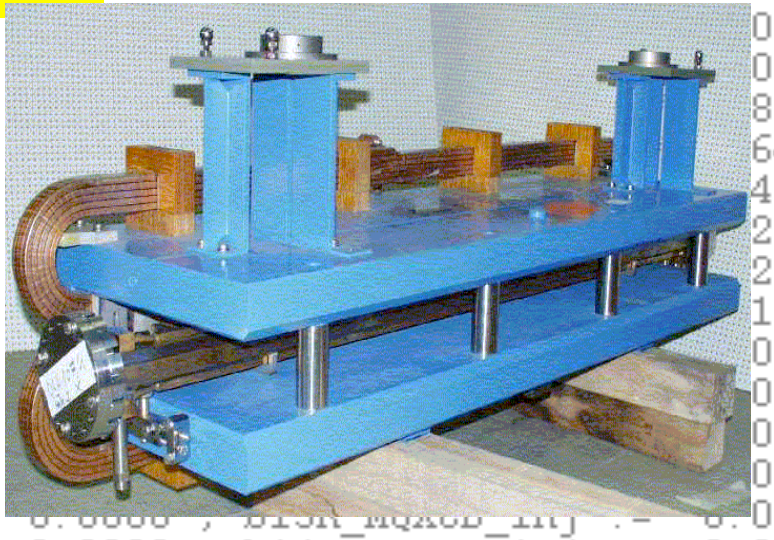
$$Q'_{cell\_x} = -\frac{1}{4\pi}\left\{k_{qf}\hat{\beta}_x l_{qf} - k_{qd}\breve{\beta}_x l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

$$Q'_{cell\_y} = -\frac{1}{4\pi}\left\{-k_{qf}\breve{\beta}_y l_{qf} + k_{qd}\hat{\beta}_y l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

Are there any Problems ???
sure there are !!!

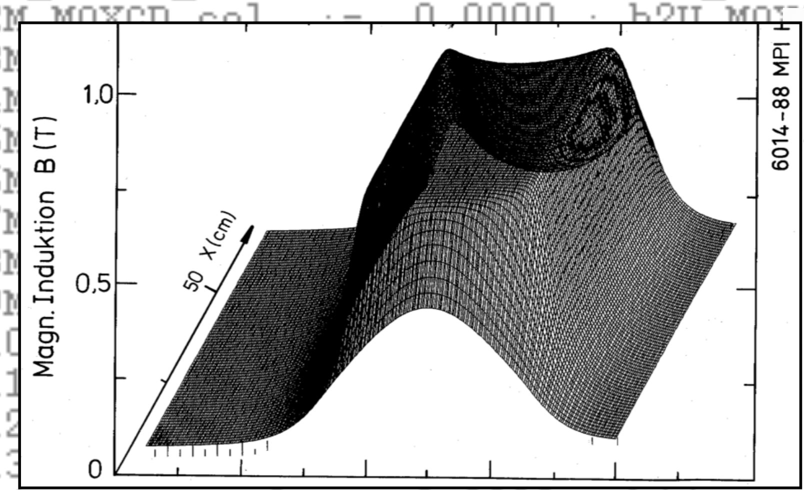
```
-----------------------------
et Quad)***************
-----------------------------
b1M_MQXCD_inj   :=  0.0000 ; b1U_MQXCD_inj   :=                              0000
b2M_MQXCD_inj   :=  0.0000 ; b2U_MQXCD_inj   :=                              0000
b3M_MQXCD_inj   :=  0.0000 ; b3U_MQXCD_inj   :=                              8900
b4                                           :=                              6400
b5                                           :=                              4600
b6                                           :=                              2800
b7                                           :=                              2100
b8M_MQXCD_inj   :=  0.0000 ; b8U_MQXCD_inj   :=                              1600
b9M_MQXCD_inj   :=  0.0000 ; b9U_MQXCD_inj   :=                              0800
b10M_MQXCD_inj  :=  0.5000 ; b10U_MQXCD_inj  :=                              0600
b11M_MQXCD_inj  :=  0.0000 ; b11U_MQXCD_inj  :=                              0300
b12                          12U_MQXCD_inj   :=                              0200
b13                          13U_MQXCD_inj   :=                              0100
b14M_MQXCD_inj  := -0.2700 ; b14U_MQXCD_inj  :=  0.0300 ; b14R_MQXCD_inj  :=  0.0100
b15M_MQXCD_                  b15U_MQXCD_inj  :=  0.0000 ; b15R_MQXCD_inj  :=  0.0000

!                            bn in collision
b1M_MQXCD_c                  b1U_MQXCD_col   :=  0.0000 ; b1R_MQXCD_col   :=  0.0000
b2M_MQXCD_col   :=  0.0000 ; b2U_MQ              0.0000 ; b2R_MQXCD_col   :=  0.0000
b3
b4
b5
b6
b7
b8
b9
b10
b11
b12
b13
b14M_MQXCD_col  :=  0.0000 ; b14U_M
b15M_MQXCD_col  :=  0.0000 ; b15U_MQXCD_col  :=  0.0000 ; b15R_MQXCD_col  :=  0.0000
```

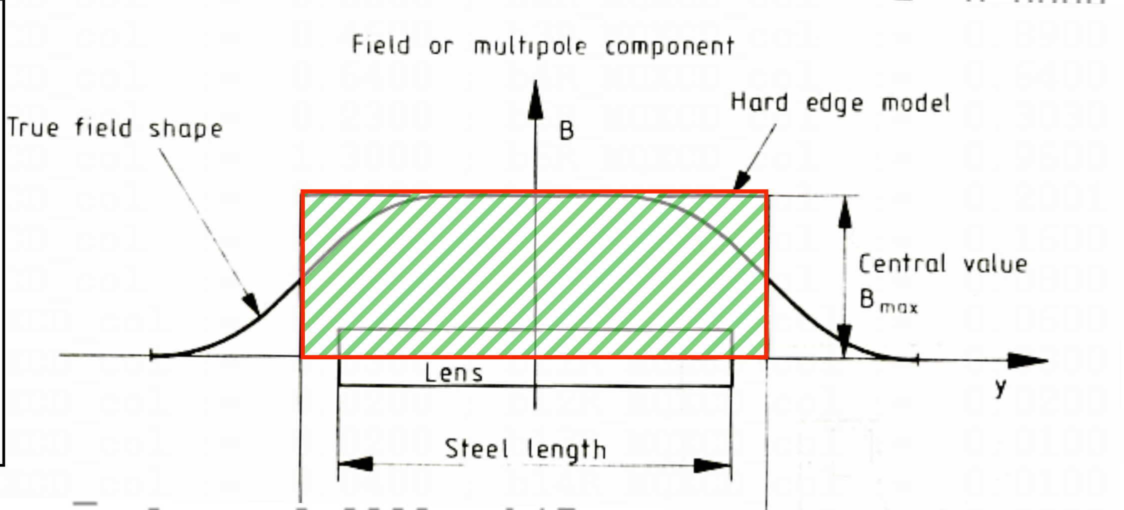
$$B_y + iB_x = B_{ref} * \sum_{n=1}^{\infty} \left(b_n + ia_n\right) \left(\frac{x+iy}{r_0}\right)^{n-1}$$



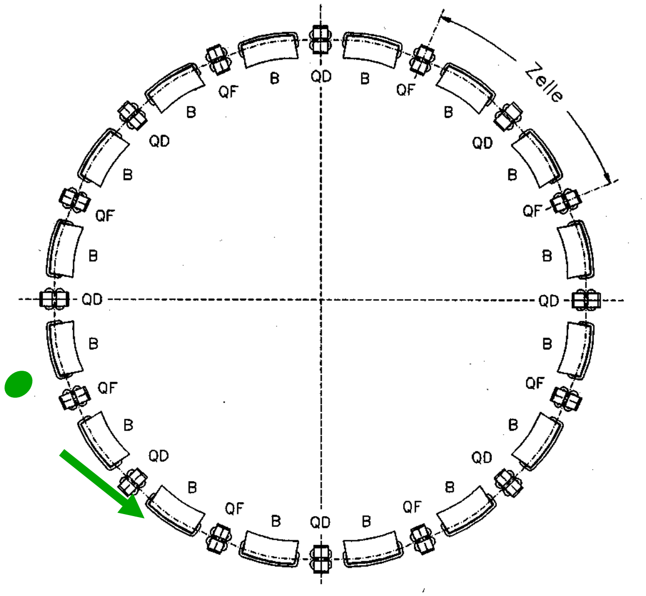
"effective magnetic length"

$$B * l_{eff} = \int_0^{l_{mag}} B\,ds$$





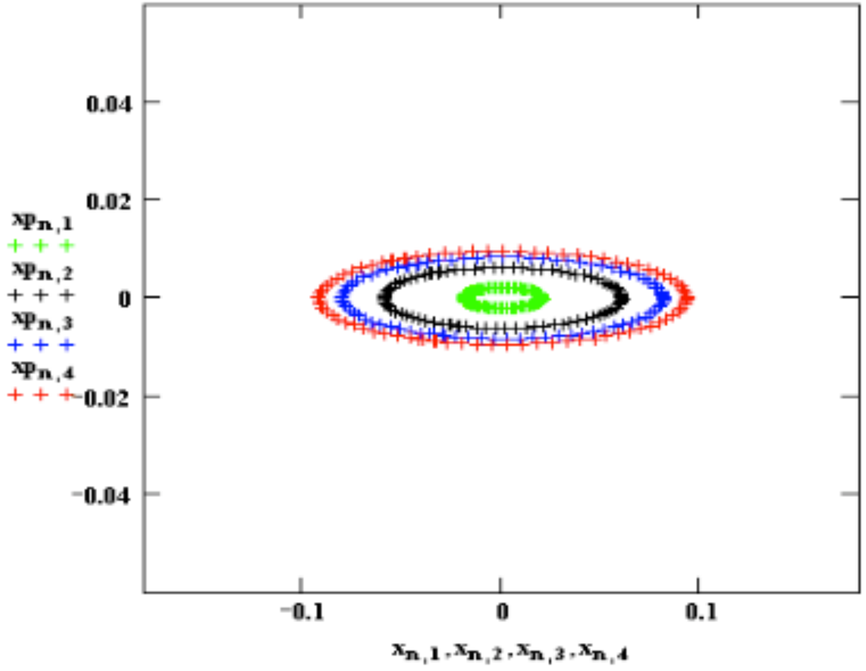
Clearly there is another problem ...
... if it were easy everybody could do it



Again: the phase space ellipse

for each turn write down – at a given position „s“ in the ring – the single partilce amplitude x and the angle x´... and plot it.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{turn} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

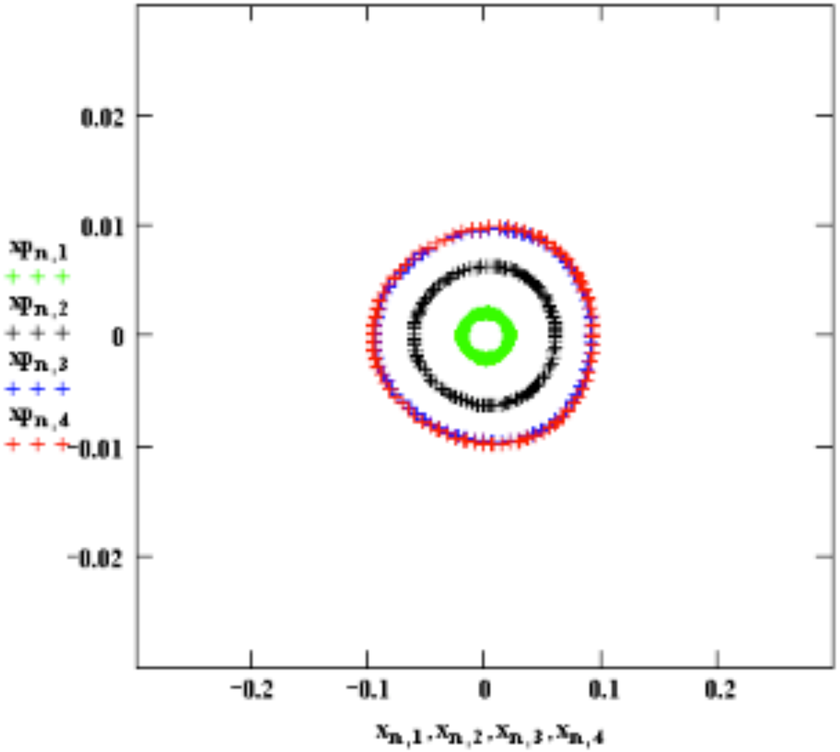


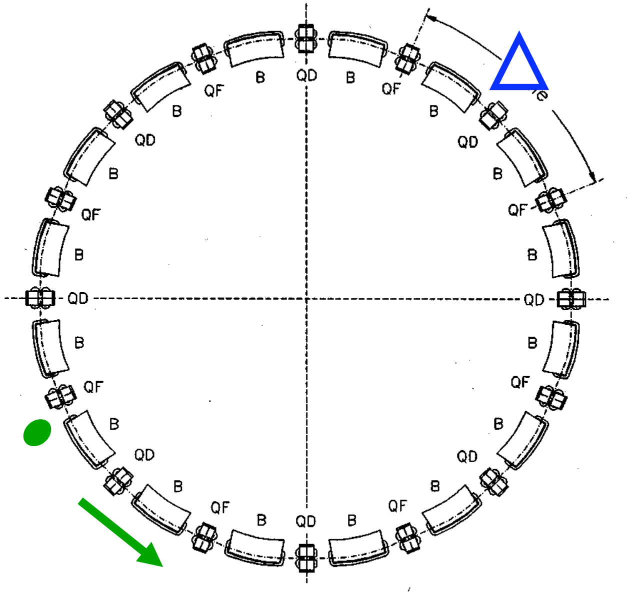
A beam of 4 particles

– each having a slightly different emittance:

Installation of a weak (!!!) sextupole magnet

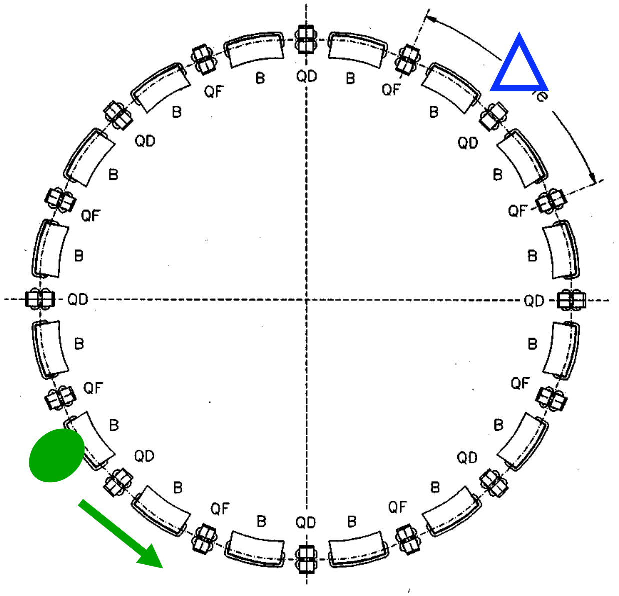
The good news: sextupole fields in accelerators
cannot be treated analytically anymore.
→ no equatiuons; instead: Computer simulation
„ particle tracking "

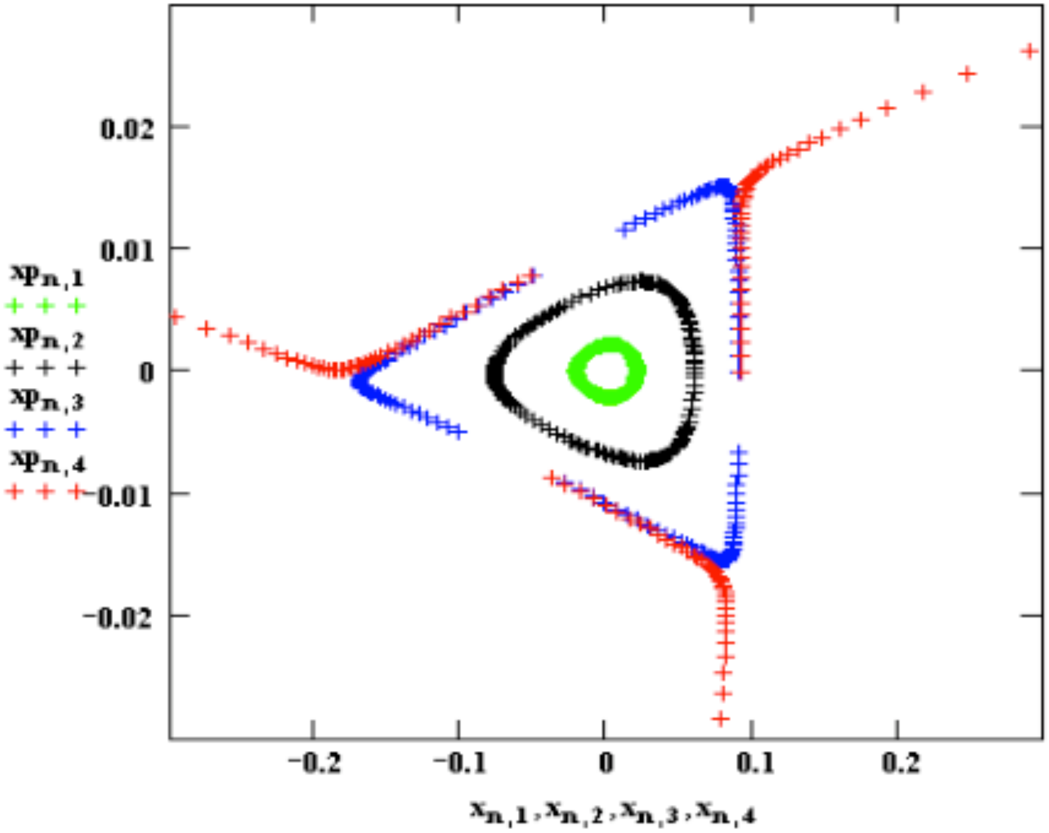




Effect of a strong (!!!) Sextupole …

→ *Catastrophy !*



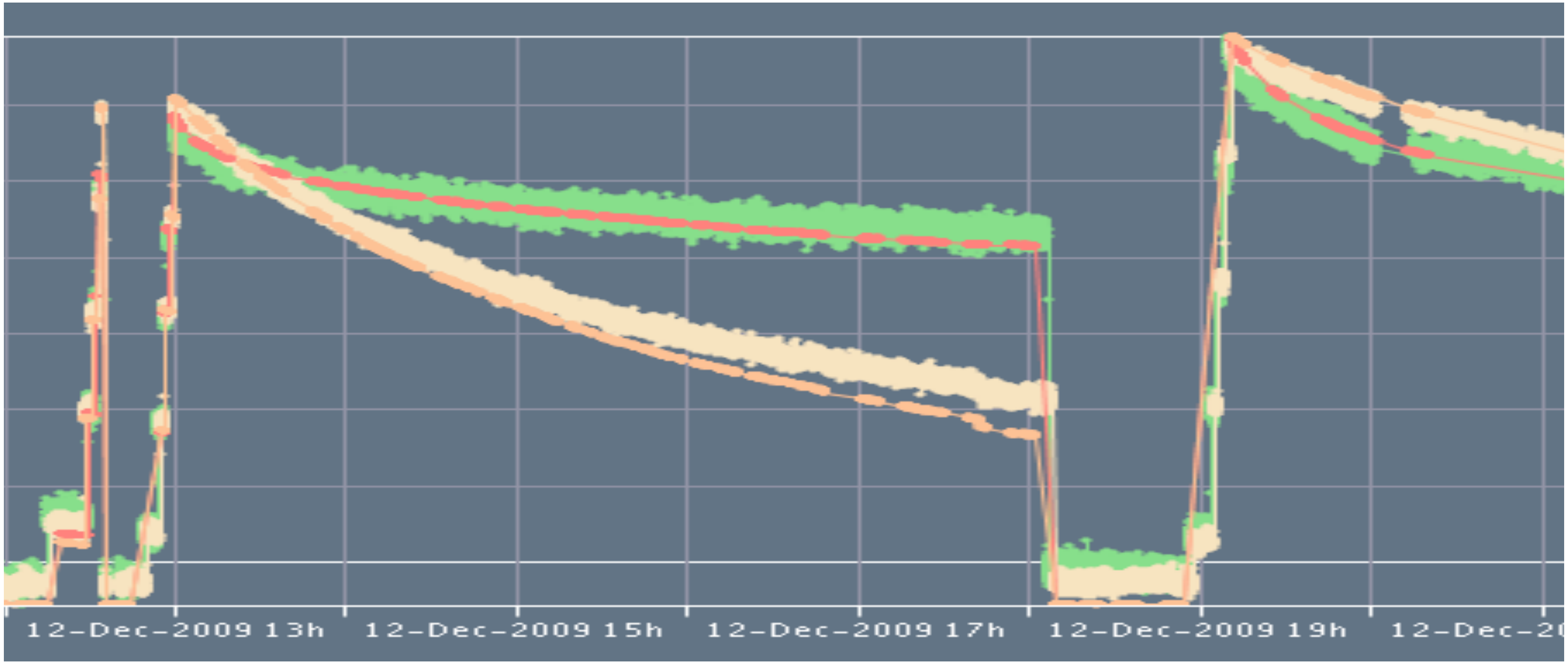


„dynamic aperture“

Golden Rule: COURAGE

... somehow and unexpectedly

these machines are running nevertheless.

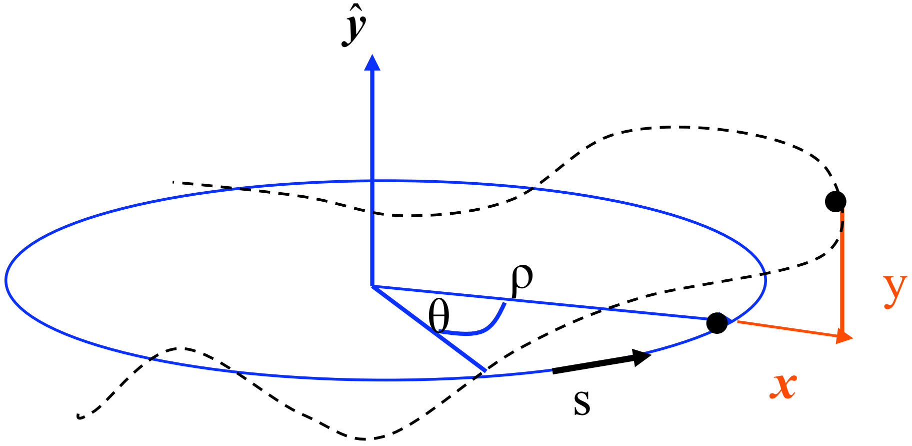


thank'x for your attention

Accelerator Physics is exciting!

- We already know a lot, but many open issues

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

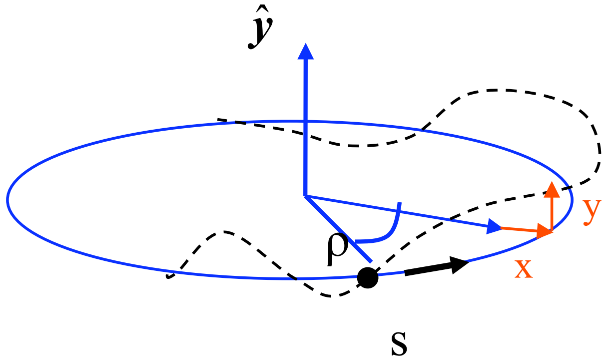
$$F = mv^2 / \rho$$

general trajectory: $\rho \rightarrow \rho + x$

$$\boldsymbol{F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} \;=\; e\,B_y\,v}$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

① ②



① $$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$$ *… as ρ = const*

② *remember: x ≈ mm , ρ ≈ m … → develop for small x*

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}(1-\frac{x}{\rho})$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \quad \ldots$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_y v$$

guide field in linear approx.

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad \Bigg| \;:\; m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

independent variable: $t \rightarrow s$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''\,v^2 + \cancel{\frac{dx}{ds}\frac{dv}{ds}}\,v$$

$$x''v^2 - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \qquad \Bigg| \;:\; v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e\ B_0}{mv} + \frac{e\ x\,g}{mv}$$

$m\ v = p$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\,g}{p/e}$$

normalize to momentum of particle

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

$$\frac{g}{p/e} = k$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\ x$$

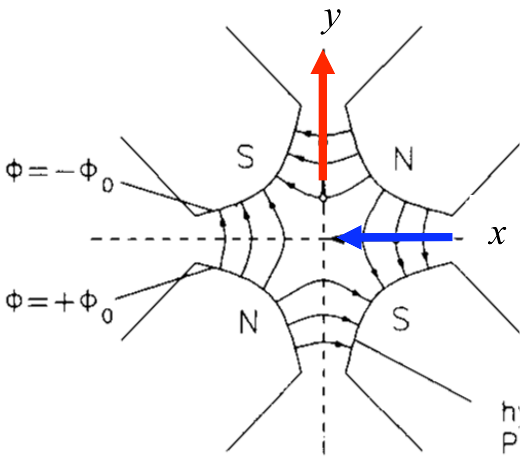
$$\boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = 0}$$

* ***Equation for the vertical motion:***

$\frac{1}{\rho^2} = 0$ *no dipoles … in general …*

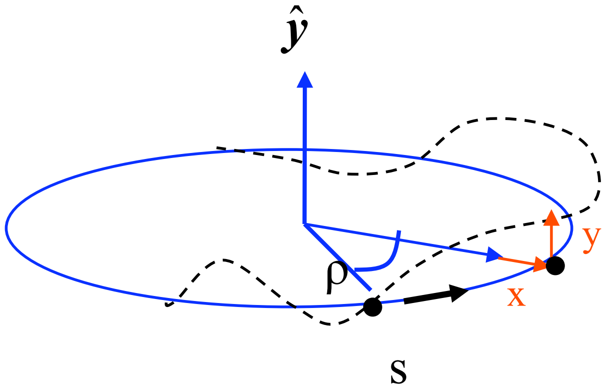
$k \quad \leftrightarrow \quad -k$ *quadrupole field changes sign*

$$y'' + k\ y = 0$$



16.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: do you remember last session, page 12 ? … sure you do



Force acting on the particle

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

remember: x ≈ mm , ρ ≈ m … → develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_y v$$

consider only linear fields, and change independent variable: $t \rightarrow s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = \frac{e\ B_0}{mv} + \frac{e\ x\,g}{mv}$$

$p=p_0+\Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\ B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k * x} - \underbrace{xeg \frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} \frac{1}{\rho} \longrightarrow \boxed{x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p_0} \frac{1}{\rho}}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
→ inhomogeneous differential equation.

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\left\{ \begin{array}{l} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{array} \right.$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

*** is that special orbit, an ideal particle would have for Δp/p = 1***

*** the orbit of any particle is the sum of the well known x_β and the dispersion***

*** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***