Introduction to the Standard Model

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Fermions	1^{st} generation	2^{nd} generation	3^{rd} generation
$\mathrm{Q} \left\{ \begin{matrix} \mathrm{U} \\ \mathrm{D} \end{matrix} \right.$	$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{matrix} u_R \\ d_R \end{matrix}$	$\begin{pmatrix} c \\ s \end{pmatrix}_L \quad \begin{matrix} c_R \\ s_R \end{matrix}$	$\begin{pmatrix} t \\ b \end{pmatrix}_L \quad \begin{matrix} t_R \\ b_R \end{matrix}$
$\mathrm{L} \left\{ \begin{matrix} \mathrm{E} \\ \mathrm{N} \end{matrix} \right.$	$\begin{pmatrix} e \\ \nu_e \end{pmatrix}_L \quad e_R$	$\begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}_L \quad \mu_R$	$\begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}_L \quad \tau_R$

Gauge bosons | γ , $W^{\pm}$, Z , G^a

plus the Higgs boson H.

$$\mathcal{L}_{\rm SM} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm D} + \mathcal{L}_{\rm Yukawa} + \mathcal{L}_{\rm Higgs}$$

The kinetic part of the gauge fields

The Yukawa sector (interactions between the Higgs doublet and fermions)

The Dirac fermions

Higgs dynamics and EWSB

In this course we shall not consider possible gauge-fixing and ghost field contributions (which may result from other choices of gauge)

Local Gauge Invariance

Let us consider transformations which do depend on the space-time coordinates ($\theta^a = \theta^a(x)$). One speaks in this case of local or gauge symmetries (Weyl 1929).

The advantage of gauge symmetries is that from a free theory invariant under global transformations it is possible to construct a theory invariant under local transformations (gauge transformations) by adding interaction terms and one or more vector fields (gauge fields).

How to introduce these terms is not arbitrary but the imposition of the invariance of the Lagrangian under gauge transformations allows us to "generate" interactions and introduce vector fields which are the mediators of forces.

Example: Electromagnetism

Consider, as a starting point, the Dirac equation for a free electron

$$\mathcal{L}_D = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

which is invariant under global $U(1)$ transformations:

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{-i\alpha}\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) &= e^{i\alpha}\bar{\psi}(x)\ . \end{aligned}$$

The corresponding local symmetry is:

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{-i\alpha(x)}\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) &= e^{i\alpha(x)}\bar{\psi}(x)\ . \end{aligned}$$

The mass term of the Lagrangian is invariant under the local transformation, but the derivative term is not:

$$\bar{\psi}(x)\partial_\mu\psi(x) \rightarrow \bar{\psi}(x)\partial_\mu\psi(x) - i\bar{\psi}(x)[\partial_\mu\alpha(x)]\psi(x)\ .$$

To offset this additional term one can define a covariant derivative with the property:

$$D_\mu\psi(x) \rightarrow e^{-i\alpha(x)}D_\mu\psi(x)$$

which provides an invariant term in the Lagrangian $\bar{\psi}(x)D_\mu\psi(x)$. The covariant derivative is obtained with the introduction of a vector field (gauge field) $a_\mu(x)$:

$$D_\mu\psi(x) = (\partial_\mu + iea_\mu)\psi(x)$$

where the gauge field transforms under $U(1)$ by:

$$a_\mu(x) \rightarrow a'_\mu(x) = a_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)\ .$$

The gauge field is not currently a dynamic field, it can be eliminated using the equation of motion. To make it physical we must add a kinetic term. A term which is gauge invariant and derived from the field a_μ which is also renormalisable such as $f_{\mu\nu}(x)f^{\mu\nu}(x)$ where

$$f_{\mu\nu}(x) = \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) \ .$$

With the usual normalisation for the kinetic term, the Lagrangian deduced from Dirac's Lagrangian with the application of local invariance is

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) - \frac{1}{4} f_{\mu\nu}(x) f^{\mu\nu}(x) \ ,$$

which is the Lagrangian of quantum electrodynamics (QED).

It may be noted that there is an absence of a mass term for the field a_μ. The reason for this is that the mass term $m a_\mu a^\mu$ is not gauge invariant. The photon is therefore massless.

Example: The Yang-Mills theory

Electromagnetism can be generalised (Yang and Mills 1954) to rotation by a phase where the phase is a matrix:

$$\psi \to S\psi$$

with S being a special unitary matrix, for example $S \in SU(2)$ and ψ a doublet. The invariance of physics in relation to local rotations of $SU(2)$

$$S(x) = e^{-i\theta^a(x)\sigma^a/2} \qquad a = 1, 2, 3$$

the σ^a being the Pauli matrices, can be done in analogy if we consider an infinitesimal transformation of $SU(2)$

$$S(x) \simeq 1 - i\frac{\theta^a(x)\sigma^a}{2} \ ,$$

where the transformation of the vector field $A_\mu(x)$ is

$$A^i_\mu(x) \to A^i_\mu(x) - \frac{1}{g}\partial_\mu \theta^i + \epsilon^{ijk}\theta^j(x) A^k_\mu(x) \, .$$

Compared to the Abelian case we have an ϵ^{ijk} term and A^i_μ transforms as a triplet in the adjoint representation of $SU(2)$. So the fields A^i_μ are charged against the charge of $SU(2)$ whilst for $U(1)$ we had a neutral field (the photon) compared to the charge of $U(1)$ (electric charge). The tensor $F_{\mu\nu}(x)$:

$$F^i_{\mu\nu}(x) = \partial_\nu A^i_\mu(x) - \partial_\mu A^i_\nu(x) + g\epsilon^{ijk} A^j_\mu(x) A^k_\nu(x)$$

is a triplet under the gauge transformation of $SU(2)$:

$$F^i_{\mu\nu}(x) \to F^i_{\mu\nu}(x) + \epsilon^{ijk}\theta^j(x)F^k_{\mu\nu} \ .$$

The tensor $F^i_{\mu\nu}(x)$ is not gauge invariant, however, the product

$$Tr\left[(\sigma^a F^a_{\mu\nu}(x))(\sigma^b F^{b\mu\nu}(x))\right] \propto F^i_{\mu\nu}(x)F^{i\mu\nu}(x) \ ,$$

which we will use in the Lagrangian, is invariant. In terms of some of the other differences with the Abelian theory is the presence of self-interaction terms for the gauge fields in the kinetic term.

Symmetry breaking

Before progressing with a closer analysis of this Lagrangian and its components, we first need some background theory. We shall start with a look at symmetries, where in quantum mechanics an exact (unbroken) symmetry T has the property of transforming the states of a system:

$$T : \phi \rightarrow \phi'$$

such that the transition probabilities do not change

$$|\langle \phi, \psi \rangle|^2 = |\langle \phi', \psi' \rangle|^2 \ .$$

The operator U of the transformation is unitary or anti-unitary, and in terms of an observable A

$$T : A \to A' = U A U^{-1} \ .$$

Such a transformation preserves the commutation relations and more general algebraic relations, especially any equations of motion in the theory do not change under the transformation T.

Conversely one may ask whether a symmetry of the equations of motion implies an exact symmetry.

The answer is yes for a system with a finite number of degrees of freedom.

If the number of degrees of freedom of the theory is infinite (as in field theory) the answer is no. The reason is the presence of nonequivalent representations of canonical commutation relations. The symmetry of the equations of motion may not give rise to transformations of system states which preserve the transition probability. One speaks in this case of spontaneously broken symmetries.

As an example consider a non-relativistic system in the limit of infinite volume (which allows for an infinite number of degrees of freedom). The Lagrangian of a scalar field ϕ in this case is

$$\mathcal{L} = i\phi^\dagger \frac{\partial \phi}{\partial t} - \frac{1}{2m} \frac{\partial \phi^\dagger}{\partial x_i} \frac{\partial \phi}{\partial x_i} \; .$$

The equations of motion correspond to the Schrödinger equation

$$i \frac{\partial \phi}{\partial t} + \frac{1}{2m} \Delta \phi = 0 \; .$$

The Lagrangian and the equations of motion are invariant under $U(1)$ the transformation

$$\phi \to e^{-i\theta} \phi \qquad\qquad \phi^\dagger \to e^{i\theta} \phi^\dagger \; .$$

The general solution of the Schrödinger equation is

$$\phi(\mathbf{x},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} a_{\mathbf{k}} e^{-i(\epsilon(k)t - i\mathbf{k}\cdot\mathbf{x})}$$

with the dispersion relation $\epsilon(k) = \dfrac{\mathbf{k}^2}{2m}$ and $\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}\right] = \delta_{\mathbf{k}\mathbf{k}'}$. The Hamiltonian and number operator are

$$H = \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2m} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \qquad N = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$$

with $[H,N]=0$, which expresses the conservation of particle number.

The fundamental state of n particles (the vacuum of our theory) is

$$|n\rangle = \frac{(a_0^\dagger)^n}{\sqrt{n!}}|0\rangle \ ,$$

where $a_0^\dagger = a_{\mathbf{k}=0}^\dagger$ and $|0\rangle$ is the vacuum for zero particles $a_{\mathbf{k}}|0\rangle = 0$. The vacuum $|n\rangle$ is unique, an eigenvector of the operator N. The vacuum explicitly has the $U(1)$ symmetry.

Now consider the limit $V \to \infty$ with a constant density of particles $\rho = N/V$. We will see that the symmetry $U(1)$ is spontaneously broken in this limit. To prove this just consider the vacuum state for the system.

A vacuum in the limit of infinite volume is

$$|\theta\rangle = \exp(-n/2)\ \exp(\sqrt{n}e^{i\theta}a_0^\dagger)|0\rangle$$

and under the $U(1)$ symmetry this is not invariant

$$|\theta\rangle \to U(\alpha)|\theta\rangle = |\theta + \alpha\rangle$$

where the unitary operator of transformation is

$$U(\alpha) = e^{i\alpha N}\ .$$

To be convinced that this represents a vacuum for the theory it suffices to verify some properties: First, because the relation $H|\theta\rangle = 0$ for any $|\theta\rangle$ all these statements have the same energy $E = 0$.

However, there are no eigenvectors of the operator N, but the average number of particles corresponds to n:

$$\langle\theta|N|\theta\rangle = n$$

all these states $|\theta\rangle$ are orthogonal and normalised in the limit of infinite volume (to keep the density ρ constant n must also tend to infinity):

$$\langle\alpha|\theta\rangle = \exp\{n[\cos(\theta-\alpha) - 1 + i\sin(\theta-\alpha)]\} \rightarrow \delta_{\theta\alpha}$$

when $n \rightarrow \infty$.

The presence of degenerate vacuums implies the existence of excitations in the zero energy of the system, a result related to the dispersion relationship $\epsilon(k) \to 0$ when $k \to 0$.

In the physics of a relativistic system, the dispersion relation is determined by the Poincaré transformations

$$\epsilon(k) = \sqrt{k^2 + m^2}$$

and $\epsilon(k) \to 0$ when $k \to 0$ behaviour is possible only for a massless particle. This result is known as the Goldstone theorem.

But more on that later.

Spontaneously broken discrete symmetries

We have seen that the basis of spontaneous symmetry breaking was that an invariance of the theory (in the Lagrangian) leads to a ground state of the theory (the vacuum) which is degenerate.

That is, when it is transformed under the symmetry group it is not invariant.

One of the simplest examples is the Lagrangian a real scalar field ϕ invariant under parity transformations:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi^2)$$

with

$$P: \quad \phi \to -\phi \qquad\qquad P^2 = 1\ .$$

For the potential $V(\phi^2)$ we choose the form:

$$V(\phi^2) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \ .$$

In order to have a lower limit on the potential (that is, an energy requirement for the system) λ must be a positive constant. If μ^2 is positive the potential has its minimum at $\phi = 0$, and as a results of that the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\partial_0\phi\partial^0\phi + \frac{1}{2}\partial_i\phi\partial^i\phi + V(\phi^2)$$

commutes with the parity operator

$$P|0\rangle = |0\rangle \ .$$

As such $P|0\rangle$ and $|0\rangle$ have the same energy and coincide. Therefore the vacuum state of the scalar field is zero:

$$\langle 0|\phi|0\rangle = \langle 0|P^{-1}P\phi P^{-1}P|0\rangle = \langle 0|P\phi P^{-1}|0\rangle = -\langle 0|\phi|0\rangle \ ,$$

this being the only possible solution: $\langle 0|\phi|0\rangle = 0$.

If $\mu^2 < 0$ though, the potential V has two minima for

$$\phi = \pm\sqrt{\frac{-\mu^2}{\lambda}} \equiv \pm v \ .$$

If we call $|D\rangle$ and $|G\rangle$ the two quantum states that correspond to the standard configuration $\phi = \pm v$, parity operations are a skip from one to the other

$$P|D\rangle = |G\rangle \neq |D\rangle \ .$$

The expectation value of the vacuum $|D\rangle$ or $|G\rangle$ of the scalar field:

$$\langle D|\phi|D\rangle = \langle D|P^{-1}P\phi P^{-1}P|D\rangle = -\langle G|\phi|G\rangle$$

are no longer necessarily zero. It has a Lagrangian which is symmetric and two degenerate vacuum state which are not, the parity is spontaneously broken.

Our system therefore has a potential which is a double well. It can be surprising to obtain two degenerate vacuum solutions when quantum tunnelling could remove this degeneracy.

The difference between the result of quantum mechanics and that of field theory is due to the infinite number of degrees of freedom in the second case.

To see this behaviour in detail we will consider a double well potential in quantum mechanics and take the limit to infinite volume.

The tunnelling gives a transition probability between the non-zero $|D\rangle$ and $|G\rangle$, where the Hamiltonian has the form

$$\begin{pmatrix} E & \epsilon \\ \epsilon & E \end{pmatrix}$$

and two eigenvalues

$$\lambda_1 = E - \epsilon \qquad \lambda_2 = E + \epsilon$$

corresponding to the eigenvectors

$$|1\rangle = \frac{1}{\sqrt{2}}\left(|D\rangle - |G\rangle\right) \text{ and } |2\rangle = \frac{1}{\sqrt{2}}\left(|D\rangle + |G\rangle\right)$$

respectively. The degeneracy is removed and the vacuum of the theory is $|1\rangle$ with energy λ_1.

If at time $t = 0$ we are in the minimum of the potential, the evolution time t given by quantum mechanics is

$$|D\rangle = \frac{1}{\sqrt{2}} e^{-i\lambda_2 t}\left(|2\rangle + e^{2it\epsilon}|1\rangle\right)$$

and the period of oscillation between the two minima of the potential is $T = \pi/\epsilon$. For our scalar potential, the width of the barrier is fixed and is $2v$, the distance between two minima.

The height of the barrier is the energy difference between the maximum potential $\phi = 0$ and a minimum in $\phi = \pm v$:

$$\begin{aligned} H(\phi = 0) - H(\phi = v) &= -\int_{\mathrm{Vol}} d^3x \, \left(\frac{\mu^2}{2} v^2 + \frac{\lambda}{4} v^4 \right) \\ &= \frac{\mu^4}{4\lambda} \int_{\mathrm{Vol}} d^3x = \frac{\mu^4}{4\lambda} \mathrm{Vol}\,. \end{aligned}$$

H is the Hamiltonian of the scalar field, which is obtained by integrating the Hamiltonian density. In the infinite volume limit the height of the potential is infinite and the difference in the energy is $2\epsilon \to 0$.

The transition between the two states $|D\rangle$ and $|G\rangle$ is impossible in the limit of infinite volume and this allows us to have the spontaneous breaking of symmetry.

Spontaneously broken continuous symmetries

As we build up to the Goldstone theorem, consider a scalar theory with an $O(3)$ symmetry

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \; .$$

The notation is compact, $\phi \equiv (\phi_1, \phi_2, \phi_3)$ is a vector of $O(3)$ and ϕ^2 is the scalar product $\phi \cdot \phi$, the fourth power of ϕ means $\phi^4 = (\phi \cdot \phi)^2$.

An infinitesimal rotation through the angle θ in the direction of the vector n (with $|n|^2 = 1$) can be written as

$$\phi \rightarrow \phi + \theta \, \phi \wedge n \; .$$

Since a rotation leaves the length of a vector invariant, for an infinitesimal rotation we can write

$$|\phi|^2 \longrightarrow |\phi + \delta\phi|^2 = |\phi|^2 + 2\phi \cdot \delta\phi + \mathcal{O}(\delta\phi^2)$$

and conclude that ϕ and $\delta\phi$ are orthogonal $\phi \cdot \delta\phi = 0$ in order to keep the vector invariant.

By definition of the vector product a rotation around the direction n, $\delta\phi$ must also be orthogonal to n, as follows by comparing the above formulas

$$\delta\phi = \theta\ \phi \wedge n\ .$$

For example if $n \equiv (0,\ 0,\ 1)$ one finds

$$\delta\phi_1 = \theta\ \phi_2 \qquad \delta\phi_2 = -\theta\ \phi_1 \qquad \delta\phi_3 = 0\ .$$

The minimum of the potential is given by

$$\frac{\partial V}{\partial \phi_i} = \mu^2 \phi_i + \lambda \phi_i |\phi|^2 = 0$$

with two possible solutions $\phi_i = 0$, or $|\phi|^2 = v^2$ with $v = \sqrt{\frac{-\mu^2}{\lambda}}$.

The minimum is found by examining the second derivative

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_j} = \delta_{ij}(\mu^2 + \lambda |\phi|^2) + 2\lambda \phi_i \phi_j \ .$$

According to the sign of μ^2 we have the following two possibilities:

$$\begin{aligned} \mu^2 &> 0 & \qquad \phi = 0 \,, \\ \mu^2 &< 0 & \qquad |\phi|^2 = v^2 \,. \end{aligned}$$

If $\mu^2 > 0$ we have a single real minimum $\phi = 0$.

In the case $\mu^2 < 0$ we have an infinite number of degenerate minima, the points on the sphere $|\phi|^2 = v^2$.

By choosing one of these points, for example $\phi_i = \delta_{i3} v$, we can be develop an expansion around the minimum

$$V(\phi) = V|_{min} + \frac{1}{2} \frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\bigg|_{min} (\phi_i - \delta_{i3}\, v)\,(\phi_j - \delta_{j3}\, v)$$

and use the differences $(\phi_i - \delta_{i3} v)$ as new fields to be treated as physical around that minimum. The previous formula indicates the mass of the field after breaking the $O(3)$ symmetry:

$$M_{ij}^2 = \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{min} = -2\mu^2 \delta_{i3}\delta_{j3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\mu^2 \end{pmatrix} .$$

So the masses of the fields ϕ_1 and ϕ_2 are zero, by the conservation properties of the field $\chi = \phi_3 - v$ is nonzero:

$$m_{\phi_1}^2 = m_{\phi_2}^2 = 0 , \qquad\qquad m_\chi^2 = -2\mu^2 .$$

The potential in terms of the new fields shows explicitly how the $O(3)$ symmetry is broken:

$$
\begin{aligned}
V &= -\frac{m_\chi^4}{16\lambda} + \frac{1}{2} m_\chi^2 \chi^2 + \sqrt{\frac{m_\chi^2 \lambda}{2}} \left(\phi_1^2 + \phi_2^2 + \chi^2\right) \chi \\
&\quad + \frac{\lambda}{4} \left(\phi_1^2 + \phi_2^2 + \chi^2\right)^2 \ .
\end{aligned}
$$

It may be noted that the Lagrangian has a residual $O(2)$ symmetry, because V depends only on the combination $\phi_1^2 + \phi_2^2$ which is invariant for rotations around the axis $(0,\ 0,\ v)$.

This potential is not usually possible with the $O(2)$ symmetry, the spontaneous breaking of the $O(3)$ symmetry imposes constraints on the shape of the Lagrangian. It was also shown that we obtained a theory with two scalar bosons without mass corresponding to the symmetry breaking by the two axes 1 and 2. This has a correspondence in terms of the generators of $O(3)$:

$$T_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \quad T_2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad T_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

compared to their action on the vacuum (the state minimum we chose)

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} .$$

The vacuum is not invariant for rotations around the axes 1 and 2, whilst the $O(2)$ invariance remains with respect to rotations around axis 3:

$$T_1|0\rangle \neq 0 \qquad T_2|0\rangle \neq 0 \qquad T_3|0\rangle = 0 .$$

The Goldstone theorem

In general, if a group with an internal symmetry G is broken spontaneously into a group $H \subset G$ which corresponds to a symmetry of the vacuum state, the number of Goldstone bosons is the number of generators of G minus the number of generators of H.[1] Since the size of a group is given by number of generators we can write the number of Goldstone bosons as

$$\dim(G) - \dim(H) = \dim(G/H) \; ,$$

where G/H is called the quotient group. The physical origin of these massless particles is due to the fact that broken generators allow transitions between degenerate vacuum states (which have the same energy) and these transitions do not cost any energy to the system.

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[1] There is a peculiarity in the case of Goldstone bosons in two dimensional theories, which we will not consider here, see the Coleman-Mermin-Wagner theorem.

Spontaneously broken internal symmetries

Consider a theory with scalar fields $\phi_i(x)$ and let ϕ_0 be the constant field which minimizes the potential $V(\phi)$. By definition the minimum is

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_i(x)=\phi_0} = 0 \, ,$$

and if we expand around the minimum

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)_i \, (\phi - \phi_0)_j \, \left(\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right)_{\phi_0} + \ldots$$

The coefficient of the quadratic term is a symmetric matrix

$$\left(\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right)_{\phi_0} = m^2_{ij}$$

its eigenvalues giving the masses of the fields.

To prove the Goldstone theorem we must show that every continuous symmetry of the Lagrangian which is not a symmetry of ϕ_0 gives an eigenvalue of zero in the mass matrix.

The generators T^A of the symmetry G are spontaneously broken into two separate classes, a number $\dim(H)$ generators T^α are unbroken:

$$T^\alpha \, \phi_0 = 0$$

in the residual group H, and a number $\dim(G) - \dim(H)$ generators T^a are broken:

$$T^a \, \phi_0 \neq 0$$

in the quotient G/H.

The symmetry transformation is given by

$$\delta\phi(x) = c_A T^A \phi_0 = c_a T^a \phi_0 \ .$$

The index has A has values (α, a), the $c_A = (c_a, c_\alpha)$ are function fields.

The invariance of the potential under a symmetry transformation reads

$$V(\phi_A) = V(\phi_A + c_A T^A \phi_0)$$

or as

$$c_A T^A \frac{\partial V}{\partial \phi_A} = 0 \ ,$$

by differentiation with respect to ϕ_B with $\phi = \phi_0$ we obtain

$$0 = \left(\frac{\partial c_A T^A}{\partial \phi_B}\right)_{\phi_0} \left(\frac{\partial V}{\partial \phi_A}\right)_{\phi_0} + c_A T^A \left(\frac{\partial^2 V}{\partial \phi_A \partial \phi_B}\right)_{\phi_0} .$$

The first term is zero because ϕ_0 is a minimum of V. The second must, therefore, also vanish.

For $c_\alpha T^\alpha = 0$ our equation is satisfied without restrictions on the second derivative of V.

For $c_a(x) T^a \neq 0$ the second derivative of V must be zero. This implies the values are zero for the mass matrix in numbers equal to the number of broken generators and demonstrates the Goldstone theorem.

Exercise A

Consider a complex scalar field ϕ_i in the vector representation of $SU(n)$, which transforms as follows under infinitesimal transformations of $SU(n)$

$$\begin{aligned}\phi_i &\longrightarrow \phi_i + i\epsilon_i^j \phi_j \\ \phi^i &\longrightarrow \phi^i - i\epsilon_k^i \phi^k\end{aligned}$$

with $\phi_i^* = \phi^i$. Find an expression which is invariant under $SU(n)$ transformations and construct a renormalisable scalar potential for a general theory in 4-dimensions.

Choose a value for the vacuum of the scalar field as

$$\langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v \end{pmatrix}$$

and consider the translation of this minimum of the field to study the properties of the components of the scalar field. How many Goldstone bosons remain massless in the spectrum of the theory? What is the residual group invariance of the theory?

Doing the same exercise with two complex scalar fields ϕ_{1i} and ϕ_{2i} in the vector representation of $SU(n)$, where they transform in the same way that ϕ_i previously did. Build the scalar potential and do not forget to also consider the terms which mix the two fields.

Select vacuum expectation values

$$\langle 0|\phi_1|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v_1 \end{pmatrix} \qquad \langle 0|\phi_2|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_2 \\ v_3 \end{pmatrix}$$

and study the symmetry breaking.

The Higgs mechanism

The Goldstone theorem is a problem rather than a solution for generating masses. In the spontaneous breaking of a symmetry we obtained massless particles. When one spontaneously breaks a gauge theory the results are very different. The reason is that the Goldstone theorem does not apply to a gauge symmetry because it is impossible to quantify a gauge theory, keeping at the same time the covariance of the theory and that the norm of the Hilbert space remain positive.

In the case of a spontaneously broken gauge theory the gauge bosons corresponding to the broken symmetries have mass and the corresponding Goldstone bosons disappear. We call this phenomenon the Higgs mechanism.

Example: $O(2)$

We can consider the example of a theory with an $O(2)$ symmetry

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \ ,$$

with vector fields ϕ having two real components. The symmetry $O(2)$ is not a gauge symmetry and we can repeat the analysis of the previous section. If $\mu^2 < 0$ we can choose the vacuum

$$\phi = (v, 0) \ , \qquad v = \sqrt{-\frac{\mu^2}{\lambda}}$$

and make a translation of the field ϕ_1 to $\phi_1 = \chi + v$, with the potential becoming

$$V = -\frac{m_\chi^4}{16\lambda} + \frac{1}{2} m_\chi^2 \chi^2 + \sqrt{\frac{m_\chi^2 \lambda}{2}} \left(\phi_2^2 + \chi^2\right) \chi + \frac{\lambda}{4} \left(\phi_2^2 + \chi^2\right)^2 .$$

The Goldstone boson ϕ_2 remains massless and the continuous symmetry, $O(2)$, is completely broken (except for a discrete symmetry $\phi_2 \to -\phi_2$). The infinitesimal transformation under $O(2)$ of the field ϕ is given by

$$\delta\phi_1 = -\alpha\phi_2 \,, \qquad \delta\phi_2 = \alpha\phi_1$$

and in terms of the new fields

$$\delta\chi = -\alpha\phi_2 \,, \qquad \delta\phi_2 = \alpha\chi + \alpha v \,.$$

Thus the Goldstone boson, in terms of new variables, becomes a rotation plus a translation. The invariance of the field to translation makes the potential V flat in this direction, and this in turn means that the translation does not cost any energy and the particle is massless.

We will now analyze the same model in the case of a local symmetry (gauge symmetry). The invariance under transformations of the Goldstone boson becomes

$$\delta\phi_2(x) = \alpha(x)\chi(x) + \alpha(x)v$$

and since $\alpha(x)$ is an arbitrary function of space-time there can be a choice of how to eliminate ϕ_2.

To see these details we can transform to polar coordinates

$$\rho = \sqrt{\phi_1^2 + \phi_2^2}\,, \qquad\qquad \theta = \arcsin\frac{\phi_2}{\rho}\ ,$$

where the transformation under finite rotations is

$$\rho \longrightarrow \rho\,, \qquad\qquad \theta \longrightarrow \theta + \alpha\ .$$

In the case of an infinitesimal systems of two coordinates coinciding:

$$\rho = \sqrt{\phi_2^2 + \chi^2 + 2v\chi + v^2} \sim v + \chi\,, \qquad\qquad \theta \sim \frac{\phi_2}{\chi + v} \sim \frac{\phi_2}{v}\ .$$

we make the theory invariant under local transformations

$$\theta(x) \longrightarrow \theta(x) + \alpha(x)$$

with the choice $\alpha(x) = -\theta(x)$ and the field $\theta(x)$ can be completely eliminated from the theory.

To explicitly construct the local theory with invariance we must introduce a gauge field and covariant derivatives. It is easier to change by writing the dipole field real scalar in terms of a complex field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \ , \qquad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

and rotations of $O(2)$ become phase transformations for the complex field ϕ

$$\phi \to e^{i\alpha}\phi \ .$$

The Lagrangian model is written in the new variables

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \ .$$

To make the Lagrangian invariant under the local transformations we must introduce covariant derivatives

$$\partial_\mu \phi \to (\partial_\mu - igA_\mu)\,\phi = D_\mu \phi$$

and the kinetic term for the gauge field A_μ. Therefore

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu + igA_\mu)\,\phi^\dagger\,(\partial^\mu - igA^\mu)\,\phi - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 .$$

It is difficult to read directly the masses of the particles from this Lagrangian because we have a mixing term $A_\mu \partial^\mu \theta$ between the Goldstone boson θ and the gauge boson A_μ.

It is possible, by a gauge transformation to eliminate the mixing term because we saw how to completely eliminate the Goldstone boson from the Lagrangian earlier. In polar coordinates

$$\phi = \frac{1}{\sqrt{2}}\,\rho\,e^{i\theta}\,, \qquad \phi^\dagger = \frac{1}{\sqrt{2}}\,\rho\,e^{-i\theta}\,.$$

The gauge transformation that eliminates θ is $\phi \to \phi\,e^{-i\theta}$ for the scalar field, and $A_\mu \to A_\mu - \frac{1}{g}\partial_\mu\theta$ for the gauge field.

The Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + igA_\mu)\rho(\partial^\mu - igA^\mu)\rho \\ &\quad -\frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4\,.\end{aligned}$$

We have to perform the translation in order to be around the minimum $\rho = \chi + v$, and one can see that the covariant derivative term generates a mass term for the gauge field

$$\frac{1}{2} g^2 v^2 A_\mu A^\mu \ ,$$

thus the gauge field has mass

$$m_A^2 = g^2 v^2$$

and the Goldstone boson has disappeared from the theory. The choice of gauge where the Goldstone boson vanishes is the *gauge* unit. Note that the number of degrees of freedom of the theory has not changed: Initially we had two real scalar fields and two components of a massless gauge boson. After the gauge transformation we had a single real scalar field and three components of a massive boson.

In general, if the global symmetry group of the Lagrangian is G, $H \subset G$ is the invariance group of the vacuum, and $G_W \subset G$ the local gauge symmetry (with $K = H \cap G_W \neq 0$) the broken generators of G can be separated into two categories: $T_K \in K$ are the generators associated with the massive gauge bosons, and the other broken generators correspond to massless Goldstone bosons. The unbroken generator G_W corresponds to massless gauge bosons.

Exercise B

Consider the Lagrangian of the local $O(3)$ symmetry

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu)_{ij}\phi_j(D^\mu)_{ik}\phi_k - \frac{\mu^2}{2}\phi_i\phi_i - \frac{\lambda}{4}(\phi_i\phi_i)^2$$

with covariant derivative

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - ig\,(T_a)_{ij}\,W_\mu^a$$

and $(T_a)_{ij} = -i\epsilon_{aij}$. Choose the solution with spontaneous symmetry breaking ($\mu^2 < 0$) and the vacuum of the theory along the 3 direction:

$$\phi_i = v\,\delta_{i3}$$

and show that both gauge fields W_1^μ and W_2^μ associated with broken generators T_1 and T_2 have weight g^2v^2. Also show that W_3^μ has zero mass.