Lattice simulations for nuclei, ultracold atoms, and ions

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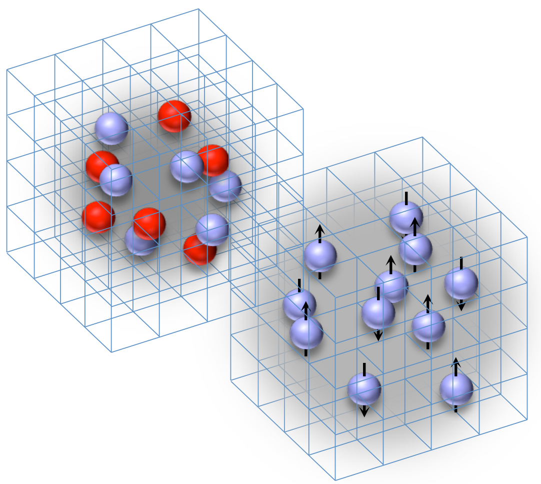


24th European Conference on
Few-Body Problems in Physics
University of Surrey
September 3, 2019

Outline

Lattice effective field theory

A tale of two interactions

Essential elements for nuclear binding

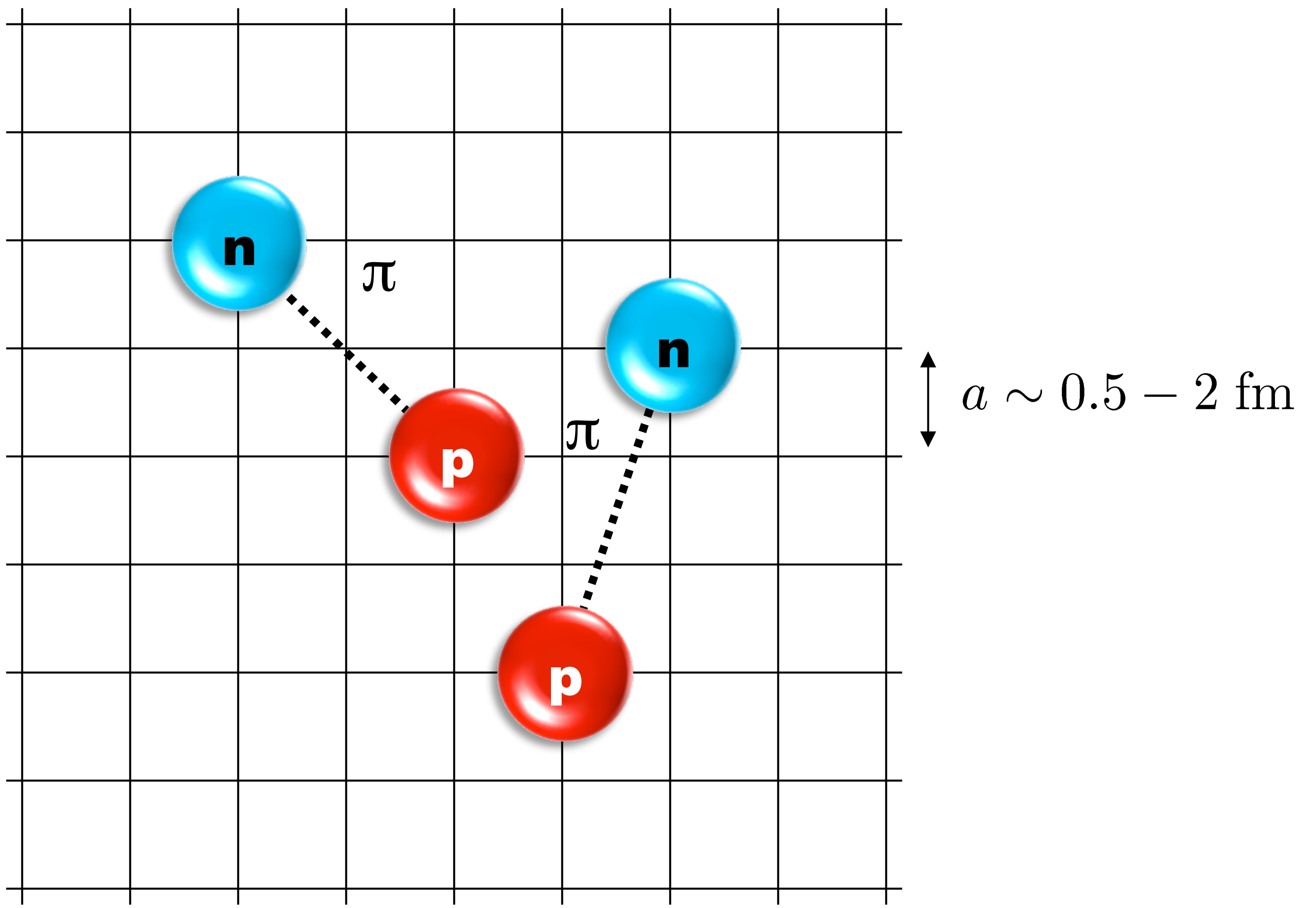
Quantum scale anomalies with trapped ions

Discrete scale invariance for two bosons

Time fractals

Summary and outlook

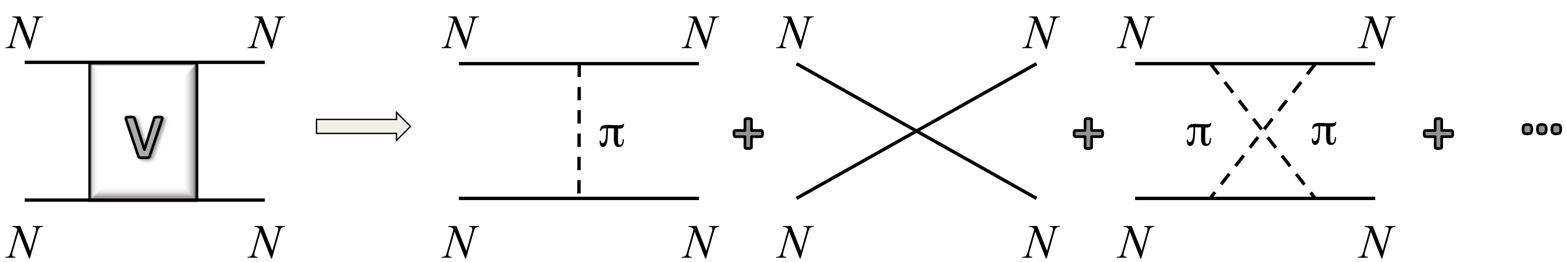
Lattice effective field theory



Review: D.L, Prog. Part. Nucl. Phys. 63 117-154 (2009)
Springer Lecture Notes: Lähde, Meißner, "Nuclear Lattice Effective Field Theory" (2019)

Chiral effective field theory

Construct the effective potential order by order



V^{OPEP} Contact interactions V^{TPEP}

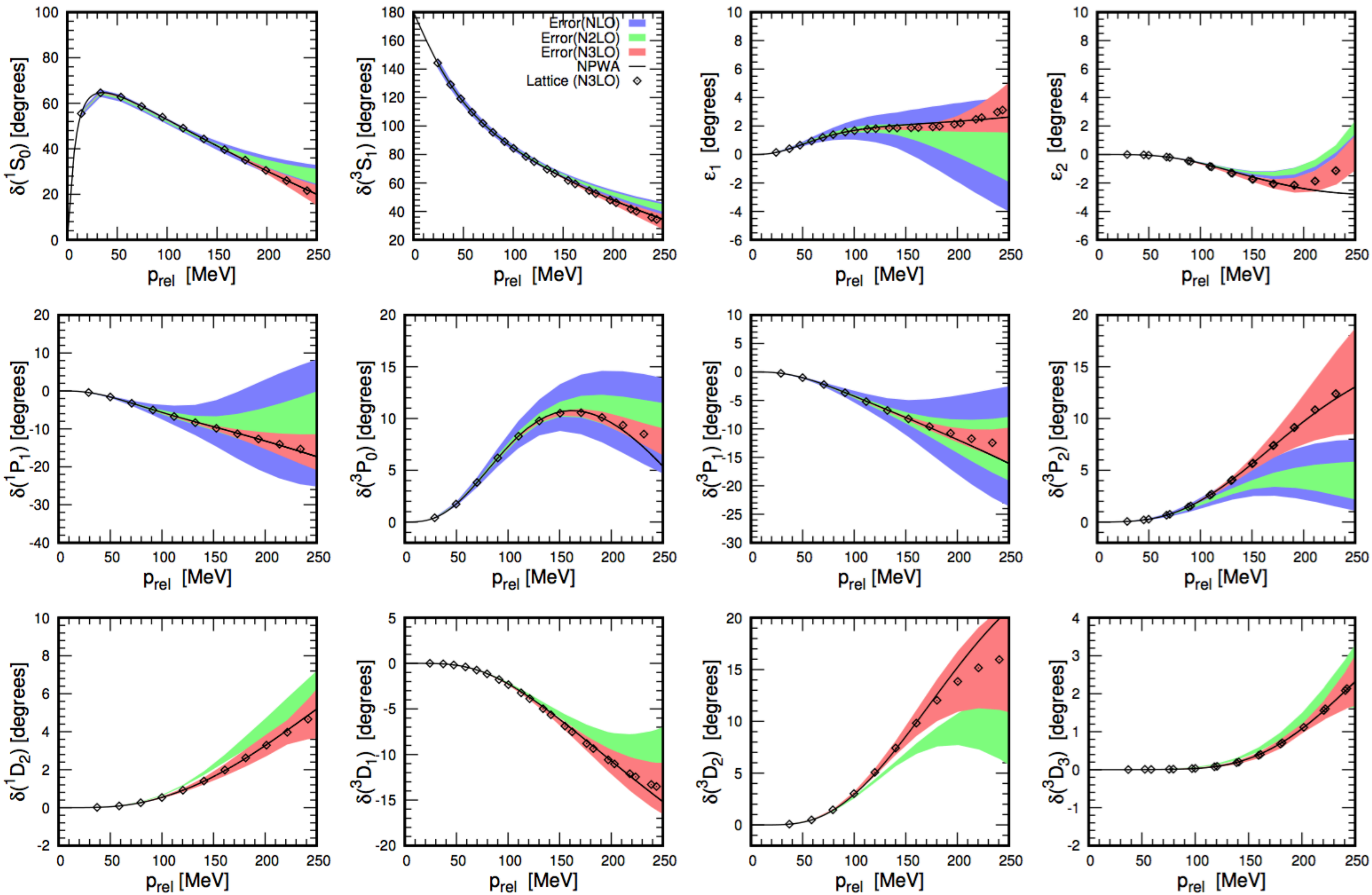
Leading order (LO) Next-to-leading order (NLO)

Related:

See the talk by Lukas Bovermann on Friday afternoon at 14:55 in the Atoms and Molecules Session. He will discuss the spherical wall method and calculating phase shifts and mixing angles on the lattice for an arbitrary number of coupled channels.

$a = 1.315\,\mathrm{fm}$

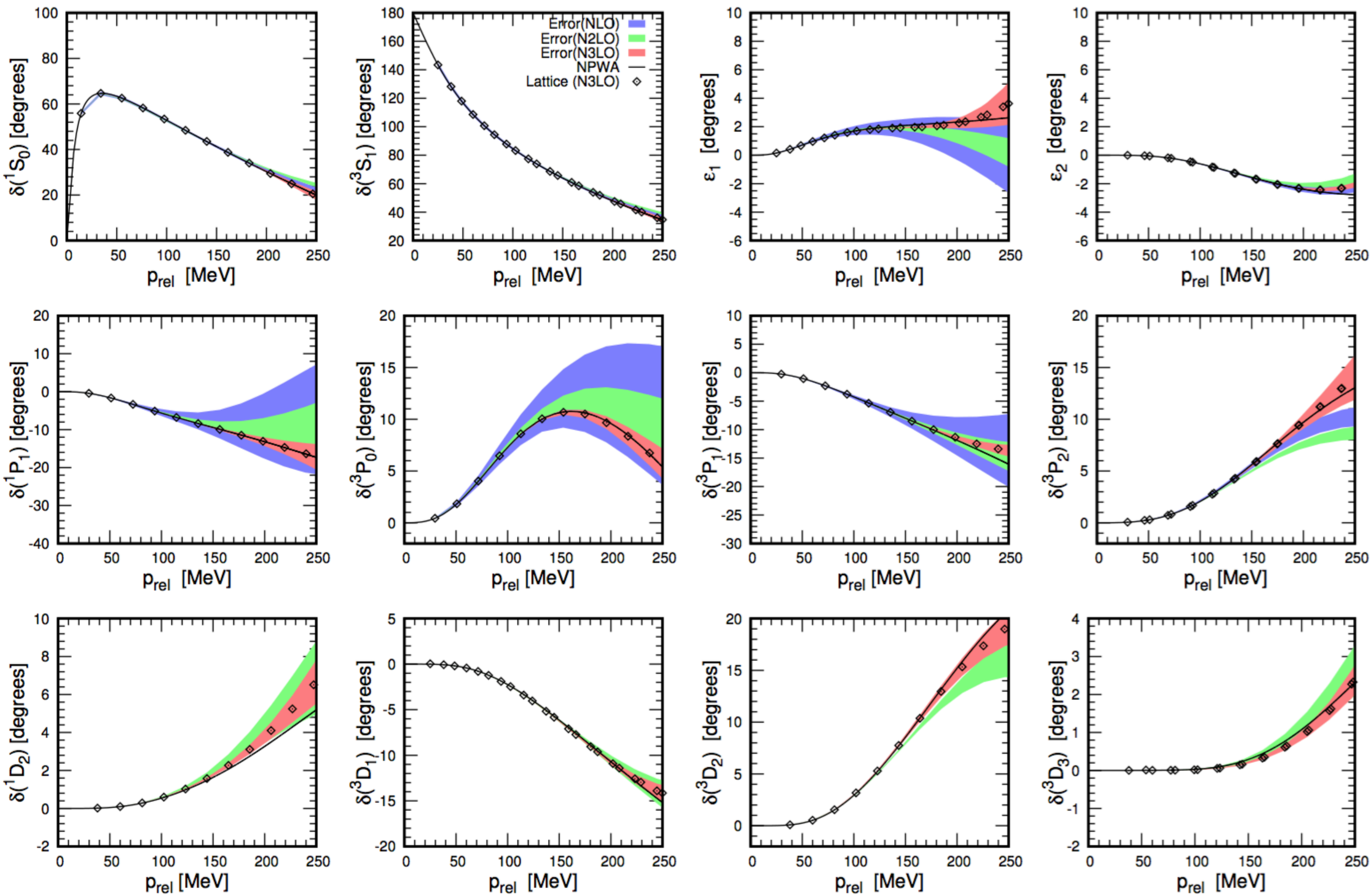
Figures by Ning Li



Li, Elhatisari, Epelbaum, D.L., Lu, Meißner, PRC 98, 044002 (2018)

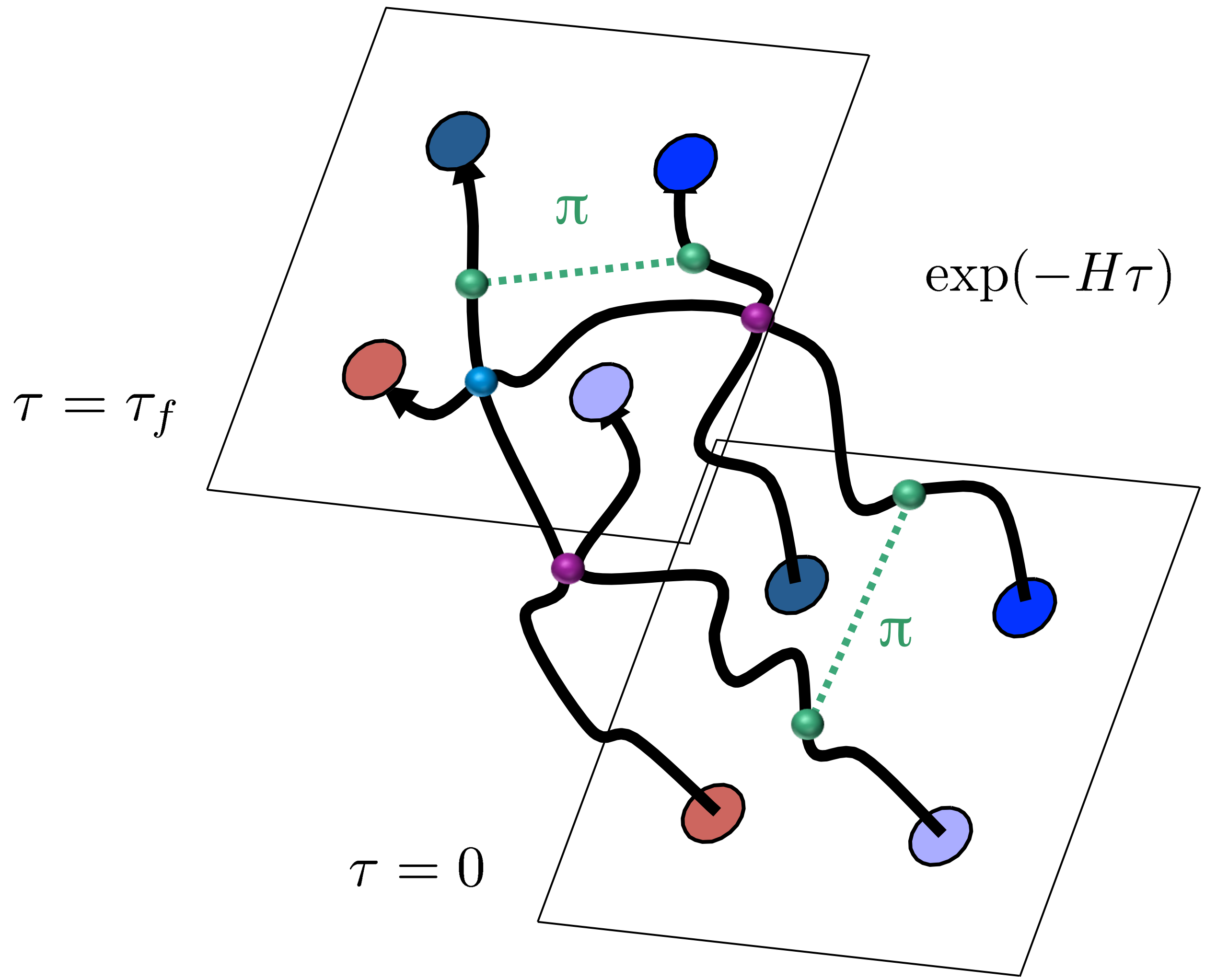
$a = 0.987\,\mathrm{fm}$

Figures by Ning Li



Li, Elhatisari, Epelbaum, D.L., Lu, Meißner, PRC 98, 044002 (2018)

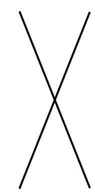
Euclidean time projection



Auxiliary field method

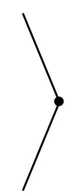
We can write exponentials of the interaction using a Gaussian integral identity

$$\exp\left[-\frac{C}{2}(N^\dagger N)^2\right]$$



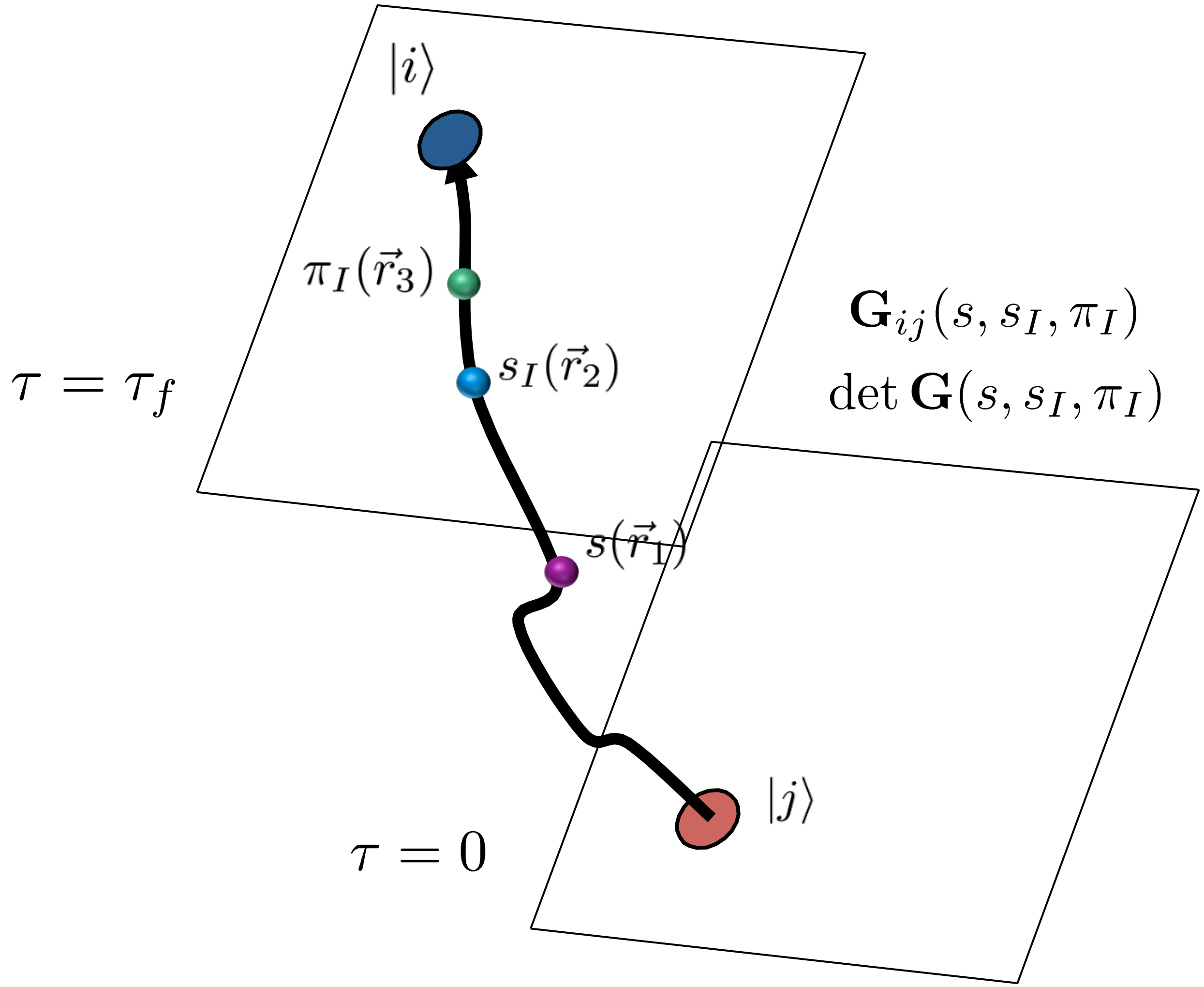
$(N^\dagger N)^2$

$$= \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} ds \exp\left[-\frac{1}{2}s^2 + \sqrt{-C}\, s(N^\dagger N)\right]$$



$sN^\dagger N$

We remove the interaction between nucleons and replace it with the interactions of each nucleon with a background field.


$|i\rangle$
$\pi_I(\vec{r}_3)$
$s_I(\vec{r}_2)$
$\tau = \tau_f$
$\mathbf{G}_{ij}(s, s_I, \pi_I)$
$\det \mathbf{G}(s, s_I, \pi_I)$
$s(\vec{r}_1)$
$|j\rangle$
$\tau = 0$

A tale of two interactions

Two LO interactions, A and B, have nearly identical nucleon-nucleon phase shifts and well as three- and four-nucleon bound states

Nucleus	A (LO)	B (LO)	A (LO + Coulomb)	B (LO + Coulomb)	Experiment
^{8}Be	$-58.61(14)$	$-59.73(6)$	$-56.51(14)$	$-57.29(7)$	-56.591
^{12}C	$-88.2(3)$	$-95.0(5)$	$-84.0(3)$	$-89.9(5)$	-92.162
^{16}O	$-117.5(6)$	$-135.4(7)$	$-110.5(6)$	$-126.0(7)$	-127.619
^{20}Ne	$-148(1)$	$-178(1)$	$-137(1)$	$-164(1)$	-160.645

Elhatisari, Li, Rokash, Alarcon, Du, Klein, Lu, Meißner, Epelbaum, Krebs, Lähde, D.L., Rupak, PRL 117, 132501 (2016)

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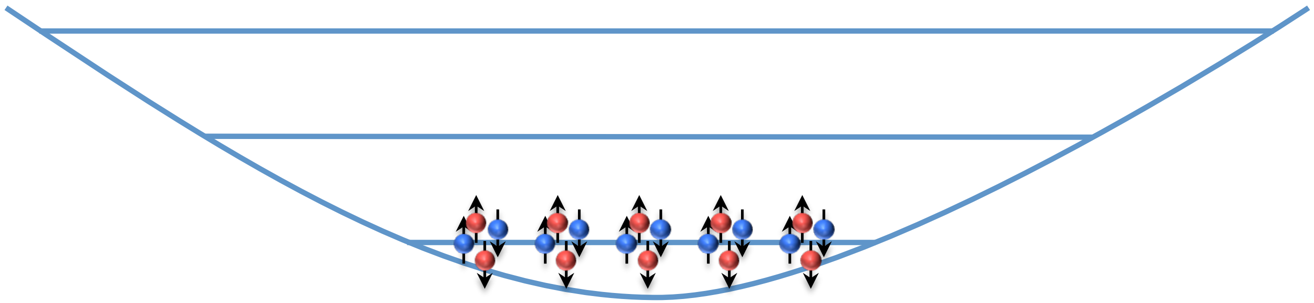
$$\frac{E_{^8\mathrm{Be}}}{E_{^4\mathrm{He}}} = 1.997(6)$$

$$\frac{E_{^{12}\mathrm{C}}}{E_{^4\mathrm{He}}} = 3.00(1)$$

$$\frac{E_{^{16}\mathrm{O}}}{E_{^4\mathrm{He}}} = 4.00(2)$$

$$\frac{E_{^{20}\mathrm{Ne}}}{E_{^4\mathrm{He}}} = 5.03(3)$$

Bose condensate of alpha particles!



Nucleus	A (LO)	B (LO)	A (LO + Coulomb)	B (LO + Coulomb)	Experiment
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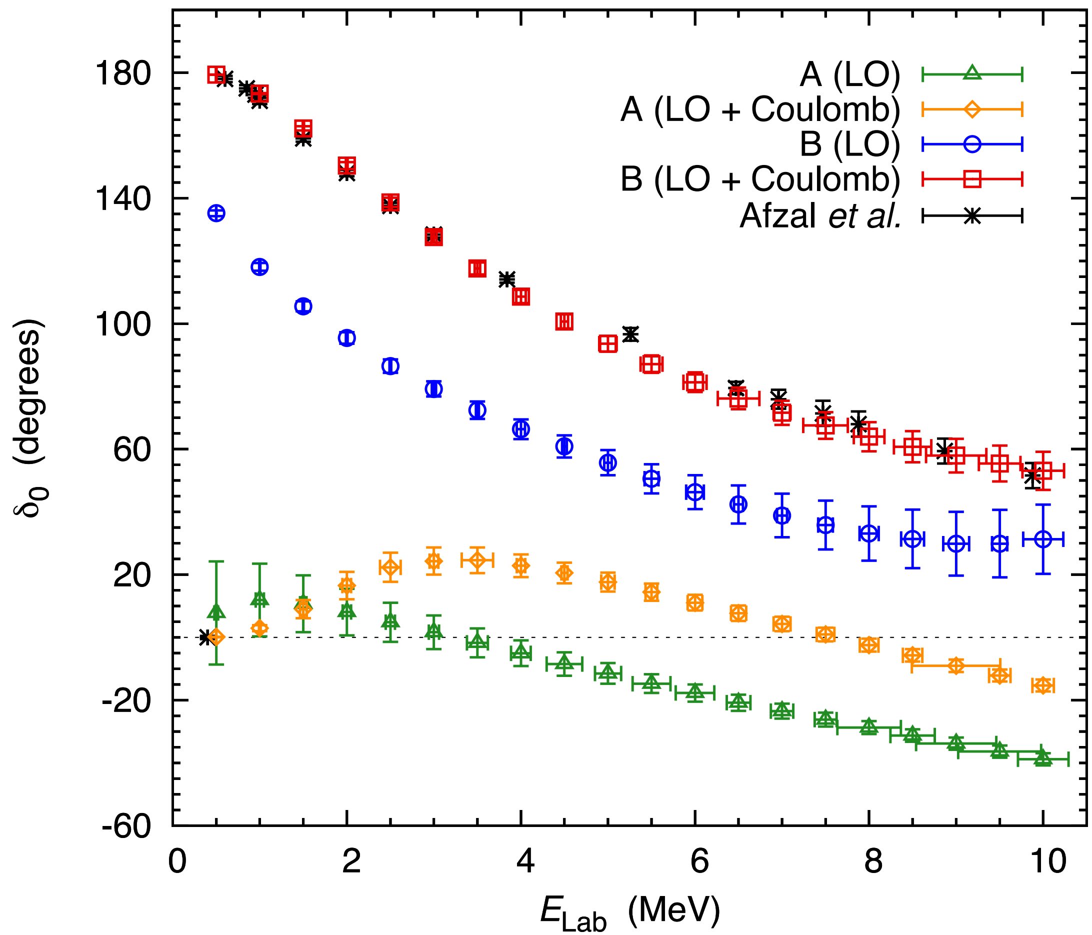
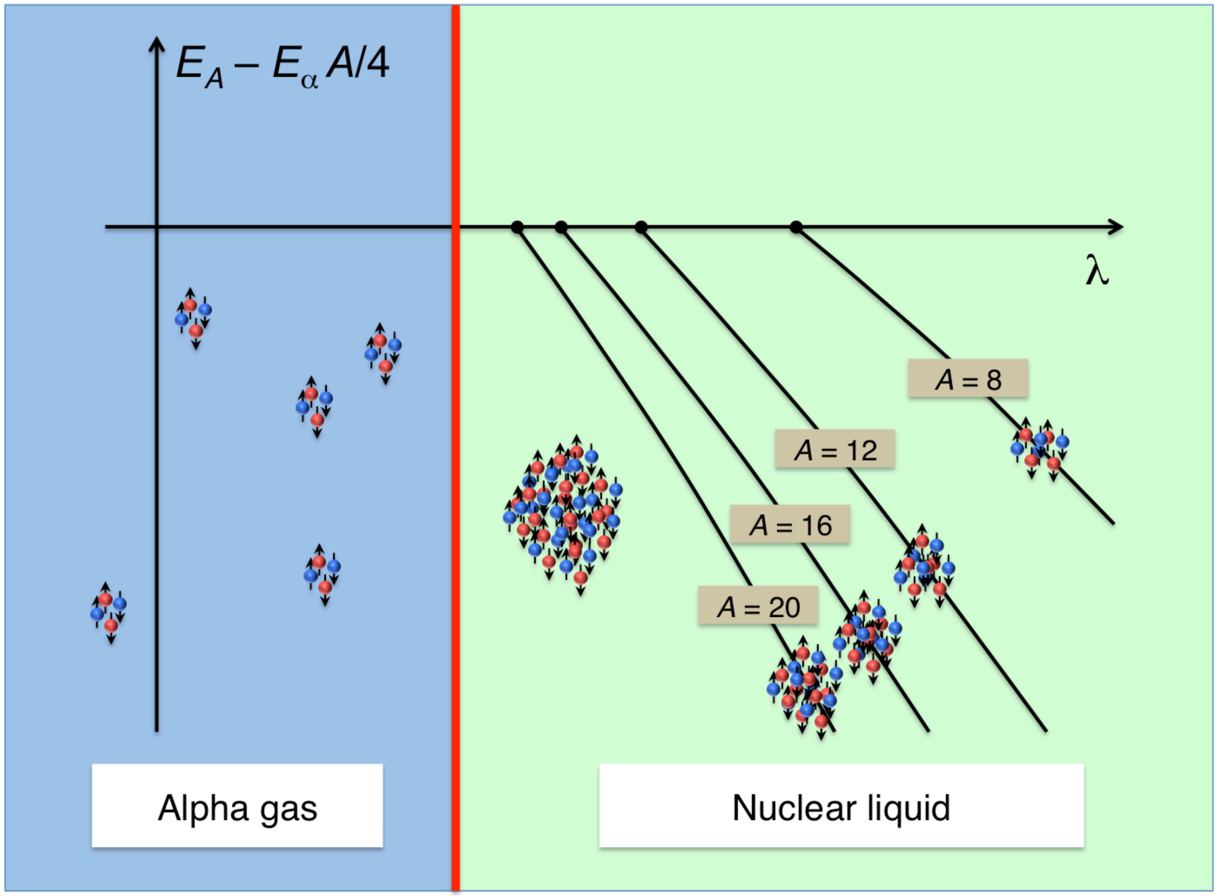
Alpha-alpha scattering



Calculated using the adiabatic projection method

Elhatisari, D.L., Rupak, Epelbaum, Krebs, Lähde, Luu, Meißner, Nature 528, 111 (2015)

Control parameters: Sensitivity to interaction range and locality



Elhatisari, Li, Rokash, Alarcon, Du, Klein, Lu, Meißner, Epelbaum, Krebs, Lähde, D.L., Rupak,
PRL 117, 132501 (2016)

Essential elements for nuclear binding

What is the minimal nuclear interaction that can reproduce the ground state properties of light nuclei, medium-mass nuclei, and neutron matter simultaneously with no more than a few percent error in the energies and charge radii?

We construct an interaction with only four parameters.

1. Strength of the two-nucleon S-wave interaction
2. Range of the two-nucleon S-wave interaction
3. Strength of three-nucleon contact interaction
4. Range of the local part of the two-nucleon interaction

Lu, Li, Elhatisari, D.L., Epelbaum, Meißner, arXiv:1812.10928, PLB in press

Related:

See the talk by Alejandro Kievsky that took place on Monday afternoon at 14:30 in the Few-Nucleons Session, and also the talk by Sebastian König on Thursday afternoon at 12:05 in the Young Researcher Award Ceremony.

SU(4)-invariant pionless interaction

$$H = H_{\text{free}} + V_{\text{SU}(4)} + V_{\text{Coulomb}}$$

$$V_{\text{SU}(4)} = \frac{1}{2!} C_2 \sum_{\boldsymbol{n}} \tilde{\rho}(\boldsymbol{n})^2 + \frac{1}{3!} C_3 \sum_{\boldsymbol{n}} \tilde{\rho}(\boldsymbol{n})^3$$

$$\tilde{\rho}(\boldsymbol{n}) = \sum_i \tilde{a}_i^\dagger(\boldsymbol{n}) \tilde{a}_i(\boldsymbol{n}) + s_L \sum_{|\boldsymbol{n}'-\boldsymbol{n}|=1} \sum_i \tilde{a}_i^\dagger(\boldsymbol{n}') \tilde{a}_i(\boldsymbol{n}')$$

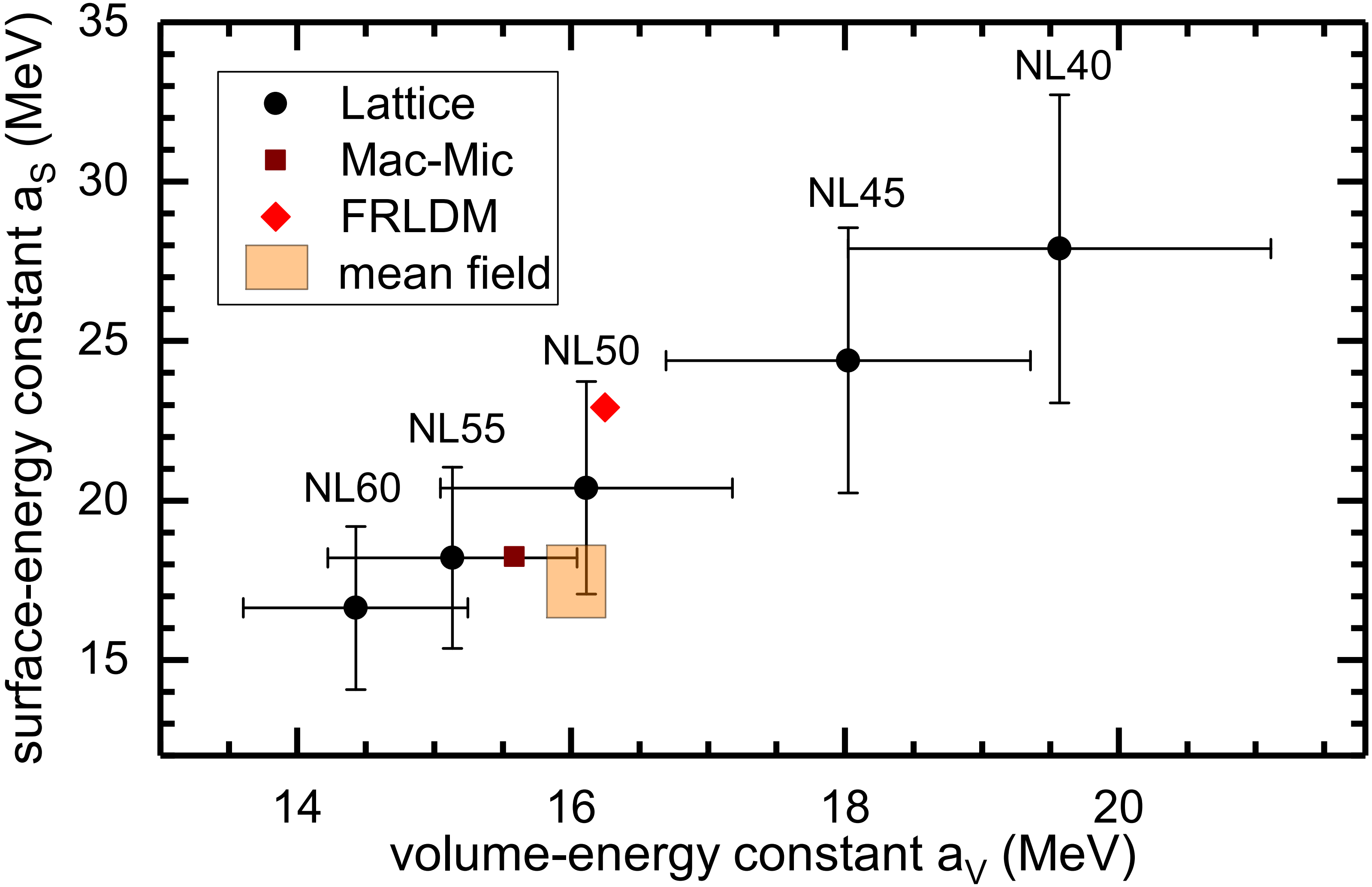
$$\tilde{a}_i(\boldsymbol{n}) = a_i(\boldsymbol{n}) + s_{NL} \sum_{|\boldsymbol{n}'-\boldsymbol{n}|=1} a_i(\boldsymbol{n}')$$

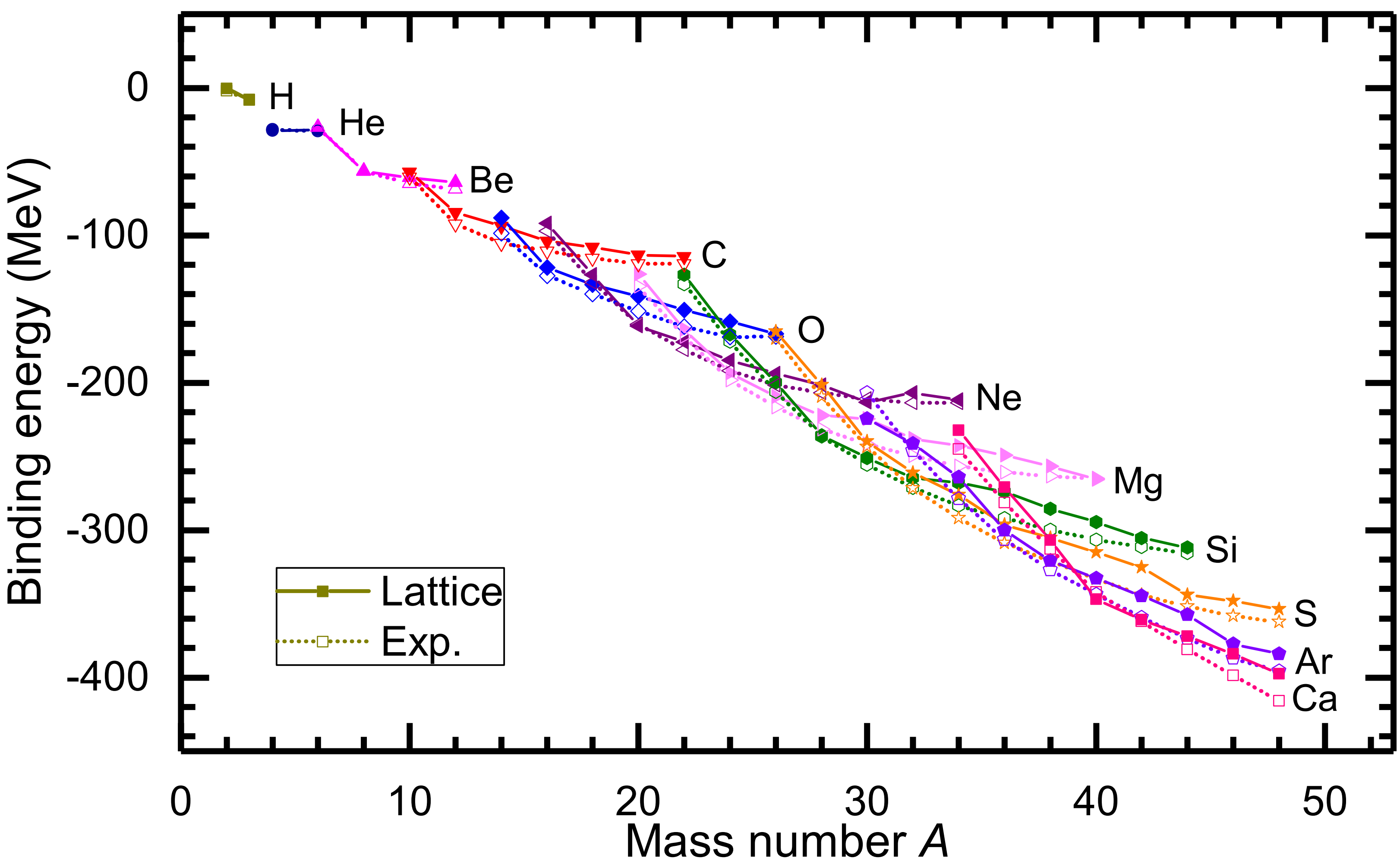
S-wave parameters: $a_0 = 9.1$ fm, $r_0 = 2.2$ fm

Triton binding energy: $B(^3\text{H}) = 8.48$ MeV

lattice spacing $= 1.32$ fm

$$s_{NL} = 0.40, 0.45, 0.50, 0.55, 0.60$$

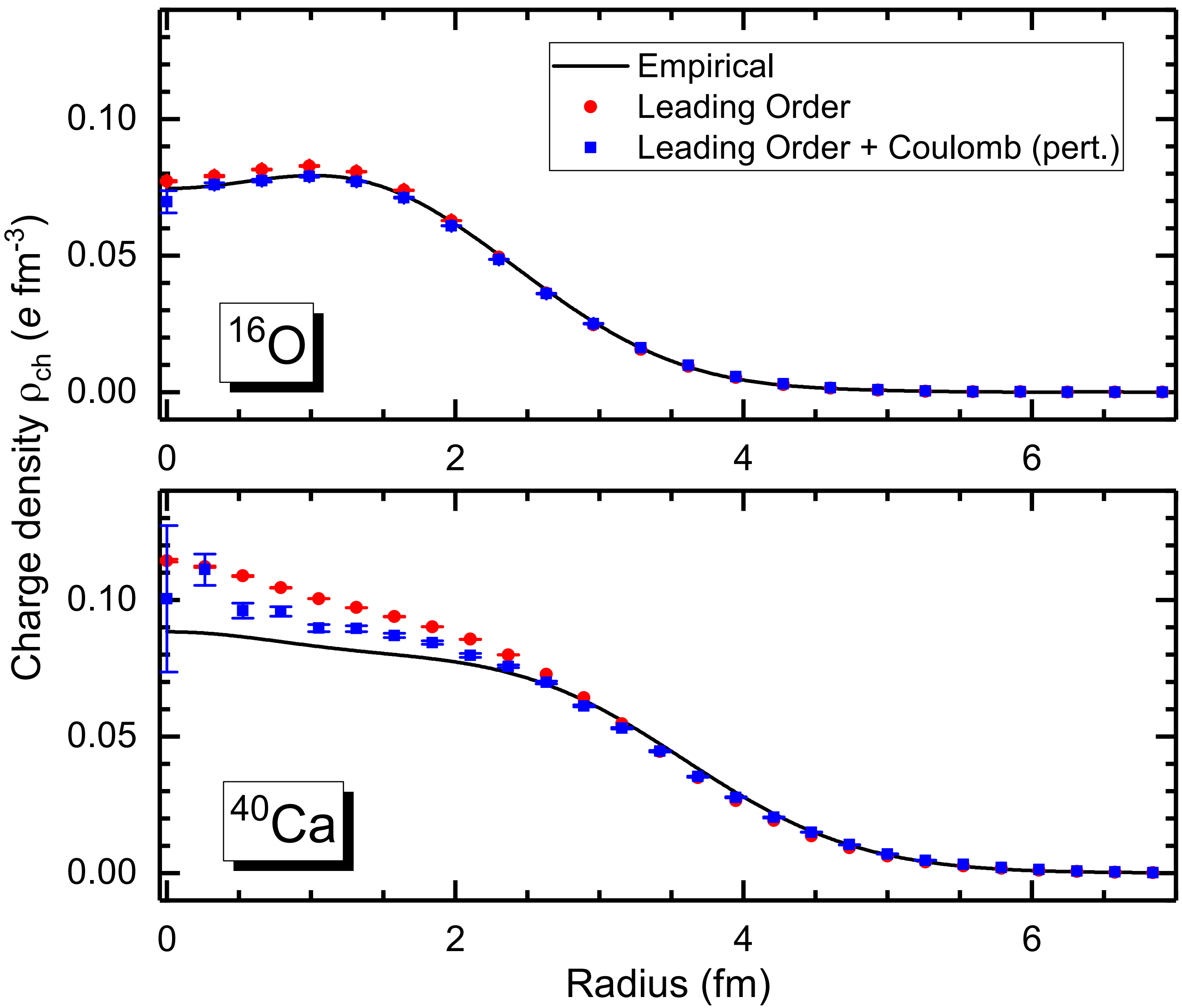




Lu, Li, Elhatisari, D.L., Epelbaum, Meißner, arXiv:1812.10928, PLB in press

	B	Exp.	R_{ch}	Exp.
^{3}H	8.48(2)(0)	8.48	1.90(1)(1)	1.76
^{3}He	7.75(2)(0)	7.72	1.99(1)(1)	1.97
^{4}He	28.89(1)(1)	28.3	1.72(1)(3)	1.68
^{16}O	121.9(1)(3)	127.6	2.74(1)(1)	2.70
^{20}Ne	161.6(1)(1)	160.6	2.95(1)(1)	3.01
^{24}Mg	193.5(02)(17)	198.3	3.13(1)(2)	3.06
^{28}Si	235.8(04)(17)	236.5	3.26(1)(1)	3.12
^{40}Ca	346.8(6)(5)	342.1	3.42(1)(3)	3.48

Lu, Li, Elhatisari, D.L., Epelbaum, Meißner, arXiv:1812.10928, PLB in press



Lu, Li, Elhatisari, D.L., Epelbaum, Meißner, arXiv:1812.10928, PLB in press

Quantum scale anomalies with trapped ions

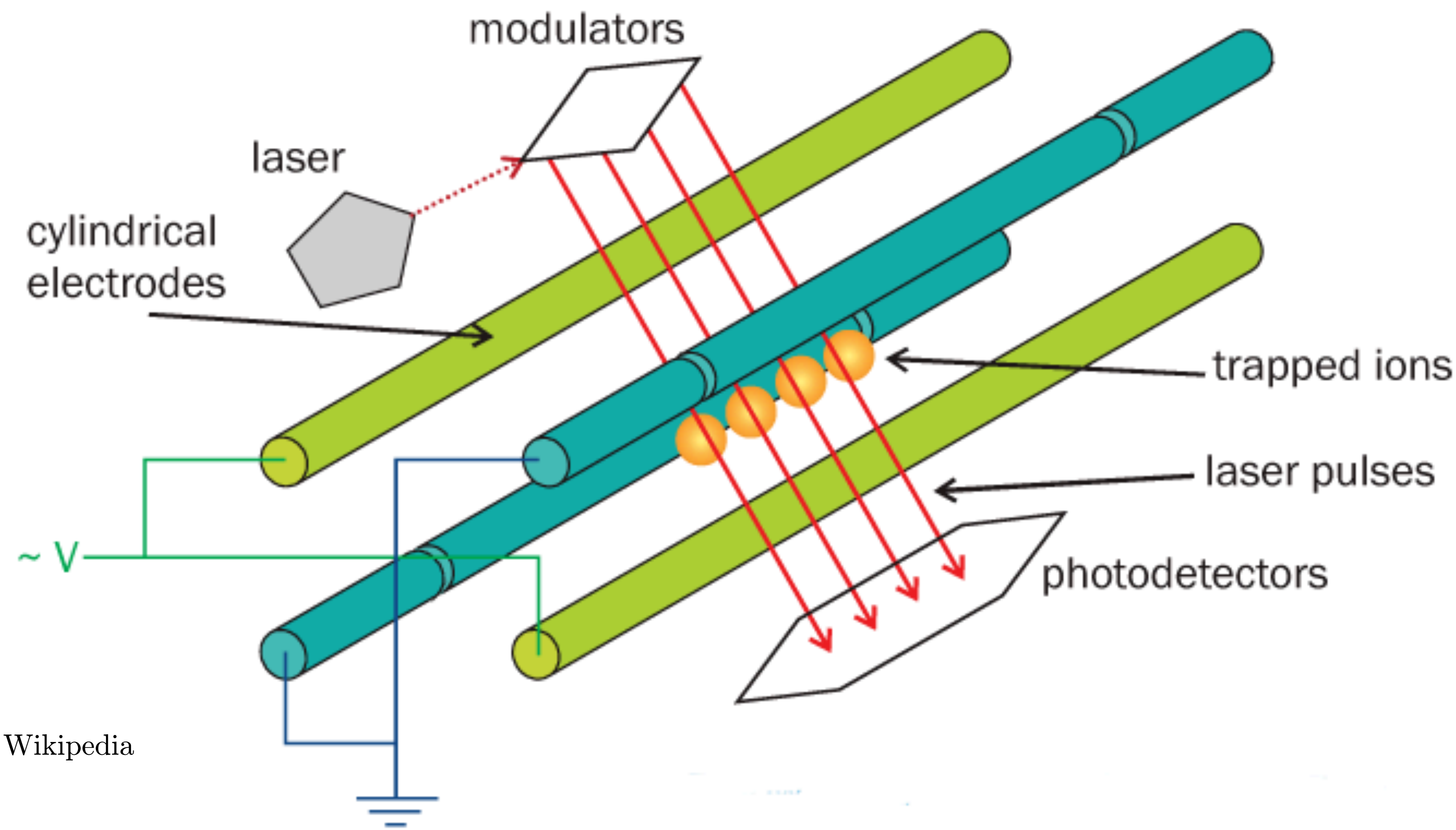
In quantum mechanics and quantum field theory, scale invariance can be spoiled by quantum scale anomalies. This happens when there are bound states, which necessarily correspond to discrete energy levels.

Nevertheless it may happen that a discrete subgroup of the scale symmetry is preserved for the dynamics of certain sectors of the Hilbert space.

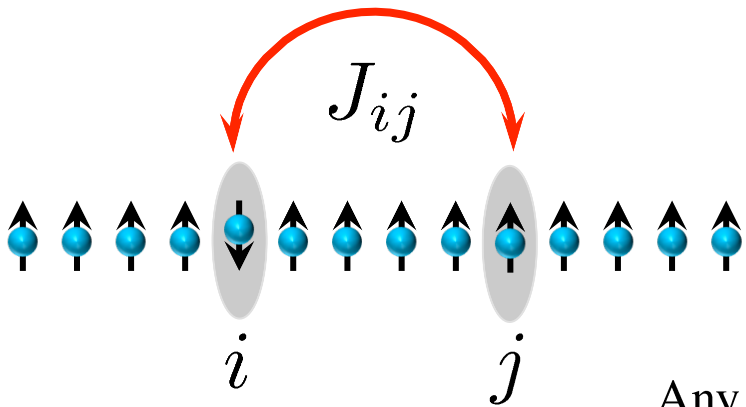
This phenomenon was first noted by Efimov for bound states of three bosons when the two-body interactions are pointlike and the interaction strength is tuned to produce a zero-energy two-body resonance.

Efimov, Sov. J. Nucl. Phys. 12, 589 (1971); Efimov, Phys. Rev. C47 1876 (1993)

Realization with trapped ions

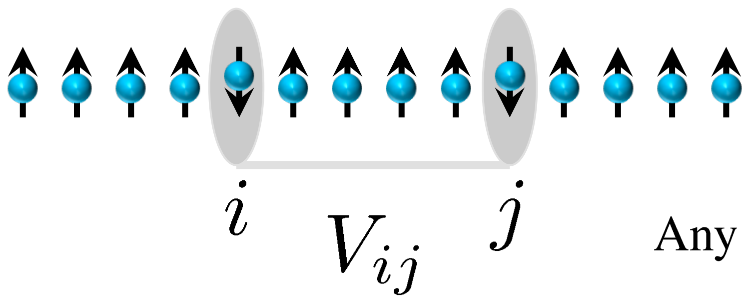


Wikipedia



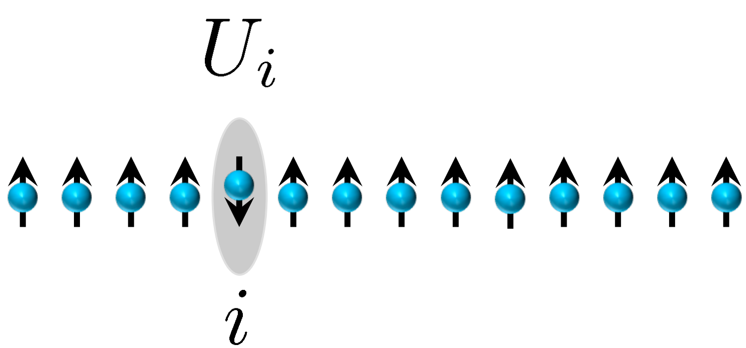
$$J_{ij} = \frac{J_0}{|r_i - r_j|^{\alpha}}$$

Any $0 < \alpha < 3$, but $0.5 < \alpha < 1.8$ favored.



$$V_{ij} = \frac{V_0}{|r_i - r_j|^{\beta}}$$

Any $0 < \beta < 3$, but $0.5 < \beta < 1.8$ favored.



Zhang *et al.*, Nature 543, 217 (2017), Zhang *et al.*, Nature 551, 601 (2017)

We define the state with all spins pointing up as the vacuum state. Then down-spin sites can be viewed as identical hardcore boson excitations.

We can write our Hamiltonian in terms of hardcore boson annihilation and creation operators.

$$H = \frac{1}{2}\sum_i \sum_{j \neq i} J_{ij}[b_i^\dagger b_j + b_j^\dagger b_i] + \frac{1}{2}\sum_i \sum_{j \neq i} V_{ij} b_i^\dagger b_i b_j^\dagger b_j + \sum_i U_i b_i^\dagger b_i + C$$

Let us now take

$$U_i = -\sum_{j \neq i} J_{ij} = -\sum_{j \neq i} \frac{J_0}{|r_i - r_j|^\alpha}$$

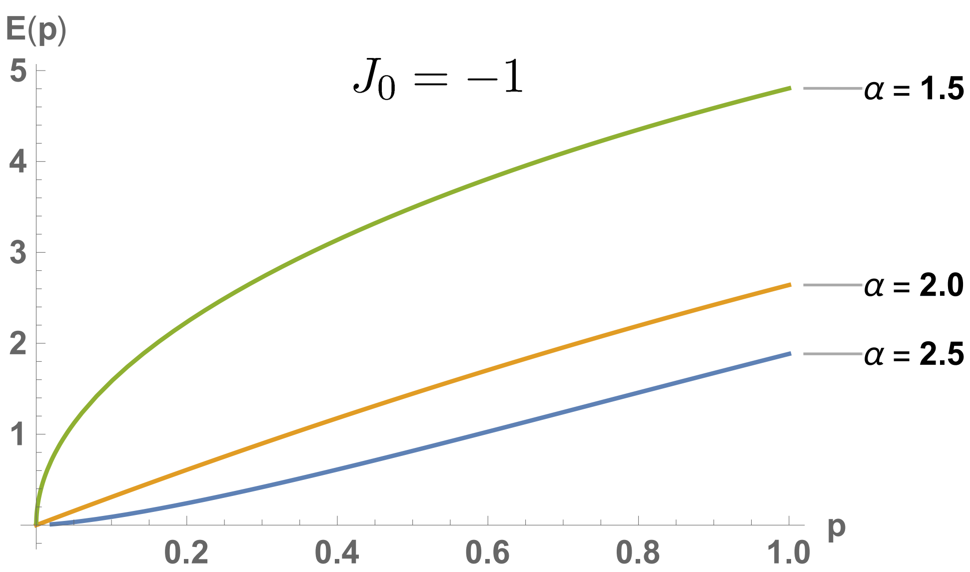
This choice ensures that a zero-momentum boson has zero energy. We now consider the dispersion relation for one boson.

For a boson with momentum p, the energy is

$$E(p) = 2J_0 \sum_{n>0} \frac{\cos(pn) - 1}{n^\alpha} = J_0 \left[\mathrm{Li}_\alpha(e^{ip}) + \mathrm{Li}_\alpha(e^{-ip}) - 2\mathrm{Li}_\alpha(1)\right]$$

At low momenta, this can be simplified as

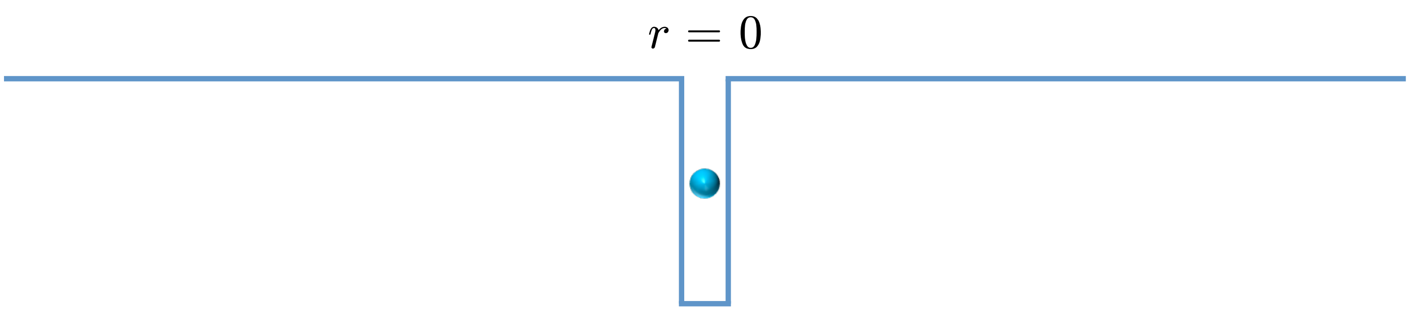
$$E(p) = 2J_0 \sin(\alpha\pi/2)\Gamma(1-\alpha)|p|^{\alpha-1} + J_0\zeta(\alpha-2)p^2 + O(p^4) \quad \text{for } \alpha < 3$$



We now introduce a single-site deep trapping potential that traps one boson at some site i_0

$$U_i = -\sum_{j \neq i} \frac{J_0}{|r_i - r_j|^\alpha} - u\delta_{i,i_0} \qquad \text{for } u \gg 1$$

We choose the position of site i_0 to be $r = 0$. We subtract a constant from the Hamiltonian so that the energy of this state is exactly zero.



Discrete scale invariance for two bosons

We now add one more boson to the system. We regard the immobile boson at $r = 0$ as a static source.

The low-energy effective Hamiltonian for the mobile boson is

$$H(p,r) = 2J_0 \sin(\alpha\pi/2)\Gamma(1-\alpha)|p|^{\alpha-1} + \frac{V_0}{|r|^{\beta}}$$

where we have dropped terms of $O(p^2)$. We will consider the case where both J_0 and V_0 are negative. In order that the Hamiltonian have classical scale invariance, we take $\beta = \alpha - 1$.

Therefore

$$H(p,r) = 2J_0 \sin(\alpha\pi/2)\Gamma(1-\alpha)|p|^{\alpha-1} + \frac{V_0}{|r|^{\alpha-1}}$$

D.L., Watkins, Frame, Given, He, Li, Lu, Sarkar, PRA 100, 011403(R) (2019)

In the limit of zero energy, the bound-state wave functions have the following forms for even and odd parity

$$\psi_+(r) = \frac{1}{2}\left(|r|^{i\delta_+} + |r|^{-i\bar{\delta}_+}\right)$$
$$\psi_-(r) = \frac{1}{2}\,\mathrm{sgn}(r)\left(|r|^{i\delta_-} + |r|^{-i\bar{\delta}_-}\right)$$

where

$$2J_0\delta_+\Gamma(1-\alpha)\sin(\alpha\pi/2)\Gamma(i\delta_+)\sinh(\delta_+\pi/2) = V_0\Gamma(2-\alpha+i\delta_+)\cos((\alpha-i\delta_+)\pi/2)$$

$$2J_0\delta_-\Gamma(1-\alpha)\sin(\alpha\pi/2)\Gamma(i\delta_-)\cosh(\delta_-\pi/2) = iV_0\Gamma(2-\alpha+i\delta_-)\sin((\alpha-i\delta_-)\pi/2)$$

The case $\alpha = 2$ corresponds to a Hamiltonian of the form

$$H(p,r) = -\pi J_0|p| + \frac{V_0}{|r|}$$

For the case $\alpha = 2$,

$$\delta_+ = \frac{V_0}{J_0 \pi} \coth(\delta_+ \pi/2), \quad \delta_- = \frac{V_0}{J_0 \pi} \tanh(\delta_- \pi/2)$$

We can rewrite the zero-energy bound-state solutions as

$$\psi_+(r) = \cos[\delta_+ \ln(|r|)], \quad \psi_-(r) = \mathrm{sgn}(r) \cos[\delta_- \ln(|r|)]$$

Under the scale transformations $r \to \lambda_\pm r$,

$$\psi_+(r) \to \cos[\delta_+ \ln(|r|) + \delta_+ \ln(\lambda_+)]$$

$$\psi_-(r) \to \mathrm{sgn}(r) \cos[\delta_- \ln(|r|) + \delta_- \ln(\lambda_-)]$$

The wave functions exhibit discrete scale invariance when the scale factors are

$$\lambda_+ = \exp(\pi/\delta_+), \quad \lambda_- = \exp(\pi/\delta_-)$$

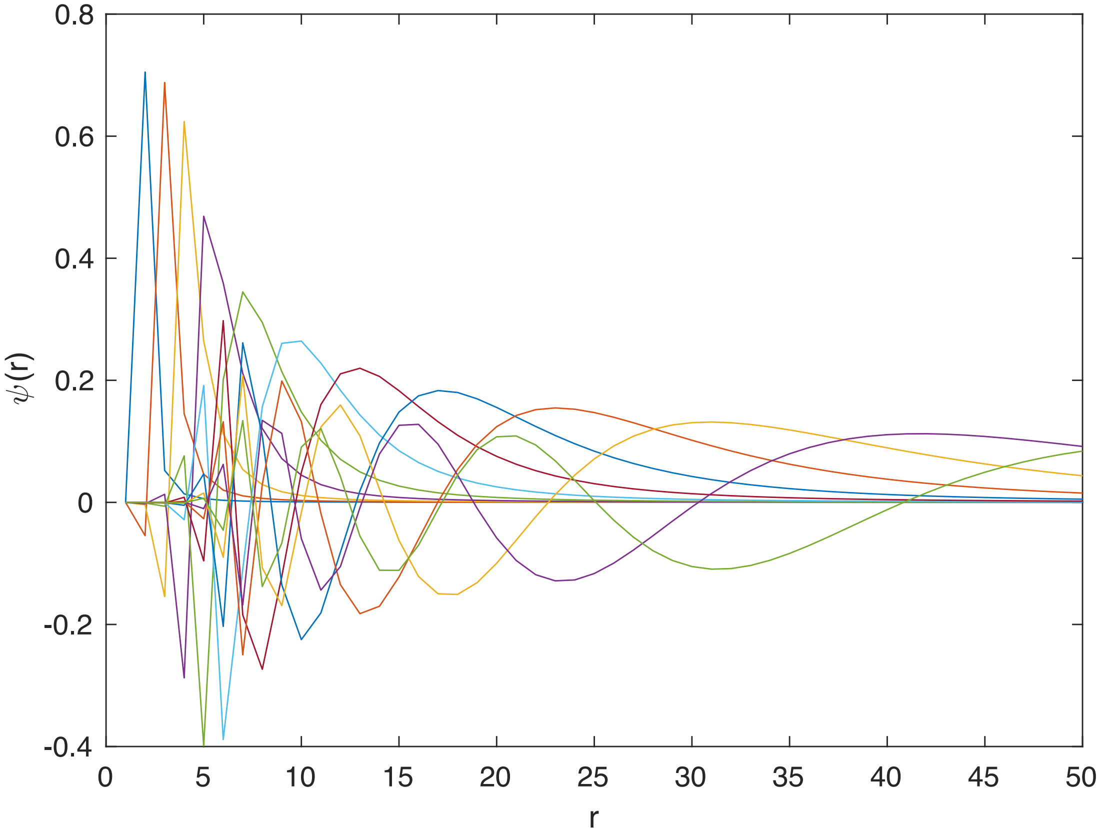
The bound state energies form a geometric progression

$$E_+^{(n)} = \epsilon_+ \lambda_+^{-n}, \quad E_-^{(n)} = \epsilon_- \lambda_-^{-n}$$

The general formulae are

$$\lambda_+ = \exp(\pi/\mathrm{Re}\,\delta_+), \quad \lambda_- = \exp(\pi/\mathrm{Re}\,\delta_-)$$
$$E_+^{(n)} = \epsilon_+ \lambda_+^{-(\alpha-1)n}, \quad E_-^{(n)} = \epsilon_- \lambda_-^{-(\alpha-1)n}$$

The first twelve even-parity bound-state wave functions:



$\alpha = 2$, $\beta = 1$, $J_0 = -1$, and $V_0 = -30$

D.L., Watkins, Frame, Given, He, Li, Lu, Sarkar, PRA 100, 011403(R) (2019)

n	$E_+^{(n)}$	$E_+^{(n-1)}/E_+^{(n)}$	$E_-^{(n)}$	$E_-^{(n-1)}/E_+^{(n)}$
0	-27.05304149	–	-26.5188669	–
1	-11.93067205	2.267520336	-11.79861873	2.247624701
2	-6.977774689	1.709810446	-6.919891389	1.705029468
3	-4.553270276	1.5324754	-4.521425357	1.530466798
4	-3.139972298	1.450098869	-3.120231851	1.449067112
5	-2.233327278	1.405961557	-2.220194049	1.405386998
6	-1.617052389	1.381110033	-1.607920414	1.380786033
7	-1.182654461	1.367307563	-1.176124883	1.367134084
8	-0.869406941	1.360300229	-0.864656962	1.360221377
9	-0.640405903	1.357587332	-0.636916042	1.357568195
10	-0.471738446	1.357544438	-0.469161911	1.357561276
11	-0.347112043	1.359037968	-0.345207121	1.359073675
12	-0.254996818	1.361240684	-0.253589633	1.361282464
13	-0.187011843	1.363532996	-0.18597462	1.363571189
theory	–	$\lambda_+ = 1.3895595319$	–	$\lambda_- = 1.3895595319$

Time fractals

We use a phase convention where all of the bound-state wave functions are real valued. Let us construct a coherent superposition of the first N even-parity bound states, where N is large.

$$|S\rangle = \sum_{n=0}^{N-1} |\psi_+^{(n)}\rangle$$

We could have just as easily chosen odd-parity bound states. We now consider the amplitude

$$A(t) = \mathrm{Re}[Z(t)], \quad Z(t) = \langle S| \exp(-iHt)|S\rangle$$

Aside from corrections of relative size $1/N$ from endpoint terms at $n = 0$ and $n = N - 1$, the amplitude is invariant under the discrete rescaling of time.

$$t \to \lambda_+^{\alpha-1} t$$

$$A(t) \to A(\lambda_+^{\alpha-1} t) = A(t) \cdot [1 + O(1/N)]$$

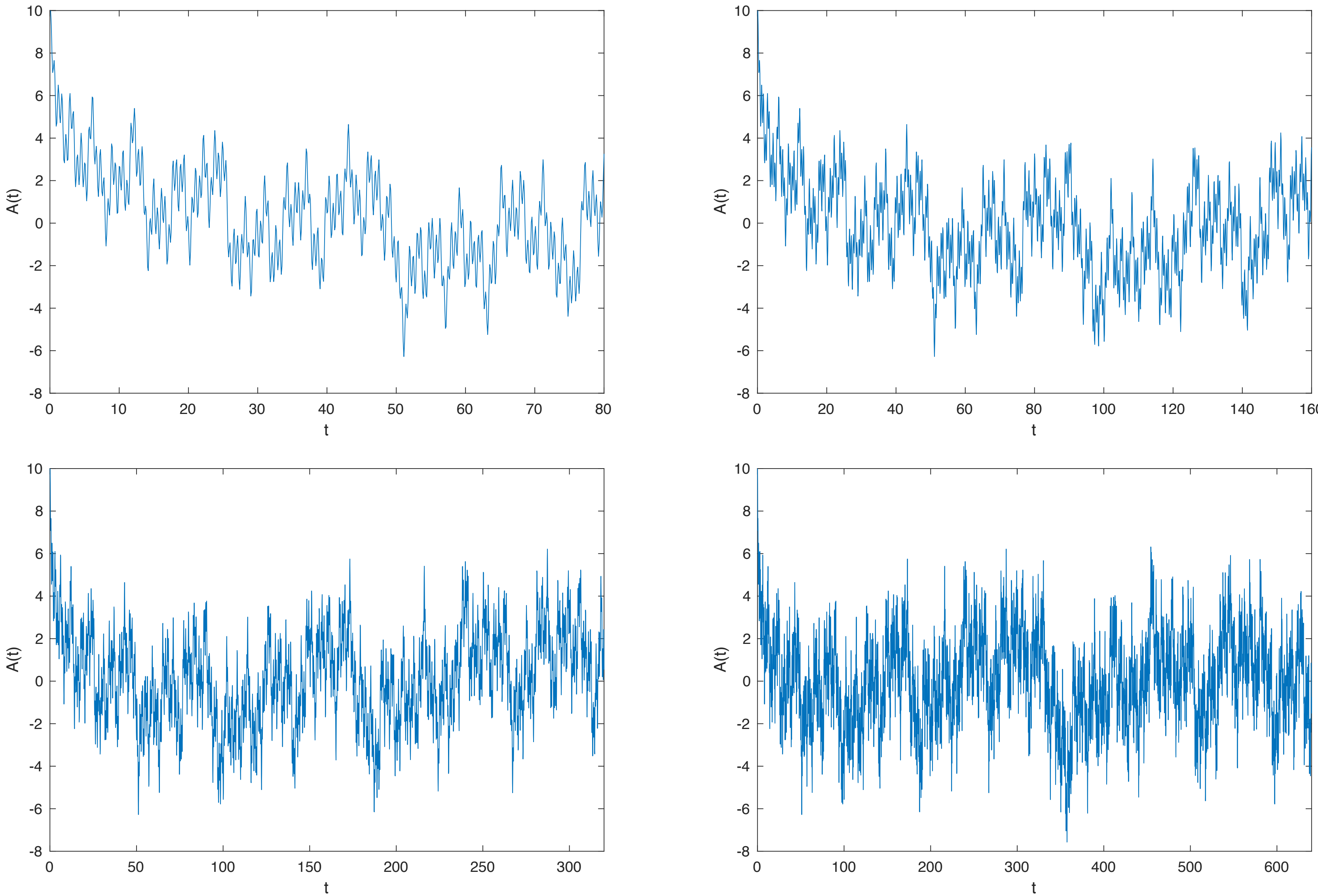
Given the self-replicating behavior of the amplitude under time rescaling, we call it a time fractal.

We choose an integer time scaling factor

$$\lambda_+^{\alpha-1} = 2$$

by taking the parameters

$$\alpha = 2, \quad J_0 = -1, \quad V_0 = -14.2388293$$



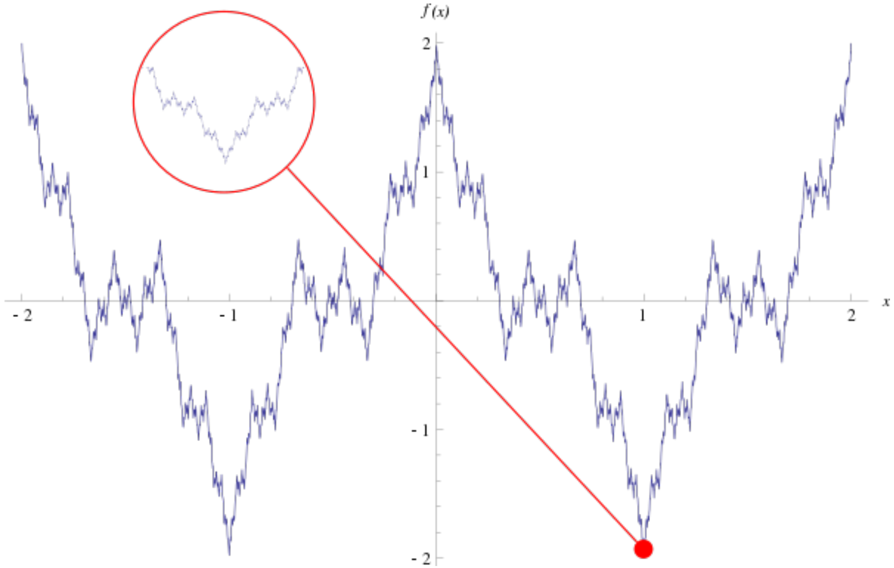
D.L., Watkins, Frame, Given, He, Li, Lu, Sarkar, PRA 100, 011403(R) (2019)

This is a particular case of the Weierstrass function,

$$w(t) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t) \qquad 0 < a < 1 < b \text{ with } ab \geq 1$$

In our case we take $a \to 1$, $b = 2$ and truncate after a finite number of terms. The next slide shows a picture of the Weierstrass function for $a = 0.5$, $b = 3$.

$$a = 0.5, \quad b = 3$$



The Weierstrass function has fractal dimension

$$D = 2 + \frac{\log a}{\log b}$$

Hardy, Trans. Amer. Math. Soc. 17, 301 (1916)
Hunt, Proc. Amer. Math. Soc. 126, 791 (1998)

Summary and Outlook

We have constructed a minimal nuclear interaction that can reproduce the ground state properties of light nuclei, medium-mass nuclei, and neutron matter simultaneously with no more than a few percent error in the energies and charge radii.

This SU(4)-invariant interaction has consequences for the computational reach of future nuclear lattice simulations. Opens the possibility of simulations with as many as one or two hundred nucleons.

Trapped ion quantum simulators offer a new way to produce quantum scale anomalies and do so quite naturally. Using several adjustable parameters, one can study a broad class of systems with discrete scale invariance and fractal-like time dependence.

There are many interesting questions that remain to be explored. For example, the N-boson systems exhibit multi-halo structures with heterogeneous discrete scale invariance.

We are using the quantum scale anomaly system as a testing ground for new quantum computing algorithms.