Few nucleons and other stories

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24th European Few-Body Conference

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TECHNISCHE UNIVERSITÄT DARMSTADT

Thanks...

...to my mentors and collaborators

- P. Klos, J. Lynn, H.-W. Hammer, A. Schwenk, K. Hebeler (TU Darmstadt)
- H.W. Grießhammer (G. Washington U.), U. van Kolck (IPN Orsay, U. of Arizona)
- D. Lee, S. Bogner (FRIB, Michigan State U.)
- R.J. Furnstahl (Ohio State U), T. Papenprock (UTK)
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...for funding and computing time:

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- Jülich Supercomputing Centre
- Lichtenberg Cluster (Darmstadt)

Outline

[...] the Few-Body Systems Award for Young Researchers

has been assigned to you "For contributions to

effective field theories

and

finite-volume techniques

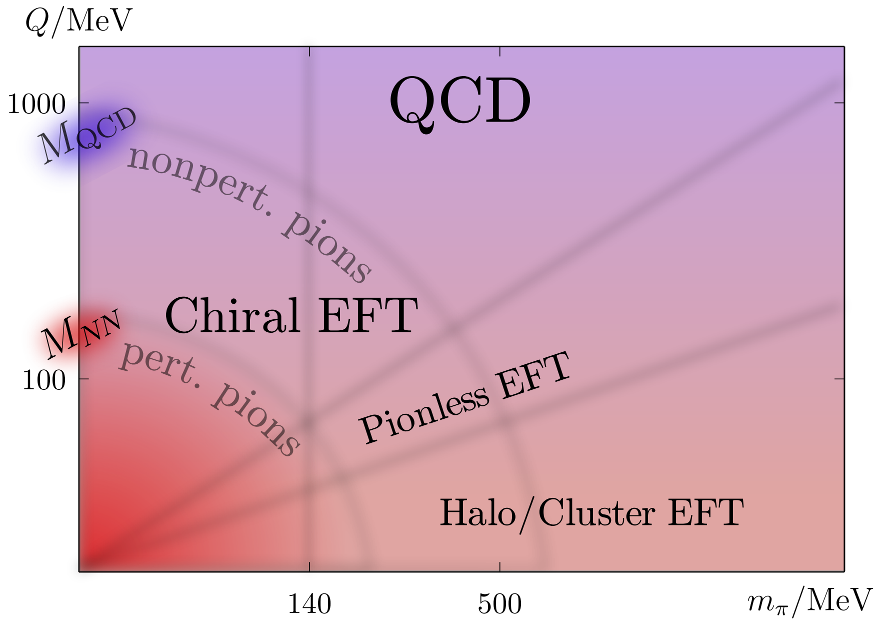
in the description of few-body systems."

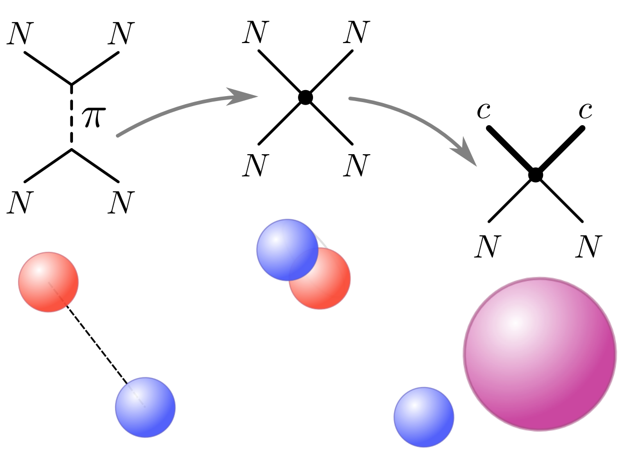
Part I

The unitarity expansion

Effective field theory

- choose **degrees of freedom** approriate to energy scale
- only restricted by **symmetry**, ordered by **power counting**





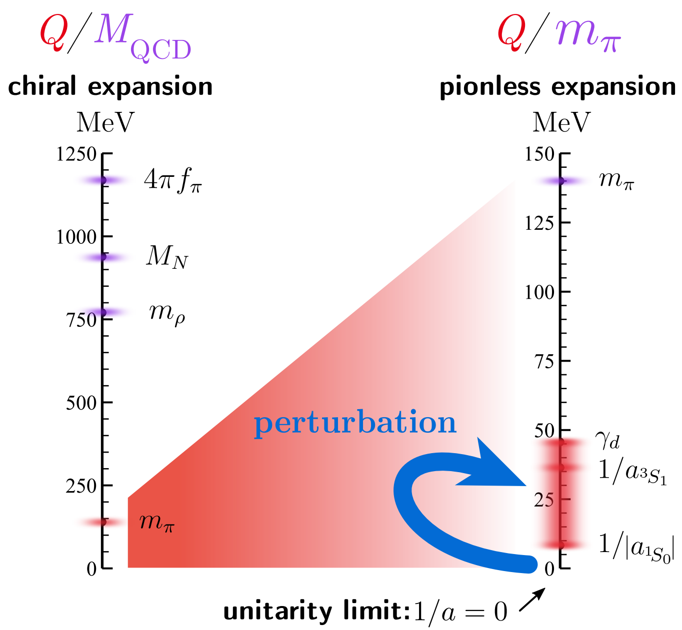
- unified discussion in recent review
- even more effective d.o.f.: rotations, vibrations

Hammer, SK, van Kolck, arXiv:1906:12122

Papenbrock, NPA **852** 36 (2011); ...

The unitarity expansion

Capture gross features at leading order, build up the rest as perturbative "fine structure!"



- **take unitarity limit as leading order**
  - infinite S-wave scattering lengths
  - deuteron at zero energy
- shift focus away from two-body details
- **physics in universality regime**

The unitarity expansion

Capture gross features at leading order, build up the rest as perturbative "fine structure!"

Nuclear sweet spot

- $1/a < Q_A < 1/R \sim m_\pi$
- $Q_A = \sqrt{2M_N B_A/A}$

SK et al. PRL **118** 202501 (2017)

A	2	3	4	$\cdots$	56
$\mathbf{Q_A R}$	0.3	0.5	0.8	$\cdots$	0.9

$\hookrightarrow$ iron not much different from ^{4}He

van Kolck (2018)

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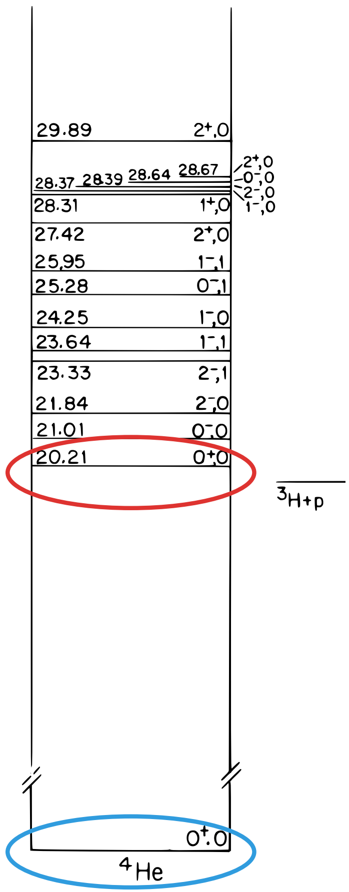
van Kolck (2018)

- **discrete scale invariance as guiding principle (Efimov effect!)**
  - near equivalence to bosonic clusters, exact SU(4) symmetry

Wigner, Phys. Rev. **51** 106 (1937); Mehen et al., PRL **83** 931 (1999); Bedaque et al., NPA **676** 357 (2000)
Vanasse+Phillips, FB Syst. **58** 26 (2017)

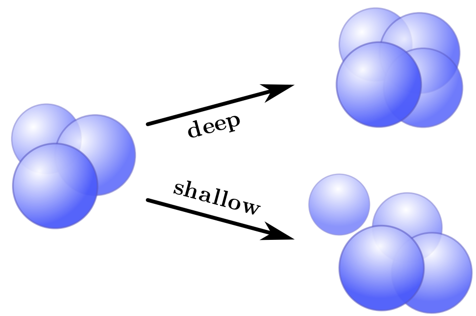
cf. also Kievsky+Gattobigio, EPJ Web Conf. **113** 03001 (2016), ...
A. Kievsky, talk on Monday; D. Lee, talk on Tuesday

Efimov trimers and tetramers



TUNL nuclear data

- ^{3}H as Efimov state   Efimov, PLB **33** 563 (1970); Bedaque et al.(2000)
- **two associated tetramers for each Efimov state**
  Hammer+Platter, EPJA **32** 13 (2007); von Stecher, JPB **43** 101002 (2010); ...



- **at unitarity**
  - $B_4/B_3 \simeq 4.611$, $B_{4*}/B_3 \simeq 1.002$ Deltuva, PRA **82** 040701 (2010)
- **in ^{4}He**
  - ground state at $B_\alpha/B_H \simeq 3.66$
  - resonance at $B_{\alpha*}/B_H \simeq 1.05$ (where $B_H = 7.72$)

Unitarity prescription

SK et al., PRL **118** 202501 (2017)

(1) describe strong force with contact interaction

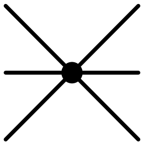
$$C_0 = \underbrace{C_0^{(0)}}_{\text{leading order (LO)}} + C_0^{(1)} + \cdots$$

- momentum cutoff Λ gives "smearing"
- fit $C_0^{(0)}$ to get $a = \infty$ in both NN S-wave channels

(2) fix Efimov spectrum to physical triton energy

- pionless LO three-body force
- **triton as "anchor" at each order**

Bedaque et al., NPA **676** 357 (2000)



(3) include in perturbation theory

- finite a, Coulomb
- range corrections
- all further higher-order corrections

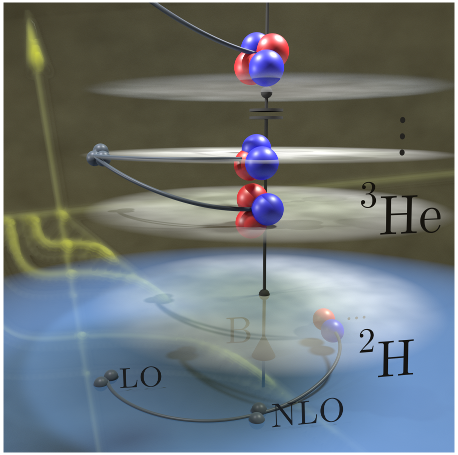
Leading order has a single parameter,
all the rest is a perturbation!

Unitarity expansion summary

- **novel approach to few-nucleon systems**
  - leading order at unitarity limit (infinite scattering length)
  - everything else as perturbative fine structure

SK et al., PRL **118** 202501 (2017); SK, JPG **44** 064007 (2017)

	LO	NLO*	N^2LO	exp.
^{2}H	0	0	1.4(1.1)	**2.22**
^{3}H	8.48	8.48	8.48	**8.48**
^{3}He	8.48	7.6(2)	7.72	**7.72**
^{4}He	39(12)	30(9)		**28.3**



- **part of greater nuclear simplification trend**

C. Elster, talk on Monday; D. Lee, talk on Tuesday

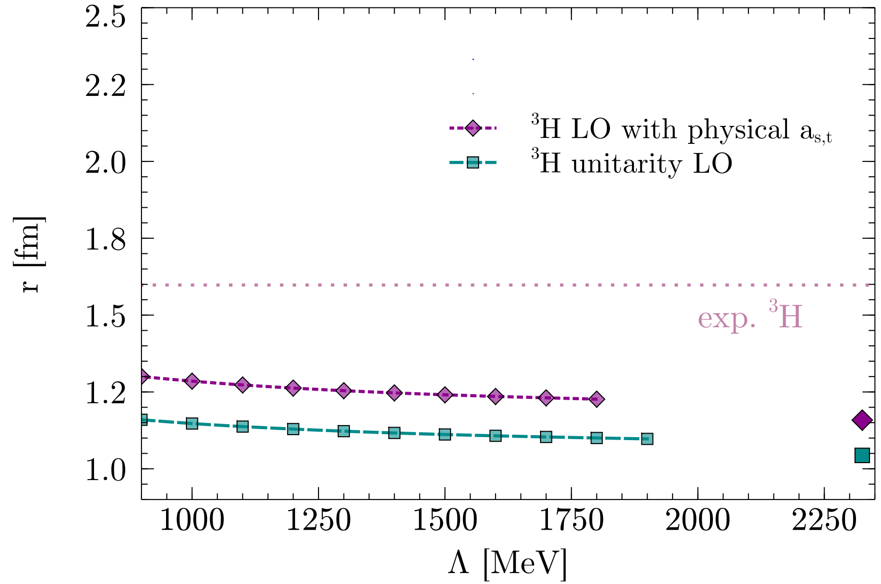
- demonstrates feasability of perturbative EFT calculations for $A > 3$
  - unified Faddeev/Faddeev-Yakubowsky framework

Kamada + Glöckle, NPA **548** 205 (1992); Platter (2005)

Recent developments

Charge radii

- **calculate charge form factors** $F_C(q) = \langle\Psi|\rho(q)|\Psi\rangle \rightsquigarrow \langle r^2\rangle = -\frac{1}{6}\frac{d^2}{dq^2}F_C\,,\ q \to 0$



- point charge radii: subtract effects from r_p and r_n
- **triton result in excellent agreement with previous pionless calculations**
  - range corrections known to be large

Vanasse, PRC **95** 024002 (2017); Vanasse+Phillips, FB Syst. **58** 26 (2017)

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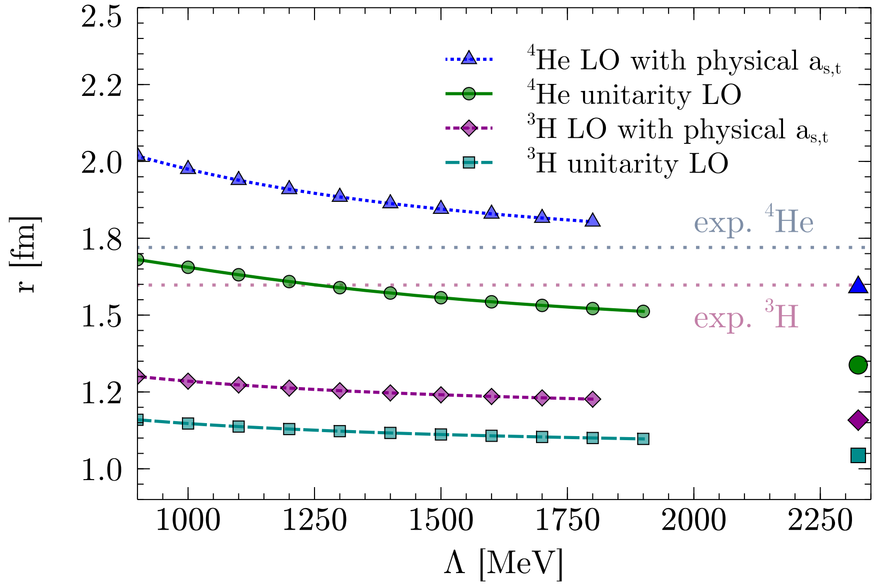

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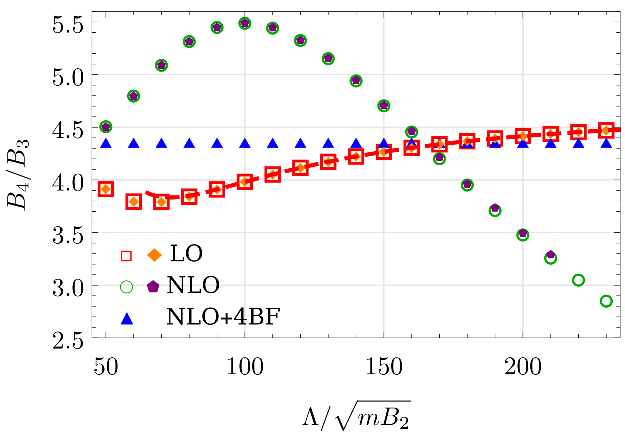
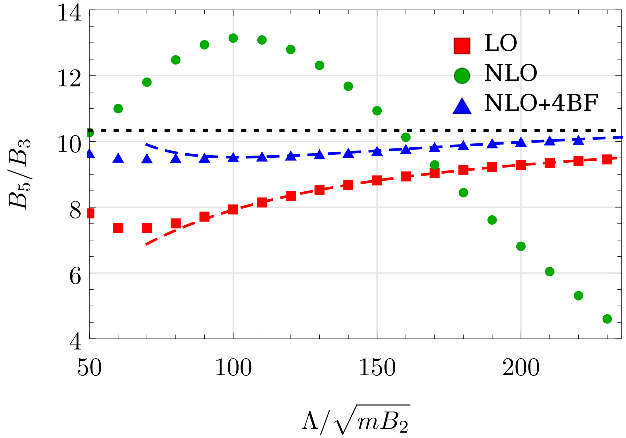
Vanasse, PRC **95** 024002 (2017); Vanasse+Phillips, FB Syst. **58** 26 (2017)

NLO four-body force

- full next-to-leading order includes range corrections $\sim C_2\,(k^2 + k'^2)$
- cancel in trinucleon energy splitting, but not in general
- four-boson energy does not converge with cutoff
- **promotion of four-body force to NLO**

Bazak, Kirscher, SK et al., PRL **122** 143001 (2019)





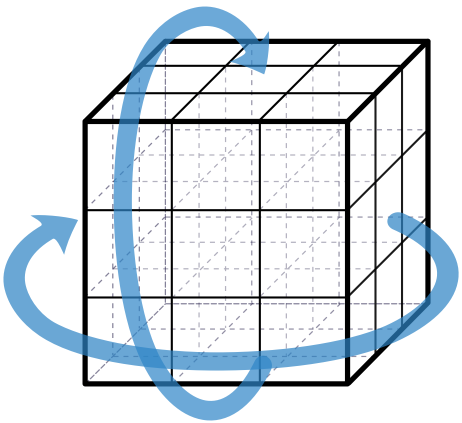
- **inclusion of four-body force stabilized five- and six-body system as well**
- general prediction for promotion of many-body forces (for bosons!)

see talk by J. Kirscher tomorrow

Part II

Few-body states in finite volume

Finite periodic boxes



- physical system enclosed in finite volume (box)
- typically used: periodic boundary conditions
- **leads to volume-dependent energies**

Lüscher formalism

- physical properties encoded in the vol.-dependent energy levels!
- infinite-volume S-matrix governs discrete finite-volume spectrum
- **finite volume used as theoretical tool**

Lüscher, Commun. Math. Phys. **104** 177 (1986); ...

Few-body states in finite volume

Bound states

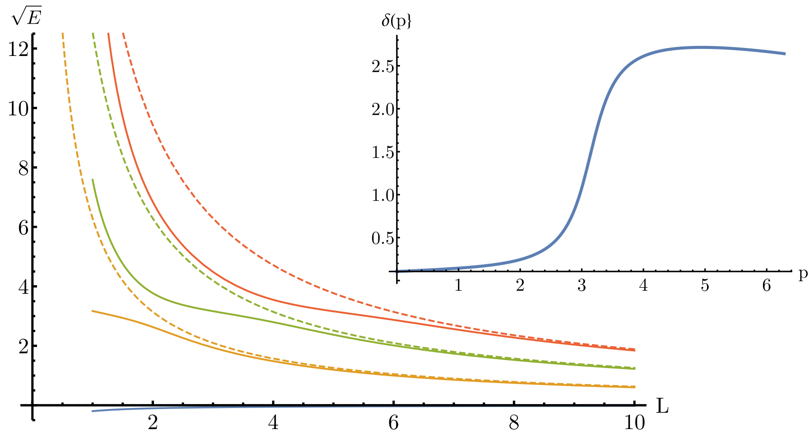
- **exponential volume dependence**
- physics encoded in prefactor and fall-off scale

Lüscher, Commun. Math. Phys. **104** 177 (1986); ...

Resonances

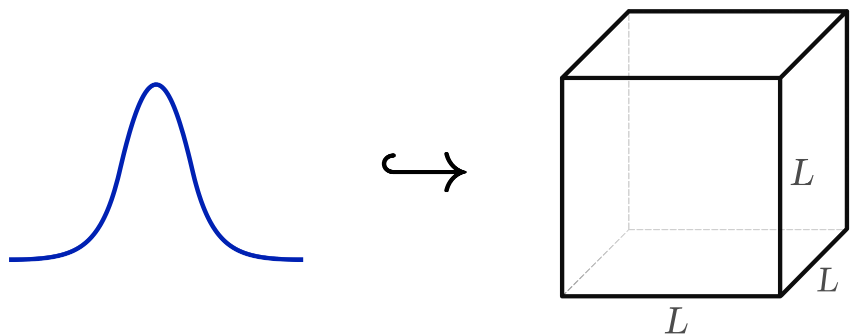
- continuum discretized into states with power-law volume dependence
- resonances show up as **avoided crossings**

Wiese, NPB (Proc. Suppl.) **9** 609 (1989); ...



Two-body bound states

- consider bound state with energy $E_B = -\kappa^2/(2\mu)$
- binding momentum κ corresponds to intrinsic length scale
- **finite volume affects the binding energy:** $E_B(L)$



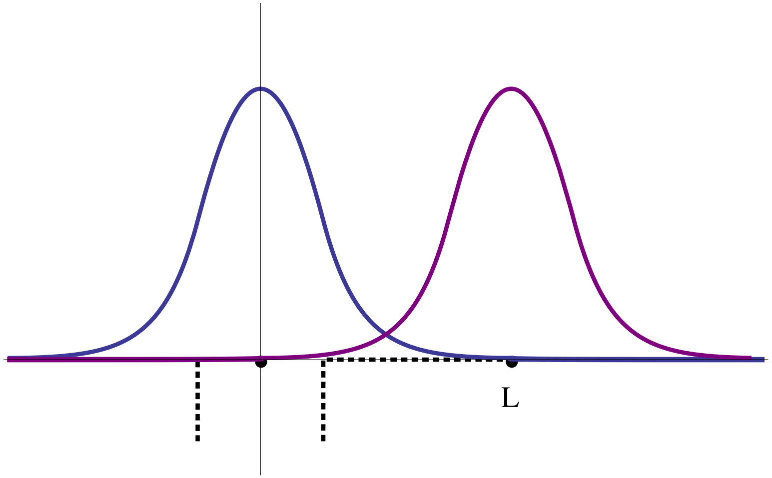
- for S-wave states, one finds $\Delta B(L) \sim -|\gamma|^2 \exp\big(-\kappa L\big)/L + \cdots$, $\boldsymbol{\gamma}$ **= ANC**

  Lüscher, Commun. Math. Phys. **104** 177 (1986); ...
- in general, the prefactor is a polynomial in $1/\kappa L$

  SK et al., PRL **107** 112001 (2011); Annals Phys. 327, 1450 (2012)
- **asymptotic wavefunction determines volume dependence**

Two-body bound states

$$\Delta B(L) = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3 r \, \psi_B^*(\mathbf{r}) \, V(\mathbf{r}) \, \psi_B(\mathbf{r} + \mathbf{n}L) + \mathcal{O}(\mathrm{e}^{-\sqrt{2}\kappa L})$$



- peridic boundary: sum over nearest neigbour overlapping wavefunctions
- contribution to integral only inside potential range
- use Schrödinger equation to eliminate potential: $\psi(\mathbf{r})V(\mathbf{r}) = [\Delta_r - \kappa^2]\psi(\mathbf{r})$

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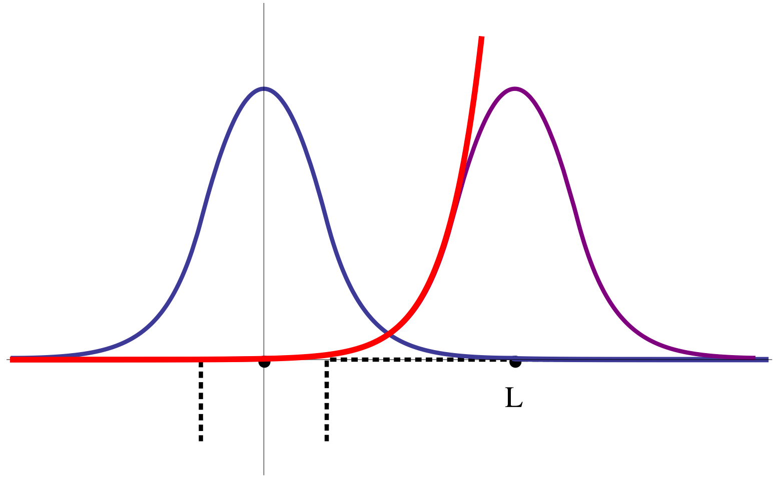

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- only asymptotic tail matters: $\psi(\mathbf{r}) \sim \gamma \times \exp\left(-\kappa r\right)/r$

Now consider the general case

N **particles**

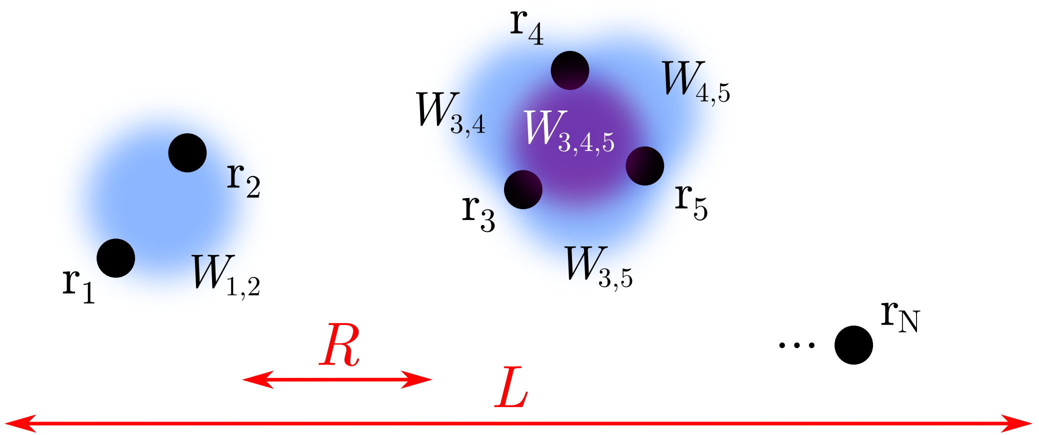
d **spatial dimensions**

N-body setup

- 2- up to N-body interactions, can be local or non-local

$$V_{1\cdots N}(\mathbf{r}_1,\cdots\mathbf{r}_N;\mathbf{r}'_1,\cdots\mathbf{r}'_N)=\sum_{i<j}W_{i,j}(\mathbf{r}_i,\mathbf{r}_j;\mathbf{r}'_i,\mathbf{r}'_j)1_{\not{i},\not{j}}+\cdots$$

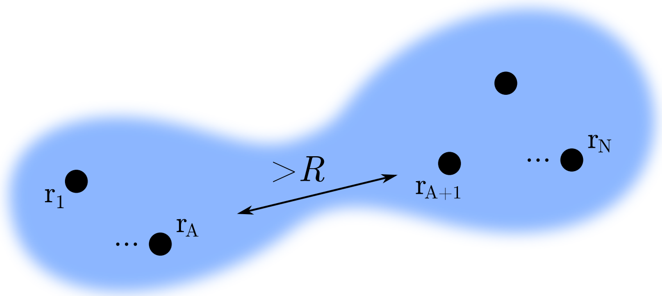
- all with finite range, set $R=\max\{R_{i,j},\cdots\}$
- assume asymptotically large volume: $L \gg R$



General result

Separate A particles and factorize wavefunction

SK + Lee, PLB **779** 9 (2018)



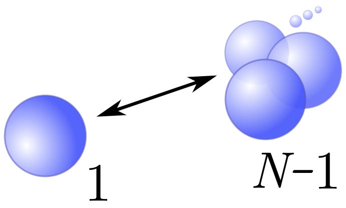
$$\psi_N^B(\mathbf{r}_1,\cdots\mathbf{r}_N) \sim \psi_A^B(\mathbf{r}_1,\cdots\mathbf{r}_A)\psi_{N-A}^B(\mathbf{r}_{A+1},\cdots\mathbf{r}_N)$$
$$\times\left(\kappa_{A|N-A}r_{A|N-A}\right)^{1-d/2} K_{d/2-1}\left(\kappa_{A|N-A}r_{A|N-A}\right)$$

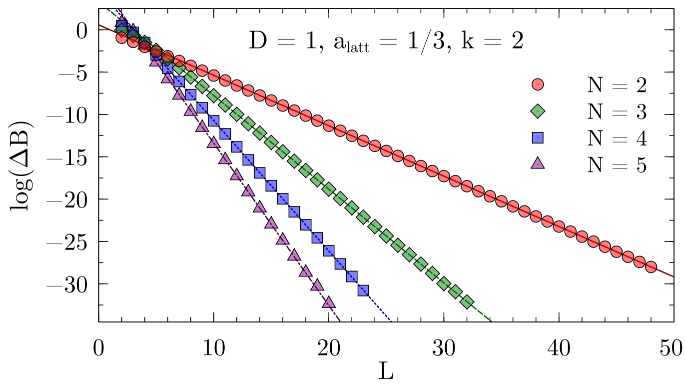
$$\Delta B_N(L) \propto (\kappa_{A|N-A}L)^{1-d/2}\,K_{d/2-1}(\kappa_{A|N-A}L)\ \sim \exp\left(-\kappa_{A|N-A}L\right)/L^{(d-1)/2}$$

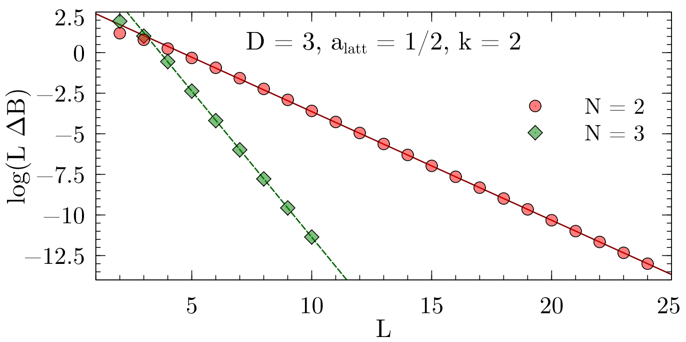
- **smallest** $\kappa_{A|N-A} = \sqrt{2\mu_{A|N-A}(B_N - B_A - B_{N-A})}$ **governs volume dependence**
- this assumes both clusters to be bound (otherwise: power-law correction factors)
- prefactor determined by asymptotic normalization constant (ANC)

Numerical results

- consider one particle separated from the rest







- diagonalization of discretized Hamiltonian
- interaction = short-range Gaussian two-body potentials
  - $L \gg R$ well satisfied for $L \gtrsim 5$
- all quantities in natural units with mass = 1
- **straight lines indicate excellent agreement with analytical result**

Finally: Few-body resonances

Finite-volume resonance signatures

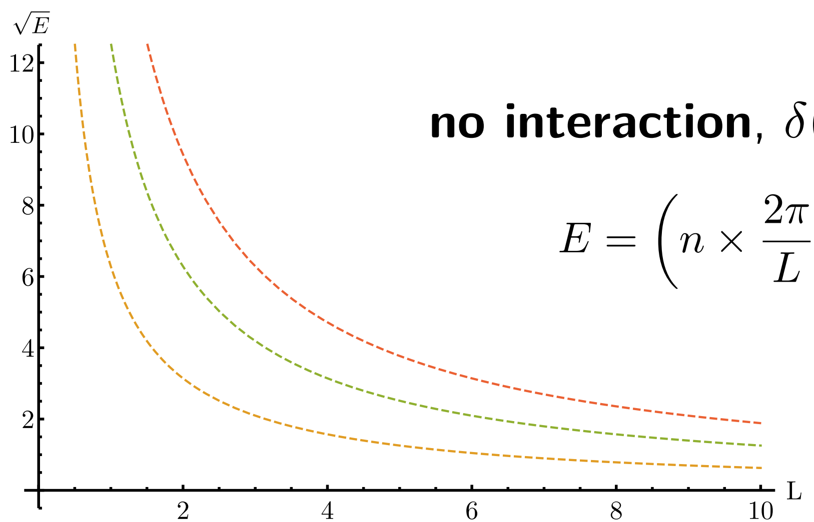
Lüscher formalism

$$p \cot \delta_0(p) = \frac{1}{\pi L} S(\eta) \;\; , \;\; \eta = \left(\frac{Lp}{2\pi}\right)^2 \;\; , \;\; p = p\big(E(L)\big)$$

Lüscher, NPB **354** 531 (1991); ...

- finite volume $\to$ momentum quantization $\to$ discrete energy levels
- **resonance contribution** $\rightsquigarrow$ **avoided level crossing**

Wiese, NPB (Proc. Suppl.) **9** 609 (1989); ...



no interaction, $\delta(p) = 0$

$$E = \left(n \times \frac{2\pi}{L}\right)^2$$

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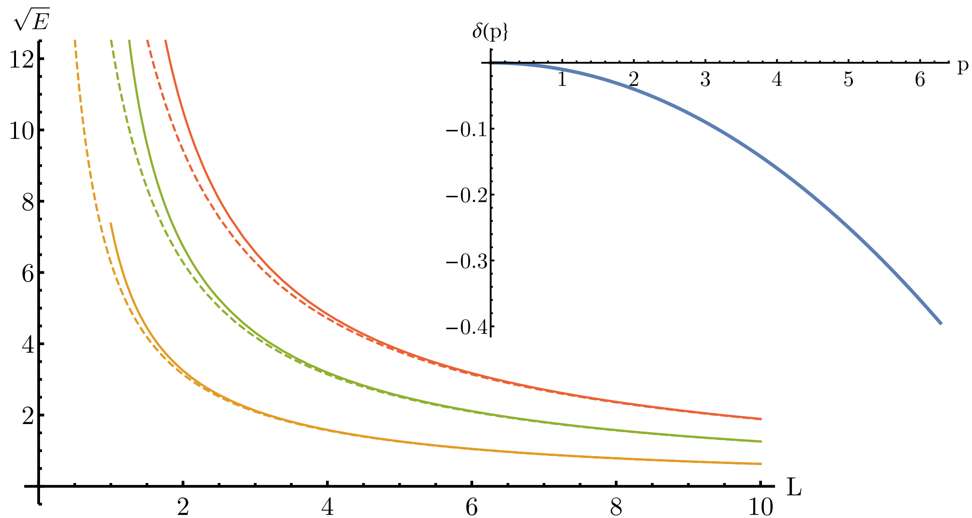

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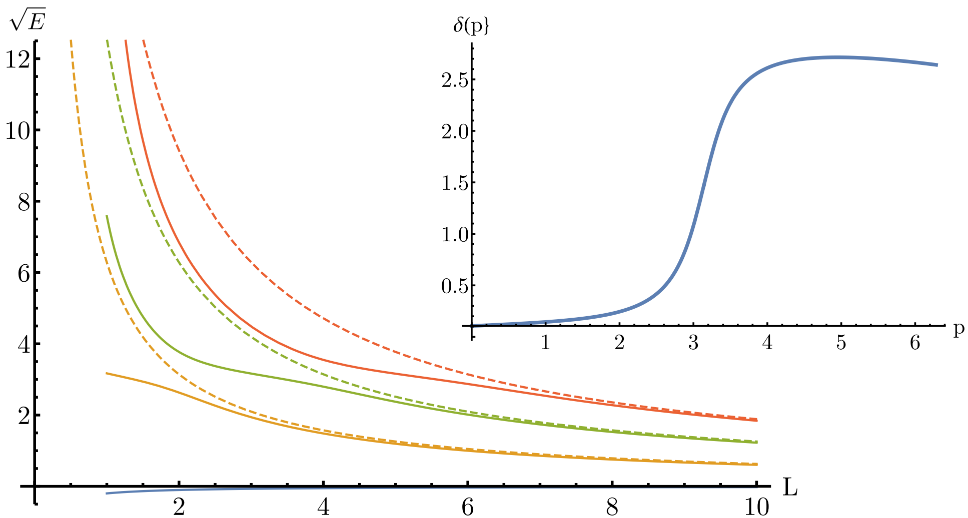


Discrete variable representation

Need calculation of several few-body energy levels

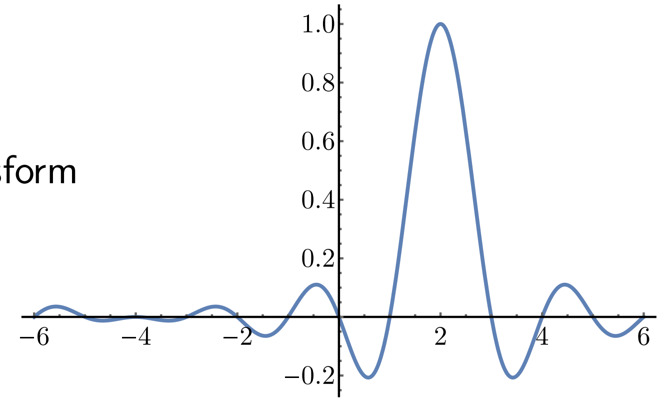
- difficult to achieve with QMC methods

Klos et al., PRC **94** 054005 (2016)

- **use a Discrete Variable Representation (DVR)**

well established in quantum chemistry, suggested for nuclear physics by Bulgac+Forbes, PRC **87** 051301 (2013)

- basis functions localized at grid points
- potential energy matrix diagonal
- kinetic energy matrix sparse...
- ...or implemented via Fast Fourier Transform



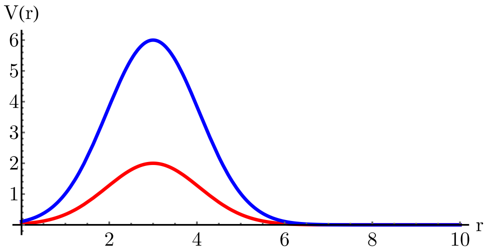
- periodic boundary condistions ↔ plane waves as starting point
- **implementation for large-scale calculations**
  - numerical framework scales from laptop to HPC clusters

SK et al., PRC **98** 034004 (2018)

Two-body check

Use model potential to produce S-wave resonance

- shifted gaussian barrier
  - $V(r) = V_0 \exp\left(-\left(\frac{r-a}{R_0}\right)^2\right)$
- tune parameters to generate resonances



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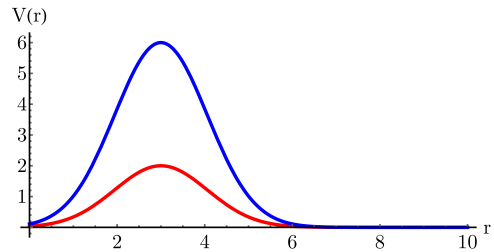
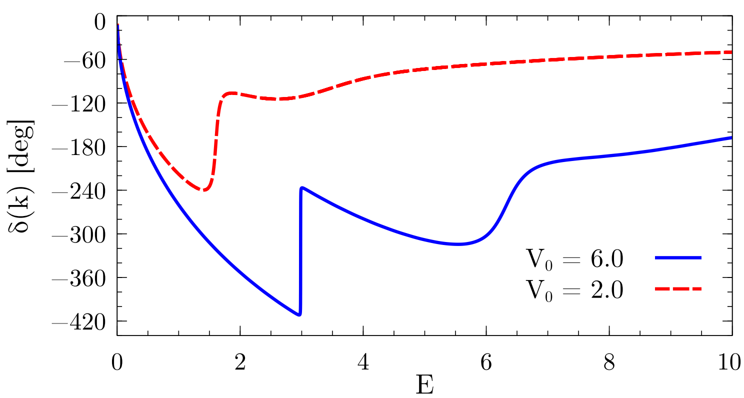
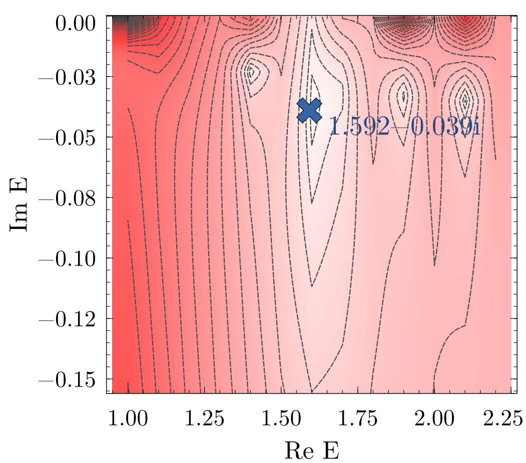


Phase shifts and S-matrix pole





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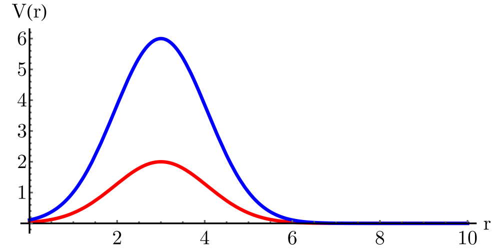
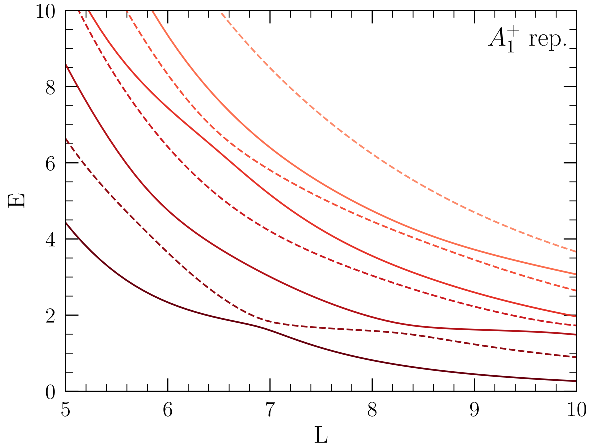
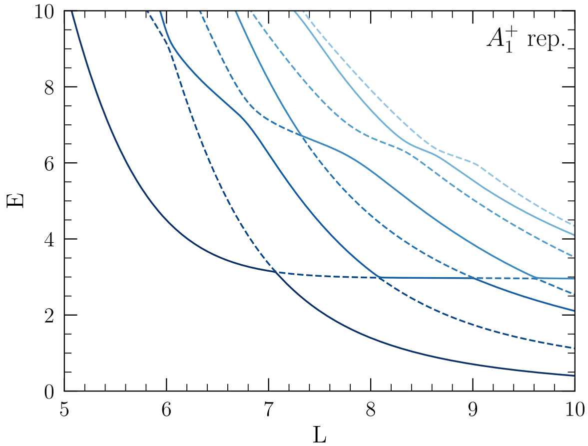


Finite-volume spectra



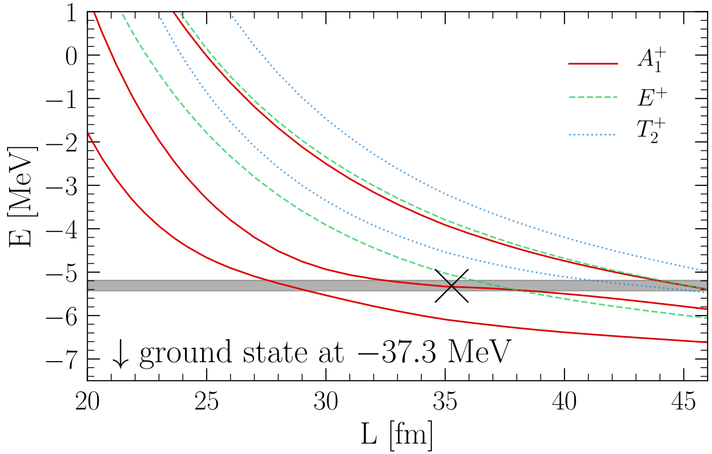


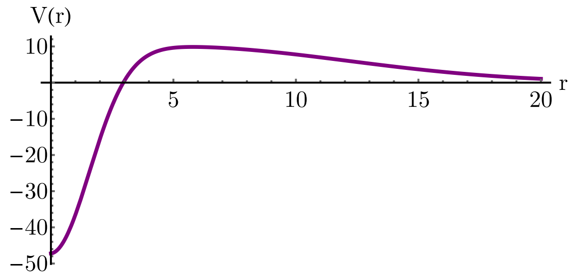
Three-body check

Study established three-body resonance from literature

Fedorov et al., Few-Body Syst. **33** 153 (2003); Blandon et al., PRA **75** 042508 (2007)

- three bosons with mass $m = 939.0$ MeV, potential = sum of two Gaussians
- **three-body resonance at**
  - $-5.31 - i0.12$ MeV (Blandon et al.)
  - $-5.96 - i0.40$ MeV (Fedorov et al.)





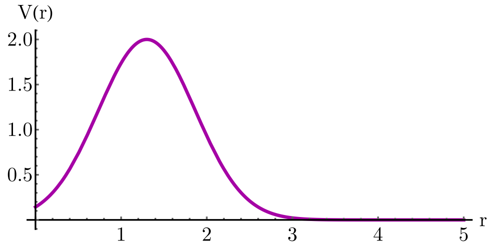
- **fit inflection point(s) to extract resonance energy:** $E_R = -5.32(1)$ MeV

SK et al., PRC **98** 034004 (2018)

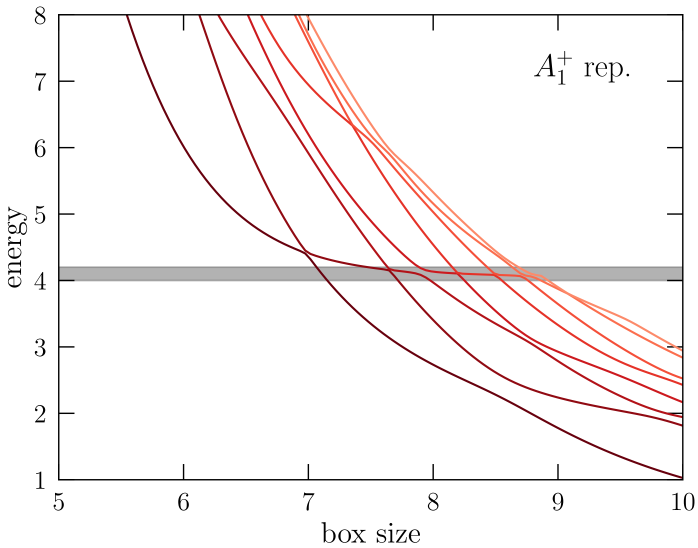
Genuine three-body resonance

Three-boson system

- shifted Gaussian two-body potential
- **no two-body bound state!**
- add short-range three-body force



SK et al., PRC **98** 034004 (2018)



Genuine three-body resonance

Three-boson system

- shifted Gaussian two-body potential
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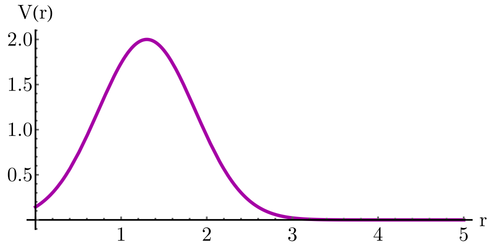

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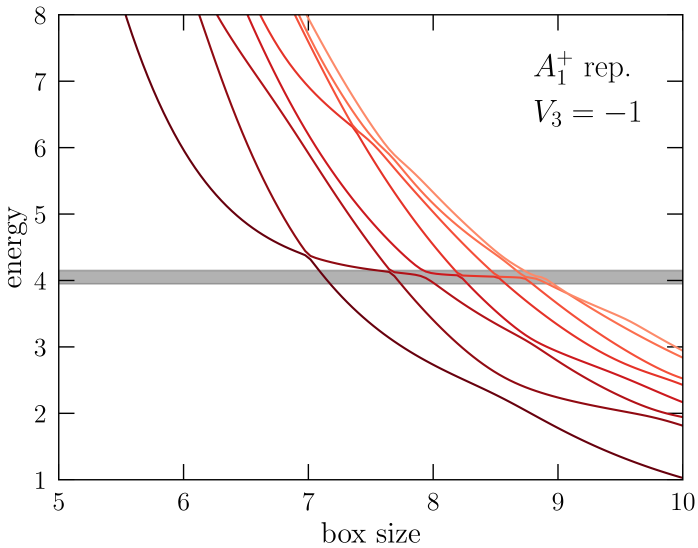

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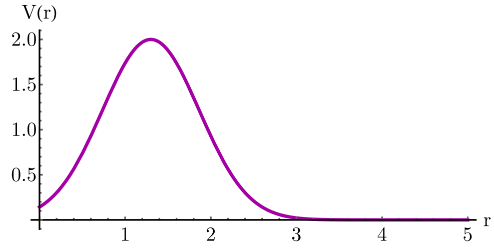
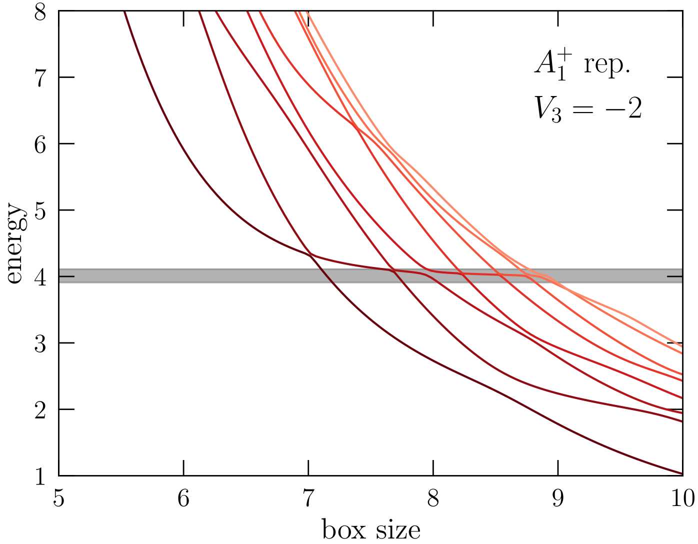


SK et al., PRC **98** 034004 (2018)



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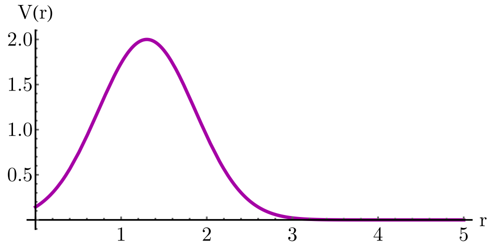
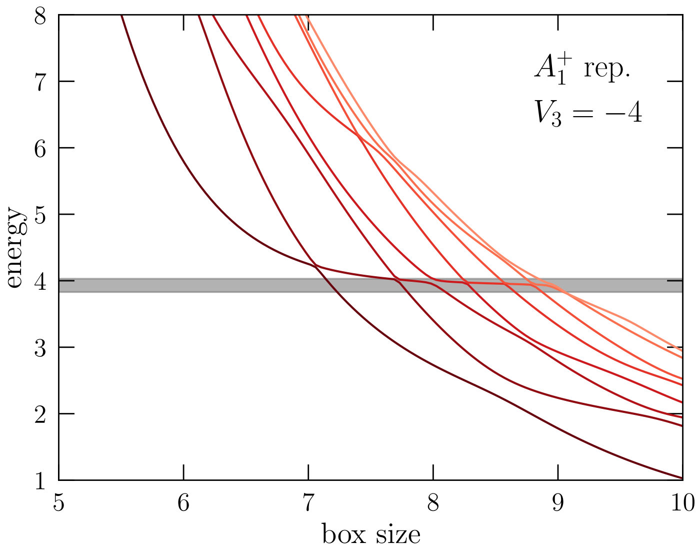


SK et al., PRC **98** 034004 (2018)



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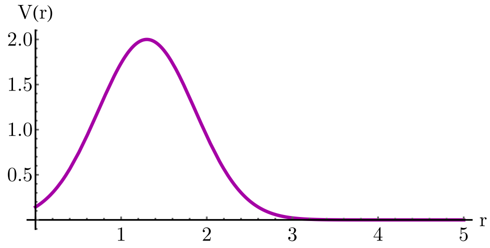
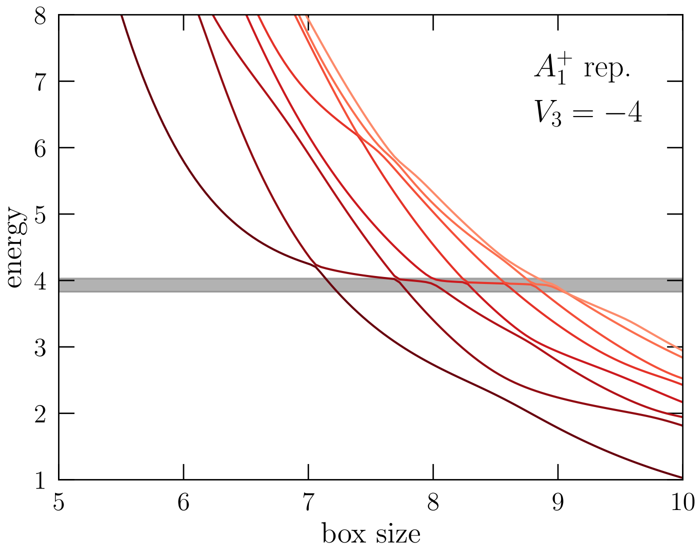


SK et al., PRC **98** 034004 (2018)



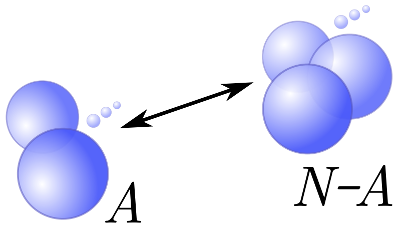
- **possible to move three-body state ↔ spatially localized wavefunction**

Finite-volume summary

Bound states

SK + Lee, PLB **779** 9 (2018)

- leading volume dependence known for **arbitrary bound states**
- reproduces known results, **checked numerically**
- calculate ANCs, **single-volume extrapolations possible!**
- applications to lattice QCD, EFT, cold-atomic systems



Resonances

SK et al., PRC **98** 034004 (2018)

- explicit proof of concept for **up to four particles**
- efficient **large-scale numerical implementation (DVR)**
- different concrete **physics applications**
  - **few-neutron systems**, alpha clusters, atomic resonances, ...

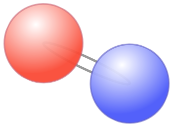
Thank you

Backup slides

Implementation

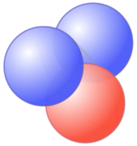
Unified (2-, 3-, 4-body) numerical framework

Two-nucleon system



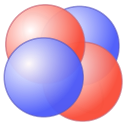
- separable regulator for contact interactions: $V = C_0|g\rangle\langle g|$
- can be solved analytically to get scattering amplitudes

Three-nucleon system



- Faddeev equations: $|\psi\rangle = G_0 t_2 P|\psi\rangle + G_0 t_2|\psi_3\rangle$
- **used to fit three-body force**

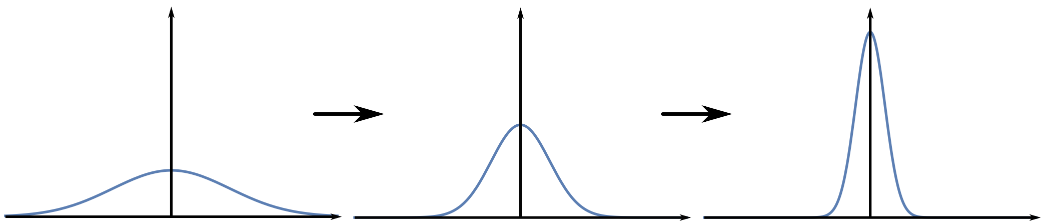
Four-nucleon system



- Faddeev-Yakubowsky equations: two components $|\psi_{A,B}\rangle$
- need full wavefunction for perturbation theory:
  - $|\Psi\rangle = (1{-}P_{34}{-}PP_{34})(1+P)|\psi_A\rangle + (1{+}P)(1{+}\tilde{P})|\psi_B\rangle$

The cutoff

- increasing the momentum cutoff Λ decreases interaction range
- **RG invariance:** fix $C = C(\Lambda)$ to keep input observables invariant

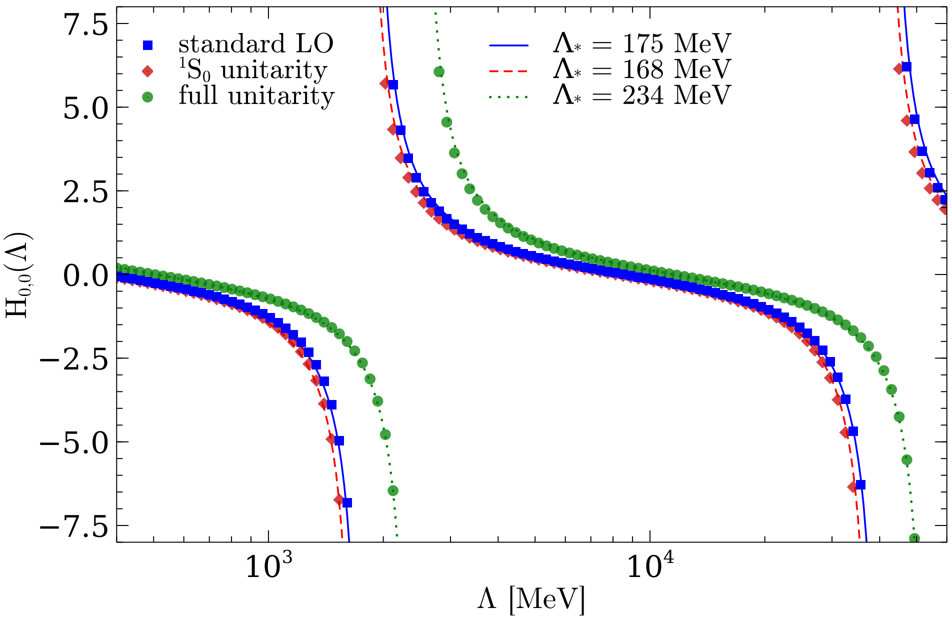


- **predicted observables should converge as Λ increases...**
- **...but individual contributions generally do not, e.g.:**

Λ / MeV	800	1000	1200	1400
E_{kin} / MeV	$+113.67$	$+140.58$	$+168.44$	$+197.09$
E_{pot} / MeV	-139.77	-167.47	-195.76	-224.62

Three-body force running

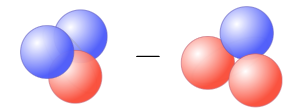
- three-body force $D_0^{(0)}(\Lambda)$ depends on a single parameter Λ_*
- fit to triton binding energy (in this case)

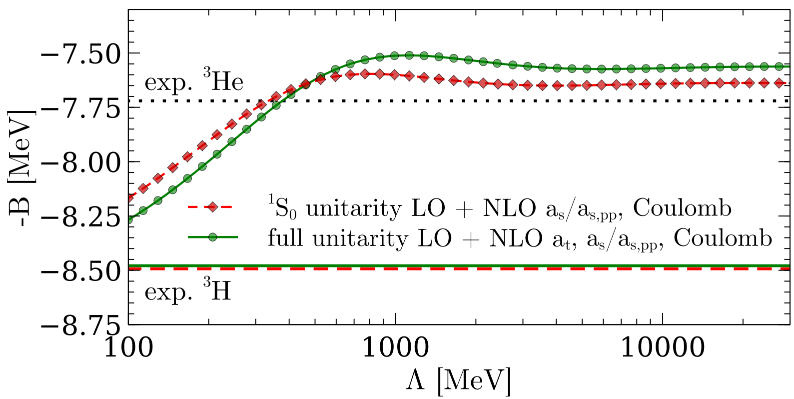


- **not much shift in Λ_* due to unitarity limit**

Trinucleon energy difference

- at LO ^{3}H and ^{3}He are degenerate (exact isospin symmetry)
- Coulomb correction enters together with $1/a_{s,pp}$ at NLO
- predict binding energy difference

ΔE_3 : 



	LO	NLO	exp.
^{3}H	8.48	8.48	**8.48**
^{3}He	8.48	7.6(2)	**7.72**

(red = input)

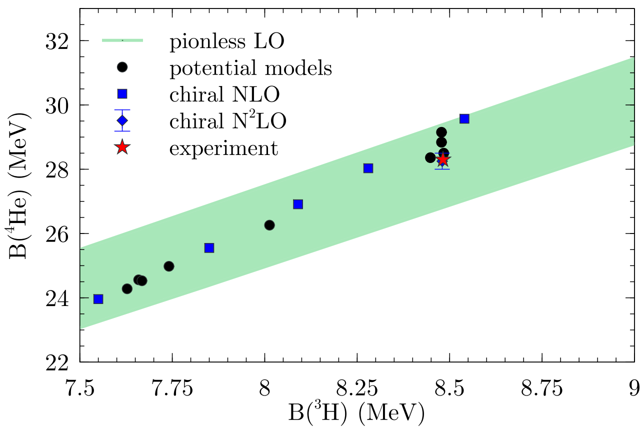
SK et al. PRL **118** 202501 (2017)

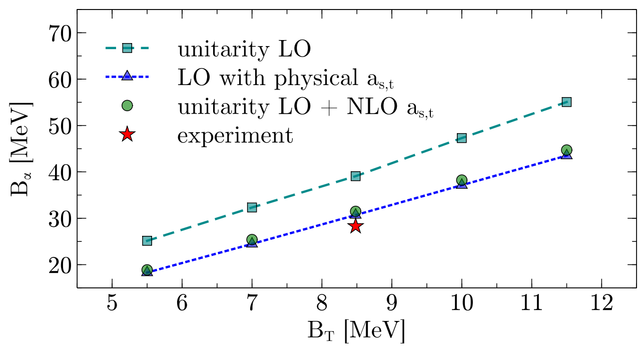
- **range corrections cancel at NLO**
  - leading order is isospin symmetric
  - small isospin breaking $r_{pp} \neq r_{np}$ (5%) relegated to next higher order

SK et al. JPG **43** 055106 (2016)

Tjon line

Correlation between ^{3}H and ^{4}He binding energies



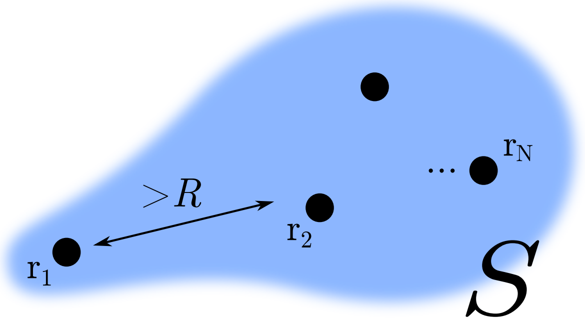


- originally observed comparing different potential models
- **well reproduced by unitarity expansion**
- **perturbativeness of $1/a$ persists off the physical point**

Tjon, PLB **56** 217 (1975)

Cluster separation

- **consider one particle separated from all others**



$$S = \{(\mathbf{r}_1, \cdots \mathbf{r}_N) : |\mathbf{r}_1 - \mathbf{r}_i| > R \quad \forall i = 2, \cdots N\}$$

- **look at Hamiltonian restricted to S**

$$\hat{H}\big|_S = \sum_{i=2}^{N} \left[\hat{K}_i - \hat{K}^{\mathrm{CM}}_{2\cdots N} + \hat{V}_{2\cdots N}\right] + \hat{K}^{\mathrm{rel}}_{1|N-1}$$

no interaction $\hat{V}_{1\ldots}$!

- **sepration ansatz:** $\Psi(\mathbf{r}_1, \cdots \mathbf{r}_N) = \sum_{\alpha} f_\alpha(\mathbf{r}_2, \cdots \mathbf{r}_N)\; g_\alpha(\mathbf{r}_{1|N-1})$
  - lowest f_0 is eigenstate of sub-Hamiltonian with energy $-B_{N-1}$
  - g_0 is Bessel function with scale set by $B_N - B_{N-1}$

Analytical examples

Three bosons at unitarity

- two-body interaction with zero range and infinite scattering length

$$\Delta B_3(L) \propto (\kappa_{1|2}L)^{-1/2} K_{1/2}(\kappa_{1|2}L) P(\kappa_{1|2}L)$$
$$\sim \exp\left(-\sqrt{\tfrac{4mB_3}{3}}L\right)\left(\sqrt{\tfrac{4mB_3}{3}}L\right)^{-1} P(\kappa_{1|2}L)$$

- same exp. dependence as exact result ✓
- by comparison, power-law factor $P(x) = x^{-1/2}$

Meißner et al., PRL **114** 091602 (2015)

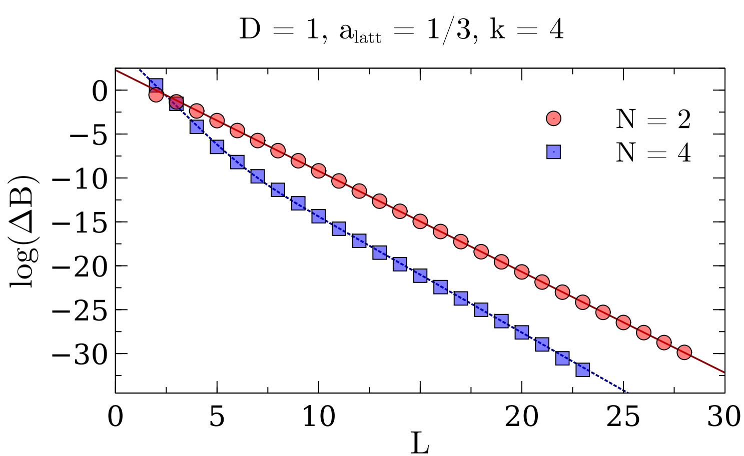
N particles with N-body interaction only

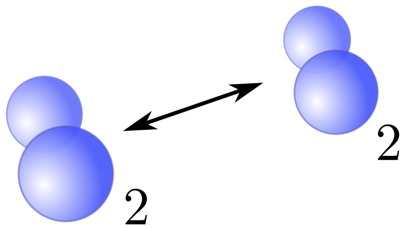
- spinless N-particle bound state with only an N-particle interaction
- $\psi(\mathbf{r}_1, \cdots) \propto (\kappa_{1|N-1} r_{1|N-1})^{1-d(N-1)/2} K_{d(N-1)/2-1}(\kappa_{1|N-1} r_{1|N-1})$
- again leads exactly to expected exp. dependence ✓
- read off power-law factor $P(x) = x^{-d(N-2)/2}$

More complicated example

Typically one channel dominates, but not necessarily...

- take an attractive two-body force $\rightsquigarrow B_2 < 0$
- add a repulsive three-body force $\rightsquigarrow$ no three-body bound state
- add attractive four-body force $\rightsquigarrow B_4 < 0$





$\kappa_{\mathrm{fit}} = 1.318$ $\kappa_{2|2} = 1.282$

- contributions from two channels clearly visible
- **asymptotic slope in good agreement with $2|2$ separation**

Further possibilities

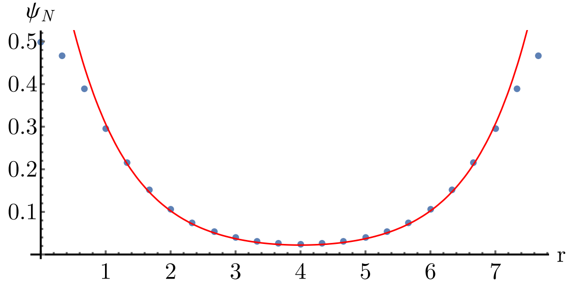
(a) Extract ANC from volume dependence

$$\Delta B_N(L) = \frac{(-1)^{\ell+1}\sqrt{\frac{2}{\pi}}f(d)|\gamma|^2}{\mu_{A|N-A}}\kappa_{A|N-A}^{2-d/2}L^{1-d/2}K_{d/2-1}(\kappa_{A|N-A}L)$$

- ANC γ relevant for low-energy capture reactions
- notoriously difficult to measure experimentally (especially for charged systems)
- **possible to extract ANC for arbitrary cluster systems**

(b) Extrapolate from single-volume calculations

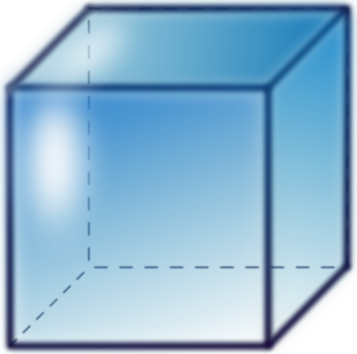
- get N-body and $(N-A)$-body wavefunctions
  - look along fixed direction
  - mind periodic boundary condition
- $\kappa_{A|N-A}$ obtained from same fit
- $\hookrightarrow$ **directly calculate $\Delta B_N(L)$**



Broken symmetry

- **the finite volume breaks the spherical symmetry of the system**

$\rightarrow$



rotation group $SO(3)$ **cubic group O**

- **irreducible representations of $SO(3)$ are reducible with respect to O**

  - finite subgroup of $SO(3)$
  - number of elements $= 24$
  - five irreducible representations

Γ	A_1	A_2	E	T_1	T_2
$\dim\Gamma$	1	1	2	3	3

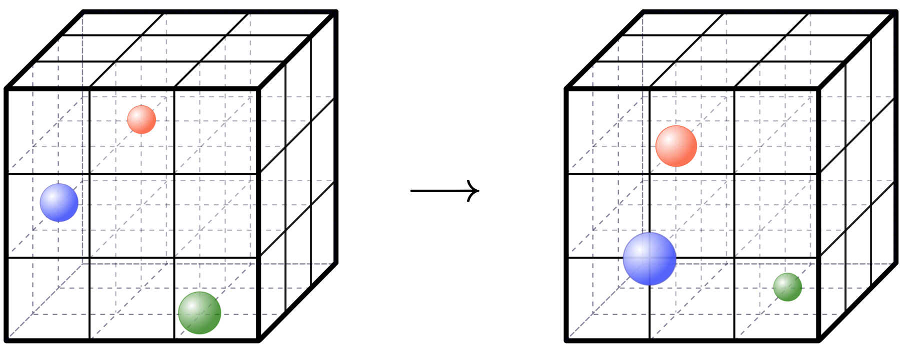
Cubic projection

Cubic projector

$$\mathcal{P}_\Gamma = \frac{\dim\Gamma}{24} \sum_{R \in \mathcal{OO}} \chi_\Gamma(R) D_n(R)\,,\ \chi_\Gamma(R) = \text{character}$$

Johnson, PLB **114** 147 (1982)

- $D_n(R)$ realizes a cubic rotation R on the n-body DVR basis
- $\rightsquigarrow$ permutation/inversion of relative coordinate components
- indices are wrappen back into range $-N/2, \ldots, N/2-1$

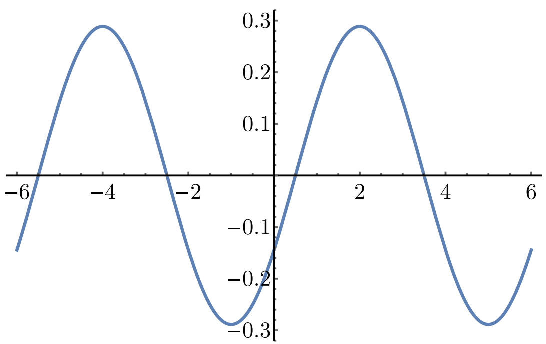


- **numerical implementation:** $\hat{H} \to \hat{H} + \lambda(1 - \mathcal{P}_\Gamma)$, $\lambda \gg E$

DVR construction

Basic idea

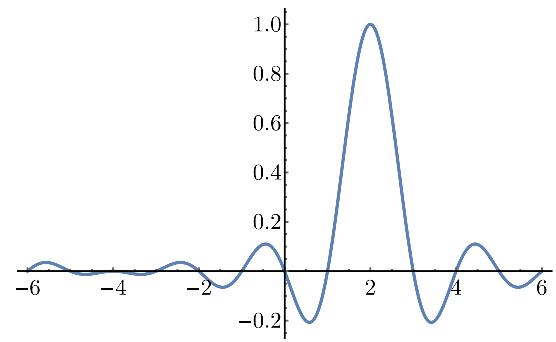
- start with some initial basis; here: plane waves $\phi_i(x) = \frac{1}{\sqrt{L}} \exp\left(\mathrm{i}\frac{2\pi i}{L}x\right)$
- consider (x_k, w_k) such that $\sum_{k=-N/2}^{N/2-1} w_k\, \phi_i^*(x_k)\phi_j(x_k) = \delta_{ij}$



unitary trans.

$\longrightarrow$

$\mathcal{U}_{ki} = \sqrt{w_k}\phi_i(x_k)$



DVR states

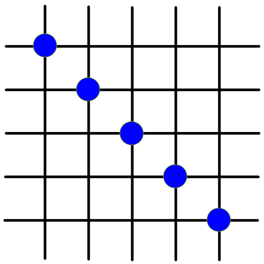
- $\psi_k(x)$ localized at x_k, $\psi_k(x_j) = \delta_{kj}/\sqrt{w_k}$
- **note duality:** momentum mode $\phi_i \leftrightarrow$ spatial mode ψ_k

DVR features

Potential energy is diagonal

$$\langle\psi_k|V|\psi_l\rangle = \int \mathrm{d}x\,\psi_k(x)\,V(x)\,\psi_l(x)$$

$$\approx \sum_{n=-N/2}^{N/2-1} w_n\,\psi_k(x_n)\,V(x_n)\,\psi_l(x_n) = V(x_k)\delta_{kl}$$



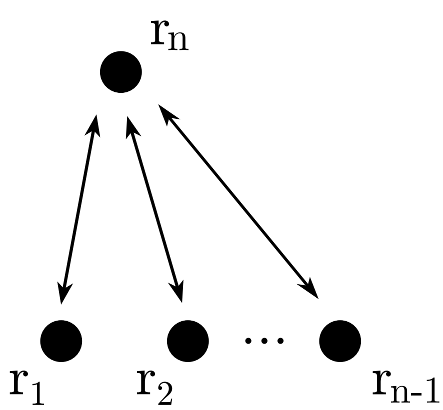
- no need to evaluate integrals
- number N of DVR states controls quadrature approximation

Kinetic energy is simple (via FFT) or sparse (in $d > 1$)

- plane waves are momentum eigenstates
  - evaluate kinetic energy in momentum space: $\hat{T}|\psi_k\rangle \sim \mathcal{F}^{-1} \otimes \hat{p}^2 \otimes \mathcal{F}|\psi_k\rangle$
- $\langle\psi_k|T|\psi_l\rangle =$ known in closed form
  - replicated for each coordinate, with Kronecker deltas for the rest

DVR basis states

- construct DVR basis in simple relative coordinates
  - because Jacobi coord. would complicate the boundary conditions
- separate center-of-mass energy (choose $\mathbf{P} = \mathbf{0}$)
- mixed derivatives in kinetic energy operator



$$\mathbf{x}_i = \sum_{i=1}^{n} U_{ij} \mathbf{r}_i$$

$$U_{ij} = \begin{cases} \delta_{ij} & \text{for} \quad i, j < n \\ -1 & \text{for} \quad i < n,\ j = n \\ 1/n & \text{for} \quad i = n \end{cases}$$

- **general DVR state for n particles in d dimensions**
  - $|s\rangle = |(k_{1,1}, \cdots, k_{1,d}), \cdots, (k_{n-1,1}, \cdots); \text{spins}\rangle \in B$
- basis size: $\dim B = (2S+1)^n \times N^{d\times(n-1)}$

(Anti-)symmetrization and parity

Permutation symmetry

- for each $|s\rangle \in B$, construct $|s\rangle_{\mathcal{A}} = \mathcal{N} \sum_{p \in S_n} \operatorname{sgn}(p)\, D_n(p)|s\rangle$
  - then $|s\rangle_{\mathcal{A}}$ is antisymmetric: $\mathcal{A}|s\rangle_{\mathcal{A}} = |s\rangle_{\mathcal{A}}$
  - for bosons, leave out $\operatorname{sgn}(p) \rightsquigarrow$ symmetric state
  - $D_n(p)|s\rangle =$ some other $|s'\rangle \in B$, modulo periodic boundary

This operation partitions the orginal basis!

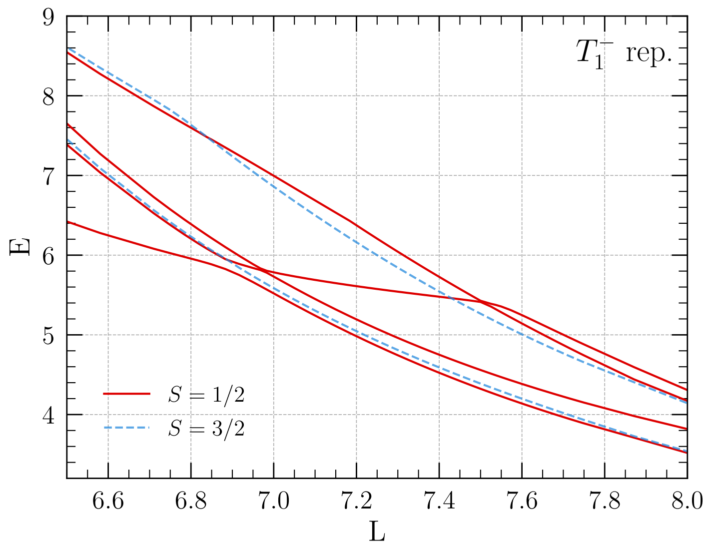
Reduced basis

- **each state appears in at most one (anti-)symmetric combination**
  - no need for expensive symmetry eigenspace determination
- significant reduction of basis size: $N \to N_{\text{reduced}} \approx N/n!$
- parity (with projector $\mathcal{P}_{\pm} = 1 \pm \mathcal{P}$) can be handled analogously

Three fermions

Consider shifted Gaussian potential now for three fermions

- add spin d.o.f., but no spin dependence in potential
  - total spin S good quantum number (fix S_z to filter states)
  - can also still consider simple cubic irreps.

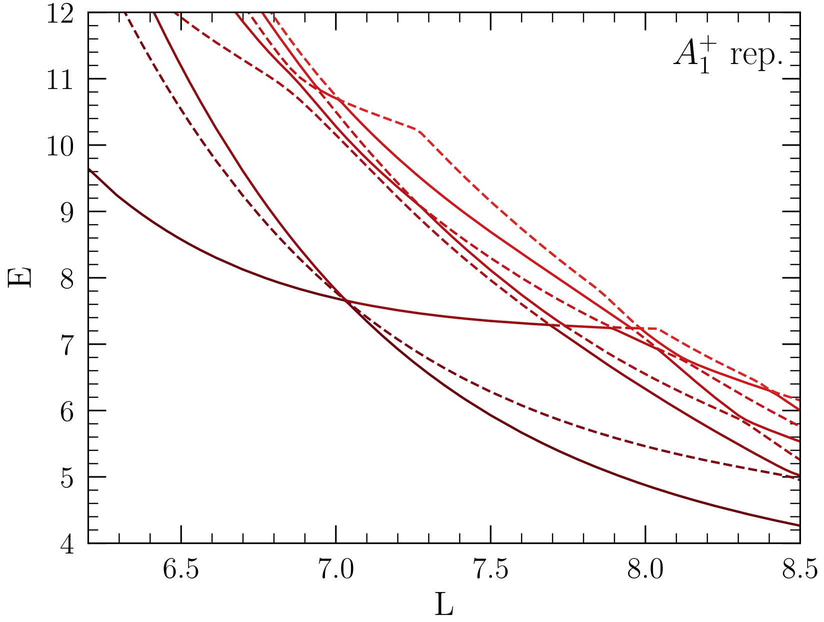


- all lowest states found to be in T_1^- irrep. ($\sim$ P-wave state)
- **extract $S = 1/2$ resonance at $E_R = 5.7(2)$**

Four-boson resonance

Now look at four bosons...

- still the same shifted Gaussian potential $(V_0 = 2.0)$



- **clear horizontal sequence of avoided level crossings**
  - (supposedly narrow) resonance at $E_R = 7.31(8)$