QED in the Clothed-Particle Representation (CPR): a fresh look at positronium properties treatment

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1 Introductory remarks

We have extended our previous applications of the method of unitary clothing transformations (UCTs) in mesodynamics [1, 2] to quantum electrodynamics (QED) [3, 4]. Starting from the primary canonical interaction between electromagnetic and electron-positron fields, the QED Hamiltonian has been expressed through a new family of the Hermitian and energy independent interaction operators built up in the e^2-order for the clothed electrons and positrons. In this context, we show the QED Hamiltonian $H_{\text{qed}}(\boldsymbol{\alpha}) = H_F(\boldsymbol{\alpha}) + V_{\text{qed}}(\boldsymbol{\alpha})$ in the bare particle representation (see, e.g., the monograph [5]), where V_{qed} is given by

$$V_{\text{qed}} = \int d\boldsymbol{x} j_k(\boldsymbol{x}) a^k(\boldsymbol{x}) + V_{\text{Coul}} = V^{(1)} + V_{\text{Coul}}\,, \tag{1}$$

with the electron-positron current density operator

$$j_\mu(\boldsymbol{x}) = e\bar{\psi}(\boldsymbol{x})\gamma_\mu\psi(\boldsymbol{x}) \tag{2}$$

and the Coulomb part

$$V_{Coul} = :\frac{1}{2}\int d\boldsymbol{x}\int d\boldsymbol{y}\frac{j^0(\boldsymbol{x})j^0(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}e^{-\lambda|\boldsymbol{x}-\boldsymbol{y}|} : . \tag{3}$$

In the CPR

$$H_{\text{qed}}(\alpha) = H_F(\alpha) + V_{\text{qed}}(\alpha) \equiv K(\alpha_c) = K_F(\alpha_c) + K_I(\alpha_c). \tag{4}$$

Admittedly the exponential factor is introduced to deal with infrared divergences with the parameter $\lambda > 0$ set to zero at the end of all calculations. Here α_c denotes the set of all creation and destruction operators for the clothed particles included. Note also that we use the Coulomb gauge (CG), where the photon field a_μ being transverse, has two independent polarizations.
It is proved, that in the e^2-order the interaction part $K_I(\alpha_c)$ is approximated by

$$K_I^{(2)}(\alpha_c) = K_{e^-e^-\to e^-e^-} + K_{e^+e^+\to e^+e^+} + K_{e^+e^-\to e^+e^-} + K_{e^+e^-\to\gamma\gamma} + K_{\gamma\gamma\to e^+e^-} + K_{e^-\gamma\to e^-\gamma} + K_{e^+\gamma\to e^+\gamma}\,, \tag{5}$$

where the separate contributions in the r.h.s are responsible for the different physical processes in this system of interacting photons and leptons.

2 Analytical expressions

A distinctive feature of our approach is that all expressions in the r.h.s of (5) are obtained simultaneously with mass and vertex renormalizations from the commutator of V from (1) with the generator of the first unitary clothing operator [1]. In particular, we present the interaction operator between clothed electrons and positrons

$$K_{e^-e^+\to e^-e^+} = \int \frac{d\boldsymbol{p}_1'}{p_1'^0}\frac{d\boldsymbol{p}_2'}{p_2'^0}\frac{d\boldsymbol{p}_1}{p_1^0}\frac{d\boldsymbol{p}_2}{p_2^0} V_{e^-e^+}(p_1', p_2'; p_1, p_2)\times b^\dagger(p_1')d^\dagger(p_2')b(p_1)d(p_2), \tag{6}$$

with

$$V_{e^-e^+}(p_1', p_2'; p_1, p_2) = \frac{e^2m^2}{(2\pi)^3}\delta(\boldsymbol{p}_2' + \boldsymbol{p}_1' - \boldsymbol{p}_2 - \boldsymbol{p}_1)\times\left[\upsilon_S(p_1', p_2'; p_1, p_2) + \upsilon_A(p_1', p_2'; p_1, p_2)\right], \tag{7}$$

where m the physical electron (positron) mass, $b(d)$ is the destruction operator for the clothed electron (positron). Henceforth, we omit polarization indices where it does not lead to confusion. In addition, we have introduced the decomposition into the so-called scattering and annihilation contributions υ_S and υ_A. Each of them has the structure

$$\upsilon_{S/A} = \upsilon_{S/A}^{\text{Feynman-like}} + \upsilon_{S/A}^{\text{off-energy-shell}}, \tag{8}$$

$$\upsilon_S^{\text{Feynman-like}} = -\bar{u}(p_1')\gamma^\mu u(p_1)\frac{1}{2}\left\{\frac{1}{(p_1'-p_1)^2} + \frac{1}{(p_2'-p_2)^2}\right\}\bar{\upsilon}(p_2)\gamma_\mu\upsilon(p_2'),$$

$$\upsilon_S^{\text{off-energy-shell}} = \frac{(p_1'+p_2'-p_1-p_2)}{(\boldsymbol{p}_1'-\boldsymbol{p}_1)^2+\lambda^2}\bar{u}(p_1')\gamma^0 u(p_1)\times \frac{1}{2}\left\{\frac{(p_1'-p_1)}{(p_1'-p_1)^2} + \frac{(p_2'-p_2)}{(p_2'-p_2)^2}\right\}\bar{\upsilon}(p_2)\gamma^0\upsilon(p_2'),$$

$$\upsilon_A^{\text{Feynman-like}} = \bar{u}(p_1')\gamma^\mu\upsilon(p_2')\frac{1}{2}\left\{\frac{1}{(p_1+p_2)^2} + \frac{1}{(p_1'+p_2')^2}\right\}\bar{\upsilon}(p_2)\gamma_\mu u(p_1),$$

$$\upsilon_A^{\text{off-energy-shell}} = \frac{(p_1'+p_2'-p_1-p_2)}{(\boldsymbol{p}_1'+\boldsymbol{p}_2')^2+\lambda^2}\bar{u}(p_1')\gamma^0\upsilon(p_2')\times \frac{1}{2}\left\{\frac{(p_1'+p_2')}{(p_1'+p_1)^2} - \frac{(p_1+p_2)}{(p_1+p_2)^2}\right\}\bar{\upsilon}(p_2)\gamma^0 u(p_1).$$

Such a decomposition implies that only the Feynman-like part survives on the energy shell, i.e., on the condition $p_1'^0 + p_2'^0 = p_1^0 + p_2^0$. Of course, all momenta included are defined on the mass-shell: $p^2 = p_0^2 - \boldsymbol{p}^2 = m^2$.
Furthermore, we present the operator of the process of the annihilation of clothed electron and positron to two photons

$$K_{e^-e^+\to\gamma\gamma} = \int \frac{d\boldsymbol{k}_1}{k_1^0}\frac{d\boldsymbol{k}_2}{k_2^0}\frac{d\boldsymbol{p}_1}{p_1^0}\frac{d\boldsymbol{p}_2}{p_2^0} V_{e^-e^+\gamma\gamma}(k_2, k_1; p_2, p_1)\times c^\dagger(k_2)c^\dagger(k_1)b(p_2)d(p_1), \tag{9}$$

where $c^\dagger$ is the creation operator for the clothed photon. Similarly to (8), we separate off-energy-shell part which goes to zero if energy conservation law satisfied.

$$V_{e^-e^+\gamma\gamma}(k_2, k_1; p_2, p_1) = \frac{e^2m}{2(2\pi)^3}\delta(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{k}_1 - \boldsymbol{k}_2)\times\left[\upsilon^{\text{Feynman}} + \upsilon^{\text{off-energy-shell}}\right], \tag{10}$$

$$\upsilon^{\text{Feynman}} = \frac{\bar{\upsilon}(p_1)\not{e}(k_1)\not{e}(k_2)u(p_2)}{\not{p}_1 - \not{k}_1 + m},$$

$$\upsilon^{\text{off-energy-shell}} = -\frac{1}{2}\left[\frac{\bar{\upsilon}(p_1)\not{e}(k_1)\not{e}(k_2)u(p_2)}{\not{p}_1 - \not{k}_1 + m} + \frac{\bar{\upsilon}(p_1)\not{e}(k_2)\not{e}(k_1)u(p_2)}{\not{p}_2 - \not{k}_1 - m}\right].$$

3 Correction to the positronium ground state energy

The problem of describing the bound states in QED in case of the positronium (Ps) has been considered by using the new interaction (6). Positronium consisting of an electron and a positron is the simplest bound system in QED. Its ground state (g.s.) has two possible configurations with total spin values S=0, 1. The singlet (triplet) lowest-energy state with S=0 (S=1) is known as the para-positronium (ortho-positronium). For this exposition, we will restrict ourselves to the consideration of the para-positronium (p-Ps) system.
As noted in [6], the Fock subspace of all the clothed states can be divided into several sectors (two electrons sector, photon-electron sector, etc.) such that $K^{(2)}(\alpha_c)$ leaves each of them to be invariant, i.e., for any state vector $|\Phi\rangle$ of such sector $K^{(2)}(\alpha_c)|\Phi\rangle$ belongs to the same sector. Here we make an assumption that the Ps state belongs only to electron-positron sector. The corresponding g.s., being the H eigenvector, viz.,

$$H|\text{p-Ps}(\boldsymbol{P})\rangle = E|\text{p-Ps}(\boldsymbol{P})\rangle, \tag{11}$$

can be represented as

$$|\boldsymbol{P}; \text{p-Ps}\rangle = \int \frac{d\boldsymbol{p}_1}{p_1^0}\frac{d\boldsymbol{p}_2}{p_2^0}\Psi_{00}(\boldsymbol{P}; \boldsymbol{p}_1, \boldsymbol{p}_2)b^\dagger(p_1)d^\dagger(p_2)|\Omega\rangle. \tag{12}$$

In the p-Ps rest system or center mass system (c.m.s.) the eigenvalue equation has the form

$$2p_0\Psi_{00}(\boldsymbol{p}) + \int \frac{d\boldsymbol{p}'}{p_0p_0'}\bar{V}(p', p)\Psi_{00}(\boldsymbol{p}') = m_{\text{p-Ps}}\Psi_{00}(\boldsymbol{p}), \tag{13}$$

where $m_{\text{p-Ps}} = m_{e^-} + m_{e^+} + \varepsilon_{\text{p-Ps}}$ the para-positronium mass and $\varepsilon_{\text{p-Ps}}$ its binding energy, $\Psi_{00}(\boldsymbol{p}) \equiv \Psi_{00}(\boldsymbol{P} = 0; \boldsymbol{p}, -\boldsymbol{p})$ (since we work in c.m.s.) and $\bar{V}(p', p)$ gets out from (6) in c.m.s. ($\boldsymbol{p} \equiv \boldsymbol{p}_1 = -\boldsymbol{p}_2$, $\boldsymbol{p}' \equiv \boldsymbol{p}_1' = -\boldsymbol{p}_2'$)

$$\bar{V}(p_1', p_2', p_1, p_2) = \langle\Omega|d(p_2')b(p_1')K_{e^-e^+\to e^-e^+}b^\dagger(p_1)d^\dagger(p_2)|\Omega\rangle. \tag{14}$$

In the non-relativistic limit ($p_0 = p_0' = m$) the eigenvalue equation reduces to the ordinary Schrödinger equation for the Coulomb potential in momentum space. Therefore we come to the well-known Coulomb problem with the g.s. energy $\varepsilon_{\text{g.s.}} \approx -6.8eV$. By considering the difference between $\bar{V}(p', p)$ and the Coulomb potential as a perturbation (it is not evident) and using the non-perturbative wave function of the ground state

$$\Psi_{00}(\boldsymbol{p}) = \frac{2}{\pi}\frac{\sqrt{2a^3}}{(1+a^2p^2)^2}, \tag{15}$$

from Appendix C in [7] we have computed the energy shift

$$\Delta\varepsilon = -4.7325\cdot 10^{-4}\ eV.$$

This value surprisingly coincides with those estimations given in [7] (see formula (1.1) therein). In order to verify such a coincidence beyond the perturbation theory, we are addressing the partial wave decomposition of the positronium eigenvectors that has been successful when finding the u and w components of the deuteron wave function (WF) [8].

4 The partial eigenvalue equation for para-positronium

In this context, we derive the partial eigenvalue equation for the para-positronium WFs that belong to the total angular momentum J, viz.,

$$2p_0\Psi^J(p) + \int_0^\infty \frac{p'^2dp'}{p_0p_0'}\Psi^J(p')\bar{V}^J(p, p') = m_{\text{p-Ps}}\Psi^J(p). \tag{16}$$

Here $\bar{V}^J(p, p')$ is the partial electron-positron quasipotential derived in the momentum representation from the new e^-e^+-interaction operator. In turn, we have

$$\bar{V}^J(p, p') = \bar{\upsilon}^J(\text{Feynman-like}) + \bar{\upsilon}^J(\text{off-energy-shell}). \tag{17}$$

Such a separation implies that only the Feynman-like part survives on the energy shell, where $p_0' = \sqrt{p'^2 + m^2} = p_0 = \sqrt{p^2 + m^2}$. The task of solving the eigenvalue equation and obtaining the corresponding positronium states in the CPR is underway (see Appendix C in [9]).

5 Decay rates for Ps$\to\gamma\gamma$

The positronium decay to two photons has considered. The corresponding decay rate is given by (see formula (9.337) in [10])

$$\Gamma = \sum_{\sigma_1\sigma_2}\int\frac{d\boldsymbol{k}_1}{k_1^0}\int\frac{d\boldsymbol{k}_2}{k_2^0}\pi\delta(k_1^0 + k_2^0 - E_{\text{Ps}})\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{P})|T_{fi}|^2, \tag{18}$$

where the T-matrix element $T_{fi} = \langle\Omega|c(k_1\sigma_1)c(k_2\sigma_2)T|Ps\rangle$ from the initial Ps ground state to the final state of the two photons respectively with the momenta $k_1 = (k_1^0, \boldsymbol{k}_1)$ and $k_2 = (k_2^0, \boldsymbol{k}_2)$ and their polarizations σ_1, σ_2. In this connection, we note an equivalence theorem proved in [11], that allows us to use a recipe for calculating the S-matrix (T-matrix) in the CPR.
In the rest frame of positronium ($\boldsymbol{P} = 0$) one can do integration with both δ-functions

$$\Gamma = \frac{\pi}{2}\sum_{\sigma_1\sigma_2}\int d\hat{\boldsymbol{k}}_1|T_{fi}|^2, \tag{19}$$

where $\hat{\boldsymbol{k}}_1$ is unit vector along vector $\boldsymbol{k}_1$. Due to δ-functions in this expression should be done the replacements: $\boldsymbol{k} \equiv \boldsymbol{k}_1 = -\boldsymbol{k}_2$ and $k_0 = \frac{1}{2}E_{Ps}$.
The positronium state-vector with spin S can be presented in the representation of total spin as

$$|Ps(S)\rangle = \sum_{M_S}\int\frac{d\boldsymbol{p}}{p^0}\Psi_{SM_S}(\boldsymbol{p})|p\,SM_S\rangle, \tag{20}$$

where $|p\,SM_S\rangle = \sum_{\mu_1\mu_2}(\frac{1}{2}\mu_1\frac{1}{2}\mu_2|SM_S)b^\dagger(p_-\mu_2)d^\dagger(p\mu_1)|\Omega\rangle$ the electron-positron state-vector in total spin representation and $p_- = (p_0, -\boldsymbol{p})$. In Born approximation we suppose $T \approx K_{e^-e^+\to\gamma\gamma}$. Thus our formula for para-positronium decay in Born approximation is

$$\Gamma = \frac{\pi}{2}\sum_{\sigma_1\sigma_2}\int d\hat{\boldsymbol{k}}|\langle k\sigma_1\sigma_2|K_{e^-e^+\to\gamma\gamma}|\text{p-Ps}\rangle|^2,$$
$$\langle k\sigma_1\sigma_2|K_{e^-e^+\to\gamma\gamma}|\text{p-Ps}\rangle = \int\frac{d\boldsymbol{p}}{p_0}\Psi_{00}(\boldsymbol{p})\langle k\sigma_1\sigma_2|K_{e^-e^+\to\gamma\gamma}|p\,0\,0\rangle, \tag{21}$$

where $\langle k\sigma_1\sigma_2| \equiv \langle\Omega|c(k\sigma_1)c(k_-\sigma_2)$. Here we use g.s. Coulomb wave function as an approximation of positronium g.s. wave function.
The interaction in an arbitrary frame from (9) has the form

$$\langle\Omega|c(k_1\sigma_1)c(k_2\sigma_2)K_{e^-e^+\to\gamma\gamma}b^\dagger(p_2\mu_2)d^\dagger(p_1\mu_1)|\Omega\rangle = \frac{\alpha m}{4\pi^2}\bar{\upsilon}_{e^-e^+\gamma\gamma}, \tag{22}$$

$$\bar{\upsilon}_{e^-e^+\gamma\gamma} = \bar{\upsilon}(p_1\mu_1)\not{e}(k_1\sigma_1)\frac{1}{2}\left\{\frac{1}{\not{p}_2 - \not{k}_2 - m} - \frac{1}{\not{p}_1 - \not{k}_1 + m}\right\}\not{e}(k_2\sigma_2)u(p_2\mu_2) + (k_1, \sigma_1 \longleftrightarrow k_2, \sigma_2), \tag{23}$$

where $(k_1, \sigma_1 \longleftrightarrow k_2, \sigma_2)$ means the same expression but with the replacements. It can be noted that in the static limit ($\boldsymbol{p}_{1,2} = 0$) we get the well known Pirenne-Wheeler result $\Gamma = \frac{1}{2}\alpha^5 m \approx 8.0325\cdot 10^9$ sec^{-1} [12, 13].
After going to c.m.s. and performing the summation over fermions polarizations (μ_1, μ_2) we get

$$\langle k\sigma_1\sigma_2|K_{e^-e^+\to\gamma\gamma}|\text{p-Ps}\rangle = i\frac{\alpha m a^{\frac{3}{2}}}{4\pi^3}\int\frac{d\boldsymbol{p}}{p_0}\frac{1}{(1+a^2\boldsymbol{p}^2)^2}\times e_i(k\sigma_1)e_j(k_-\sigma_2)[W_{ij}(\boldsymbol{p}, \boldsymbol{k}) + W_{ji}(\boldsymbol{p}, -\boldsymbol{k})]\,,$$
$$W_{ij}(\boldsymbol{p}, \boldsymbol{k}) = \frac{1}{p_0k_0 - \boldsymbol{pk}}\left\{\mathcal{E}_{ikl}\frac{p_jp_kk_l}{(p_0+m)m} - \mathcal{E}_{jkl}\frac{p_ip_kk_l}{(p_0+m)m} + \mathcal{E}_{ij\kappa}\left[\left(\frac{p_0-m}{m} + \frac{\boldsymbol{p}(\boldsymbol{k}-\boldsymbol{p})}{(p_0+m)m}\right)p_\kappa - \frac{p_0+m}{m}k_\kappa\right]\right\}, \tag{24}$$

where $p \equiv p_1 = -p_2$, $k \equiv k_1 = -k_2$ and indices $i, j, \kappa, l = 1, 2, 3$. Here we use Coulomb gauge thus polarization vectors satisfy the properties: $e_0(k\sigma) = 0$, and $\boldsymbol{ke}(k\sigma) = 0$. Then after the summation over photons polarizations (σ_1, σ_2) we obtain

$$\Gamma = \frac{128m}{\alpha\pi^2}\text{I}^2. \tag{25}$$

An interesting fact is that the dependence on the module of the photon momentum has completely disappeared, i.e., our rate will be independent on positronium binding energy. The integral I in (25)

$$\text{I} = \int_0^\infty d\rho\frac{\rho}{(1+\frac{4}{\alpha^2}\rho^2)^2}\frac{1}{\sqrt{\rho^2+1}}\ln\frac{\sqrt{\rho^2+1}+\rho}{\sqrt{\rho^2+1}-\rho} \approx 7.5865\cdot 10^{-8} \tag{26}$$

has no singularities and can be calculated numerically with any precision. It means that the decay rate of para-positronium into two photons

$$\Gamma = 7.9411\cdot 10^9\ \text{sec}^{-1}. \tag{27}$$

Recall that the experimental result for this value [14] is: $7.9909 \pm 0.0017\cdot 10^9$ sec^{-1}. Of interest is to what extend the estimation (27) can be changed with the off-shell interaction included. It has turned out that the off-energy-shell part on the r.h.s. of (10) does not contribute to the decay rate. Probably it is due to the fact that in our approximation the p-Ps wave function (15) is independent of fermion polarizations.

Conclusions

We have shown that the UCT method can be successfully applied to the treatment of the bound states in QED. Our consideration gives one more application of a well-forgotten concept on the clothed particles in quantum field theory, put forward by Greenberg and Schweber [15]. We have seen that our approach leads to new hermitian and energy independent interactions between clothed particles including the off-energy-shell and recoil effects (the latter in all orders of the v^2/c^2 - expansion).

Acknowledgements

Two of us, Y.K. and A.A. are very grateful to the Organizing Committee of the 24th European Conference on Few-Body Problems in Physics and the International Relations Office of V. N. Karazin Kharkiv National University for the financial assistance, of our participation in this conference.

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