Meson interactions at Large $N_c$ from Lattice QCD

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In collaboration with: A. Donini, P. Hernández & C. Pena


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Overview of meson interactions at Large $N_c$

1. Motivation
2. Ensembles at Large $N_c$
3. $M_\pi$ and $F_\pi$ at Large $N_c$
4. Scattering at Large $N_c$
5. $K \rightarrow \pi\pi$ at Large $N_c$
6. Summary
Motivation
Let \( \text{QCD} \) be a \( SU(N_c) \) gauge theory

- \( N_c \rightarrow \infty \)
- \( \alpha_s N_c = \text{constant} \)
The Large $N_c$ limit ('t Hooft limit)

- Let QCD be a $SU(N_c)$ gauge theory
  - $N_c \to \infty$
  - $\alpha_s N_c = \text{constant}$

- QCD is dominated by gluon loops and keeps relevant features (i.e. confinement, spontaneous chiral symmetry breaking...)

$\pi \to \pi\pi$
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**Example of $N_c$ counting:**

1. Gluon loops $\sim N_c^2$

2. Quark loops $\sim N_c$
Chiral Perturbation Theory

ChPT is an EFT for QCD at low energies in terms of mesons.

\[ \phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}} \eta & \sqrt{2} \pi^+ & \sqrt{2} K^+ \\ \sqrt{2} \pi^- & -\pi^0 + \frac{1}{\sqrt{3}} \eta & \sqrt{2} K^0 \\ \sqrt{2} K^- & \bar{K}^0 & -\frac{2}{\sqrt{3}} \eta \end{pmatrix}, \quad U = e^{i \phi / F_0}, \]

\( U \) transforms as:

\[ U \rightarrow L^\dagger UR, \text{ with } L, R \in SU(N_f) \]

The lowest order (\( p^2 \sim M \)) Lagrangian with the QCD symmetries:

\[ \mathcal{L}_2 = \frac{F_0^2}{4} \text{tr} \left( \partial_\mu U^\dagger \partial^\mu U \right) + \frac{F_0^2 B}{2} \text{tr} \left( MU + M^\dagger U^\dagger \right), \]

with \( M = \text{diag} (m_u, m_d, m_s) \).
The LO Lagrangian ($\mathcal{L}_2$) of ChPT is very predictive with few parameters: $F_0$, $B$ and quark masses.
Chiral Perturbation Theory at NLO and Large $N_c$

1. The LO Lagrangian ($\mathcal{L}_2$) of ChPT is very predictive with few parameters: $F_0$, $B$ and quark masses.
2. At NLO, there are more terms in the Lagrangian with additional couplings (Low Energy Constants):

$$\mathcal{L}_4 = \sum_{i=0}^{10} L_i \mathcal{O}_i.$$
Motivation

Ensembles

$M_\pi$ & $F_\pi$

Scattering

$K \to \pi\pi$

Summary

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- LECs encode the high energy physics information
- They have different Large $N_c$ behaviour

<table>
<thead>
<tr>
<th>$L_i$</th>
<th>Value</th>
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</tr>
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<tbody>
<tr>
<td>$2L_1 - L_2$</td>
<td>$-0.4 \pm 0.2$</td>
<td>1</td>
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<td>$L_4$</td>
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<td>$N_c$</td>
</tr>
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<tr>
<td>$L_5$</td>
<td>$1.2 \pm 0.1$</td>
<td>$N_c$</td>
</tr>
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<td>$L_8$</td>
<td>$0.5 \pm 0.2$</td>
<td>$N_c$</td>
</tr>
<tr>
<td>$L_9$</td>
<td>$6.9 \pm 0.7$</td>
<td>$N_c$</td>
</tr>
<tr>
<td>$L_{10}$</td>
<td>$-5.2 \pm 0.1$</td>
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Bijnens & Ecker, 2014
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- LECs encode the high energy physics information
- They have different Large $N_c$ behaviour
- Large $N_c$ allows for important simplifications!

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The **Large** $N_c$ limit in Phenomenology

“**Large** $N_c$-inspired” approximations are usual in phenomenology.

**Updated Standard Model Prediction for $\varepsilon'/\varepsilon$**

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Apt. Correus 22085, E-46071 València, Spain

6. The SM prediction for $\varepsilon'/\varepsilon$

Taking into account all computed corrections in Eq. (7), our SM prediction for $\varepsilon'/\varepsilon$ is

\[
\text{Re}(\varepsilon'/\varepsilon) = \left(15 \pm 2\mu \pm 2m_s \pm 2\Omega_{\text{eff}} \pm \frac{61}{N_c}\right) \times 10^{-4}
\]

- Uncertainties from **Large** $N_c$ are hard to estimate.
- Can Lattice QCD improve this?
Properties of light resonances from unitarized Chiral perturbation theory: \(N_c\) behavior and quark mass dependence

J. R. Peláez\(^a\,*\) J. Nebreda\(^a\), G. Ríos\(^a\)

\(^a\)Dept. de Física Teórica II. Universidad Complutense. 28040 Madrid. Spain
Properties of light resonances from unitarized Chiral perturbation theory: $N_c$ behavior and quark mass dependence

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Lattice QCD at Large $N_c$

→ The Large $N_c$ is often very useful. However:

1. Systematic errors are hard to estimate.
2. It fails for few observables: $K \to \pi\pi$ (∋later!)

Lattice QCD can help!

Topics that we aim to address at Large $N_c$ from lattice simulations:

- Meson masses and decay constants
- Meson scattering (isospin-2, isospin-1/$\rho$, isospin-0)
- $K \to \pi$ and $K \to \pi\pi$
- $\eta'$ meson
- Tetraquarks
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Ensembles at Large $N_c$
Our Large $N_c$ Ensembles with $N_f = 4$

- Iwasaki gauge action and $O(a)$ improved Wilson fermions.

<table>
<thead>
<tr>
<th>Ensemble</th>
<th>$N_c$</th>
<th>$L \times T$</th>
<th>$\beta$</th>
<th>$m_0$</th>
<th>$aM$</th>
<th>$M$ (MeV)</th>
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<td>3</td>
<td>$20 \times 36$</td>
<td>-</td>
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<tr>
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<td>4</td>
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<tr>
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<td>450</td>
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<td>-0.3340</td>
<td>0.1354(7)</td>
<td>360</td>
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Generated with HiRep, M. Hansen arXiv:1705.11010
Scale Setting at Large $N_c$

Use observables from Gradient Flow (Lüscher):

$$
\langle t^2 E(t) \rangle = \frac{3}{128\pi^2} \frac{N_c^2 - 1}{N_c} \lambda_{GF}(\mu)
$$

with $\mu = 1/\sqrt{8t}$ and $\lambda_{GF} = N_c g_{GF}^2$ ('t Hooft coupling).

For QCD, $t_0$ is defined through the implicit equation:

$$
\langle t^2 E(t) \rangle \bigg|_{t=t_0} = 0.3.
$$

Input $\rightarrow \sqrt{t_0}$ from other lattice simulations

**Generalization for arbitrary $N_c$:**

$$
\langle t^2 E(t) \rangle \bigg|_{t=t_0} = 0.1125 \frac{N_c^2 - 1}{N_c}, \quad (M\sqrt{t_0}) \bigg|_{M=420 \text{ MeV}} = 0.3090(83)
$$
Mass dependence of $t_0$

$$t_0(M^2) = t_0^{\text{chiral}} (1 + kM^2) + O(M^4) \xrightarrow{\text{Large } N_c} t_0^{\text{chiral}}$$
Spectrum: $N_c = 6, \, M_\pi = 560$ MeV

\[ m_\pi + m_{\eta'} \]

\[ 2m_\pi \]

\[ (\pi\pi)_{I=2} \]

\[ (\pi\pi)_{AA} \]

\[ \rho \]

\[ \eta' \]

\[ \pi \]
$M_\pi$ and $F_\pi$ at Large $N_c$
Large $N_c$ scaling of meson masses and decay constants

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¹IFIC (CSIC-UVEG), Edificio Institutos Investigación, Apt. 22085, E-46071 Valencia, Spain
²Departamento de Física Teórica and Instituto de Física Teórica UAM-CSIC,
Universidad Autónoma de Madrid, E-28049 Madrid, Spain

(Dated: July 29, 2019)

We perform an *ab initio* calculation of the $N_c$ scaling of the low-energy couplings of the chiral
Lagrangian of low-energy strong interactions, extracted from the mass dependence of meson masses
and decay constants. We compute these observables on the lattice with four degenerate fermions,
$N_f = 4$, and varying number of colours, $N_c = 3 - 6$, at a lattice spacing of $a \simeq 0.075$ fm. We
find good agreement with the expected $N_c$ scaling and measure the coefficients of the leading and
subleading terms in the large $N_c$ expansion. From the subleading $N_c$ corrections, we can also infer
the $N_f$ dependence, that we use to extract the value of the low-energy couplings for different values
of $N_f$. We find agreement with previous determinations at $N_c = 3$ and $N_f = 2,3$ and also, our
results support a strong paramagnetic suppression of the chiral condensate in moving from $N_f = 2$
to $N_f = 3$. 

Meson decay constant

- The decay constant parametrizes the hadronic part of $\pi^+ \to \ell^+ + \nu$. 
- It is defined as:
  \[
  \langle 0|A_\mu(0)|\pi^+(q)\rangle = -iq_\mu \sqrt{2}F_\pi
  \]
  with $A_0 = \bar{q}\gamma_0\gamma_5q$.
- On the lattice:
  \[
  C_A(t) = \langle A_0(0)A_0(t)\rangle \propto F_\pi^2 e^{-M_\pi t}.
  \]
- By simple counting of traces,
  \[
  F_\pi^2 = O(N_c)
  \]
In Chiral Perturbation Theory:

\[
F_\pi = F \left[ 1 + \frac{M_\pi^2}{F_\pi^2} \left( 4L_5(\mu) + 4N_fL_4(\mu) \right) \right. \\
\left. + \frac{N_f}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \log \frac{M_\pi^2}{\mu^2} \right]
\]

1. \(F_\pi^2 = O(N_c)\)
2. \(L_5 = O(N_c)\)
3. \(L_4 = O(1)\)  \(\xrightarrow{\text{Large } N_c}\)  \(F_\pi = F \left[ 1 + 4 \frac{M_\pi^2}{F_\pi^2} L_5 \right]\)
Large $N_c$ Scaling of Low Energy Constants for $F_\pi$

\[
F_\pi = F \left[ 1 + \frac{M_\pi^2}{F_\pi^2} \left( 4L_F \right) + \frac{N_f}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \log \frac{M_\pi^2}{\mu^2} \right]
\]

\[
F = \sqrt{N_c} \left( F_0 + \frac{F_1}{N_c} \right)
\]

$L_F = N_c L_F^{(0)} + L_F^{(1)}$
$F_\pi$ at Large $N_c$

Simultaneous chiral and $N_c$ fit.
Large $N_c$ scaling of Low Energy Constants for $M_\pi$

$$M_\pi^2 = 2Bm \left[ 1 + \frac{M_\pi^2}{F_\pi^2} \left( 8L_M \right) + \frac{1}{N_f} \frac{M_\pi^2}{(4\pi F_\pi)^2} \log \frac{M_\pi^2}{\mu^2} \right]$$

$$B = B_0 + \frac{B_1}{N_c}$$

$L_M = N_c L_M^{(0)} + L_M^{(1)}$
Simultaneous chiral and $N_c$ fit.
Selected Results

From the fit, we can infer the $N_c$ and $N_f$ dependence:

$$\frac{F}{\sqrt{N_c}} = \left[ 67(3) - 26(4) \frac{N_f}{N_c} \right] \text{ MeV}$$

$N_f = 2 \rightarrow F = 86(3) \text{ MeV}$

$N_f = 3 \rightarrow F = 71(3) \text{ MeV}$
Selected Results

- Predict the decay constant LECs at $N_f = 2$
  \[ \bar{\ell}_4 = 5.1(3), \text{ vs. } \text{FLAG 2019} \bar{\ell}_4 = 4.40(28) \]

- Predict the chiral condensates at $N_f = 3$
  \[ \Sigma^{1/3}(N_f = 3) = 223(9) \text{ MeV vs. [Fukaya et al.]} 214(6)(24) \text{ MeV} \]

- Predict the ratio of chiral condensates at $N_f = 2, 3$
  \[ \frac{\Sigma(N_f = 2)}{\Sigma(N_f = 3)} = 1.49(10) \text{ vs. [Bernard et al.]} 1.51(11) \]

- We find that the subleading parts to the LECs are sizeable.
  For the meson mass LEC we obtain:
  \[ \frac{L^{N_f=4}}{N_c} \cdot 10^3 = -0.2(2) + \frac{2.9(6)}{N_c} + O(N_c^{-2}). \]
Scattering at Large $N_c$
Scattering in infinite volume

- Define asymptotic states: $|\phi_{\text{IN}}\rangle$, $|\phi_{\text{OUT}}\rangle$
- The Scattering Matrix relates these states:
  $$|\phi_{\text{OUT}}\rangle = \hat{S} |\phi_{\text{IN}}\rangle$$
- The Phase Shifts parametrize the S Matrix
  $$\langle \vec{k}\ell m | \hat{S} | \vec{p}\ell m \rangle = S_{\ell} = e^{2\delta_{\ell}(k)} \delta \left( |\vec{k}| - |\vec{p}| \right)$$
- Effective range expansion (s-wave):
  $$k \cot \delta_0 = -\frac{1}{a_0} + \frac{1}{2} rk^2 + O(k^4)$$
Scattering in finite volume

Lüscher method

Make use of finite volume artefacts to study interactions

\[ \det [\cot \delta + M] = 0 \]

\[ k \cot \delta_0 = -\frac{1}{a_0} + \frac{1}{2} rk^2 + O(k^4) \]

S Matrix $\leftrightarrow$ kinematical quantity
Scattering in finite volume

**Lüscher method**

Make use of finite volume artefacts to study interactions

\[ \text{det} [\cot \delta + M] = 0 \]

\[ k \cot \delta_0 = -\frac{1}{a_0} + \frac{1}{2} rk^2 + O(k^4) \]

**S Matrix** \(\leftrightarrow\) **kinematical quantity** \(\rightarrow\) **scattering length**

For the ground state, a simpler formula is available:

(Huang & Yang, Lüscher, Hansen & Sharpe)

\[ E - 2m = \frac{4\pi a_0}{mL^3} \left( 1 + c_1 \frac{a_0}{L} + c_2 \left( \frac{a_0}{L} \right)^2 + c_3 \left( \frac{a_0}{L} \right)^3 + \frac{2\pi r (a_0)^2}{L^3} - \frac{\pi a_0}{m^2 L^3} \right) \]

\[ \Rightarrow \text{At order } L^{-5}, \text{ the ground state is solely explained by } a_0 \]
Isospin 2 $\pi\pi$ scattering

$\pi^+\pi^+ \to \pi^+\pi^+$ is the simplest application to QCD

Phenomenological value (Ynduráin, 2002)

$$M_\pi a_0^{l=2} = 0.0422(22)$$

- Weakly coupled
- Less noisy
- Extract LECs of ChPT

$$C_{\pi\pi} = \langle \pi\pi(0)\pi^\dagger\pi^\dagger(t) \rangle$$
**Motivation**

Ensembles

$M_\pi$ & $F_\pi$

Scattering

$K \rightarrow \pi\pi$

Summary

---

**Ensemble**

---

**M**

---

**π** & **F**

---

**Scattering**

---

**K** → **π**π

---

**Summary**

---

**l=2** $\pi\pi$ scattering at Large $N_c$

---

In Chiral Perturbation Theory with $N_f = 4$ (Bijnens et al.)

\[
M_\pi a_0^{l=2} = -\frac{M_\pi^2}{16\pi F_\pi^2} \left[ 1 - \frac{16M_\pi^2}{F_\pi^2} L_{\pi\pi}(\mu) - \frac{M_\pi^2}{32\pi^2 F_\pi^2} \left( \frac{13}{4} \log \frac{M_\pi^2}{\mu^2} - \frac{3}{4} \right) \right]
\]

with $L_{\pi\pi} = L_0 + 2L_1 + 2L_2 + L_3 - 2L_4 - L_5 + 2L_6 + L_8$

---

$F_\pi^2 = O(N_c)$

$L_0, L_5, L_3, L_8 = O(N_c)$ \(\xrightarrow{\text{Large } N_c}\) $a_0^{l=2} \propto \frac{1}{N_c} (1 + \text{LECs})$

$L_1, L_2, L_4, L_6 = O(1)$

---
Preliminary results for Isospin 2 $\pi \pi$ scattering

\[ \frac{N_c}{3} M_\pi a_{I=2} \]

- Physical value
- $N_c = 3$
- $N_c = 4$
- $N_c = 5$
- $N_c = 6$
Other scattering channels

In a general theory $N_f > 3$, there are more scattering channels (Bijnens et al.)

\[
\begin{align*}
    a^I_{0,\text{tree}} &= \frac{M^2_\pi}{16\pi F^2_\pi} \left( 2N - \frac{1}{N} \right), \\
    a^S_{0,\text{tree}} &= \frac{M^2_\pi}{16\pi F^2_\pi} \left( N - \frac{2}{N} \right), \\
    a^A_{1,\text{tree}} &= \frac{M^2_\pi}{48\pi F^2_\pi} N, \\
    a^{SA}_{1,\text{tree}} &= a^{AS}_{1,\text{tree}} = 0, \\
    a^{SS}_{0,\text{tree}} &= -\frac{M^2_\pi}{16\pi F^2_\pi}, \\
    a^{AA}_{0,\text{tree}} &= \frac{M^2_\pi}{16\pi F^2_\pi}.
\end{align*}
\]
Preliminary results for $\pi D_s - KD$ scattering

\[ \frac{N_c}{3} M_\pi a_0^{AA} \]

\[ \sqrt{\frac{N_c}{3} \frac{M_\pi}{F_\pi}} \]
$K \to \pi\pi$ at Large $N_c$
Large $N_c$ limit for $K \rightarrow \pi\pi$

Weak decay with two isospin final states, $I = 0, 2 \rightarrow A_0, A_2$
Large $N_c$ limit for $K \rightarrow \pi\pi$

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- Diag.(a) has 2 quark loops $\to O(N_c^2)$
- Diag.(b) has 1 quark loop $\to O(N_c)$
- Diag.(c) has 2 quark loops and 4 vertices $\to O(\alpha_s^2 N_c^2) = O(1)$
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$\Rightarrow$ For neutral particles, Diag.(a) is not present:

$A(K^0 \to \pi^0\pi^0) = A_0 - \sqrt{2}A_2 = 0$ at Large $N_c$
Large $N_c$ limit for $K \rightarrow \pi\pi$

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\begin{align*}
\Rightarrow & \text{For neutral particles, Diag.(a) is not present:} \\
A(K^0 \rightarrow \pi^0\pi^0) = A_0 - \sqrt{2}A_2 = 0 \text{ at Large } N_c
\end{align*}

The Large $N_c$ prediction for $K \rightarrow \pi\pi$ is:

$$\frac{\text{Re } A_0}{\text{Re } A_2} = \sqrt{2}$$
Large $N_c$ limit for $K \to \pi\pi$

- The Large $N_c$ prediction for $K \to \pi\pi$ is (Manohar, Large $N$ QCD):
  \[
  \frac{\text{Re } A_0}{\text{Re } A_2} = \sqrt{2}
  \]

- Experimental values for $K \to \pi\pi$ are very well measured in two isospin channels, $I = 0, 2$ and Large $N_c$ fails.
  \[
  \frac{\text{Re } A_0}{\text{Re } A_2} \approx 22 \gg \sqrt{2}
  \]

- State of the art result by RBC-UKQCD, 2015:
  \[
  \frac{\text{Re } A_0}{\text{Re } A_2} = 31(11),
  \]

- Why does Large $N_c$ fail? Very large $1/N_c$ corrections? Can Lattice QCD help?
Motivation

Ensembles

$M_\pi \& F_\pi$

Scattering

$K \rightarrow \pi \pi$

Summary

Relating $K \rightarrow \pi$ to $A_2$ and $A_0$

$\mathcal{H}^{SM}(W^\mu, u, d...) \rightarrow \mathcal{H}_W^{N_f=4}(u, d, c, s) \rightarrow \mathcal{H}_W^{ChPT}(\pi, K, D, \eta)$

$\mathcal{H}_W^{ChPT} \propto g^+ O^+ + g^- O^-$

The tree level result in ChPT for the ratio is:

$$\frac{A_0}{A_2} = \frac{1}{2\sqrt{2}} \left( 1 + 3\frac{g^-}{g^+} \right) \xrightarrow{\text{Large } N_c} \sqrt{2}$$

Determine $g^\pm$ from Lattice QCD:

$$A^\pm = \langle K | O^\pm | \pi \rangle \xrightarrow{M_\pi \rightarrow 0} g^\pm$$

$g^\pm \propto$

Color-disconnected $O(N_c^2)$

Color-connected $O(N_c)$
Preliminary results for $K \to \pi$ at Large $N_c$


\[ R^+ = 1 + 1.42/N_c + 2.07/N_c^2 \]
\[ R^+ = 1 + 1.04/N_c + 8.15/N_c^2 \]
\[ R^- = 1 - 1.39/N_c + 1.34/N_c^2 \]
\[ R^- = 1 - 1.53/N_c - 0.05/N_c^2 \]
Preliminary results for $K \rightarrow \pi$ at Large $N_c$


$$R^+ = 1 + 1.42/N_c + 2.07/N_c^2$$
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Quenching effects at order $1/N_c^2$
Preliminary results for $K \rightarrow \pi$ at Large $N_c$


\[
R^+ = 1 + \frac{1.42}{N_c} + \frac{2.07}{N_c^2} \\
R^- = 1 + \frac{1.04}{N_c} + \frac{8.15}{N_c^2}
\]

\[
\frac{A_0}{A_2} \bigg|_{N_c=3}^{N_f=4} = 5.7(3)_{\text{stat}}
\]

Quenching effects at order $1/N_c^2$
Summary
Outlook

- Meson masses and decay constants
- Meson scattering (isospin-2, isospin-1/ρ, isospin-0)
- $K \rightarrow \pi$ and $K \rightarrow \pi\pi$
- $\eta'$ meson
- Tetraquarks

1. Finished
2. Started
3. Planned
Outlook

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Final goal: getting some understanding of QCD ...
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Final goal: getting some understanding of QCD ...
Thanks for your attention!

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Framework for $K \rightarrow \pi \pi$: weak interactions in ChPT

1. Take the lowest order Lagrangian is written as:
   $$\mathcal{L}_2 = \frac{F^2}{4} \text{tr} \left( \partial_\mu U^\dagger \partial^\mu U \right) + \frac{F^2 B}{2} \text{tr} \left( MU + M^\dagger U^\dagger \right),$$

2. Use covariant derivative with an external left-handed source:
   $$\partial_\mu U \rightarrow D_\mu U = \partial_\mu U + T^a A^a_\mu U,$$

3. Define the left current as:
   $$\mathcal{J}^a_\mu = \frac{\delta \mathcal{L}}{\delta A^a_\mu} = F^2 \text{Tr} \left( T_a U \partial_\mu U^\dagger \right) \leftrightarrow \bar{q} T^a \gamma^L_\mu q$$

4. Build operators with the right irreducible representation of the $SU(4)$ flavour group ($O_{\Gamma=20}, O_{\Gamma=84}$). One needs 4 left indices:
   $$O_\Gamma = t_{ijkl} (U \partial_\mu U^\dagger)_{ij} (U \partial_\mu U^\dagger)_{kl}$$

The electroweak Hamiltonian in ChPT is:

$$\mathcal{H}_W = g^+ O^+_{\Gamma=84} + g^- O^-_{\Gamma=20}$$
U($N_f$) Chiral Perturbation Theory

At Large $N_c$, the flavour singlet becomes a Goldstone boson. A consistent power counting is:

$$\mathcal{O}(\delta) \sim \mathcal{O}(p^2) \sim \mathcal{O}(m_q) \sim \mathcal{O}(m_{\pi}^2) \sim \mathcal{O}(N_c^{-1}).$$

At NNLO for the mass:

$$M_{\pi}^2 = 2m \left( B_0 + \frac{B_1}{N_c} + \frac{B_2}{N_c^2} \right) \left[ 1 + \frac{1}{N_f} \frac{M_{\pi}^2}{(4\pi F_{\pi})^2} \log \frac{M_{\pi}^2}{\mu^2} \right. \right.

\left. - \frac{1}{N_f} \frac{M_{\eta'}^2}{(4\pi F_{\pi})^2} \log \frac{M_{\eta'}^2}{\mu^2} + 8 \frac{M_{\pi}^2}{F_{\pi}^2} (N_c L^{(0)}_M + L^{(1)}_M) + N_c^2 K^{(0)}_M \left( \frac{M_{\pi}^2}{F_{\pi}^2} \right)^2 \right]$$

Matching of SU($N_f$) and U($N_f$):

$$[B]_{SU(N_f)} = [B]_{U(N_f)} \left( 1 - \frac{1}{N_f} \frac{M_0^2}{(4\pi F_{\pi})^2} \log \frac{M_0^2}{\mu^2} \right),$$

$$\left[ L^{(1)}_M \right]_{SU(N_f)} = \left[ L^{(1)}_M \right]_{U(N_f)} - \frac{1}{8N_f(4\pi)^2} \left( \log \frac{M_0^2}{\mu^2} + 1 \right),$$