Application of the gradient flow to the energy–momentum tensor

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Yang–Mills gradient flow (Narayanan–Neuberger hep-th/0601210, Lüscher, arXiv:1006.4518)

- One-parameter $t \geq 0$ (the flow time) deformation of the gauge field $A_\mu(x)$,

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according to (the flow equation)

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$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\text{YM}}[B]}{\delta B_\mu(t,x)} = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots ,$$

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- Since

$$D_\mu = \partial_\mu + [B_\mu, \cdot], \quad G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)],$$

this is a diffusion-type equation with the diffusion length,

$$x \sim \sqrt{8t}.$$

The flow time t has the mass dimension -2.

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- Applications in lattice gauge theory,
  - Topological charge
  - Scale setting
  - Non-perturbative gauge coupling constant
  - Chiral condensate
  - Various renormalized operators, including the energy–momentum tensor
  - Supersymmetric theory
  - ...

Finiteness of the gradient flow (Lüscher, Weisz, arXiv:1101.0963)

- Correlation function of the flowed gauge field,

$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu \, B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \, e^{-S_{\text{YM}}[A]},$$

when expressed in terms of renormalized coupling,

$$g^2 = g_0^2 \mu^{-2\varepsilon} Z^{-1},$$

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- This is quite contrast to the conventional gauge field, for which

$$\langle A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) \rangle \,,$$

  requires the wave function renormalization

$$(A_R)_\mu^a = Z^{-1/2} Z_3^{-1/2} A_\mu^a.$$

Finiteness of the gradient flow

- This finiteness persists even for the equal-point product,

$$\langle B_{\mu_1}(t_1, x_1) B_{\mu_2}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle , \qquad t_1 > 0, \ldots, t_n > 0.$$

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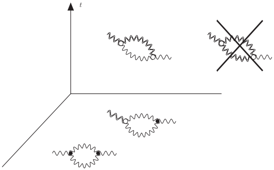
- All order proof of the finiteness uses a local $D+1$-dimensional field theory:



- Because of the gaussian damping factor $\sim e^{-tp^2}$ in the propagator $\Rightarrow$ No bulk ($t>0$) counterterm.
- BRS symmetry $\Rightarrow$ No boundary ($t=0$) counterterm besides Yang–Mills ones.

Fermion flow (Lüscher, arXiv:1302.5246) and the ringed variables (Makino, H.S, arXiv:1403.4772)

- Also for the fermion fields, we introduce the flow

$$\partial_t \chi(t,x) = (\partial_\mu + B_\mu)^2 \chi(t,x), \qquad \chi(t=0,x) = \psi(x),$$
$$\partial_t \bar{\chi}(t,x) = \bar{\chi}(t,x)(\overleftarrow{\partial}_\mu - B_\mu)^2, \qquad \bar{\chi}(t=0,x) = \bar{\psi}(x).$$

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- It turns out that the flowed fermion field requires the wave function renormalization ($Z_\chi=1+[g^2/(4\pi)^2]C_F3(1/\epsilon)+O(g^4)$):

$$\chi_R(t,x)=Z_\chi^{1/2}\chi(t,x),\qquad \bar{\chi}_R(t,x)=Z_\chi^{1/2}\bar{\chi}(t,x).$$

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$$\mathring{\chi}(t,x) = \mathcal{C}\frac{\chi(t,x)}{\sqrt{t^2\left\langle\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle}}, \qquad \mathcal{C} \equiv \sqrt{\frac{-2\dim(R)}{(4\pi)^2}}.$$

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Lattice gauge theory (LGT) and the energy–momentum tensor (EMT)

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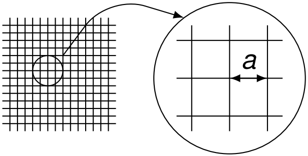

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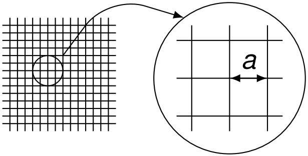

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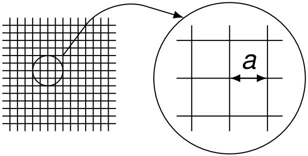

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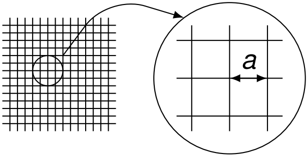

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- For $a \neq 0$, one cannot define the Noether current associated with the translational invariance, EMT $T_{\mu\nu}(x)$.
- Even for the continuum limit $a \to 0$, this is difficult, because EMT is a composite operator which generally contains UV divergences:

$$a \times \frac{1}{a} \stackrel{a \to 0}{\to} 1.$$

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- The correct EMT is characterized by the translation Ward–Takahashi relation

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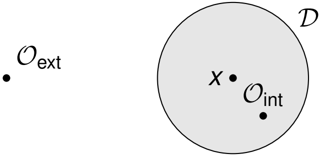

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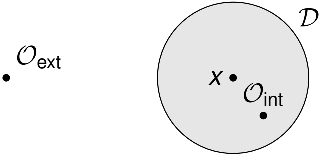

- This contains the correct normalization and the conservation law.

Traditional approach (Caracciolo et al., 1989–)

- Under the hypercubic symmetry, the operator reproducing the correct EMT of QCD for $a \to 0$ is given by

$$T_{\mu\nu}(x) = \sum_{i=1}^{7} Z_i \mathcal{O}_{i\mu\nu}(x)\big|_{\text{lattice}} - \text{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho,\sigma} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x),$$

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- Modern ideas:
  Del Debbio, Patella, Rago, arXiv:1306.1173, gradient flow,
  Giusti, Meyer, arXiv:1011.2727, shifted boundary conditions,
  Dalla Brida, Giusti, Pepe, arXiv:1904.00896 moving frame.

Our approach (arXiv:1304.0533)

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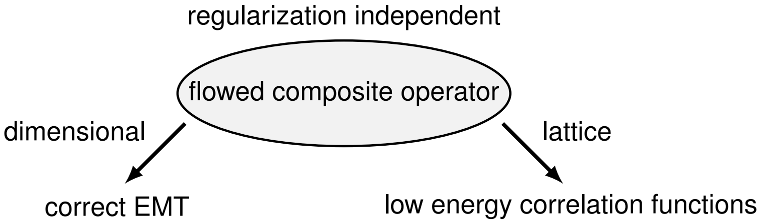
- Schematically,



EMT in dimensional regularization

- Vector-like gauge theory:

$$S = -\frac{1}{2g_0^2}\int d^Dx\,\mathrm{tr}\,[F_{\mu\nu}(x)F_{\mu\nu}(x)] + \int d^Dx\,\bar{\psi}(x)(\not{D} + m_0)\psi(x).$$

- By the Noether method,

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left\{\mathcal{O}_{1\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2\mu\nu}(x)\right\} + \frac{1}{4}\mathcal{O}_{3\mu\nu}(x) - \frac{1}{2}\mathcal{O}_{4\mu\nu}(x) - \mathcal{O}_{5\mu\nu}(x) - \mathrm{VEV},$$

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- Under the dimensional regularization, this simple combination is the correct EMT.

Small flow-time expansion (Lüscher, Weisz, arXiv:1101.0963)

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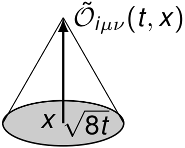

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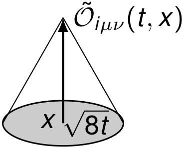

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- This is quite analogous to the OPE, but the continuous flow time t appears more flexible for LGT.

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- Inverting this,

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  we have a representation of the (renormalized) operator in terms of flowed field.
- Furthermore, the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$ can be determined by perturbation theory, thanks to the asymptotic freedom (cf. OPE).
- We use these facts to find a universal representation of the EMT.

EMT from the gradient flow (Makino, H.S., arXiv:1403.4772)

- We take following composite operators of flowed fields:

$$\tilde{\mathcal{O}}_{1\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x),$$
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$$\tilde{\mathcal{O}}_{3\mu\nu}(t,x) \equiv \mathring{\bar{\chi}}(t,x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{4\mu\nu}(t,x) \equiv \delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{5\mu\nu}(t,x) \equiv \delta_{\mu\nu}m\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x),$$

and then the small flow-time expansion reads,

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- We compute $\zeta_{ij}(t)$ with dimensional regularization. We then substitute

$$\mathcal{O}_{i\mu\nu}(x) - \left\langle\mathcal{O}_{i\mu\nu}(x)\right\rangle \mathbb{1} = \lim_{t\to 0}\left\{\sum_j \left(\zeta^{-1}\right)_{ij}(t)\left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle\tilde{\mathcal{O}}_{j\mu\nu}(t,x)\right\rangle \mathbb{1}\right]\right\},$$

  in the expression of the EMT in dimensional regularization.

Universal formula for EMT (Makino, H.S., arXiv:1403.4772)

- In this way,

$$T_{\mu\nu}(x) = \lim_{t\to 0}\Big\{c_1(t)\left[\tilde{\mathcal{O}}_{1,\mu\nu}(t,x) - \frac{1}{4}\tilde{\mathcal{O}}_{2,\mu\nu}(t,x)\right] + c_2(t)\tilde{\mathcal{O}}_{2,\mu\nu}(t,x)$$
$$+ c_3(t)\left[\tilde{\mathcal{O}}_{3,\mu\nu}(t,x) - 2\tilde{\mathcal{O}}_{4,\mu\nu}(t,x)\right]$$
$$+ c_4(t)\tilde{\mathcal{O}}_{4,\mu\nu}(t,x) + c_5(t)\tilde{\mathcal{O}}_{5,\mu\nu}(t,x) - \mathrm{VEV}\Big\},$$

where, to the one-loop order ($T_F = (1/2)n_f$)

$$c_1(t) = \frac{1}{g(\mu)^2} + \left[-\beta_0 L(\mu,t) - \frac{7}{3}C_A + \frac{3}{2}T_F\right]\frac{1}{(4\pi)^2},$$

$$c_2(t) = \frac{1}{4}\left(\frac{11}{6}C_A + \frac{11}{6}T_F\right)\frac{1}{(4\pi)^2},$$

$$c_3(t) = \frac{1}{4} + \left[\frac{1}{4}\left(\frac{3}{2} + \ln 432\right)C_F\right]\frac{g(\mu)^2}{(4\pi)^2},$$

$$c_4(t) = \frac{3}{4}C_F\frac{g(\mu)^2}{(4\pi)^2},$$

$$c_5(t) = -1 - \left[3L(\mu,t) + \frac{7}{2} + \ln 432\right]C_F\frac{g(\mu)^2}{(4\pi)^2},$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F$ and $L(\mu,t) = \ln(2\mu^2 t) + \gamma_E$. We set $\mu \propto 1/\sqrt{t} \to \infty$.

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- We need the extrapolation of the lattice data to $t \to 0$ (see below) and this can be the source of systematic error.
- This is similar to the scale problem in the non-perturbative renormalization... possible usage of the finite size and/or step scaling?

Two-loop coefficients! (Harlander, Kluth, Lange, arXiv:1808.09837, Artz, Harlander, Lange, Neumann, Prausa, arXiv:1905.00882)

- For

$$T_{\mu\nu}(x) = \lim_{t\to 0}\left[\check{c}_1(t)\tilde{\mathcal{O}}_{1,\mu\nu}(t,x) + \check{c}_2(t)\tilde{\mathcal{O}}_{2,\mu\nu}(t,x) + \mathring{\check{c}}_3(t)\tilde{\mathcal{O}}_{3,\mu\nu}(t,x) + \mathring{\check{c}}_4(t)\tilde{\mathcal{O}}_{4,\mu\nu}(t,x) - \mathrm{VEV}\right],$$

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- They obtained

$$\begin{aligned}
\check{c}_1(t) &= \frac{1}{g(\mu)^2}\Bigg(1 + \frac{g(\mu)^2}{(4\pi)^2}\left[-\beta_0 L(\mu,t) - \frac{7}{3}C_A + \frac{3}{2}T_F\right] \\
&+ \frac{g(\mu)^4}{(4\pi)^4}\Bigg\{-\beta_1 L(\mu,t) + C_A^2\left(-\frac{14482}{405} - \frac{16546}{135}\ln 2 + \frac{1187}{10}\ln 3\right) \\
&\qquad + C_A T_F\left[\frac{59}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{10873}{810} + \frac{73}{54}\pi^2 - \frac{2773}{135}\ln 2 + \frac{302}{45}\ln 3\right] \\
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($\mathrm{Li}_2(z)$ is the dilogarithm function) and similar expressions for $\check{c}_2(t)$, ...

- Here, the equation of motion (EoM),

$$\bar{\psi}(x)\left(\frac{1}{2}\overleftrightarrow{\not{D}} + m_0\right)\psi(x) = 0,$$

is used to eliminate $\tilde{\mathcal{O}}_5(t,x)$.

Application to the equation of state (EoS) of QCD

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  and this is very interesting.

- The traceless part gives the entropy density:

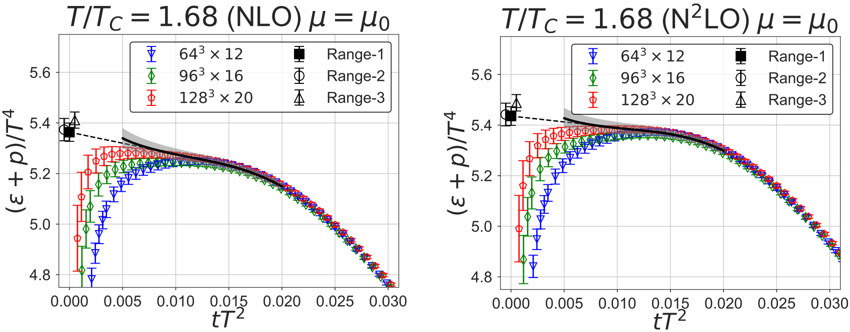
$$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\nu}(x)\right\rangle_T\,,$$

  and the trace part gives the "trace anomaly":

$$\varepsilon - 3p = -\langle T_{\mu\mu}(x) - \mathrm{VEV}\rangle_T\,.$$

Quenched QCD (Iritani, Kitazawa, H.S., Takaura, arXiv:1812.06444)

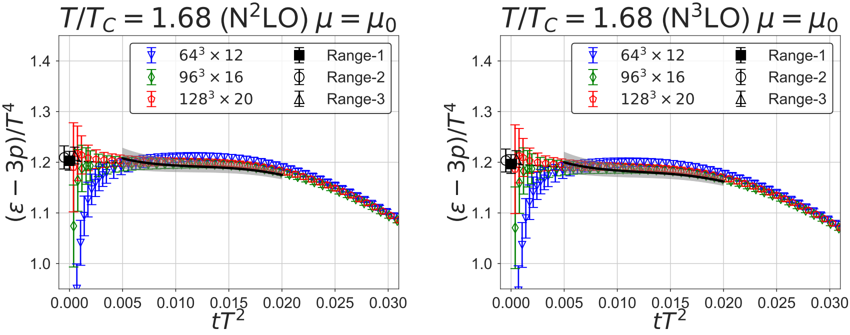
- For the entropy density ($\mu_0 \equiv 1/\sqrt{2e^{\gamma_E} t}$),



- The higher order coefficients render the behavior more stable $\Rightarrow$ Less sensitive to the method of the $t \to 0$ extrapolation.

Quenched QCD

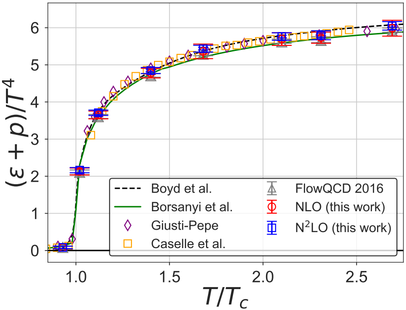
- For the trace anomaly,



- For the trace anomaly, the two-loop coefficients already give well-stable behavior.

Quenched QCD

- Already a field of precise determination:



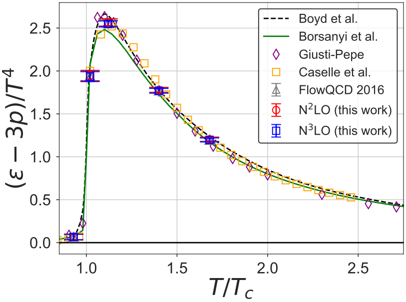
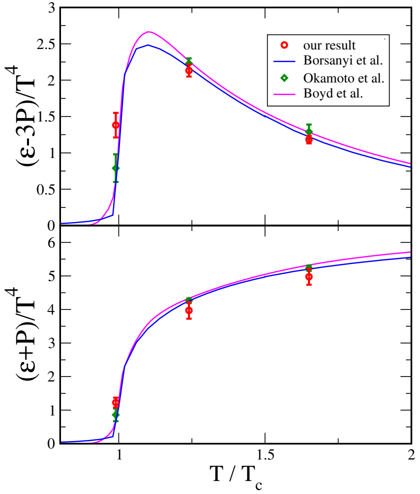


Figure: Boyd et al., Borsanyi et al.: Integral method, Giusti, Pepe: Moving frame method, Caselle et al.: Jarzynski's equality.

5 years ago, at this same place, I reported the first trial. . .

- FlowQCD Collaboration, arXiv:1312.7492:

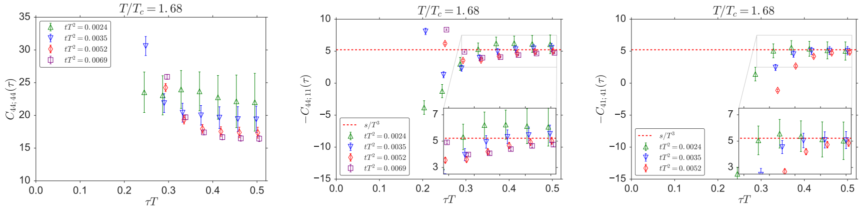


Two point function in the quenched QCD (Kitazawa, Iritani, Asakawa, Hatsuda, arXiv:1708.01415)

- The connected part

$$C_{\mu\nu;\rho\sigma}(\tau) \equiv \frac{1}{T^5} \int_V d^3x \, \langle \delta T_{\mu\nu}(x) \delta T_{\rho\sigma}(0) \rangle \,,$$

where $\delta T_{\mu\nu}(x) \equiv T_{\mu\nu}(x) - \langle T_{\mu\nu}(x) \rangle$.



- Indicating the conservation law of the EMT, $\partial_\tau C_{\mu\nu;\rho\sigma}(\tau) = 0$.
- Confirms the linear response relations, s.t,

$$\frac{\varepsilon + p}{T^4} = \frac{1}{T^3} \frac{dp}{dT} = -C_{44,;11}(\tau).$$

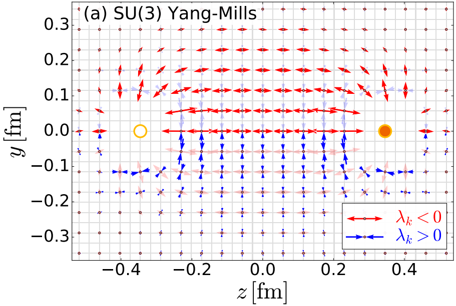
Stress tensor distribution around the static quark–anti-quark pair (Yanagihara, Iritani, Kitazawa, Asakawa, Hatsuda, arXiv:1803.05656)

- The EMT around the static quark–anti-quark pair:

$$\mathcal{T}_{\mu\nu}(x) \equiv \langle T_{\mu\nu}(x) \rangle_{Q\bar{Q}} = \lim_{T\to\infty} \frac{\langle T_{\mu\nu}(x) W(R,T) \rangle}{\langle W(R,T) \rangle}.$$

- Eigenvectors:

$$\mathcal{T}_{ij} n_j^{(k)} = \lambda_k n_j^{(k)}$$



Full QCD (WHOT-QCD Collaboration: Baba, Ejiri, Iwami, Kanaya, Kitazawa, Shimojo, Shirogane, A. Suzuki, H.S., Taniguchi, Umeda)

- $N_f = 2 + 1$ QCD by the NP $O(a)$-improved Wilson quark action and the RG improved Iwasaki gauge action.

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- $N_f = 2 + 1$ QCD by the NP $O(a)$-improved Wilson quark action and the RG improved Iwasaki gauge action.
- Somewhat heavy ud quarks ($m_\pi/m_\rho \simeq 0.63$, $m_{\eta_{ss}}/m_\phi \simeq 0.74$)
  - $a = 0.0701(29)$ fm, $28^3 \times 56$ (JLQCD), $32^3 \times N_t$ ($N_t = 6, 8, \ldots, 16$)
  - $a = 0.0970(26)$ fm, $32^3 \times 40$, $32^3 \times N_t$ ($N_t = 8, 10, 11, 12$)
  - [$a = 0.04976$ fm, $40^3 \times 80$]

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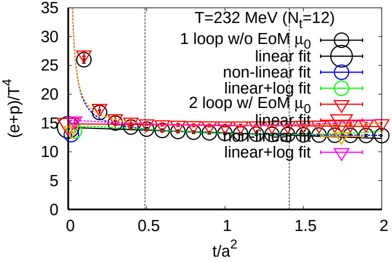
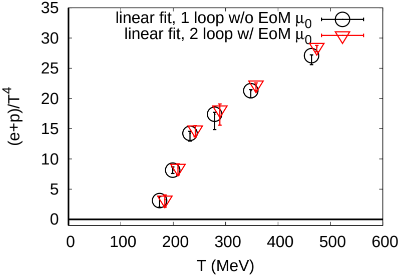
- Physical mass ud quarks
  - $a = 0.08995(40)$ fm, $32^3 \times 64$ (PACS-CS), $32^3 \times N_t$ ($N_t = 4, 5, 6, \ldots, 14, 15, 16, 18$)
- Physical prediction on EoS etc. . .

Somewhat heavy ud quarks, $a \simeq 0.07$ fm (WHOT-QCD Collaboration, arXiv:1609.01417 and preliminary results)

- For the entropy density,





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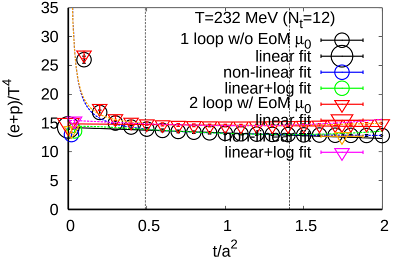

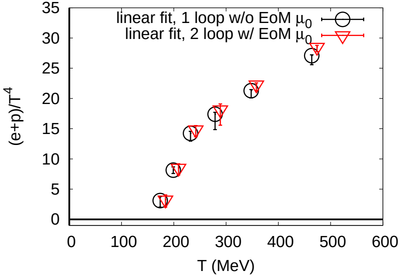

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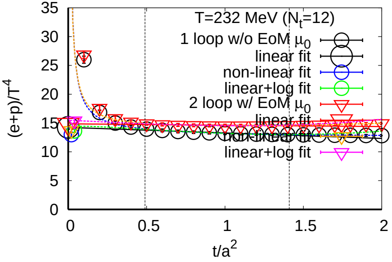

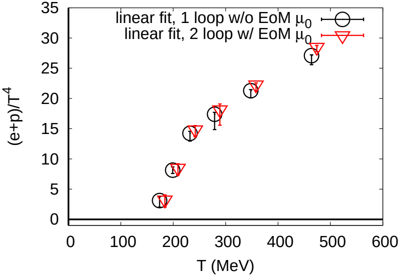

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- Note that the entropy density is the traceless part and does not contain the EoM.

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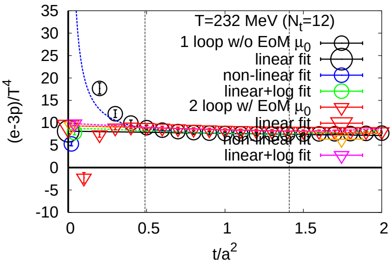

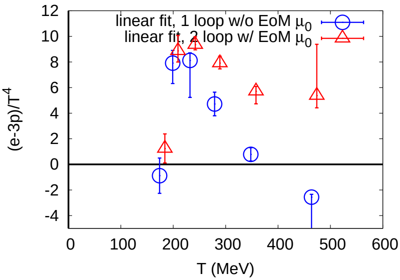

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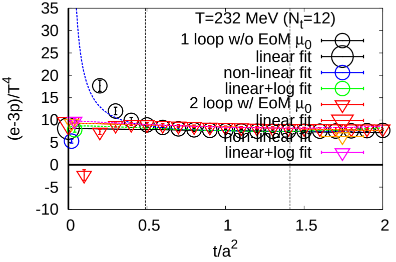
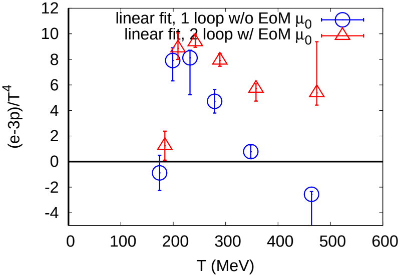
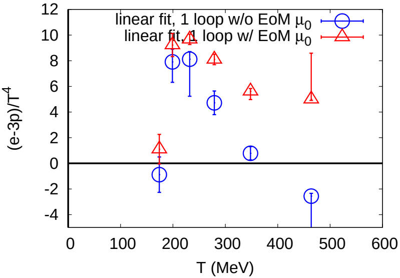




- It seems that the EoM suffers from very large lattice artifact for $N_t \lesssim 10$ $(a = 1/(N_t T))$...

Somewhat heavy ud quarks, $a \simeq 0.07\,\mathrm{fm}$

- In fact, even in the 1 loop level,



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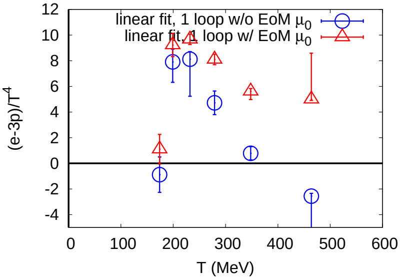
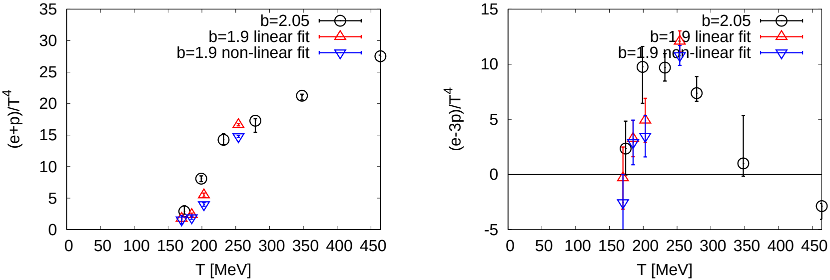


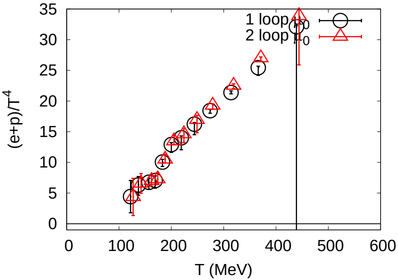
- This must be a lattice artifact and expected to (or should) disappear in the continuum limit.

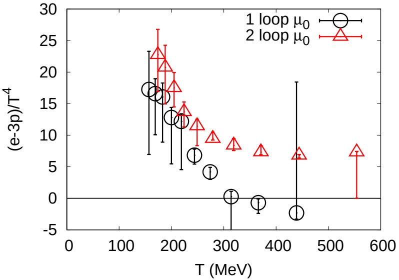
Somewhat heavy ud quarks, $a \simeq 0.07$ fm and $a \simeq 0.097$ fm (Preliminary)



- It appears that the dependence on the lattice spacing is fairly small.

Physical mass ud, $a \simeq 0.09$ fm, (WHOT-QCD Collaboration, arXiv:1710.10015, and preliminary results)





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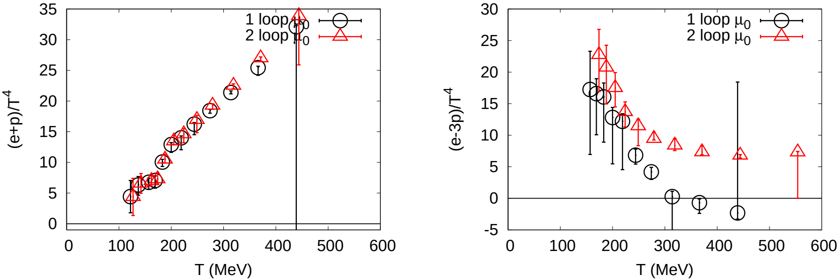

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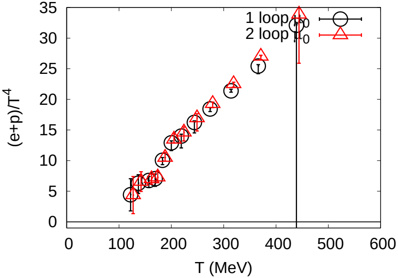

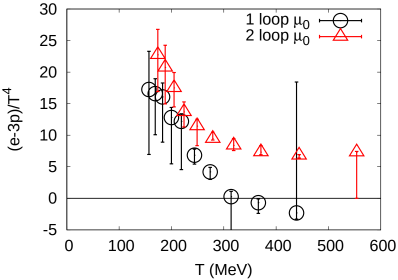

- Entropy density seems to be consistent with that obtained by the staggered quarks (Borsányi et. al; HotQCD Collaboration).
- Trace anomaly seems much larger than that by the staggered quarks, although the error bars are still rather large.
- More statistics and finer lattices are ongoing. is a future problem.

Two point function in $N_f = 2+1$ full QCD and viscosities (for the somewhat heavy ud quarks, $a \simeq 0.07\,\mathrm{fm}$) (WHOT-QCD Collaboration, arXiv:1901.01666)

- Euclidean EMT two point function and the spectral function are related by

$$\int d^3x \left\langle T_{ij}(-i\tau,\vec{x})T_{kl}(0,\vec{0})\right\rangle_T = \int_0^\infty \frac{dk_0}{2\pi}\frac{\cosh k_0(\tau-\beta/2)}{\sinh k_0\beta/2}\rho_{ij;kl}(k_0,\vec{0}).$$

and the shear and the bulk viscosities are given by

$$\eta = \lim_{k_0\to 0}\frac{\rho_{ij;ij}(k_0,\vec{0})}{2k_0}, \qquad \zeta = \lim_{k_0\to 0}\frac{\rho_{ii;ii}(k_0,\vec{0})}{2k_0}.$$

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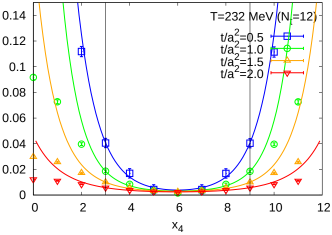
- To obtain $\rho(k_0)$ from lattice data is an ill-posed problem and, for instance, we adopt the Breit–Wigner type ansatz:

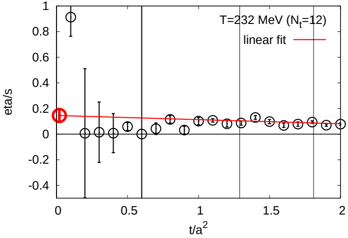
$$\frac{\rho_{\text{BW}}(k_0)}{2k_0} = \frac{F}{1 + b^2(k_0 - \omega_0)^2} + \frac{F}{1 + b^2(k_0 + \omega_0)^2},$$

  and use F, b, and ω_0 as fit parameters.

Two point function in $N_f = 2 + 1$ full QCD and viscosities (WHOT-QCD Collaboration, arXiv:1901.01666)

- A case the BW fit works fairly well ($N_f = 12$, $T = 232\,\text{MeV}$):





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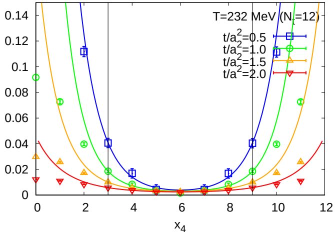
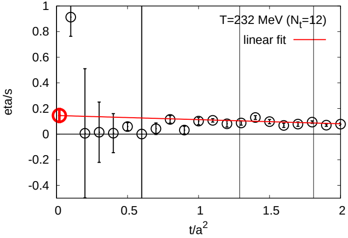
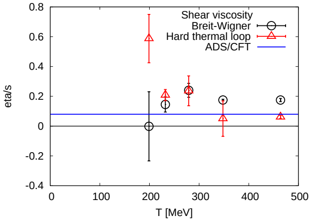




- Shear viscosity η/s vs. T (still preliminary)



3D scalar theory (Morikawa, Sonoda, H.S., work in progress)

- 3D *N*-component scalar theory

$$S = \int d^D x \left[\frac{1}{2} \partial_\mu \phi^I \partial_\mu \phi^I + \frac{m_0^2}{2} \phi^I \phi^I + \frac{\lambda_0}{8N} \left(\phi^I \phi^I \right)^2 \right]$$

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- Universal formula for EMT ($\mathcal{C} = 3.844365111074$):

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu \varphi^I \partial_\nu \varphi^I - \delta_{\mu\nu} \left[\frac{1}{2} \partial_\rho \varphi^I \partial_\rho \varphi^I + \frac{m^2}{2} \varphi^I \varphi^I + \frac{\lambda}{8N} \left(\varphi^I \varphi^I \right)^2 \right] \\ &- \delta_{\mu\nu} \left(\frac{\lambda}{4\pi} \left(1 + \frac{2}{N} \right) \left(-\frac{1}{3} \right) (8\pi t)^{-1/2} \right. \\ &\left. + \frac{\lambda^2}{(4\pi)^2} \left\{ \left(1 + \frac{2}{N} \right)^2 \left(-\frac{1}{4\pi} \right) + \frac{1}{N} \left(1 + \frac{2}{N} \right) \left(-\frac{1}{8} \right) \left[\ln(8\pi\mu^2 t) - \frac{1}{3} + \mathcal{C} \right] \right\} \right) \varphi^I \varphi^I. \end{aligned}$$

3D scalar theory (Morikawa, Sonoda, H.S., work in progress)

- The theory around the Wilson–Fisher fixed point can be realized as the long-distance limit,

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle_{g_E} = \lim_{\tau\to\infty} e^{nx_h\tau}\,\langle\phi(e^\tau x_1)\ldots\phi(e^\tau x_n)\rangle_{m^2,\lambda}\,,$$

where

$$m^2 = m^2_{\text{cr}}(\lambda) + g_E e^{-y_E\tau}.$$

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- cf. in the large N limit,

$$x_h = \frac{1}{2}, \qquad y_E = 1,$$

  and

$$m^2_{\mathrm{cr}}(\lambda) = 0.$$

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- The formula can actually be used in non-perturbative lattice simulations and numerical experiments so far show encouraging results, besides the "EoM problem" with the present lattice spacings.
- Yet, we have the scale problem,

$$a \lesssim \sqrt{8t}.$$

Step scaling or something analogous???

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Summary and prospects

- Asymptotic form in $t \to 0$? (work in progress with Takaura).
- Push applications further: EoS of QCD, viscosities in gauge theory, momentum/spin structure of baryons, critical exponents in low-energy conformal field theory, dilaton physics, ...
- Further theoretical understanding on the equal-point correction. The axial $U(1)_A$ anomaly in gravitational field is not automatically reproduced (Morikawa, H.S., arXiv:1803.04132),

$$\partial_\alpha^x \langle j_{5\alpha}(x) T_{\mu\nu}(y) T_{\rho\sigma}(z) \rangle$$
$$\neq \int_{p,q} e^{ip(x-y)} e^{iq(x-z)} \frac{1}{(4\pi)^2} \frac{1}{6} \epsilon_{\mu\rho\beta\gamma} p_\beta q_\gamma (q_\nu p_\sigma - \delta_{\nu\sigma} pq) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma),$$

but requires a correction by a "local counterterm" $\propto \delta(x-y)\delta(x-z)$.