Application of the gradient flow to the energy–momentum tensor

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- One-parameter $t \geq 0$ (the flow time) deformation of the gauge field $A_\mu(x)$,

$$A_\mu(x) \rightarrow B_\mu(t, x), \quad B_\mu(t = 0, x) = A_\mu(x),$$

according to (the flow equation)

$$\partial_t B_\mu(t, x) = -g_0^2 \frac{\delta S_{YM}[B]}{\delta B_\mu(t, x)},$$

Here, $S_{YM}$ is the Yang–Mills action and the RHS is the gradient in functional space. So the name of the Yang–Mills gradient flow.

Since $D = \partial + [B; \quad G(t, x)]$,

$$G(t, x) = \partial B(t, x) + [B(t, x) ; \quad B(t, x)]$$

this is a diffusion-type equation with the diffusion length, $x_p \sqrt{t}$.

The flow time $t$ has the mass dimension $2$. 

Hiroshi Suzuki (Kyushu University)
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according to (the flow equation)

$$\partial_t B_\mu(t, x) = -g_0^2 \frac{\delta S_{YM}[B]}{\delta B_\mu(t, x)} = D_{\nu} G_{\nu\mu}(t, x) = \Delta B_\mu(t, x) + \cdots,$$

Here, $S_{YM}$ is the Yang–Mills action and the RHS is the gradient in functional space. So the name of the Yang–Mills gradient flow.

Since

$$D_\mu = \partial_\mu + [B_\mu, \cdot], \quad G_{\mu\nu}(t, x) = \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) + [B_\mu(t, x), B_\nu(t, x)],$$

this is a diffusion-type equation with the diffusion length,

$$x \sim \sqrt{8t}.$$
Yang–Mills gradient flow (continuum)

\[ \partial_t B_\mu(t, x) = -g_0^2 \frac{\delta S_{YM}[B]}{\delta B_\mu(t, x)}, \quad B_\mu(t = 0, x) = A_\mu(x). \]
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- Wilson flow (lattice)

\[
\partial_t V(t, x, \mu)V(t, x, \mu)^{-1} = -g_0^2 \partial_{x, \mu} S_{\text{Wilson}}[V], \quad V(t = 0, x, \mu) = U(x, \mu).
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Applications in lattice gauge theory,
- Topological charge
- Scale setting
- Non-perturbative gauge coupling constant
- Chiral condensate
- Various renormalized operators, including the energy–momentum tensor
- Supersymmetric theory
- ...
Correlation function of the flowed gauge field,

\[ \langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle = \frac{1}{Z} \int \mathcal{D}A_{\mu} B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) e^{-S_{YM}[A]}, \]

when expressed in terms of renormalized coupling,

\[ g^2 = g_0^2 \mu^{-2\varepsilon} Z^{-1}, \]

is UV finite without the wave function renormalization.
Correlation function of the flowed gauge field,

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is UV finite without the wave function renormalization.

This is quite contrast to the conventional gauge field, for which

\[ \langle A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) \rangle, \]

requires the wave function renormalization

\[ (A_R)^a_\mu = Z^{-1/2} Z_3^{-1/2} A^a_\mu. \]
This finiteness persists even for the equal-point product,

\[ \langle B_{\mu_1}(t_1, x_1) B_{\mu_2}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle, \quad t_1 > 0, \ldots, t_n > 0. \]
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All order proof of the finiteness uses a local \( D + 1 \)-dimensional field theory:

Because of the gaussian damping factor \( \sim e^{-tp^2} \) in the propagator \( \Rightarrow \) No bulk \( (t > 0) \) counterterm.

BRS symmetry \( \Rightarrow \) No boundary \( (t = 0) \) counterterm besides Yang–Mills ones.
Also for the fermion fields, we introduce the flow
\[
\partial_t \chi(t, x) = (\partial_\mu + B_\mu)^2 \chi(t, x), \quad \chi(t = 0, x) = \psi(x),
\]
\[
\partial_t \bar{\chi}(t, x) = \bar{\chi}(t, x)(\bar{\partial}_\mu - B_\mu)^2, \quad \bar{\chi}(t = 0, x) = \bar{\psi}(x).
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\]

It turns out that the flowed fermion field requires the wave function renormalization (\(Z^\chi = 1 + [g^2/(4\pi)^2]C_F3(1/\epsilon) + O(g^4))\):
\[
\chi_R(t, x) = Z^{1/2}_\chi \chi(t, x), \quad \bar{\chi}_R(t, x) = Z^{1/2}_\chi \bar{\chi}(t, x).
\]
Fermion flow (Lüscher, arXiv:1302.5246) and the ringed variables (Makino, H.S, arXiv:1403.4772)

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- Still, any composite operators of \( \chi_R(t, x) \) are UV finite.
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Still, any composite operators of \( \chi_R(t, x) \) are UV finite.

So we introduce the ringed variable (similarly for \( \bar{\chi}(t, x) \)),

\[
\hat{\chi}(t, x) = C \frac{\chi(t, x)}{\sqrt{t^2 \langle \bar{\chi}(t, x) \overleftarrow{D} \chi(t, x) \rangle}}, \quad C \equiv \sqrt{-2 \text{dim}(R)} \left(\frac{4\pi}{4\pi} \right)^2.
\]
Also for the fermion fields, we introduce the flow

\[ \partial_t \chi(t, x) = (\partial_\mu + B_\mu)^2 \chi(t, x), \quad \chi(t = 0, x) = \psi(x), \]
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Any composite operators of \(\hat{\chi}(t, x)\) and \(\hat{\bar{\chi}}(t, x)\) are UV finite.
Lattice gauge theory (LGT) and the energy–momentum tensor (EMT)

LGT preserves exact gauge symmetry.

For $a \neq 0$, one cannot define the Noether current associated with the translational invariance, $\text{EMT}_T(x)$. Even for the continuum limit $a \to 0$, this is difficult, because EMT is a composite operator which generally contains UV divergences:
Lattice gauge theory (LGT) and the energy–momentum tensor (EMT)

- LGT preserves exact gauge symmetry.
- This however breaks spacetime symmetries (translation, Poincaré, SUSY, ...) for $a \neq 0$. 

![Diagram showing lattice and circle with parameter $a$.]
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- Even for the continuum limit \( a \to 0 \), this is difficult, because EMT is a composite operator which generally contains UV divergences:

\[
a \times \frac{1}{a} \xrightarrow{a \to 0} 1.
\]
EMT in LGT?

We want to construct EMT in LGT, which becomes the correct EMT (automatically) in the continuum limit $a \rightarrow 0$. 

\[
\langle O_{\text{ext}} \partial_o O_{\text{int}} \rangle = \langle O_{\text{ext}} \partial_o O_{\text{int}} \rangle 
\]

This contains the correct normalization and the conservation law.
We want to construct EMT in LGT, which becomes the correct EMT (automatically) in the continuum limit $a \to 0$.

Applications to physics related to spacetime symmetries: QCD thermodynamics, transport coefficients in gauge theory, momentum/spin structure of baryons, conformal field theory, dilaton physics, ...
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- The correct EMT is characterized by the translation Ward–Takahashi relation

$$\left\langle \mathcal{O}_{\text{ext}} \int_{\mathcal{D}} d^D x \, \partial_\mu T_{\mu\nu}(x) \mathcal{O}_{\text{int}} \right\rangle = - \left\langle \mathcal{O}_{\text{ext}} \partial_\nu \mathcal{O}_{\text{int}} \right\rangle.$$
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This contains the correct normalization and the conservation law.
Under the hypercubic symmetry, the operator reproducing the correct EMT of QCD for $a \to 0$ is given by

$$T_{\mu\nu}(x) = \sum_{i=1}^{7} Z_i O_{i\mu\nu}(x)|_{\text{lattice}} - \text{VEV},$$

where

$$O_{1\mu\nu}(x) \equiv \sum_{\rho} F_{\mu\rho}^a(x) F_{\nu\rho}^a(x),$$

$$O_{2\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho,\sigma} F_{\rho\sigma}^a(x) F_{\rho\sigma}^a(x),$$

$$O_{3\mu\nu}(x) \equiv \bar{\psi}(x) \left( \gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \psi(x),$$

$$O_{4\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}(x) \overleftrightarrow{D} \psi(x),$$

$$O_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x) \psi(x),$$

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Modern ideas:
Del Debbio, Patella, Rago, arXiv:1306.1173, gradient flow,
Giusti, Meyer, arXiv:1011.2727, shifted boundary conditions,
Our approach (arXiv:1304.0533)

- We bridge **lattice** regularization and **dimensional** regularization, which preserves the translational invariance, by the gradient flow.
Our approach (arXiv:1304.0533)

- We bridge lattice regularization and dimensional regularization, which preserves the translational invariance, by the gradient flow.
- Schematically,
EMT in dimensional regularization

- Vector-like gauge theory:
  \[
  S = -\frac{1}{2g_0^2} \int d^D x \ tr [F_{\mu\nu}(x)F_{\mu\nu}(x)] + \int d^D x \overline{\psi}(x) (\not{\!D} + m_0)\psi(x).
  \]

- By the Noether method,
  \[
  T_{\mu\nu}(x) = \frac{1}{g_0^2} \left\{ O_{1\mu\nu}(x) - \frac{1}{4} O_{2\mu\nu}(x) \right\} + \frac{1}{4} O_{3\mu\nu}(x) - \frac{1}{2} O_{4\mu\nu}(x) - O_{5\mu\nu}(x) - \text{VEV},
  \]
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\[ T_{\mu\nu}(x) = \frac{1}{g_0^2} \left\{ \mathcal{O}_{1\mu\nu}(x) - \frac{1}{4} \mathcal{O}_{2\mu\nu}(x) \right\} + \frac{1}{4} \mathcal{O}_{3\mu\nu}(x) - \frac{1}{2} \mathcal{O}_{4\mu\nu}(x) - \mathcal{O}_{5\mu\nu}(x) - \text{VEV}, \]

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\[ \mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x)\psi(x). \]

- Under the dimensional regularization, this simple combination is the correct EMT.
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The relation becomes tractable, however, in the small flow time limit $t \to 0$. 

\[
\text{Small flow-time expansion} \quad (\text{Lüscher, Weisz, arXiv:1101.0963})
\]

\begin{itemize}
  \item \[
  \langle \mathcal{O}_i(t; x) \rangle 
  \approx O_i(t; x) = \langle \mathcal{O}_i(t; x) \rangle / x^{31} + \sum_j i_{ij}(t) [O_R(t) + \text{VEV}] + O(t)
  \]
  \end{itemize}

This is quite analogous to the OPE, but the continuous flow time $t$ appears more flexible for LGT.
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**Small flow-time expansion**

$$\tilde{O}_{i\mu\nu}(t, x) = \langle \tilde{O}_{i\mu\nu}(t, x) \rangle \mathbb{1} + \sum_j \zeta_{ij}(t) [O_{R_{i\mu\nu}}(x) - \text{VEV}] + O(t).$$
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Inverting this,

\[ O_{Ri\mu\nu}(x) - \text{VEV} = \lim_{t \to 0} \left\{ \sum_j (\zeta^{-1})_{ij}(t) [\tilde{O}_{j\mu\nu}(t, x) - \langle \tilde{O}_{j\mu\nu}(t, x) \rangle \mathbb{1}] \right\}, \]

we have a representation of the (renormalized) operator in terms of flowed field.
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\]

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O_{Ri\mu\nu}(x) - VEV = \lim_{t \to 0} \left\{ \sum_j (\zeta^{-1})_{ij}(t) \left[ \tilde{O}_{j\mu\nu}(t, x) - \langle \tilde{O}_{j\mu\nu}(t, x) \rangle \mathbb{1} \right] \right\},
\]

we have a representation of the (renormalized) operator in terms of flowed field.

- Furthermore, the \( t \to 0 \) behavior of the coefficients \( \zeta_{ij}(t) \) can be determined by perturbation theory, thanks to the asymptotic freedom (cf. OPE).
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\tilde{\mathcal{O}}_{\mu\nu}(t, x) = \langle \tilde{\mathcal{O}}_{\mu\nu}(t, x) \rangle \mathbb{1} + \sum_j \zeta_{ij}(t) [\mathcal{O}_{Rj\mu\nu}(x) - \text{VEV}] + O(t).
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we have a representation of the (renormalized) operator in terms of flowed field.

Furthermore, the \( t \to 0 \) behavior of the coefficients \( \zeta_{ij}(t) \) can be determined by perturbation theory, thanks to the asymptotic freedom (cf. OPE).

We use these facts to find a universal representation of the EMT.
We take following composite operators of flowed fields:

\[
\tilde{O}_{1\mu\nu}(t, x) \equiv G^a_{\mu\rho}(t, x)G^a_{\nu\rho}(t, x),
\]

\[
\tilde{O}_{2\mu\nu}(t, x) \equiv \delta_{\mu\nu}G^a_{\rho\sigma}(t, x)G^a_{\rho\sigma}(t, x),
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\[
\tilde{O}_{3\mu\nu}(t, x) \equiv \tilde{\chi}(t, x) \left( \gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \tilde{\chi}(t, x),
\]

\[
\tilde{O}_{4\mu\nu}(t, x) \equiv \delta_{\mu\nu}\tilde{\chi}(t, x) \overleftrightarrow{D} \tilde{\chi}(t, x),
\]

\[
\tilde{O}_{5\mu\nu}(t, x) \equiv \delta_{\mu\nu}m\tilde{\chi}(t, x)\tilde{\chi}(t, x),
\]

and then the small flow-time expansion reads,

\[
\tilde{O}_{i\mu\nu}(t, x) = \left( \tilde{O}_{i\mu\nu}(t, x) \right) \mathbb{I} + \sum_j \tilde{\zeta}_{ij}(t) \left[ \tilde{O}_{j\mu\nu}(x) - \left( \tilde{O}_{j\mu\nu}(x) \right) \mathbb{I} \right] + \mathcal{O}(t).
\]
EMT from the gradient flow (Makino, H.S., arXiv:1403.4772)

- We take following composite operators of flowed fields:
  \[
  \tilde{O}_{1\mu \nu}(t, x) \equiv G_{\mu \rho}^{a}(t, x)G_{\nu \rho}^{a}(t, x),
  \]
  \[
  \tilde{O}_{2\mu \nu}(t, x) \equiv \delta_{\mu \nu} G_{\rho \sigma}^{a}(t, x)G_{\rho \sigma}^{a}(t, x),
  \]
  \[
  \tilde{O}_{3\mu \nu}(t, x) \equiv \tilde{\chi}(t, x) \left( \gamma_{\mu} \overleftrightarrow{D}_{\nu} + \gamma_{\nu} \overleftrightarrow{D}_{\mu} \right) \tilde{\chi}(t, x),
  \]
  \[
  \tilde{O}_{4\mu \nu}(t, x) \equiv \delta_{\mu \nu} \tilde{\chi}(t, x) \overleftrightarrow{D} \tilde{\chi}(t, x),
  \]
  \[
  \tilde{O}_{5\mu \nu}(t, x) \equiv \delta_{\mu \nu} m \tilde{\chi}(t, x) \tilde{\chi}(t, x),
  \]
  and then the small flow-time expansion reads,
  \[
  \tilde{O}_{i\mu \nu}(t, x) = \langle \tilde{O}_{i\mu \nu}(t, x) \rangle \mathbb{1} + \sum_{j} \zeta_{ij}(t) \left[ O_{j\mu \nu}(x) - \langle O_{j\mu \nu}(x) \rangle \mathbb{1} \right] + O(t).
  \]

- We compute \( \zeta_{ij}(t) \) with dimensional regularization. We then substitute
  \[
  O_{i\mu \nu}(x) - \langle O_{i\mu \nu}(x) \rangle \mathbb{1} = \lim_{t \to 0} \left\{ \sum_{j} (\zeta^{-1})_{ij}(t) \left[ \tilde{O}_{j\mu \nu}(t, x) - \langle \tilde{O}_{j\mu \nu}(t, x) \rangle \mathbb{1} \right] \right\},
  \]
  in the expression of the EMT in dimensional regularization.
In this way,

\[
T_{\mu\nu}(x) = \lim_{t \to 0} \left\{ c_1(t) \left[ \tilde{O}_{1,\mu\nu}(t, x) - \frac{1}{4} \tilde{O}_{2,\mu\nu}(t, x) \right] + c_2(t) \tilde{O}_{2,\mu\nu}(t, x) + c_3(t) \left[ \tilde{O}_{3,\mu\nu}(t, x) - 2 \tilde{O}_{4,\mu\nu}(t, x) \right] + c_4(t) \tilde{O}_{4,\mu\nu}(t, x) + c_5(t) \tilde{O}_{5,\mu\nu}(t, x) - \text{VEV} \right\},
\]

where, to the one-loop order \((T_F = (1/2)n_f)\)

\[
c_1(t) = \frac{1}{g(\mu)^2} + \left[ -\beta_0 L(\mu, t) - \frac{7}{3} C_A + \frac{3}{2} T_F \right] \frac{1}{(4\pi)^2},
\]

\[
c_2(t) = \frac{1}{4} \left( \frac{11}{6} C_A + \frac{11}{6} T_F \right) \frac{1}{(4\pi)^2},
\]

\[
c_3(t) = \frac{1}{4} + \left[ \frac{1}{4} \left( \frac{3}{2} + \ln 432 \right) C_F \right] \frac{g(\mu)^2}{(4\pi)^2},
\]

\[
c_4(t) = \frac{3}{4} C_F \frac{g(\mu)^2}{(4\pi)^2},
\]

\[
c_5(t) = -1 - \left[ 3L(\mu, t) + \frac{7}{2} + \ln 432 \right] C_F \frac{g(\mu)^2}{(4\pi)^2},
\]

where \(\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F\) and \(L(\mu, t) = \ln(2\mu^2 t) + \gamma_E\). We set \(\mu \propto 1/\sqrt{t} \to \infty\).
Universal formula for EMT

- This is **manifestly finite**, as it should be so for EMT!
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- This is **universal**: \( c_i(t) \) are independent of the regularization.
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We have to first send the cutoff infinity \( a \to 0 \) and then take the flow time zero, \( t \to 0 \).
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We have to first send the cutoff infinity $a \to 0$ and then take the flow time zero, $t \to 0$.

Practically, however, we cannot simply take $a \to 0$, and $t$ is limited to

\[ a \lesssim \sqrt{8t}. \]
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We need the extrapolation of the lattice data to $t \to 0$ (see below) and this can be the source of systematic error.

This is similar to the scale problem in the non-perturbative renormalization. . . possible usage of the finite size and/or step scaling?
For 

\[ T_{\mu\nu}(x) = \lim_{t \to 0} \left[ \partial_1(t) \partial_1,\mu\nu(t, x) + \partial_2(t) \partial_2,\mu\nu(t, x) + \partial_3(t) \partial_3,\mu\nu(t, x) + \partial_4(t) \partial_4,\mu\nu(t, x) - \text{VEV} \right], \]

For

\[ T_{\mu\nu}(x) = \lim_{t \to 0} \left[ \tilde{c}_1(t)\tilde{O}_{1,\mu\nu}(t, x) + \tilde{c}_2(t)\tilde{O}_{2,\mu\nu}(t, x) + \tilde{c}_3(t)\tilde{O}_{3,\mu\nu}(t, x) + \tilde{c}_4(t)\tilde{O}_{4,\mu\nu}(t, x) - \text{VEV} \right], \]

They obtained

\[ \tilde{c}_1(t) = \frac{1}{g(\mu)^2} \left( 1 + \frac{g(\mu)^2}{(4\pi)^2} \left[ -\beta_0 L(\mu, t) - \frac{7}{3} C_A + \frac{3}{2} T_F \right] \right. \]

\[ \left. + \frac{g(\mu)^4}{(4\pi)^4} \left\{ -\beta_1 L(\mu, t) + C_A^2 \left( -\frac{14482}{405} - \frac{16546}{135} \ln 2 + \frac{1187}{10} \ln 3 \right) \right. \right. \]

\[ \left. + C_A T_F \left[ \frac{59}{9} \text{Li}_2 \left( \frac{1}{4} \right) + \frac{10873}{810} + \frac{73}{54} \pi^2 - \frac{2773}{135} \ln 2 + \frac{302}{45} \ln 3 \right] \right. \]

\[ \left. + C_F T_F \left[ -\frac{256}{9} \text{Li}_2 \left( \frac{1}{4} \right) + \frac{2587}{108} - \frac{7}{9} \pi^2 - \frac{106}{9} \ln 2 - \frac{161}{18} \ln 3 \right] \right\}, \]

(\text{Li}_2(z) is the dilogarithm function) and similar expressions for \( \tilde{c}_2(t), \ldots. \)
For $T_{\mu \nu}(x)$

$$T_{\mu \nu}(x) = \lim_{t \to 0} \left[ \tilde{c}_1(t) \tilde{O}_{1,\mu \nu}(t, x) + \tilde{c}_2(t) \tilde{O}_{2,\mu \nu}(t, x) + \tilde{c}_3(t) \tilde{O}_{3,\mu \nu}(t, x) + \tilde{c}_4(t) \tilde{O}_{4,\mu \nu}(t, x) - \text{VEV} \right],$$

They obtained

$$\tilde{c}_1(t) = \frac{1}{g(\mu)^2} \left( 1 + \frac{g(\mu)^2}{(4\pi)^2} \left[ -\beta_0 L(\mu, t) - \frac{7}{3} C_A + \frac{3}{2} T_F \right] + \frac{g(\mu)^4}{(4\pi)^4} \left\{ -\beta_1 L(\mu, t) + C_A^2 \left( -\frac{14482}{405} - \frac{16546}{135} \ln 2 + \frac{1187}{10} \ln 3 \right) + C_A T_F \left[ \frac{59}{9} \text{Li}_2 \left( \frac{1}{4} \right) + \frac{10873}{810} + \frac{73}{54} \pi^2 - \frac{2773}{135} \ln 2 + \frac{302}{45} \ln 3 \right] + C_F T_F \left[ -\frac{256}{9} \text{Li}_2 \left( \frac{1}{4} \right) + \frac{2587}{108} - \frac{7}{9} \pi^2 - \frac{106}{9} \ln 2 - \frac{161}{18} \ln 3 \right] \right\} \right),$$

($\text{Li}_2(z)$ is the dilogarithm function) and similar expressions for $\tilde{c}_2(t), \ldots$.

Here, the equation of motion (EoM),

$$\bar{\psi}(x) \left( \frac{1}{2} \not\!D + m_0 \right) \psi(x) = 0,$$

is used to eliminate $\tilde{O}_5(t, x)$. 
We cannot measure VEV (one-point function) of EMT,

\[ \langle T_{\mu\nu}(x) \rangle . \]
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We can, however, measure the expectation value at the finite temperature \( T \), 
\[ \langle T_{\mu\nu}(x) - \text{VEV} \rangle_T, \]
and this is very interesting.
We cannot measure VEV (one-point function) of EMT,
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and this is very interesting.

The traceless part gives the entropy density:
\[ \varepsilon + p = -\frac{4}{3} \langle T_{00}(x) - \frac{1}{4} T_{\mu\nu}(x) \rangle_T, \]
and the trace part gives the “trace anomaly”:
\[ \varepsilon - 3p = -\langle T_{\mu\mu}(x) - \text{VEV} \rangle_T. \]
For the entropy density \( \mu_0 \equiv 1/\sqrt{2e^{\gamma_E} t} \),

\[
\frac{T}{T_C} = 1.68 \text{ (NLO)} \quad \mu = \mu_0
\]

\[
\frac{(\epsilon + p)}{T^4} = 5.6 \quad \text{for } T/TC = 1.68 \text{ (NLO)} \quad \mu = \mu_0
\]

The higher order coefficients render the behavior more stable \( \Rightarrow \) Less sensitive to the method of the \( t \to 0 \) extrapolation.
For the trace anomaly, the two-loop coefficients already give well-stable behavior.
Quenched QCD

- Already a field of precise determination:

![Graph showing various methods for quenched QCD](image)

**Figure:** Boyd et al., Borsanyi et al.: Integral method, Giusti, Pepe: Moving frame method, Caselle et al.: Jarzynski’s equality.
5 years ago, at this same place, I reported the first trial...

- FlowQCD Collaboration, arXiv:1312.7492:
The connected part

\[ C_{\mu\nu;\rho\sigma}(\tau) \equiv \frac{1}{T^5} \int_V d^3x \left\langle \delta T_{\mu\nu}(x) \delta T_{\rho\sigma}(0) \right\rangle, \]

where \( \delta T_{\mu\nu}(x) \equiv T_{\mu\nu}(x) - \left\langle T_{\mu\nu}(x) \right\rangle. \)

Indicating the conservation law of the EMT, \( \partial_\tau C_{\mu\nu;\rho\sigma}(\tau) = 0. \)

Confirms the linear response relations, s.t,

\[ \frac{\varepsilon + p}{T^4} = \frac{1}{T^3} \frac{dp}{dT} = -C_{44;11}(\tau). \]
Stress tensor distribution around the static quark–anti-quark pair (Yanagihara, Iritani, Kitazawa, Asakawa, Hatsuda, arXiv:1803.05656)

- The EMT around the static quark–anti-quark pair:

\[ \mathcal{T}_{\mu\nu}(x) \equiv \langle T_{\mu\nu}(x) \rangle_{Q\bar{Q}} = \lim_{T \to \infty} \frac{\langle T_{\mu\nu}(x) W(R, T) \rangle}{\langle W(R, T) \rangle}. \]

- Eigenvectors:

\[ \mathcal{T}_{ij} n_j^{(k)} = \lambda_k n_j^{(k)} \]
$N_f = 2 + 1$ QCD by the NP $O(a)$-improved Wilson quark action and the RG improved Iwasaki gauge action.
- \( N_f = 2 + 1 \) QCD by the NP \( O(a) \)-improved Wilson quark action and the RG improved Iwasaki gauge action.

- Somewhat heavy ud quarks \((m_\pi/m_\rho \approx 0.63, m_{\eta_{ss}}/m_\phi \approx 0.74)\)
  - \( a = 0.0701(29) \text{ fm, } 28^3 \times 56 \text{ (JLQCD), } 32^3 \times N_t (N_t = 6, 8, \ldots, 16) \)
  - \( a = 0.0970(26) \text{ fm, } 32^3 \times 40, 32^3 \times N_t (N_t = 8, 10, 11, 12) \)
  - \([a = 0.04976 \text{ fm, } 40^3 \times 80]\)
\[ N_f = 2 + 1 \text{ QCD by the NP } O(a)-\text{improved Wilson quark action and the RG improved Iwasaki gauge action.} \]

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- Aiming at the test of the methodology, the continuum limit.
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**Aiming at the test of the methodology, the continuum limit.**

**Physical** mass ud quarks

- \( a = 0.08995(40) \text{ fm}, 32^3 \times 64 \) (PACS-CS), \( 32^3 \times N_t \) \((N_t = 4, 5, 6, \ldots, 14, 15, 16, 18)\)

**Physical prediction on EoS etc.**
Somewhat heavy ud quarks, \( a \approx 0.07 \text{ fm} \) (WHOT-QCD Collaboration, arXiv:1609.01417 and preliminary results)

- For the entropy density,
Somewhat heavy ud quarks, $a \approx 0.07$ fm (WHOT-QCD Collaboration, arXiv:1609.01417 and preliminary results)

- For the entropy density,

![Graph showing entropy density vs. $t/a^2$ and $T$](image)

- For the entropy density, 2 loop coefficients typically give stable behavior and 1 loop and 2 loop results are consistent; this is assuring.
Somewhat heavy ud quarks, $a \simeq 0.07 \text{ fm}$ (WHOT-QCD Collaboration, arXiv:1609.01417 and preliminary results)

- For the entropy density, 2 loop coefficients typically give stable behavior and 1 loop and 2 loop results are consistent; this is assuring.

- Note that the entropy density is the traceless part and does not contain the EoM.
Somewhat heavy ud quarks, \( a \approx 0.07 \text{ fm} \)

- For the trace anomaly, on the other hand,
Somewhat heavy ud quarks, \( a \approx 0.07 \text{ fm} \)

- For the trace anomaly, on the other hand,

- It seems that the EoM suffers from very large lattice artifact for \( N_t \lesssim 10 \) \( (a = 1/(N_t T)) \)...
Somewhat heavy ud quarks, $a \approx 0.07$ fm

- In fact, even in the 1 loop level,
Somewhat heavy ud quarks, $a \approx 0.07 \text{ fm}$

- In fact, even in the 1 loop level,

- This must be a lattice artifact and expected to (or should) disappear in the continuum limit.
Somewhat heavy ud quarks, $a \sim 0.07$ fm and $a \sim 0.097$ fm (Preliminary)

- It appears that the dependence on the lattice spacing is fairly small.
Physical mass $ud$, $a \approx 0.09 \text{ fm}$, (WHOT-QCD Collaboration, arXiv:1710.10015, and preliminary results)

- Entropy density seems to be consistent with that obtained by the staggered quarks (Borsányi et al; HotQCD Collaboration).
Physical mass $ud$, $a \approx 0.09 \text{ fm}$, (WHOT-QCD Collaboration, arXiv:1710.10015, and preliminary results)

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Trace anomaly seems much larger than that by the staggered quarks, although the error bars are still rather large.

More statistics and finer lattices are ongoing. is a future problem.
Two point function in $N_f = 2 + 1$ full QCD and viscosities (for the somewhat heavy ud quarks, $a \approx 0.07$ fm) (WHOT-QCD Collaboration, arXiv:1901.01666)

- Euclidean EMT two point function and the spectral function are related by

$$
\int d^3x \left\langle T_{ij}(-i\tau, \vec{x}) T_{kl}(0, \vec{0}) \right\rangle_T = \int_0^\infty dk_0 \frac{\cosh k_0(\tau - \beta/2)}{2\pi \sinh k_0\beta/2} \rho_{ij;kl}(k_0, \vec{0}).
$$

and the shear and the bulk viscosities are given by

$$
\eta = \lim_{k_0 \to 0} \frac{\rho_{ij;ij}(k_0, \vec{0})}{2k_0}, \quad \zeta = \lim_{k_0 \to 0} \frac{\rho_{ii;ii}(k_0, \vec{0})}{2k_0}.
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- To obtain $\rho(k_0)$ from lattice data is an ill-posed problem and, for instance, we adopt the Breit–Wigner type ansatz:

$$\frac{\rho_{BW}(k_0)}{2k_0} = \frac{F}{1 + b^2(k_0 - \omega_0)^2} + \frac{F}{1 + b^2(k_0 + \omega_0)^2},$$

and use $F$, $b$, and $\omega_0$ as fit parameters.
Two point function in $N_f = 2 + 1$ full QCD and viscosities (WHOT-QCD Collaboration, arXiv:1901.01666)

- A case the BW fit works fairly well ($N_f = 12$, $T = 232$ MeV):

![Graph showing the two-point function and shear viscosity vs. $T$ and $t/a^2$.](image)
Two point function in $N_f = 2 + 1$ full QCD and viscosities (WHOT-QCD Collaboration, arXiv:1901.01666)

- A case the BW fit works fairly well ($N_f = 12$, $T = 232$ MeV):

- Shear viscosity $\eta/s$ vs. $T$ (still preliminary)
3D scalar theory (Morikawa, Sonoda, H.S., work in progress)

- 3D $N$-component scalar theory

\[
S = \int d^D x \left[ \frac{1}{2} \partial_\mu \phi^I \partial_\mu \phi^I + \frac{m_0^2}{2} \phi^I \phi^I + \frac{\lambda_0}{8N} (\phi^I \phi^I)^2 \right]
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- The flow equation

$$ \partial_t \phi^I(t, x) = \partial_\mu \partial_\mu \phi^I(t, x), \quad \phi^I(t = 0, x) = \phi^I(x). $$
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\]

- **Universal formula for EMT \((C = 3.844365111074)\):**

\[
T_{\mu\nu} = \partial_\mu \phi^I \partial_\nu \phi^I - \delta_{\mu\nu} \left[ \frac{1}{2} \partial_\rho \phi^I \partial_\rho \phi^I + \frac{m^2}{2} \phi^I \phi^I + \frac{\lambda}{8N} (\phi^I \phi^I)^2 \right]

- \delta_{\mu\nu} \left( \frac{\lambda}{4\pi} \left( 1 + \frac{2}{N} \right) \left( -\frac{1}{3} \right) (8\pi t)^{-1/2} \right.

+ \frac{\lambda^2}{(4\pi)^2} \left\{ \left( 1 + \frac{2}{N} \right)^2 \left( -\frac{1}{4\pi} \right) + \frac{1}{N} \left( 1 + \frac{2}{N} \right) \left( -\frac{1}{8} \right) \left[ \ln(8\pi \mu^2 t) - \frac{1}{3} + C \right] \right\} \left. \right) \phi^I \phi^I.
\]
The theory around the Wilson–Fisher fixed point can be realized as the long-distance limit,

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle_{g_E} = \lim_{\tau \to \infty} e^{nx_h\tau} \langle \phi(e^\tau x_1) \ldots \phi(e^\tau x_n) \rangle_{m^2, \lambda},$$

where

$$m^2 = m^2_{cr}(\lambda) + g_E e^{-\gamma_E \tau}.$$ 

($m^2 = m^2_{cr}(\lambda)$ is the critical line).
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It can be interesting to explore the GF fixed point and the critical exponents by using the universal formula.
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It can be interesting to explore the GF fixed point and the critical exponents by using the universal formula.

cf. in the large \(N\) limit,

\[
x_h = \frac{1}{2}, \quad y_E = 1,
\]

and

\[
m_{cr}^2(\lambda) = 0.
\]
Summary and prospects

We wrote down a universal formula for the EMT in vector-like gauge theories by employing the gradient flow.
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Yet, we have the scale problem,

\[ a \lesssim \sqrt{8t}. \]

Step scaling or something analogous???
Summary and prospects

- Asymptotic form in $t \to 0$? (work in progress with Takaura).
Summary and prospects

- Asymptotic form in $t \to 0$? (work in progress with Takaura).
- Push applications further: EoS of QCD, viscosities in gauge theory, momentum/spin structure of baryons, critical exponents in low-energy conformal field theory, dilaton physics, . . .
Asymptotic form in $t \to 0$? (work in progress with Takaura).

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Further theoretical understanding on the equal-point correction. The axial $U(1)_A$ anomaly in gravitational field is not automatically reproduced (Morikawa, H.S., arXiv:1803.04132),

$$\partial^x_\alpha \langle j_5^\alpha(x) T_{\mu\nu}(y) T_{\rho\sigma}(z) \rangle$$

$$\neq \int_{\rho,\sigma} e^{i\rho(x-y)} e^{i\sigma(x-z)} \frac{1}{(4\pi)^2} \frac{1}{6} \epsilon_{\mu\rho\beta\gamma} p_\beta q_\gamma (q_\nu p_\sigma - \delta_{\nu\sigma} pq) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma),$$

but requires a correction by a “local counterterm” $\propto \delta(x-y)\delta(x-z)$. 