The hadronic light-by-light scattering contribution to the muon $g-2$ from lattice QCD

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DESY.

Advances in lattice gauge theory, CERN, 1 August 2019

- Magnetic moment of the muon : $\vec{\mu} = g\left(\frac{Qe}{2m}\right)\vec{s}$ Dirac equation : $g = 2$

3.6 σ discrepancy

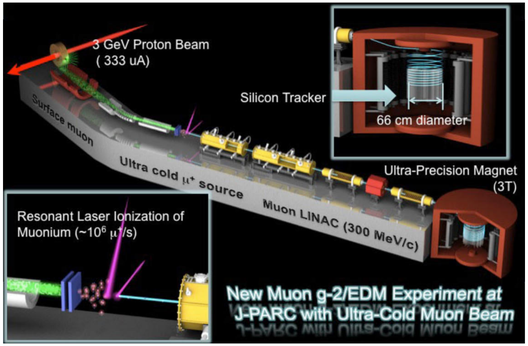
($>$ electroweak contribution !)

Theory	$a_\mu = (116\,591\,823 \pm 43) \cdot 10^{-11}$	
Experiment	$a_\mu = (116\,592\,089 \pm 63) \cdot 10^{-11}$	[BNL E821]

- Two new experiments aiming at $\times 4$ improvement

E989 - Fermilab

E34 - J-PARC

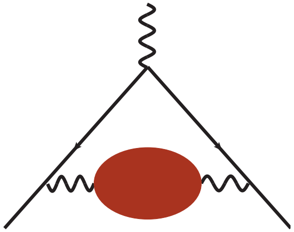


$\rightarrow$ theory error dominated by hadronic uncertainties (non-perturbative)

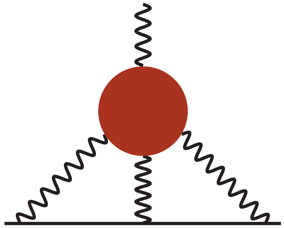
- Sensitive probe to new physics $a_\ell^{\mathrm{NP}} \propto \frac{m_\ell^2}{\Lambda^2}$

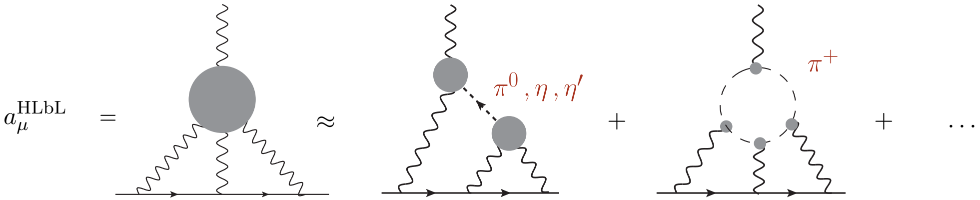
Contribution	$a_\mu \times 10^{11}$	
- QED (leptons, 5^{th} order)	$116\ 584\ 718.846 \pm 0.037$	[Aoyama et al. '12]
- Electroweak	153.6 ± 1.0	[Gnendiger et al. '13]
- Strong contributions		
HVP (LO)	$6\ 869.9 \pm 42.1$	[Jegerlehner '15]
HVP (NLO)	-98 ± 1	[Hagiwara et al. 11]
HVP (NNLO)	12.4 ± 0.1	[Kurtz et al. '14]
HLbL	102 ± 39	[Jegerlehner '15, Nyffeler '09]
Total (theory)	$116\ 591\ 811 \pm 62$	
Experiment	$116\ 592\ 089 \pm 63$	

Hadronic Vacuum Polarisation (HVP, α^2)



Hadronic Light-by-Light scattering (HLbL, α^3)





[de Rafael '94]

1) Chiral counting

2) N_c counting

[extracted from A. Nyffeler's slide], units : $a_\mu \times 10^{11}$

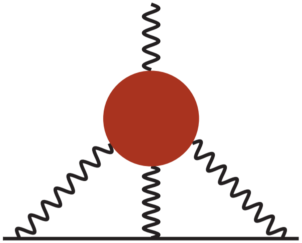
Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

1) Pseudoscalar contributions dominate numerically : transition form factors as input

2) Glasgow consensus : $a_\mu^{\rm HLbL} = (105 \pm 26) \times 10^{-11}$

3) Results are in good agreement but errors are difficult to estimate (model calculations)



Dispersive approach

[Colangelo et al. '14, '15], [Pauk, Vanderhaeghen '14]

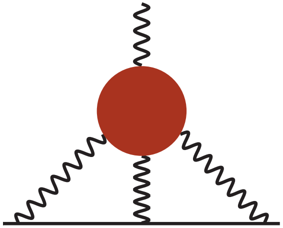
- Similar to the HVP (data driven), but :
  - $\rightarrow$ Many dispersion relations (19 vs 1 for HVP!)
  - $\rightarrow$ Experimental data are often missing
- LQCD can provide inputs
  - $\rightarrow$ pion-pole contribution (dominant) ...

Direct lattice calculation

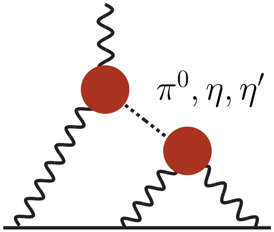
- 4-pt correlation function
  - $\rightarrow$ HVP : only a 2-pt correlation function
  - $\rightarrow$ very challenging
  - $\rightarrow$ but $\mathcal{O}(10\ \%)$ precision needed
- Two groups :

  [RBC/UKQCD] [Mainz]

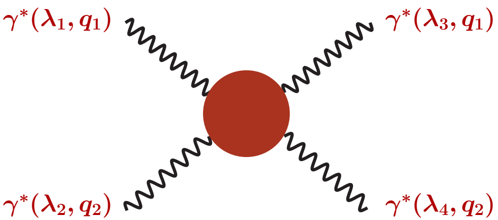
- **Direct lattice QCD calculation**
  - ↪ Only one collaboration has published results so far [Blum et. al 14', 16']
  - ↪ Difficult calculation (4-pt correlation function)
  - ↪ promising



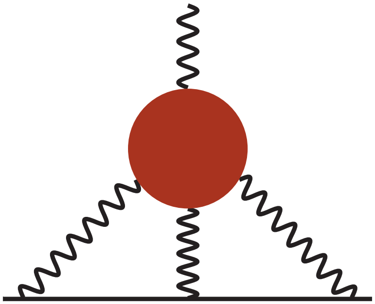
- **Pion-pole contribution on the lattice**
  - ↪ Dominant contribution to the HLbL scattering in $(g-2)_\mu$
  - ↪ Input for the dispersive approach
  - ↪ Complementary to the direct calculation



- **Hadronic Light-by-Light forward scattering amplitudes**
  - ↪ Full HLbL amplitudes contain more info than just a_μ^{HLbL}
  - ↪ Can be used to test the model (saturation)
  - ↪ Extract information about single-meson transition form factor



Strategy to compute the HLbL contribution on the lattice

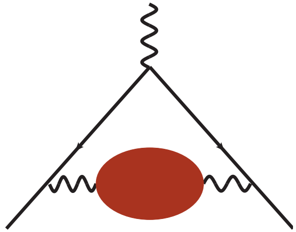


▶ Time momentum representation (TMR) [Bernecker, Meyer '12]

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int \mathrm{d}x_0 \; K(x_0) \; G(x_0)$$

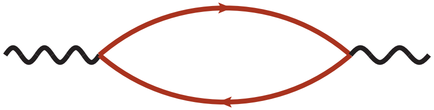
$$\rightarrow \; G(x_0) = -\frac{1}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle V_k(x) V_k(0) \rangle$$

$$\rightarrow \; K(x_0) = \text{QED kernel}$$

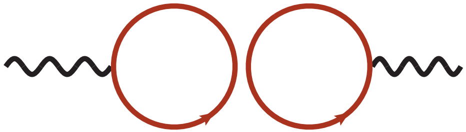


- The QED kernel is known analytically in the continuum and infinite volume
- One just needs to compute the vector two-point correlation function with $\vec{p} = \vec{0}$

connected contribution



disconnected contribution : $\mathrm{O}(1-2\%)$



- But requires a precision $< 0.25\%$

  $\rightarrow$ Sophisticated tools to reduce the noise (GEVP, ...)

  $\rightarrow$ All systematics must be addressed (FSE, chiral extrapolation, ...)

  $\rightarrow$ strong isospin-breaking and QED corrections are not negligible

Contribution	$a_\mu \times 10^{11}$	
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- Expected error (Fermilab) : $\approx 15 \times 10^{-11}$

  $\rightarrow$ 0.25% for the HVP (... reduction of the lattice error by 8 !)

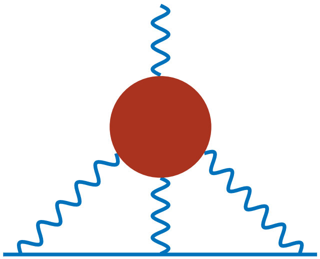
  $\rightarrow$ 10% for the HLbL

$\hookrightarrow$ some systematics can be ignored at this level of precision

- Try the same approach for the HLbL as for the HVP

[J. Green et al. '16] [N. Asmussen et al. '16 '17]

- Master formula :



$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \, \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

$$i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4z \, z_\rho \, \langle J_\mu(x) J_\nu(y) J_\sigma(z) J_\lambda(0) \rangle$$

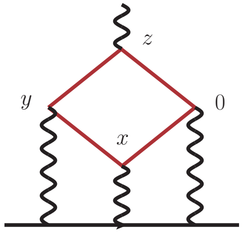
→ $\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$ is the four-point correlation function computed on the lattice

→ $\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is the QED kernel, computed semi-analytically (infra-red finite)

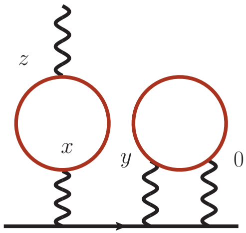
→ To compute $\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is a challenging task

→ Avoid $1/L^2$ finite-volume effects from the massless photons

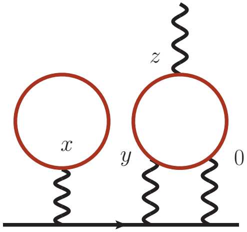
- Fully connected contribution

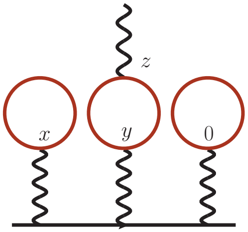


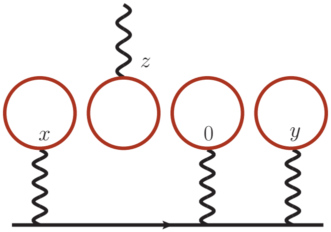
- Leading 2+2 (quark) disconnected contribution



- Sub-dominant disconnected contributions (3+1, 2+1+1, 1+1+1+1)







- 2+2 disconnected diagrams are not negligible !

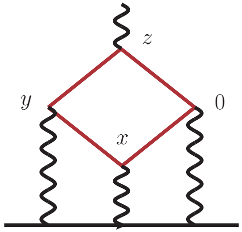
  → Large-N_c prediction : 2+2 disc ≈ - 50 % × connected [Bijnens '16]

  → Disconnected contributions : only $\mathcal{O}(1-2\ \%)$ for the HVP !

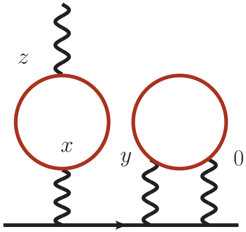
- Other diagrams vanish in the SU(3) limit (at least one quark loop which couple to a single photon)

  → Smaller contributions

- Fully connected contribution



- Leading 2+2 disconnected contribution



→ strange quark contribution is small :
  - suppressed by $1/17$ for the fully-connected contribution (charge factor)
  - but also relatively cheap to compute

- The 3+1 topology might be relevant for a 10% precision
  → first result by RBC show that much bellow 10% level
  → first preliminary results from Mainz seem to validate this result

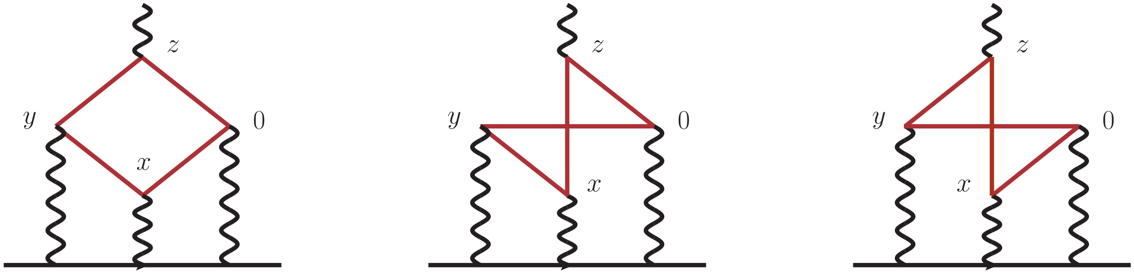
- Most relevant systematics
  → finite-size effects (can be estimated from the pion-pole contribution)
  → chiral extrapolation (the pion-pole contribution dominates ...)

- Other systematics are small (discretization effects)

- We focus on the fully-connected contribution (for simplicity)

$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \; \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

$$i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4z \, z_\rho \, \langle J_\mu(x) J_\nu(y) J_\sigma(z) J_\lambda(0) \rangle$$



→ 12D integral : integration over x and z are performed explicitly on the lattice

→ the remaining part depends only on $|y|$

→ one-dimensional integral : can be sampled using different values of $|y|$ (e.g (n,n,n,n))

→ expensive calculation : $7(N+1)$ sequential inversions for N values of $|y|$

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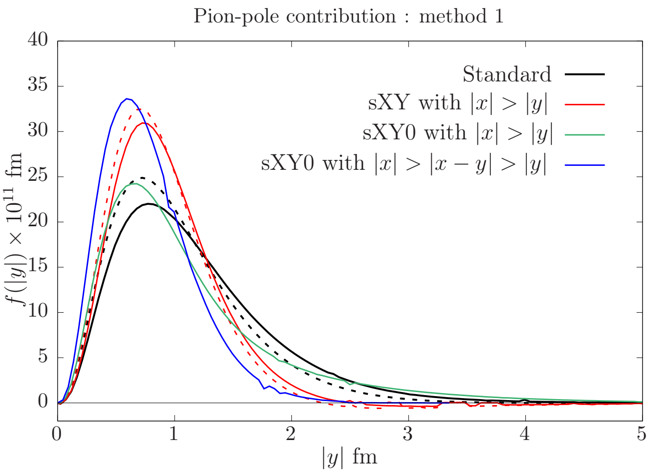
The correlation function, computed on the lattice, satisfies the Bose symmetry :

$$\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = \widehat{\Pi}_{\rho;\nu\mu\lambda\sigma}(y,x) = \widehat{\Pi}_{\rho;\lambda\nu\mu\sigma}(-x,y-x)\,,$$

→ can be enforced on the QED kernel

→ 6 equivalent sub-domains :

$$\begin{aligned} |x| &> |x-y| > |y|\,, \quad |x-y| > |x| > |y| \\ |x| &> |y| > |x-y|\,, \quad |y| > |x| > |x-y| \\ |y| &> |x-y| > |x|\,, \quad |x-y| > |y| > |x| \end{aligned}$$



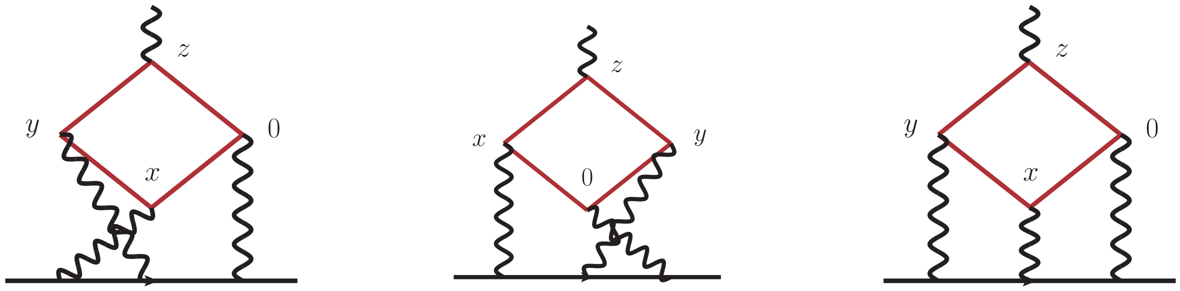
→ RBC proposed to use $r_{\min}$ such that

$r_{\min}$ = minimal distance between x, y and $x-y$

→ Equivalent to $|x| > |y|$ and $|x-y| > |y|$

→ This is just a reshuffling of the points !

- Trick : the kernel function is known for all values of x and y
- Reordering of the vertices at le level of the muon line $\Rightarrow$ only one wick contraction



$$a_\mu = \frac{me^6}{3} \int \mathrm{d}^4 y \int \mathrm{d}^4 x \Big\{ \left[\overline{\mathcal{L}}_{[\rho,\sigma],\mu\nu\lambda}(x,y) + \overline{\mathcal{L}}_{[\rho,\sigma],\nu\mu\lambda}(y,x) - \overline{\mathcal{L}}_{[\rho,\sigma],\lambda\nu\mu}(x,x-y) \right] \, i\widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) + \int \mathrm{d}^4 z \, \overline{\mathcal{L}}_{[\rho,\sigma],\lambda\nu\mu}(x,x-y) \, x_\rho \, \Pi^{(1)}_{\mu\nu\sigma\lambda}(x,y,z) \Big\}$$

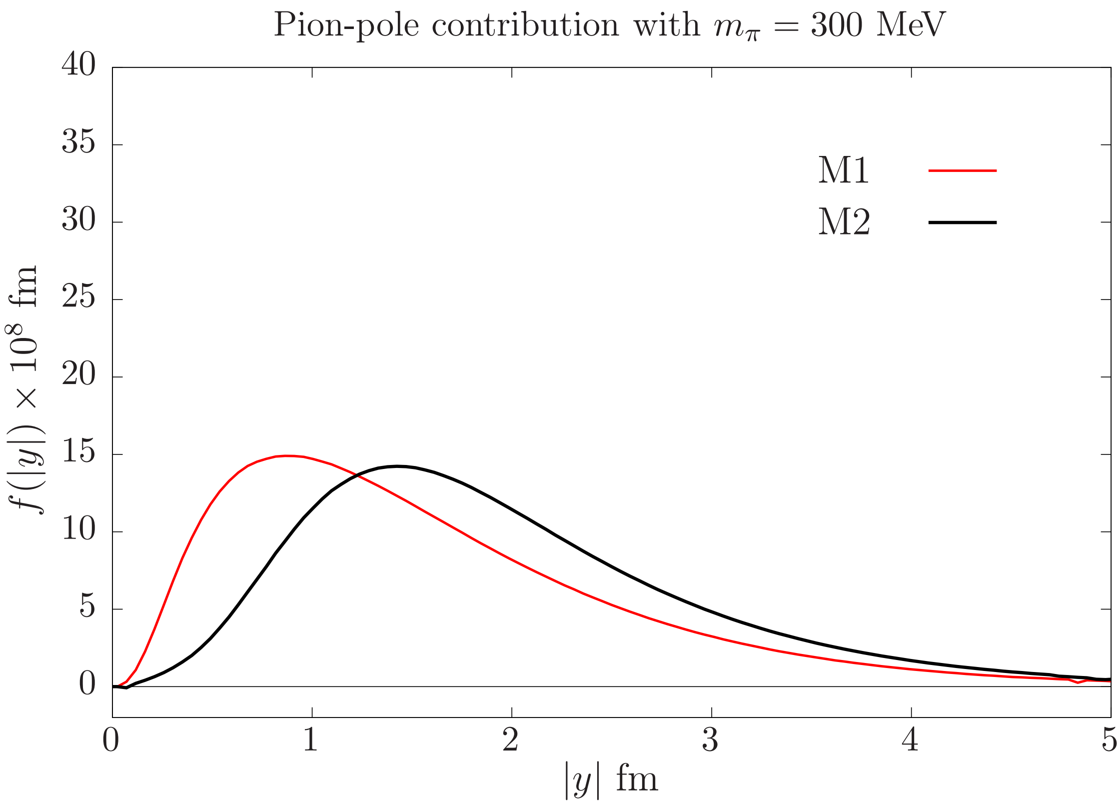
$\rightarrow$ two point sources at 0 and y : only N+1 inversions per $|y|$ (compared to $7(N+1)$)

$\rightarrow$ integration over x and z are performed explicitly on the lattice

$\rightarrow$ the integrand as a function of $|y|$ is different

$\rightarrow$ we cannot restrict the integration range anymore

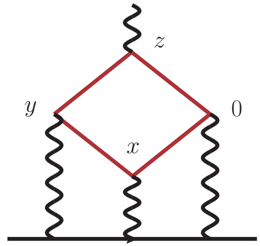
- Assuming a VMD pion transition form factor to compute $\Pi_{\mu\nu\sigma\lambda}(x, y, z)$



- M2 even more long range compared to M1 when m_π decreases

Mainz formula :

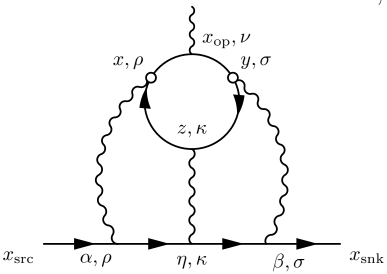
$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \, \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$



→ reduces to a 1d integral with $|y|$

RBC formula (using QED in infinite volume) :

$$\frac{a_\mu^{\mathrm{HLbL}}}{m} \frac{(\sigma_{s',s})_i}{2} = \sum_{r,\tilde{z}} 3\left(\frac{r}{2}, -\frac{r}{2}, \tilde{z}\right) \sum_{\tilde{x}_{\mathrm{op}}} \frac{1}{2} \epsilon_{i,j,k} \left(\tilde{x}_{\mathrm{op}}\right)_j \cdot i\bar{u}_{s'}(\vec{0}) \mathcal{F}_k^C \left(\frac{r}{2}, -\frac{r}{2}, \tilde{z}, \tilde{x}_{\mathrm{op}}\right) u_s(\vec{0}).$$



$$\mathcal{F}_\nu^C(x,y,z,x_{\mathrm{op}}) = (-ie)^6 \mathcal{G}_{\rho,\sigma,\kappa}(x,y,z) \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}})$$

$$r = x - y \quad , \quad w = (x+y)/2$$

$\widetilde{z} = z - w$ and $\widetilde{x}_{\mathrm{op}} = x_{\mathrm{op}} - w$

→ Sums over x_{op} and $\widetilde{z}$ are performed exactly (like us)

→ Sum over r is done stochastically [therefore, it is very useful to restrict the integration range !]

- Both results must agree in the continuum / infinite volume
- But comparison of the integrand more difficult

Freedom in the choice of the QED kernel function

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \; \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

- Conservation of the vector current : $\partial_\mu J_\mu(x) = 0 \;\Rightarrow$ The QED kernel is not unique [RBC/UKQCD '17]

$$0 = \sum_x \partial_\mu^{(x)} \left(x_\alpha \widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y) \right) = \sum_x \widehat{\Pi}_{\rho,\alpha\nu\lambda\sigma}(x,y) + \sum_x x_\alpha \, \partial_\mu^{(x)} \widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

$\rightarrow$ we can add any fonction $f(y)$ to the standard QED kernel

$\rightarrow$ same argument valid for the other variable x

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- Examples of possible subtractions (idea : subtract very short distance contributions)

$$\mathcal{L}^{(0)}(x,y) = \mathcal{L}(x,y) \qquad \Rightarrow \mathcal{L}^{(1)}(0,0) = 0$$

$$\mathcal{L}^{(1)}(x,y) = \mathcal{L}(x,y) - \frac{1}{2}\mathcal{L}(x,x) - \frac{1}{2}\mathcal{L}(y,y) \qquad \Rightarrow \mathcal{L}^{(1)}(x,x) = 0$$

$$\mathcal{L}^{(2)}(x,y) = \mathcal{L}(x,y) - \mathcal{L}(0,y) - \mathcal{L}(x,0) \qquad \Rightarrow \mathcal{L}^{(2)}(x,0) = \mathcal{L}^{(2)}(0,y) = 0$$

$$\mathcal{L}^{(3)}(x,y) = \mathcal{L}(x,y) - \mathcal{L}(0,y) - \mathcal{L}(x,x) + \mathcal{L}(0,x) \qquad \Rightarrow \mathcal{L}^{(3)}(0,y) = \mathcal{L}^{(3)}(x,x) = 0$$

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$$\mathcal{L}^{(1)}(x,y) = \mathcal{L}(x,y) - \frac{1}{2}\mathcal{L}(x,x) - \frac{1}{2}\mathcal{L}(y,y) \qquad \Rightarrow \mathcal{L}^{(1)}(x,x) = 0$$

$$\mathcal{L}^{(2)}(x,y) = \mathcal{L}(x,y) - \mathcal{L}(0,y) - \mathcal{L}(x,0) \qquad \Rightarrow \mathcal{L}^{(2)}(x,0) = \mathcal{L}^{(2)}(0,y) = 0$$

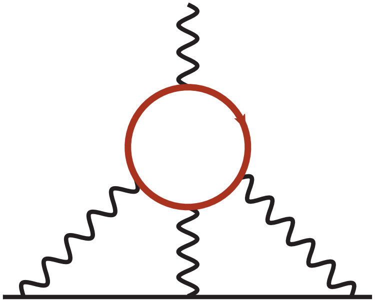
$$\mathcal{L}^{(3)}(x,y) = \mathcal{L}(x,y) - \mathcal{L}(0,y) - \mathcal{L}(x,x) + \mathcal{L}(0,x) \qquad \Rightarrow \mathcal{L}^{(3)}(0,y) = \mathcal{L}^{(3)}(x,x) = 0$$

- Different definitions may affect :

$\rightarrow$ Discretization effects / Finite-size effects / Statistical precision of the estimator

- The conditions $\mathcal{L}^{(2)}(x,0) = \mathcal{L}^{(2)}(0,y) = 0$ does not define the kernel unambiguously

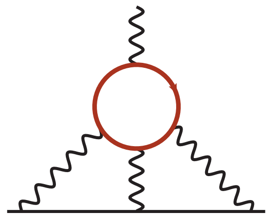
Check of the method



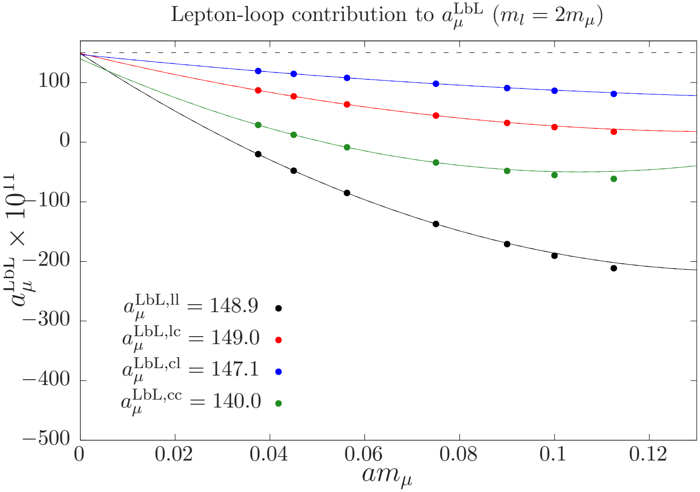
- Perform a lattice QCD calculation with unit gauge links

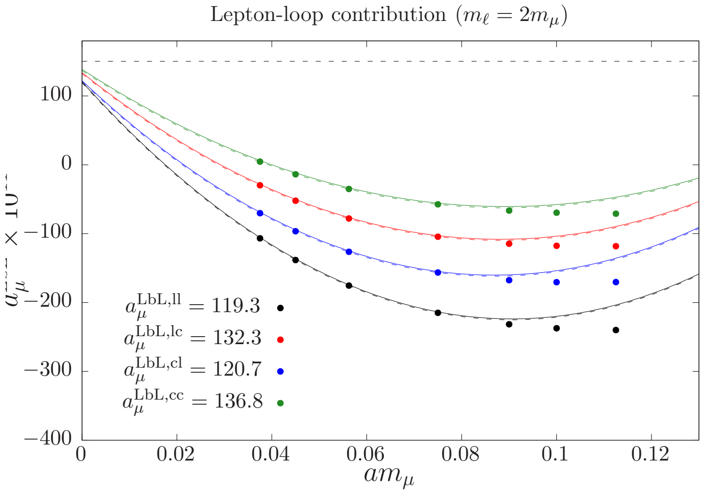
  $\rightarrow$ correspond to the well-known lepton-loop contribution (up to a trivial factor $N_c = 3$)

  $\rightarrow$ check of the QED kernel (and of the lattice implementation)



- Use both strategy : M1 (left) and M2 (right)





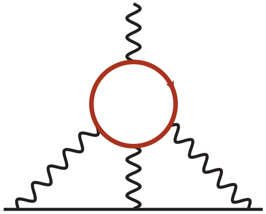
$\rightarrow$ different colors = different discretizations of the vector current

$\rightarrow$ standard kernel $\mathcal{L}^{(0)}(x,y)$: large discretization effects !

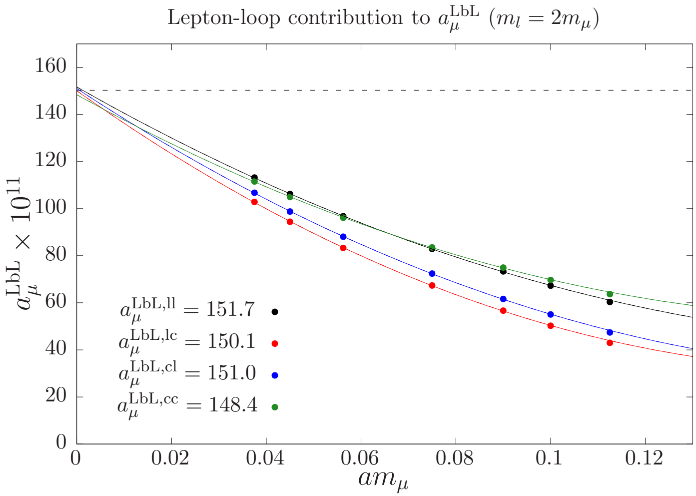
- Perform a lattice QCD calculation with unit gauge links

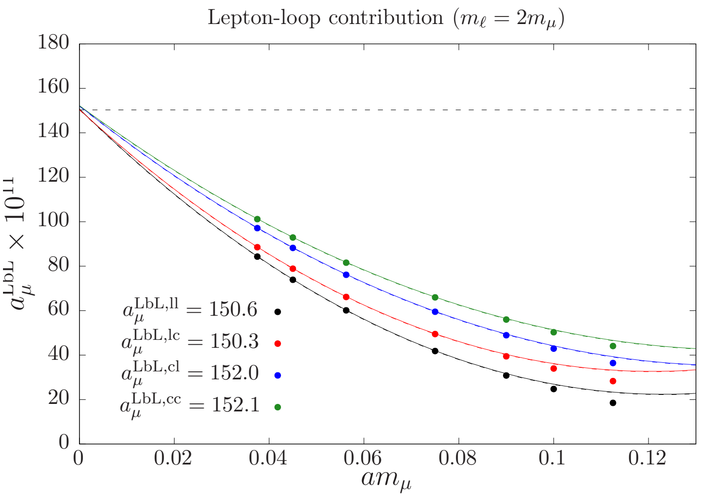
  → correspond to the well-known lepton-loop contribution (up to a trivial factor $N_c = 3$)

  → check of the QED kernel (and of the lattice implementation)



- Use both strategy : M1 (left) and M2 (right)

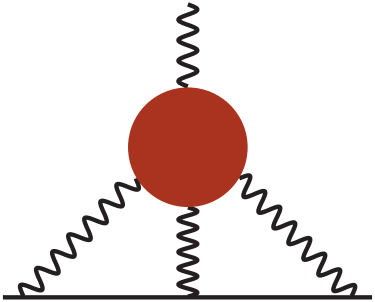




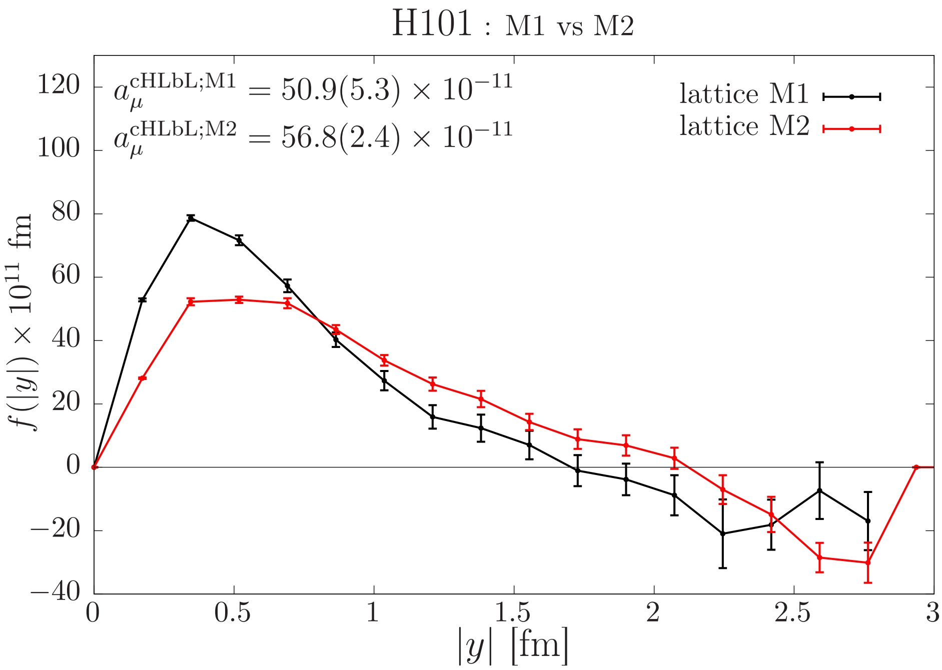
→ $\mathcal{L}^{(2)}(x,y)$ has much smaller discretization effects

→ we can reproduce the known result ($a_\mu^{\rm LbL} = 0.15031 \times 10^{-8}$) with a very good precision

The lattice QCD calculation

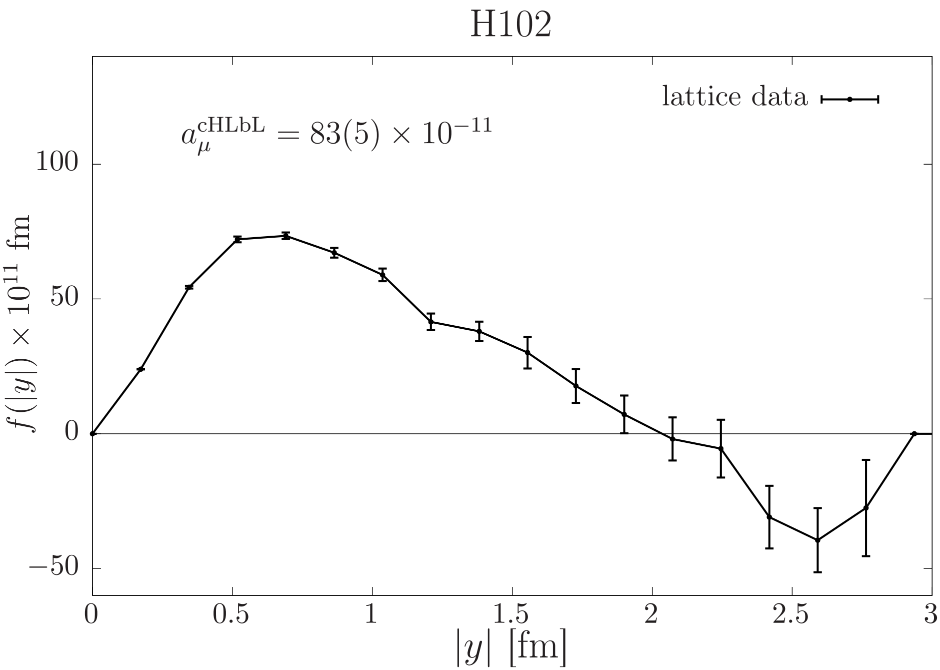


- Comparison of the two methods for a pion mass of 400 MeV
- SU(3) symmetric ensemble : $m_s = m_u = m_d$



- Method 1 is 7 times more expensive
- Method 2 : more long range

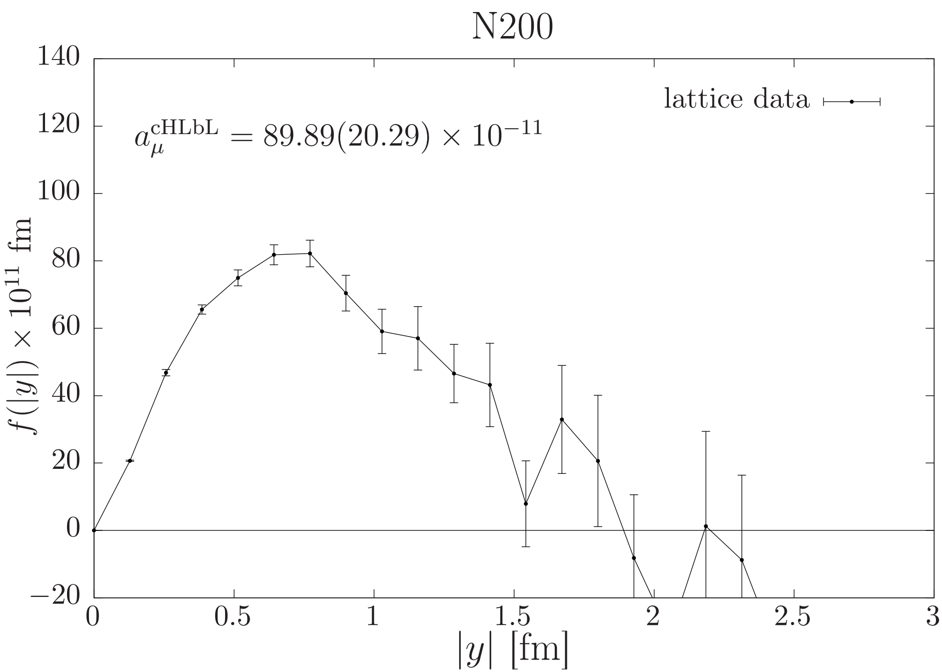
- Method 2 with $m_\pi = 340$ MeV



→ Signal starts to deteriorate at large $|y|$

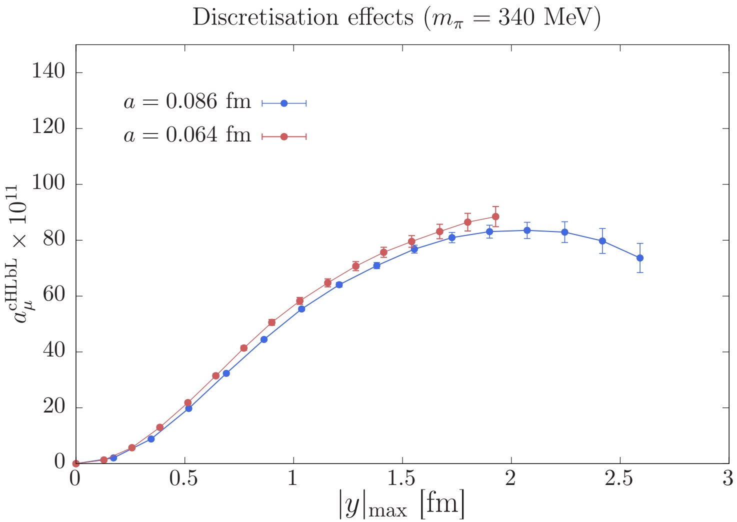
→ (unexpected) negative tail at large $|y|$ → FSE !

- Method 2 with $m_\pi = 280$ MeV



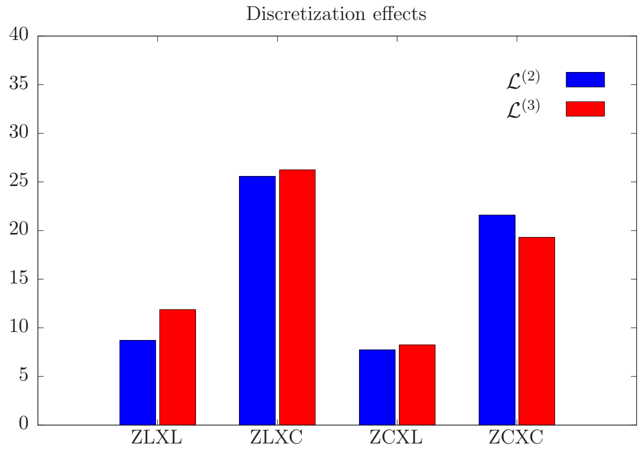
→ Signal very noisy at large $|y|$

→ long distance effects are governed by the pion



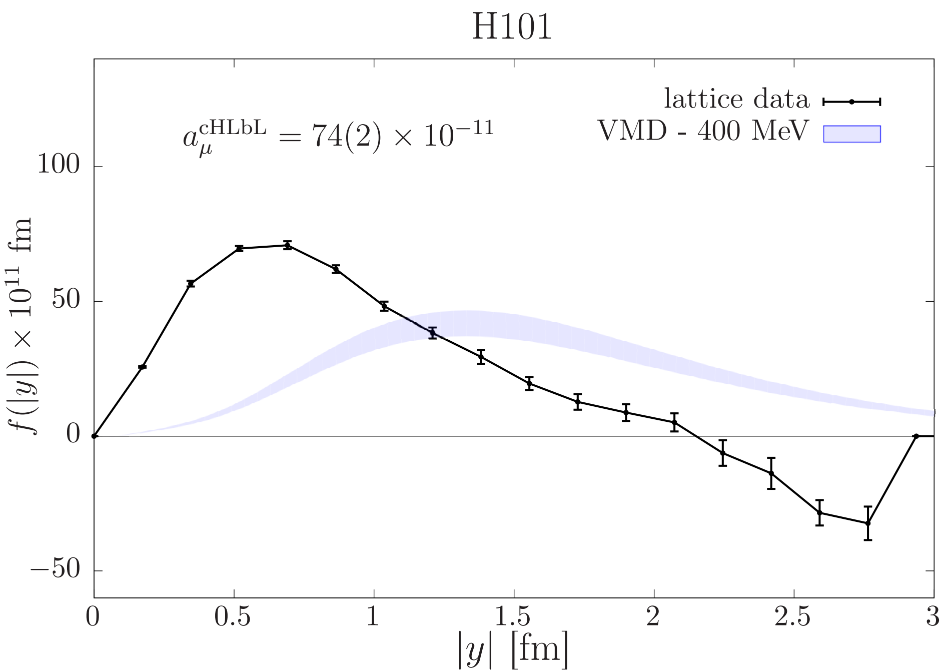
- Method 2
- conserved vector current at z
- local vector current at x

$$\Delta = \frac{a_\mu^c(0.064\ \mathrm{fm}) - a_\mu^c(0.086\ \mathrm{fm})}{a_\mu^c(0.086\ \mathrm{fm})}$$



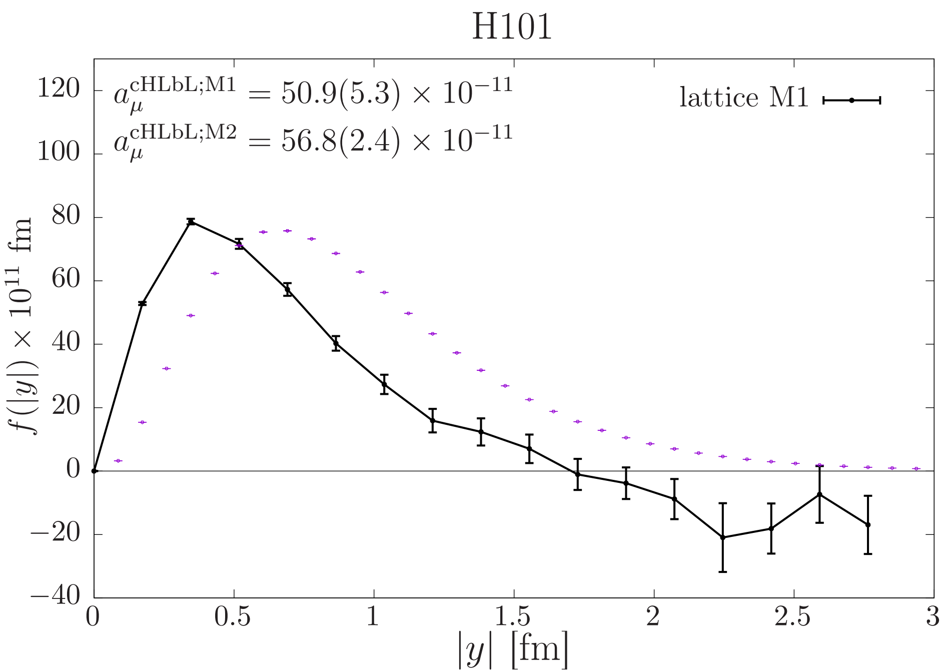
Discretization effects rather small

- Method 2 with the subtracted kernel $\mathcal{L}^{(2)}(x,y)$
- Comparison with the pion-pole prediction assuming a VMD transition form factor



FSE are important when we are close to the boundary

- Method 1 with the subtracted kernel $\mathcal{L}^{(2)}(x,y)$
- Comparison with the pion-pole prediction assuming a VMD transition form factor



similar problem with both methods

1) Discretization effects are rather small

2) Signal-to-noise problem at long distances

→ method 2 is much cheaper ($\times 7$)

→ ... but integrand long range

3) Finite-size effects are important

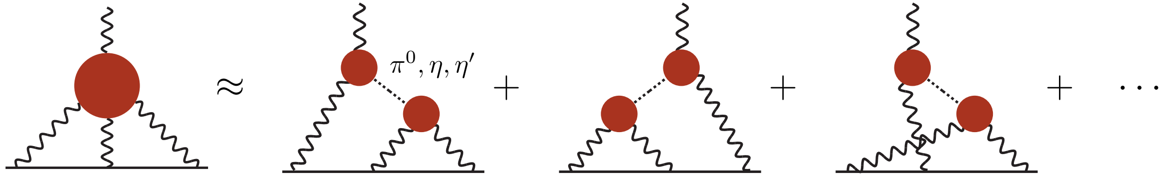
→ method 2 : integrand even more long range

- Solutions to 2) and 3) :

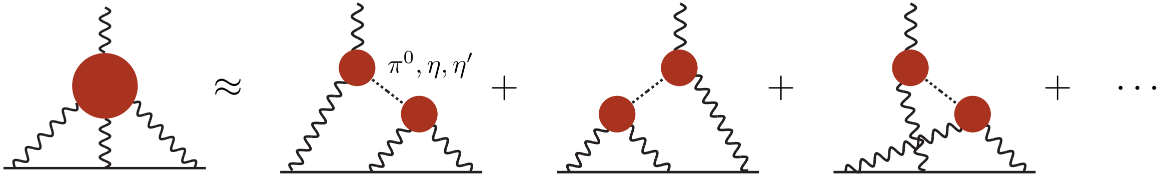
→ use an improved kernel with smaller FSE

→ we used $y = (n, n, n, n)$. Switch to $y = (\alpha n, n, n, n)$ with $\alpha = 2, 3$

→ pion transition form factor (TFF) computed on the same ensemble

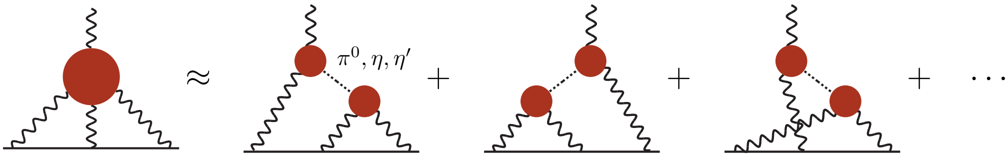


The pion-pole contribution



[Jegerlehner & Nyffeler '09]

$\tau = \cos(\theta)$
$Q_1 \cdot Q_2 = Q_1 Q_2 \cos(\theta)$

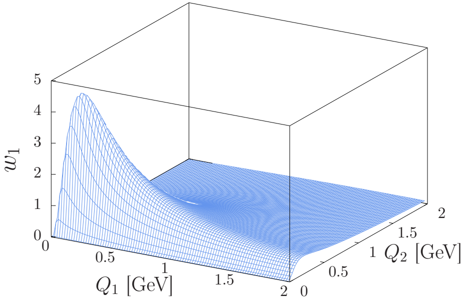


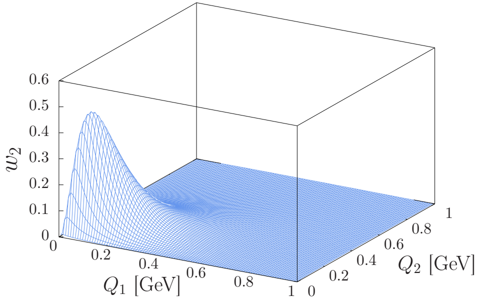
$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \; w_1(Q_1,Q_2,\tau)\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \; + \; w_2(Q_1,Q_2,\tau)\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

→ Product of one single-virtual and one double-virtual transition form factors (spacelike virtualities)

→ $w_{1,2}(Q_1, Q_2, \tau)$ are model-independent weight functions

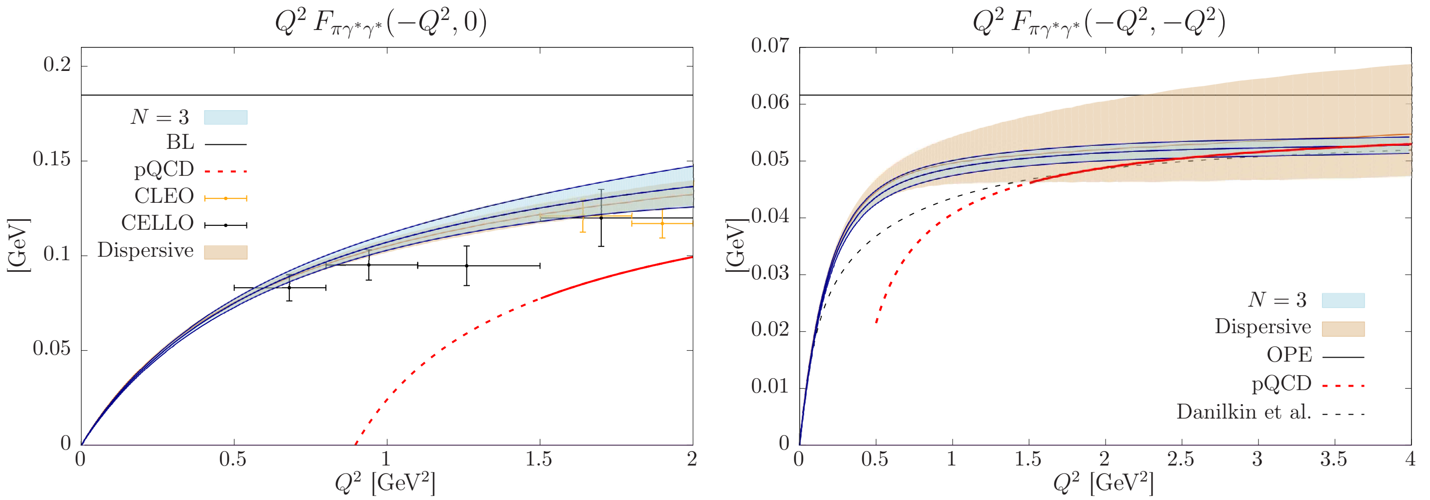
→ The weight functions are concentrated at small momenta below 1 GeV (here for $\tau = -0.5$)





↪ Need the pion TFF for arbitrary spacelike virtualities in the momentum range $[0-3]\ \mathrm{GeV}^2$

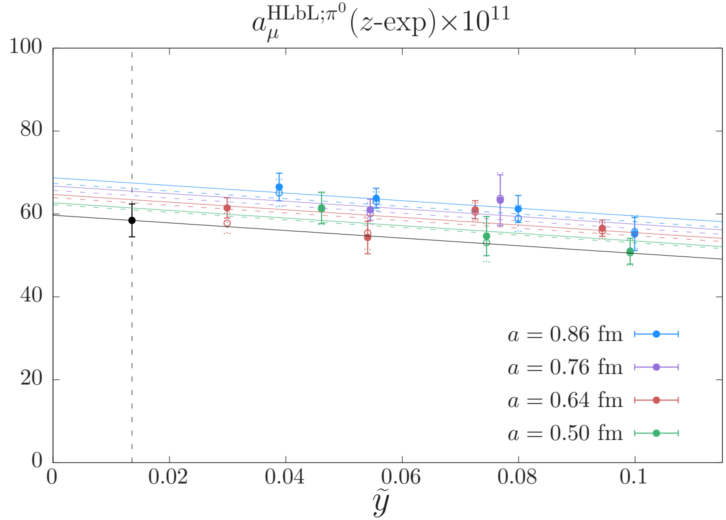
Double z−expansion [arXiv :1903.09471]



- $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = 0.278(14)\ \text{GeV}^{-1}$
  $\rightarrow$ compatible with the PRIMEX experiment (precision $\approx$ 5%)
- Results are in good agreement with experimental data
- Good agrement with the recent dispersive analysis [M. Hoferichter et al. '18]

[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \; + \; w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$



$$a_\mu^{\mathrm{HLbL};\pi^0} = (59.7 \pm 3.6) \times 10^{-11}$$

Previous model estimates :

VMD : $a_\mu^{\mathrm{HLbL};\pi^0} = 57.0 \times 10^{-11}$

LMD : $a_\mu^{\mathrm{HLbL};\pi^0} = 73.7 \times 10^{-11}$

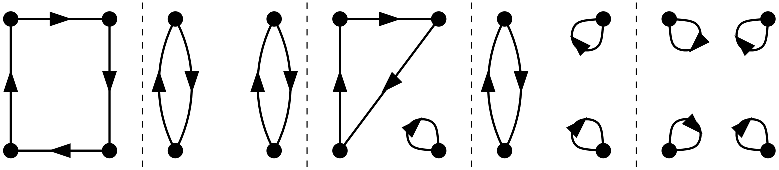
LMDV : $a_\mu^{\mathrm{HLbL};\pi^0} = 62.9 \times 10^{-11}$

→ Compatible with the $N_f = 2$ results [Gérardin et al. '16]. Error reduced by a factor 2.5

→ Compatible with the dispersive result $a_\mu^{\mathrm{HLbL}} = 62.6^{+3.0}_{-2.5} \times 10^{-11}$ [Hoferichter et al. '18]

- We have an efficient way to compute the HLbL on the lattice
  $\rightarrow$ kernel known in the continuum and infinite volume
- But we need to understand the long distance behavior of our integrand
  $\rightarrow$ new kernel with reduced FSE
  $\rightarrow$ use the pion transition form factor to estimate FSE
- Roadmap
  $\rightarrow$ study the continuum limit at the SU(3) symmetric point $m_u = m_d = m_s$
  $\rightarrow$ then, go to lighter pion masses
- This talk was mainly focused on the fully connected contribution
  $\rightarrow$ 2+2 and 3+1 disconnected contribution are underway

- There are five different topologies :



- Wick contractions lead to :

$$\Pi^{\rm HLbL} = \sum_{f} \mathcal{Q}_f^4 \, \Pi^4 + \sum_{f_1,f_2} \mathcal{Q}_{f_1}^2 \mathcal{Q}_{f_2}^2 \, \Pi^{2+2} + \sum_{f_1,f_2} \mathcal{Q}_{f_1}^3 \mathcal{Q}_{f_2} \, \Pi^{3+1} + \sum_{f_1,f_2,f_3} \mathcal{Q}_{f_1}^2 \mathcal{Q}_{f_2} \mathcal{Q}_{f_3} \, \Pi^{2+1+1} + \sum_{f_1,f_2,f_3,f_4} \mathcal{Q}_{f_1} \mathcal{Q}_{f_2} \mathcal{Q}_{f_3} \mathcal{Q}_{f_4} \, \Pi^{1+1+1+1}$$

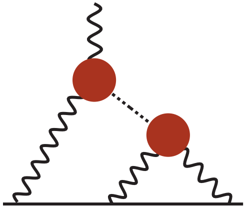
- The contribution to $\Pi^{\rm HLbL}$ of an isovector (isoscalar) resonance M_1 (M_0) can be written as

$$\Pi^{\rm HLbL}(M_1) = (\mathcal{Q}_u^2 - \mathcal{Q}_d^2)^2 \, \Pi_{M_1}$$
$$\Pi^{\rm HLbL}(M_0) = (\mathcal{Q}_u^2 + \mathcal{Q}_d^2) \, \Pi_A + (\mathcal{Q}_u + \mathcal{Q}_d)^2 (\mathcal{Q}_u^2 + \mathcal{Q}_d^2) \, \Pi_B + (\mathcal{Q}_u + \mathcal{Q}_d)^4 \, \Pi_C$$

$\rightarrow$ consequence of the isospin decomposition of the electromagnetic current $J_\mu^{\rm e.m.} = J_\mu^1 + J_\mu^0$:

$$J_\mu^1 = \frac{\mathcal{Q}_u - \mathcal{Q}_d}{2} (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d), \quad J_\mu^0 = \frac{\mathcal{Q}_u + \mathcal{Q}_d}{2} (\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)$$

$$\mathcal{F}_{M_1\gamma^*\gamma^*} = (\mathcal{Q}_u^2 - \mathcal{Q}_d^2) \, \mathcal{F} \ , \quad \mathcal{F}_{M_0\gamma^*\gamma^*} = (\mathcal{Q}_u^2 + \mathcal{Q}_d^2) \, \mathcal{F}_C + (\mathcal{Q}_u + \mathcal{Q}_d)^2 \, \mathcal{F}_D$$



- Identification of the polynomials in $\mathcal{Q}_u$ and $\mathcal{Q}_d$: leads to two sets of three equations

$$\begin{cases} \Pi_{M_1} = \Pi^4_{M_1} + \Pi^{2+2}_{M_1} + \Pi^{3+1}_{M_1} + \Pi^{2+1+1}_{M_1} + \Pi^{1+1+1+1}_{M_1} \\ -\Pi_{M_1} = 0 + \Pi^{2+2}_{M_1} + 0 + \Pi^{2+1+1}_{M_1} + 3\Pi^{1+1+1+1}_{M_1} \\ 0 = 0 + 0 + \Pi^{3+1}_{M_1} + 2\Pi^{2+1+1}_{M_1} + 4\Pi^{1+1+1+1}_{M_1} \end{cases}$$

$$\begin{cases} \Pi_A + \Pi_B + \Pi_C = \Pi^4_{M_0} + \Pi^{2+2}_{M_0} + \Pi^{3+1}_{M_0} + \Pi^{2+1+1}_{M_0} + \Pi^{1+1+1+1}_{M_0} \\ \Pi_A + \Pi_B + 3\Pi_C = 0 + \Pi^{2+2}_{M_0} + 0 + \Pi^{2+1+1}_{M_0} + 3\Pi^{1+1+1+1}_{M_0} \\ 3\Pi_B + 4\Pi_C = 0 + 0 + \Pi^{3+1}_{M_0} + 2\Pi^{2+1+1}_{M_0} + 4\Pi^{1+1+1+1}_{M_0} \end{cases}$$

- Assume that all disconnected contributions with at least one isolated quark loop are negligible (then $\Pi_A \approx \Pi_{M_0}$)

$$\begin{cases} \Pi_{M_1} + \Pi_{M_0} \approx (\Pi^4_{M_1} + \Pi^4_{M_1}) + (\Pi^{2+2}_{M_1} + \Pi^{2+2}_{M_0}) \\ -\Pi_{M_1} + \Pi_{M_0} \approx (\Pi^{2+2}_{M_1} + \Pi^{2+2}_{M_0}) \end{cases}$$

- Then the contribution to the fully connected and $2+2$ disconnected contributions read

$$(Q_u^4 + Q_d^4)\Pi^4_{M_1+M_0} \approx 2\,(Q_u^4 + Q_d^4)\Pi_{M_1} \approx 2\frac{Q_u^4 + Q_d^4}{(Q_u^2 - Q_d^2)^2}\,\Pi^{\mathrm{HLbL}}(M_1) \approx \frac{34}{9}\Pi^{\mathrm{HLbL}}(M_1)$$

$$(Q_u^2 + Q_d^2)^2\Pi^{2+2}_{M_1+M_0} \approx -\frac{(Q_u^2 + Q_d^2)^2}{(Q_u^2 - Q_d^2)^2}\,\Pi^{\mathrm{HLbL}}(M_1) + \Pi^{\mathrm{HLbL}}(M_0) \approx -\frac{25}{9}\Pi^{\mathrm{HLbL}}(M_1) + \Pi^{\mathrm{HLbL}}(M_0)$$

↪ This was already noticed by [Bijnens '16] using large-N_c arguments

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=\mathrm{I,II,III}} \mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda}\, T^{(A)}_{\alpha\beta\delta}(x,y)$$

- $\mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda}$ = traces of gamma matrices $\quad\rightarrow\quad$ sums of products of Kronecker deltas
- The tensors $T^{(A)}_{\alpha\beta\delta}$ are decomposed into a scalar S, vector V and tensor T part

$$T^{(\mathrm{I})}_{\alpha\beta\delta}(x,y) = \partial^{(x)}_{\alpha}(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})V_{\delta}(x,y)$$

$$T^{(\mathrm{II})}_{\alpha\beta\delta}(x,y) = m\partial^{(x)}_{\alpha}\left(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y)\right)$$

$$T^{(\mathrm{III})}_{\alpha\beta\delta}(x,y) = m(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})\left(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y)\right)$$

They are parametrized by six weight functions

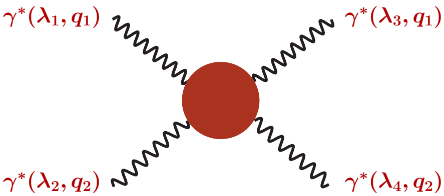
$$S(x,y) = 0$$

$$V_{\delta}(x,y) = x_{\delta}\,\bar{\mathfrak{g}}^{(1)} + y_{\delta}\,\bar{\mathfrak{g}}^{(2)}$$

$$T_{\alpha\beta}(x,y) = (x_{\alpha}x_{\beta} - \frac{x^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(1)} + (y_{\alpha}y_{\beta} - \frac{y^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(2)} + (x_{\alpha}y_{\beta} + y_{\alpha}x_{\beta} - \frac{x\cdot y}{2}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(3)}$$

- the weight functions depend on the three variables x^2, $x\cdot y = |x||y|\cos\beta$ and y^2
- Semi-analytical expressions for the weight functions have been computed to about 5 digits precision

- Forward scattering amplitudes $M_{\lambda_3\lambda_4\lambda_1\lambda_2}$: $\gamma^*(\lambda_1, q_1)\ \gamma^*(\lambda_2, q_2) \ \rightarrow\ \gamma^*(\lambda_3, q_1)\ \gamma^*(\lambda_4, q_2)$



- ▶ 81 helicity amplitudes ($\lambda_i = 0, \pm 1$)

$$\mathcal{M}_{\lambda'_1\lambda'_2\lambda_1\lambda_2} = \mathcal{M}_{\mu\nu\rho\sigma}\ \epsilon^{*\mu}(\lambda'_1)\ \epsilon^{*\nu}(\lambda'_2)\ \epsilon^{\rho}(\lambda_1)\ \epsilon^{\sigma}(\lambda_2)$$

- ▶ Photons virtualities : $Q_1^2 = -q_1^2 > 0$ and $Q_2^2 = -q_2^2 > 0$
- ▶ Crossing-symmetric variable : $\nu = q_1 \cdot q_2$

- Using parity and time invariance : only 8 independent amplitudes

$$(\mathcal{M}_{++,++} + \mathcal{M}_{+-,+-})\,,\ \mathcal{M}_{++,--}\,,\ \mathcal{M}_{00,00}\,,\ \mathcal{M}_{+0,+0}\,,\ \mathcal{M}_{0+,0+}\,,\ (\mathcal{M}_{++,00} + \mathcal{M}_{0+,-0})\,,$$
$$(\mathcal{M}_{++,++} - \mathcal{M}_{+-,+-})\,,\ (\mathcal{M}_{++,00} - \mathcal{M}_{0+,-0})$$

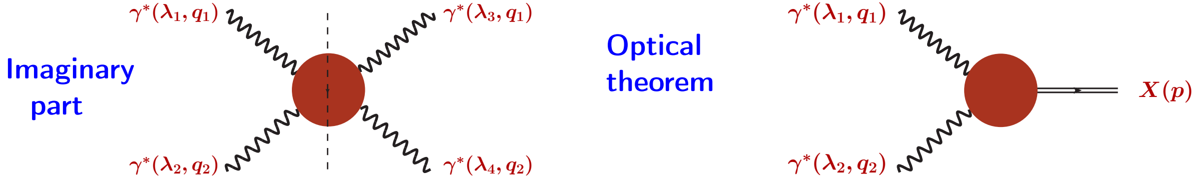
↪ Either even or odd with respect to ν

↪ The eight amplitudes have been computed on the lattice for different values of ν, Q_1^2, Q_2^2

- Relate the forward amplitudes to two-photon fusion cross sections using the optical theorem [Pascalutsa et. al '12]

↪ Eight independent dispersion relations for $\mathcal{M}_{TT}$, $\mathcal{M}^t_{TT}$, $\mathcal{M}^a_{TT}$, $\mathcal{M}_{TL}$, $\mathcal{M}_{LT}$, $\mathcal{M}^a_{TL}$, $\mathcal{M}^t_{TL}$ and $\mathcal{M}_{LL}$

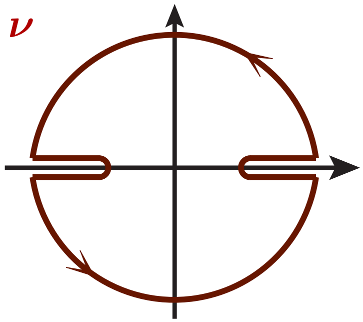
1) Optical theorem



$\hookrightarrow$ X can be any C-even final state

$$W_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \mathrm{Im}\, M_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \frac{1}{2}\int \mathrm{d}\Gamma_X (2\pi)^4 \delta(q_1 + 1_2 - p_X) \mathcal{M}_{\lambda_1\lambda_2}(q_1, q_2, p_X)\, \mathcal{M}^*_{\lambda_3\lambda_4}(q_1, q_2, p_X)$$

2) Dispersion relations [Pascalutsa et. al '12]



Once-subtracted sum rules : crossing-symmetric variable $\nu = q_1 \cdot q_2$

$$\mathcal{M}_{\mathrm{even}}(\nu) = \mathcal{M}_{\mathrm{even}}(0) + \frac{2\nu^2}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{1}{\nu'(\nu'^2 - \nu^2 - i\epsilon)} W_{\mathrm{even}}(\nu')$$

$$\mathcal{M}_{\mathrm{odd}}(\nu) = \nu\mathcal{M}_{\mathrm{odd}}(\nu) + \frac{2\nu^3}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{1}{\nu'(\nu'^2 - \nu^2 - i\epsilon)} W_{\mathrm{odd}}(\nu')$$

3) Higher mass singularities are suppressed with ν^2 :

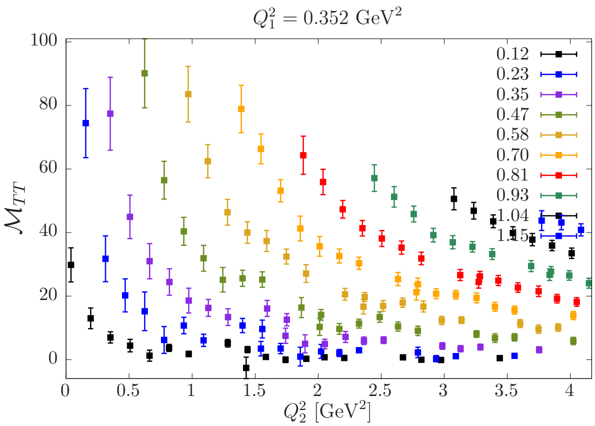
$\hookrightarrow$ Only a few states X are necessary to saturate the sum rules and reproduce the lattice data

→ For each of the eight amplitudes, we have a dispersion relation :

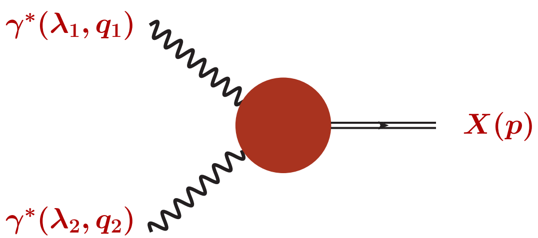
$$\overline{\mathcal{M}}_\alpha(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{X'}\,\sigma_\alpha/\tau_\alpha(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

Lattice calculation

↪ 4-pt correlation function



$\gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \to \mathrm{X}(p_\mathrm{X})$ fusion cross sections



↪ Main contribution is expected from mesons :

Pseudoscalars (0^{-+})	Axial-vectors (1^{++})
Scalar (0^{++})	Tensors (2^{++})

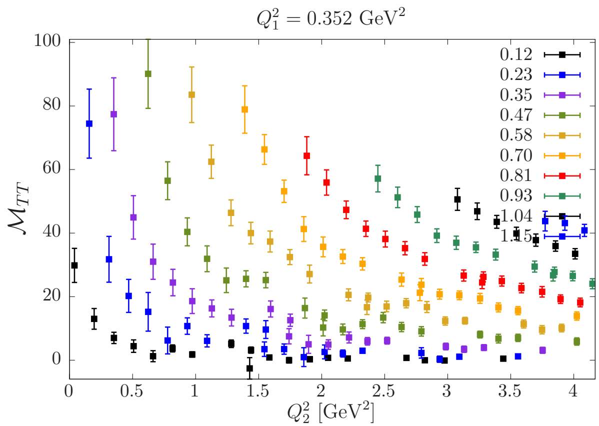
↪ Input : transition form factors

→ For each of the eight amplitudes, we have a dispersion relation :

$$\overline{\mathcal{M}}_\alpha(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{X'} \, \sigma_\alpha/\tau_\alpha(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

Lattice calculation

↪ 4-pt correlation function



$\gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \to \mathrm{X}(p_\mathrm{X})$ fusion cross sections

↪ Consider only one particle in each channel

↪ $N_f = 2$: no η meson

↪ Isospin symmetry + large-N_c approximation :

isovector only with an overall factor $34/9$

	Isovector	Isoscalar	Isoscalar
0^{-+}	π	η'	η
0^{++}	$a_0(980)$	$f_0(980)$	$f_0(600)$
1^{++}	$a_1(1260)$	$f_1(1285)$	$f_1(1420)$
2^{++}	$a_2(1320)$	$f_2(1270)$	$f_2'(1525)$