Pion Scattering to \((g - 2)_{\mu}\)
at Physical Pion Mass

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Brookhaven National Laboratory

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Advances in Lattice Gauge Theory 2019
The RBC & UKQCD collaborations

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- Chulwoo Jung
- Meifeng Lin
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- Cheng Tu

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- Norman Christ
- Duo Guo
- Christopher Kelly
- Bob Mawhinney
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- Jiyan Tu

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- Julia Kettle
- Michael Marshall
- Fionn Ó hÓgáin
- Antonin Portelli
- Tobias Tsang
- Andrew Yong
- Azusa Yamaguchi

**KEK**
- Julien Frison

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- Nicolas Garron

**MIT**
- David Murphy

**Peking University**
- Xu Feng

**University of Regensburg**
- Christoph Lehner (BNL)

**University of Southampton**
- Nils Asmussen
- Jonathan Flynn
- Ryan Hill
- Andreas Jüttner
- James Richings
- Chris Sachrajda

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- Jun-Sik Yoo
- Sergey Syritsyn (RBRC)
Outline

- Muon $g - 2$ Experiment
  - Motivation from muon $g - 2$
  - Tensions in $\pi\pi$ Scattering
  - Error Budget and LQCD Strategy

- Correlation Function Spectrum & Overlap
  - Lattice Parameters
  - GEVP Spectrum & Overlaps
  - $\pi\pi$ Scattering Phase Shift
  - $4\pi$ Correlation Functions

- Bounding Method and the Muon HVP
  - Correlation Function Reconstruction
  - (Improved) Bounding Method
  - Results

- $\pi\pi$ Scattering Phase Shifts
  - $l = 2, \ell = 0$
  - $l = 1, \ell = 1$
  - $l = 2, \ell = 2$

- Conclusions/Outlook
Introduction
Muon Anomalous Magnetic Moment Experiment

See talk by C. Lehner for detailed introduction

High-precision experiment of spin precession relative to momentum direction in storage ring

Anomalous frequency $\omega_a = \frac{g-2}{2} \frac{eB}{m} = a_\mu \frac{eB}{m}$

Sensitive to new physics, and also discrepant with experiment!
Fermilab Muon $g-2$ Experiment

Experiment has come a long way (and so has theory!)
Aiming for a $4 \times$ improvement in uncertainty over the BNL result
Muon $g−2$ Theory Error Budget

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value $\times 10^{10}$</th>
<th>Uncertainty $\times 10^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED</td>
<td>11 658 471.895</td>
<td>0.008</td>
</tr>
<tr>
<td>EW</td>
<td>15.4</td>
<td>0.1</td>
</tr>
<tr>
<td>HVP LO</td>
<td>692.5</td>
<td>2.7</td>
</tr>
<tr>
<td>HVP NLO</td>
<td>-9.84</td>
<td>0.06</td>
</tr>
<tr>
<td>HVP NNLO</td>
<td>1.24</td>
<td>0.01</td>
</tr>
<tr>
<td>Hadronic light-by-light</td>
<td>10.5</td>
<td>2.6</td>
</tr>
<tr>
<td>Total SM prediction</td>
<td>11 659 181.7</td>
<td>3.8</td>
</tr>
<tr>
<td>BNL E821 result</td>
<td>11 659 209.1</td>
<td>6.3</td>
</tr>
<tr>
<td>Fermilab E989 target</td>
<td></td>
<td>$\approx 1.6$</td>
</tr>
</tbody>
</table>

Experiment-Theory difference is $27.4(7.3) \implies 3.7\sigma$ tension!

Target measurement:
Hadronic Vacuum Polarization (HVP)

⇒ Lattice results have larger uncertainty, but systematically improve

⇒⇒ Dispersive (“R-ratio”) results more precise, but static

Aaron S. Meyer  
Section: Introduction  5/46
R-ratio data for $ee \to \pi\pi$ exclusive channel, $\sqrt{s} = 0.6 - 0.9$ GeV region
Tension between most precise measurements (BABAR/KLOE)
R-ratio $a_{\mu}^{HVP}$ uncertainty $<\text{difference in this channel}$

Avoid tension by computing precise lattice-only estimate of $a_{\mu}^{HVP}$
Use lattice QCD to inform experiment, resolve discrepancy
### Error Budget

<table>
<thead>
<tr>
<th>Contribution</th>
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<tbody>
<tr>
<td>$a_{μ,ud,conn, isospin}$</td>
<td>$202.9(1.4)<em>{S}^{C}(0.2)</em>{V}^{(0.2)}<em>{A}(0.2)</em>{Z}$</td>
<td>$649.7(14.2)<em>{S}^{C}(3.7)</em>{V}(1.5)<em>{A}(0.4)</em>{Z}(0.1)$</td>
</tr>
<tr>
<td>$a_{μ,s,conn, isospin}$</td>
<td>$27.0(0.2)<em>{S}^{C}(0.1)</em>{V}(0.0)<em>{A}(0.0)</em>{Z}$</td>
<td>$53.2(0.4)<em>{S}^{C}(0.3)</em>{V}(0.0)<em>{A}(0.0)</em>{Z}$</td>
</tr>
<tr>
<td>$a_{μ,c, conn, isospin}$</td>
<td>$3.0(0.0)<em>{S}^{C}(0.0)</em>{V}(0.0)<em>{A}(0.0)</em>{M}$</td>
<td>$14.3(0.0)<em>{S}^{C}(0.7)</em>{V}(0.1)<em>{A}(0.0)</em>{M}$</td>
</tr>
<tr>
<td>$a_{μ,uds, disc, isospin}$</td>
<td>$-1.0(0.1)<em>{S}^{C}(0.0)</em>{V}(0.0)<em>{A}(0.0)</em>{Z}$</td>
<td>$-11.2(3.3)<em>{S}^{C}(0.4)</em>{V}(2.3)_{L}$</td>
</tr>
<tr>
<td>$a_{μ,QED, conn}$</td>
<td>$0.2(0.2)<em>{S}^{C}(0.0)</em>{V}(0.0)<em>{A}(0.0)</em>{Z}(0.0)_{E}$</td>
<td>$5.9(5.7)<em>{S}^{C}(1.2)</em>{V}(0.0)<em>{A}(0.0)</em>{Z}(1.1)_{E}$</td>
</tr>
<tr>
<td>$a_{μ,QED, disc}$</td>
<td>$-0.2(0.1)<em>{S}^{C}(0.0)</em>{V}(0.0)<em>{A}(0.0)</em>{Z}(0.0)_{E}$</td>
<td>$-6.9(2.1)<em>{S}^{C}(1.4)</em>{V}(0.0)<em>{A}(0.0)</em>{Z}(1.3)_{E}$</td>
</tr>
<tr>
<td>$a_{μ,SIB}$</td>
<td>$0.1(0.2)<em>{S}^{C}(0.0)</em>{V}(0.0)<em>{A}(0.0)</em>{Z}(0.0)_{E48}$</td>
<td>$10.6(4.3)<em>{S}^{C}(6.6)</em>{V}(0.1)<em>{A}(0.0)</em>{Z}(1.3)_{E48}$</td>
</tr>
<tr>
<td>$a_{μ,udisc, isospin}$</td>
<td>$231.9(1.4)<em>{S}^{C}(0.2)</em>{V}(0.3)<em>{A}(0.2)</em>{Z}(0.0)_{M}$</td>
<td>$705.9(14.6)<em>{S}^{C}(3.7)</em>{V}(1.8)<em>{A}(0.4)</em>{Z}(2.3)<em>{L}(0.1)</em>{E48}$</td>
</tr>
<tr>
<td>$a_{μ,QED, SIB}$</td>
<td>$0.1(0.3)<em>{S}^{C}(0.0)</em>{V}(0.0)<em>{A}(0.0)</em>{Z}(0.0)<em>{E}(0.0)</em>{E48}$</td>
<td>$9.5(7.4)<em>{S}^{C}(6.9)</em>{V}(0.1)<em>{A}(0.0)</em>{Z}(1.7)<em>{E}(1.3)</em>{E48}$</td>
</tr>
<tr>
<td>$a_{μ,Ratio}$</td>
<td>$460.4(0.7)<em>{R}(2.1)</em>{R}^{S}$</td>
<td>$95.4(7.4)<em>{S}(6.9)</em>{R}(0.1)<em>{A}(0.0)</em>{Z}(1.7)<em>{E}(1.3)</em>{E48}$</td>
</tr>
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<td>$a_{μ}$</td>
<td>$692.5(1.4)<em>{S}^{C}(0.2)</em>{V}(0.3)<em>{A}(0.2)</em>{Z}(0.0)<em>{E}(0.0)</em>{E48}$</td>
<td>$715.4(16.3)<em>{S}^{C}(7.8)</em>{V}(1.9)<em>{A}(0.4)</em>{Z}(1.7)<em>{E}(2.3)</em>{L}$</td>
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<td></td>
<td></td>
<td>$715.4(16.3)<em>{S}^{C}(7.8)</em>{V}(1.9)<em>{A}(0.4)</em>{Z}(1.7)<em>{E}(2.3)</em>{L}$</td>
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**TABLE I.** Individual and summed contributions to $a_μ$ multiplied by $10^{10}$. The left column lists results for the window method with $t_0 = 0.4$ fm and $t_1 = 1$ fm. The right column shows results for the pure first-principles lattice calculation. The respective uncertainties are defined in the main text.

[Blum et al., (2018)]

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**Full program of computations to reduce uncertainties:**
| $a_{\mu}^{\text{ud, conn, isospin}}$ | 202.9(1.4)S(0.2)C(0.1)V(0.2)A(0.2)Z | 649.7(14.2)S(2.8)C(3.7)V(1.5)A(0.4)Z(0.1)E_{48}(0.1)E_{64} |
| $a_{\mu}^{s, \text{conn, isospin}}$ | 27.0(0.2)S(0.0)C(0.1)A(0.0)Z | 53.2(0.4)S(0.0)C(0.3)A(0.0)Z |
| $a_{\mu}^{c, \text{conn, isospin}}$ | 3.0(0.0)S(0.1)C(0.0)Z(0.0)M | 14.3(0.0)S(0.7)C(0.1)Z(0.0)M |
| $a_{\mu}^{\text{uds, disc, isospin}}$ | −1.0(0.1)S(0.0)C(0.0)V(0.0)A(0.0)Z | −11.2(3.3)S(0.4)V(2.3)L |
| $a_{\mu}^{\text{QED, conn}}$ | 0.2(0.2)S(0.0)C(0.0)V(0.0)A(0.0)Z(0.0)E | 5.9(5.7)S(0.3)C(1.2)V(0.0)A(0.0)Z(1.1)E |
| $a_{\mu}^{\text{QED, disc}}$ | −0.2(0.1)S(0.0)C(0.0)V(0.0)A(0.0)Z(0.0)E | −6.9(2.1)S(0.4)C(1.4)V(0.0)A(0.0)Z(1.3)E |
| $a_{\mu}^{\text{SIB}}$ | 0.1(0.2)S(0.0)C(0.2)V(0.0)A(0.0)Z(0.0)E_{48} | 10.6(4.3)S(0.6)V(0.1)A(0.0)Z(1.3)E_{48} |
| $a_{\mu}^{\text{disc, isospin}}$ | 231.9(1.4)S(0.2)C(0.1)V(0.3)A(0.2)Z(0.0)M | 705.9(14.6)S(2.9)C(3.7)V(1.8)A(0.4)Z(2.3)L(0.1)E_{48} |
| $a_{\mu}^{\text{QED, SIB}}$ | 0.1(0.3)S(0.0)C(0.2)V(0.0)A(0.0)Z(0.0)E(0.0)E_{48} | 9.5(7.4)S(0.7)C(6.9)V(0.1)A(0.0)Z(1.7)E(1.3)E_{48} |
| $a_{\mu}^{R-\text{ratio}}$ | 460.4(0.7)R_{ST}(2.1)R_{SY} | |
| $a_{\mu}$ | 692.5(1.4)S(0.2)C(0.2)V(0.3)A(0.2)Z(0.0)E(0.0)E_{48} | 715.4(16.3)S(3.0)C(7.8)V(1.9)A(0.4)Z(1.7)E(2.3)L |

| $a_{\mu}^{b(0.1)c(0.0)\Xi(0.0)\Xi^{-}(0.0)M(0.7)R_{ST}(2.1)R_{SY}}$ | |
| $a_{\mu}^{(1.5)E_{48}(0.1)E_{64}(0.3)b(0.2)c(1.1)\Xi(0.3)\Xi^{-}(0.0)M}$ | |

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Reduce statistical uncertainties on light connected contribution
## Error Budget

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<td>$a_{\mu, \text{ud, conn, isospin}}$</td>
<td>$202.9(1.4)s(0.2)c(0.1)v(0.2)\lambda(0.2)z$</td>
</tr>
<tr>
<td>$a_{\mu, \text{s, conn, isospin}}$</td>
<td>$27.0(0.2)s(0.0)c(0.1)\lambda(0.0)z$</td>
</tr>
<tr>
<td>$a_{\mu, \text{c, conn, isospin}}$</td>
<td>$3.0(0.0)s(0.1)c(0.0)\lambda(0.0)M$</td>
</tr>
<tr>
<td>$a_{\mu, \text{uds, disc, isospin}}$</td>
<td>$-1.0(0.1)s(0.0)c(0.0)v(0.0)\lambda(0.0)z$</td>
</tr>
<tr>
<td>$a_{\mu, \text{QED, conn}}$</td>
<td>$0.2(0.2)s(0.0)c(0.0)v(0.0)\lambda(0.0)z(0.0)E$</td>
</tr>
<tr>
<td>$a_{\mu, \text{QED, disc}}$</td>
<td>$-0.2(0.1)s(0.0)c(0.0)v(0.0)\lambda(0.0)z(0.0)E$</td>
</tr>
<tr>
<td>$a_{\mu, \text{SIB}}$</td>
<td>$0.1(0.2)s(0.0)c(0.2)v(0.0)\lambda(0.0)z(0.0)E_{48}$</td>
</tr>
<tr>
<td>$a_{\mu, \text{udsc, isospin}}$</td>
<td>$231.9(1.4)s(0.2)c(0.1)v(0.3)\lambda(0.2)z(0.0)M$</td>
</tr>
<tr>
<td>$a_{\mu, \text{QED, SIB}}$</td>
<td>$0.1(0.3)s(0.0)c(0.2)v(0.0)\lambda(0.0)z(0.0)E(0.0)E_{48}$</td>
</tr>
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<td>$a_{\mu, \text{R-ratio}}$</td>
<td>$460.4(0.7)<em>{\text{RST}}(2.1)</em>{\text{RSY}}$</td>
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<tr>
<td>$a_{\mu}$</td>
<td>$692.5(1.4)s(0.2)c(0.2)v(0.3)\lambda(0.2)z(0.0)E(0.0)E_{48}$</td>
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[Blum et al., (2018)]

Full program of computations to reduce uncertainties:

Reduce statistical uncertainties on light connected contribution

Compute QED contribution
Error Budget

<table>
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<tr>
<th>Error Source</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{\mu}^{ud, conn, isospin}$</td>
<td>$202.9(1.4)S(0.2)C(0.1)V(0.2)A(0.2)Z$</td>
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<td>$a_{\mu}^{s, conn, isospin}$</td>
<td>$27.0(0.2)S(0.0)C(0.1)A(0.0)Z$</td>
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<tr>
<td>$a_{\mu}^{c, conn, isospin}$</td>
<td>$3.0(0.0)S(0.1)C(0.0)Z(0.0)M$</td>
</tr>
<tr>
<td>$a_{\mu}^{uds, disc, isospin}$</td>
<td>$-1.0(0.1)S(0.0)C(0.0)V(0.0)A(0.0)Z$</td>
</tr>
<tr>
<td>$a_{\mu}^{QED, conn}$</td>
<td>$0.2(0.2)S(0.0)C(0.0)V(0.0)A(0.0)Z(0.0)E$</td>
</tr>
<tr>
<td>$a_{\mu}^{QED, disc}$</td>
<td>$-0.2(0.1)S(0.0)C(0.0)V(0.0)A(0.0)Z(0.0)E$</td>
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<td>$a_{\mu}^{SIB}$</td>
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<td>$a_{\mu}^{QED, SIB}$</td>
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<td>$a_{\mu}^{R-ratio}$</td>
<td>$460.4(0.7)<em>{RST}(2.1)</em>{RSY}$</td>
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<td>$a_{\mu}$</td>
<td>$692.5(1.4)S(0.2)C(0.2)V(0.3)A(0.2)Z(0.0)E_{48}$</td>
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[Blum et al., (2018)]

Full program of computations to reduce uncertainties:

Reduce statistical uncertainties on light connected contribution

Compute QED contribution

Improve lattice spacing determination
Full program of computations to reduce uncertainties:

Reduce statistical uncertainties on light connected contribution

Compute QED contribution

Improve lattice spacing determination

Finite volume and continuum extrapolation study

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[Blum et al., (2018)]
Exclusive Channels in the HVP

\[ C(t) = \frac{1}{3} \sum_i \langle [\bar{\psi} \gamma_i \psi]_t [\bar{\psi} \gamma_i \psi]_0 \rangle \]

Correlator has large statistical error in long-distance region, but contributions from high energy states are exponentially suppressed.

Long distance correlator dominated by two-pion states, but overlap of vector current with two-pion states is minimal.
Exclusive Channels in the HVP

Correlator has large statistical error in long-distance region, but contributions from high energy states are exponentially suppressed

Long distance correlator dominated by two-pion states, but overlap of vector current with two-pion states is minimal

Strategy:

- Construct & measure operators that overlap strongly with $\pi\pi$ states
- Correlate these operators with the local vector current
- $a_{\mu}^{HVP}$ computed by integrating with time-momentum representation kernel,

$$a_{\mu}^{HVP} = \sum_{t} w_{t} C(t) \ [\text{Bernecker et al., 1107.4388 [hep-lat]}]$$
Pion Scattering

Extraction of $\pi\pi$ exclusive channels requires many operators

Operator basis generated with distillation, perambulators

- Can access pion scattering phase shifts with data
- Study $\rho$ resonance region at physical pion mass
- Low $4\pi$ threshold, study effects on spectrum, HVP
LQCD Computation Setup
Computation Details

Computed on 2 + 1 flavor Möbius Domain Wall Fermions for valance and sea, $M_\pi$ at physical value on all ensembles

Computations using distillation setup with $N_{eig}$ eigenvectors

Results in this talk include four ensembles:

- **“24ID”**: $24^3 \times 64$ (4.8 fm), $a \approx 0.194$ fm $\approx 1.015$ GeV$^{-1}$
- **“32ID”**: $32^3 \times 64$ (6.2 fm), $a \approx 0.194$ fm $\approx 1.015$ GeV$^{-1}$
- **“48I”**: $48^3 \times 96$ (5.5 fm), $a \approx 0.114$ fm $\approx 1.730$ GeV$^{-1}$
- **“64I”**: $64^3 \times 128$ (5.3 fm), $a \approx 0.084$ fm $\approx 2.359$ GeV$^{-1}$

Future work including other ensembles for better control of extrapolations
Operators

Operators constructed in $l = 1$, $P$-wave channel to impact upon $HVP_\mu$

Designed to have strong overlap with specific target states, but all operators unavoidably couple to all states in HVP spectrum

Vector current operators:

- **Local** $O_{J\mu} = \sum_x \bar{\psi}(x) \gamma_\mu \psi(x), \mu \in \{1, 2, 3\}$
- **Smeared** $O_{j\mu} = \sum_{xyz} \bar{\psi}(x)f(x - z)\gamma_\mu f(z - y)\psi(y)$

2π operators with $O_n$ given by $\vec{p}_\pi \in \frac{2\pi}{L} \times \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 0, 0)\}$

$$O_n = \left| \sum_{xyz} \bar{\psi}(x)f(x - z)e^{-i\vec{p}_\pi \cdot \vec{z}}\gamma_5 f(z - y)\psi(y) \right|^2$$

Also test a 4π operator with $\vec{p}_\pi = \frac{2\pi}{L} \times (1, 0, 0)$:

$$O_{4\pi} = \left| \sum_{xyz} \bar{\psi}(x)f(x - z)e^{-i\vec{p}_\pi \cdot \vec{z}}\gamma_5 f(z - y)\psi(y) \right|^2 \left| \sum_{xy} \bar{\psi}(x)f(x - y)\gamma_5 \psi(y) \right|^2$$

Correlators arranged in a $N \times N$ symmetric matrix:

<table>
<thead>
<tr>
<th>$\otimes$</th>
<th>$O_{J\mu}$</th>
<th>$O_{j\mu}$</th>
<th>$O_{2\pi}$</th>
<th>$O_{4\pi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_{J\mu}$</td>
<td>$C_{J\mu} O_{J\mu}$</td>
<td>$C_{J\mu} O_{j\mu}$</td>
<td>$C_{J\mu 2\pi}$</td>
<td>$C_{J\mu 4\pi}$</td>
</tr>
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</tr>
</tbody>
</table>

$\rightarrow C(t)$
Generalized EigenValue Problem (GEVP)

Generalized EigenValue Problem to estimate overlap with vector current & energies

\[ C(t) V = C(t + \delta t) V \Lambda(\delta t) \]

\[ \Lambda_{nn}(\delta t) \sim e^{+E_n\delta t}, \quad V_{im} \propto \langle \Omega | O_i | m \rangle \]

\( C(t) \) is the matrix of correlation functions from previous slide

Compute at fixed \( \delta t \), vary \( t \): plateau for large \( t \)

From result, reconstruct exponential dependence of local vector correlation function

\[ C_{ij}^{\text{latt.}}(t) = \sum_{n}^{N} \langle \Omega | O_i | n \rangle \langle n | O_j | \Omega \rangle e^{-E_n t} \]

In theory, infinite number of states contribute to correlation function

In practice, only finite \( N \) necessary to model correlation function

\[ \implies \text{finite GEVP basis is sufficient} \]
6-operator basis on 48l ensemble: local+smeared vector, $4 \times (2\pi)$

Data points from solving GEVP at fixed $\delta t$

$$C(t_0) V = C(t_0 + \delta t) V \Lambda(\delta t), \quad \Lambda_{nn}(\delta t) \sim e^{+E_n \delta t}$$

Excited state contaminations decay as $t_0, \delta t \to \infty$

moving right on plot $\implies$ asymptote to lowest states’ spectrum & overlaps

Left: Spectrum; Right: Overlap with local vector current
An automated group theory engine has been an integral part of RBC-UKQCD's automated setup for two-pion diagrams in exclusive channel study.
An automated group theory engine has been an integral part of RBC-UKQCD’s automated setup for two-pion diagrams in exclusive channel study. Code builds a text representation of operators by performing tensor products and irrep decompositions of lattice operators with arbitrary spin & momentum.
Group Theory & Contraction Engine

This has resulted in a world-first computation of $4\pi$ to $4\pi$ correlation functions in $I = 1$ channel.
$4\pi$ Contractions
4\pi \text{ Contractions cont...}
Contractions cont... cont...
From $4\pi$ to $N\pi$ (and Beyond)

Group theory engine handles bosonic & fermionic octahedral irreps, any momentum
Easily build N-particle operators from single-particle building blocks

Strong need for multiparticle operators, as seen over past two weeks:
▶ Difficulty extracting scattering states without bilocal operators
▶ Large multiplicities of $X\pi + N$ states in nucleon correlators
▶ When simulating close to physical point, multiparticle thresholds are relevant

Nucleon matrix elements vital for neutrino community (discussion?)
Incredible work done by many people, could start tackling these problems soon
Breakdown of formalism for phase shifts $+\text{FVC}$ could occur at $4\pi$ threshold
Compute $2\pi \rightarrow 4\pi$, $4\pi \rightarrow 4\pi$ correlation functions and check explicitly
Spectrum unaffected by inclusion of $4\pi$ operator, but state is resolvable
GEVP Results - $4\pi$ Operators

Breakdown of formalism for phase shifts +FVC could occur at $4\pi$ threshold
Compute $2\pi \rightarrow 4\pi$, $4\pi \rightarrow 4\pi$ correlation functions and check explicitly
Spectrum unaffected by inclusion of $4\pi$ operator, but state is resolvable
Overlap of $4\pi$ state with local vector current unresolvable
Breakdown of formalism for phase shifts +FVC could occur at 4π threshold
Compute 2π → 4π, 4π → 4π correlation functions and check explicitly
Spectrum unaffected by inclusion of 4π operator, but state is resolvable
Overlap of 4π state with local vector current unresolvable
Overlap of state with 4π operator significant
⇒ 4π state safely negligible in local vector current
Correlator Reconstruction and Bounding
Plotted: (weight kernel) \( \times \) (correlation function); integral \( \rightarrow a^{HVP}_\mu \)

GEVP results to reconstruct long-distance behavior of
local vector correlation function needed to compute connected HVP

Explicit reconstruction good estimate of correlation function at long-distance,
missing excited states at short-distance

More states \( \Rightarrow \) better reconstruction, can replace \( C(t) \) at shorter distances
Improved Bounding Method

Use known results in spectrum to make a precise estimate of upper & lower bound on $a_{\mu}^{HVP}$ [RBC (2017)]

$$\tilde{C}(t; t_{\text{max}}, E) = \begin{cases} C(t) & t < t_{\text{max}} \\ C(t_{\text{max}})e^{-E(t-t_{\text{max}})} & t \geq t_{\text{max}} \end{cases}$$

Upper bound: $E \leq E_0$, lowest state in spectrum

Lower bound: $E \geq \log\left[\frac{C(t_{\text{max}})}{C(t_{\text{max}}+1)}\right]$

BMW Collaboration [K.Miura, Lattice2018] takes $E \to \infty$

With good control over lower states in spectrum from exclusive reconstruction, improve bounding method [RBC/UKQCD 2018 (CL@KEK Feb 2018)]:

Replace $C(t) \to C(t) - \sum_{n}^{N} |c_n|^2 e^{-E_n t}$ and apply bounding procedure for $a_{\mu} - \delta a_{\mu}$

$\implies$ Long distance convergence now $\propto e^{-E_{N+1} t}$, lower bound falls faster

$\implies$ Smaller overall contribution from neglected states

After bounding, add back $\delta a_{\mu} = \sum_{t=t_{\text{max}}}^{\infty} w_t \sum_{n}^{N} |c_n|^2 e^{-E_n t}$
Bounding Method Results - 48I

No bounding method:
Bounding method $t_{\text{max}} = 3.3$ fm, no reconstruction:

- $a_{HVP}^{\mu} = 646(38)$
- $a_{HVP}^\mu = 631(16)$

Bounding method gives factor of 2 improvement over no bounding method.

Improving the bounding method increases gain to factor of 7, including systematics.

Improvement should make all-lattice computation of $a_{HVP}^\mu$ competitive with R-ratio by 2020.
Bounding Method Results - 48I

No bounding method:

Bounding method \( t_{\text{max}} = 3.3 \text{ fm} \), no reconstruction:

Bounding method \( t_{\text{max}} = 3.0 \text{ fm} \), 1 state reconstruction:

Bounding method gives factor of 2 improvement over no bounding method

Bounding method gives factor of 7 improvement including systematics
Bounding Method Results - 481

No bounding method:
Bounding method $t_{\text{max}} = 3.3$ fm, no reconstruction:
Bounding method $t_{\text{max}} = 3.0$ fm, 1 state reconstruction:
Bounding method $t_{\text{max}} = 2.9$ fm, 2 state reconstruction:

Bounding method gives factor of 2 improvement over no bounding method

Improving the bounding method increases gain to factor of 7, including systematics

Improvement should make all-lattice computation of $a_{HVP}^\mu = 646(38)$
$a_{HVP}^\mu = 631(16)$
$a_{HVP}^\mu = 631(12)$
$a_{HVP}^\mu = 633(10)$

Preliminary
Bounding Method Results - 48I

No bounding method:
Bounding method $t_{\text{max}} = 3.3$ fm, no reconstruction: $a_{\mu}^{HVP} = 646(38)$
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Bounding method $t_{\text{max}} = 2.9$ fm, 2 state reconstruction: $a_{\mu}^{HVP} = 631(12)$
Bounding method $t_{\text{max}} = 2.2$ fm, 3 state reconstruction: $a_{\mu}^{HVP} = 633(10)$

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Bounding Method Results - 48I

No bounding method:
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Bounding method $t_{max} = 2.9$ fm, 2 state reconstruction: $a^HVP = 633(10)$
Bounding method $t_{max} = 2.2$ fm, 3 state reconstruction: $a^HVP = 624.3(7.5)$
Bounding method $t_{max} = 1.8$ fm, 4 state reconstruction: $a^HVP = 625.0(5.4)$

Bounding method gives factor of 2 improvement over no bounding method
Improving the bounding method increases gain to factor of 7, including systematics
Improvement should make all-lattice computation of $a^HVP$ competitive with R-ratio by 2020
Update to RBC-UKQCD calculation including exclusive study in preparation

$$\implies$$ could expect precision improvement $\times 3$, error on $a_{\mu}^{HVP}$ at $5 \times 10^{-10}$

$$\implies$$ to be included in $g - 2$ Theory WP before release of Fermilab first results

Further reduction will require full RBC-UKQCD program of computations

Work on the exclusive channel study using bounding method has led to 
world-first estimation of finite volume corrections to $a_{\mu}^{HVP}$ at physical $M_{\pi}$

Complete analysis with full suite of systematic improvements ongoing

$$\implies$$ precision improvement $\times 10$ over original, target error on $a_{\mu}^{HVP}$ at $1 \times 10^{-10}$

Compare to dispersive $(3 - 5) \times 10^{-10}$
$\pi\pi$ Scattering Phase Shifts
Phase shifts in continuum, infinite volume classified by partial wave, isospin channel
Bose symmetry over $\pi\pi$ states: $(-1)^{L+I} = +1$

$$\Rightarrow$$
Angular momentum conservation violated by lattice symmetries
Our analysis assumes all but lowest partial wave negligible

$$\det[e^{2i\delta(k)} - U(k)] = 0 \quad \ell=0 \quad \tan\delta(q) = \frac{q\pi^{3/2}}{Z(1, q^2)}, \quad q = \frac{kL}{2\pi}$$

<table>
<thead>
<tr>
<th>$\bar{\rho}\left(\frac{L}{2\pi}\right)$</th>
<th>Iso.</th>
<th>LQCD irrep</th>
<th>Cont. $L \leq 4$</th>
</tr>
</thead>
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<tr>
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<td>2</td>
<td>$A_1^+$</td>
<td>0, 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E^+$</td>
<td>2, 4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$T_2^+$</td>
<td>2, 4</td>
</tr>
<tr>
<td>$(0, 0, 0)$</td>
<td>1</td>
<td>$T_{1}^-$</td>
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<td>$A_1$</td>
<td>0, 2, 4</td>
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<td></td>
<td>$B_1$</td>
<td>2, 4</td>
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<td>$B_2$</td>
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<td>$B_1$</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$B_2$</td>
<td>2, 4</td>
</tr>
</tbody>
</table>
No resonances in $I = 2 \implies$ clean

Data requires subtraction periodic BC effects in time
$\implies$ handled through dedicated computation of $2\pi$ matrix elements

Breakdown of $SU(2)\chi$PT expected at around 500 MeV

Good agreement with pheno
Compute $\pi\pi$ scattering phase shifts in $I = 1$ channel from spectrum
Statistics + systematics included

Compare to simple Breit-Wigner parametrization and pheno (courtesy of M.Bruno)
Good agreement with pheno for 32ID, 48I, 64I
24ID: remnant excited state contaminations, still to be removed
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$I = 2 \ell = 2$ Phase Shift (48I)

Other lattice irreps probe higher partial waves

Phase shift small, unresolved $\Rightarrow$ limit only
Conclusions
Conclusions

Pion scattering is a simple playground to learn about multiparticle scattering in LQCD while addressing $a_{\mu}^{HVP}$:

- Computed $2\pi \rightarrow 4\pi$, $4\pi \rightarrow 4\pi$ correlation functions to show explicitly that $4\pi$ state has negligible effect on HVP

- Study of exclusive channels able to significantly reduce statistical uncertainty on an all-lattice computation of muon HVP
  
  $\Rightarrow$ factor of 4 more statistics on 48l available now
  $\Rightarrow$ expect to reach precision of $O(5 \times 10^{-10})$ by the end of year
  $\Rightarrow$ can target $O(1 \times 10^{-10})$ for all-lattice calculation

- Part of ongoing lattice study to address all lattice systematics in RBC+UKQCD HVP computation (see talk by C.Lehner)

- Computed scattering phase shifts in $I = 1, 2$
  
  $\Rightarrow$ first calculation of $\rho$ resonance at physical $M_{\pi}$!

- Scattering phase shifts to appear as part of series of papers by RBC+UKQCD

With experience from $\pi\pi$ scattering, will move on to tackle more challenging problems with $N\pi$ states and beyond

Thank you!
Neutrino oscillation experiments are of great interest to the physics community, seeking to do a high-precision measurement of neutrino oscillation parameters. Large program of research goals addressing fundamental physics questions:

- Precision measurements of oscillation parameters, including $\delta_{CP}$ & sign[$\Delta m_{13}^2$]
- Measure supernova neutrinos
- Search for proton decay

DUNE due to start installation of first detector in 2022, data collection in 2026

This experiment sets a timescale for making theory contributions!
Flux \times Cross Section

Neutrino interactions classified by their interaction products

3 general classes of interaction types:
Quasielastic (QE), Resonance (Res), Deep Inelastic Scattering (DIS)

DUNE events will be a mixture of all three; \sim \frac{1}{3} events will be resonant
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- Quasielastic (QE), Resonance (Res), Deep Inelastic Scattering (DIS)

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A dominant contribution to systematics in DUNE will be cross section uncertainties

$\implies$ Stringent requirements on cross section uncertainties, lots at stake

$\implies$ QE was dominant before; coming under control now $\implies$ next step is resonant
Discrepancies with Monte Carlo

Resonant interactions typically result in $\text{CC}1\pi$ (charge current with 1 pion) final states.

Current state of affairs for $\text{CC}1\pi$ interactions is confusing.

Lepton kinematics under control, consistent and agree well with Monte Carlo (GENIE).
Resonant interactions typically result in CC$1\pi$ (charge current with 1 pion) final states

Current state of affairs for CC$1\pi$ interactions is confusing
Lepton kinematics under control, consistent and agree well with Monte Carlo (GENIE)
Pion kinematics systematically disagree with shape
Difficult to change pion kinematics without breaking other data

⇒ Need another handle on pion kinematics!

Ideally a high-statistics $H$ or $D$ bubble chamber experiment; not likely to happen...
Nucleon to Delta Transitions

Knowledge of the $N \rightarrow \Delta$ transition is vital for neutrino oscillation experiments:

- Resonance region for neutrino scattering dominated by transitions to $\Delta$ baryon
- Four $N \rightarrow \Delta$ transition form factors, only $C_{5A}$ is relevant
- Assuming a 10% uncertainty on normalization of $C_{5A}$:
  \[
  \times \frac{1}{2} \text{ for near/far detector ratio} \\
  \times \frac{1}{3} \text{ for resonant/total interactions}
  \]
  \[
  \implies 1 - 2\% \text{ total theory uncertainty on oscillation observables}
  \]

Compared to DUNE total error budget of 1 – 3%:
\[
\implies 10\% \text{ knowledge of } C_{5A} \text{ norm would saturate the entire error budget of DUNE}
\]

Added difficulty: transition amplitudes needed to analyze same data that is used
to constrain transition amplitudes \(\implies\) likely underestimated uncertainties
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Room for improvement with Lattice QCD:

- Independent measurement to break theory/experiment degeneracy
- Providing realistic uncertainty estimates
- Improving constraints on form factor & amplitude uncertainty
Multiparticle States in Lattice QCD

Extracting properties of resonances in LQCD is complicated by finite volume:

- Resonance properties must be extracted from decay product states
- Particles in multiparticle states are not asymptotic states
- Need to enforce energy & momentum conservation with discretized momenta
- Energy eigenstates are superpositions of states with definite particle content

\[ \Rightarrow \text{avoided level crossings & complicated spectrum} \]

Argument can be turned on its head:

Finite volume as a probe of multiparticle scattering states

\[
\frac{M_\rho}{M_\pi} = 3.0
\]

\( \rho \) Unstable

Neutrino Physics to $g - 2$

Studying $N \rightarrow \Delta$ resonances requires study of $N\pi$ final states

Technically challenging:
- Fermionic spin states, unequal masses
- Many Wick contractions $\Rightarrow$ large computational cost
- Signal to noise degrades rapidly
- More than two particle scattering states, e.g. $N\pi\pi$, open up quickly

This is a long-term goal that we are developing the methodology to address

In the short term, use this formalism to address a simpler but timely physics question:
$\Rightarrow \pi\pi$ scattering in the isospin-1 channel $\Rightarrow (g - 2)_\mu$ HVP contribution

This is an ideal starting point:
- Simplest channel with resonance
- Above issues significantly mitigated/eliminated
- Consistency checks against other LQCD calculations & experiment available
Lattice QCD is ideal tool for filling in missing pieces

To have the greatest impact, must satisfy the checklist:

- Process is important for meeting experimental goals ✓
- Current precision not sufficient ✓
- Difficult/impractical to measure experimentally ✓
- Accessible to Lattice QCD ✓