CERN

Advances in Lattice Gauge Theory

Finite-volume effects in $(g-2)_\mu^{\text{HVP,LO}}$

Agostino Patella
Humboldt University, Berlin

M. T. Hansen and AP, *Finite-volume effects in* $(g-2)_\mu^{HVP,LO}$, arXiv:1904.10010 [hep-lat].
M. T. Hansen and AP, *The long one...*, in preparation.

Introduction

Euclidean $T \times L^3$ torus with periodic boundary conditions:

M. Della Morte *et al.*, *The hadronic vacuum polarization contribution to the muon $g-2$ from lattice QCD*, JHEP **1710**, 020 (2017).

$$a^{\mathrm{HVP}}(T,L) = \int_0^{T/2} dx_0\, K(x_0) \int_0^L d^3\mathbf{x} \sum_{\mu=1}^{3} \langle j_\mu(x_0,\mathbf{x}) j_\mu(0)\rangle_{T,L}$$

$K(x_0)$ continuous function in $[0,\infty)$ with

$$K(x_0) \overset{x_0\to\infty}{\propto} x_0^2\ .$$

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In the large-L limit

$$a^{\mathrm{HVP}}(T,L) = a^{\mathrm{HVP}}(\infty) + O\left(e^{-m_\pi L}\right) + O\left(e^{-\sqrt{2}m_\pi L}\right) + O\left(e^{-\sqrt{3}m_\pi L}\right) + O\left(e^{-2m_\pi L}\right)$$
$$+ O\left(e^{-m_\pi T}\right) + O\left(e^{-\frac{3}{2}m_\pi T}\right)$$
$$+ O\left(e^{-m_\pi\sqrt{T^2+L^2}}\right) + \dots$$

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arXiv:1904.10010 — next paper and this talk — next paper and maybe this talk

How to calculate finite-volume effects, $T = \infty$

General effective theory of pions with arbitrarily complicated local interactions.

M. Lüscher, *Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States*, Commun. Math. Phys. **104** (1986) 177.

Details of interactions do not matter: final formula will not depend on these details!

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$$\int d^4x \; e^{ikx} \; \langle j_\mu(x) j_\mu(0) \rangle_L =$$

$$= \sum_{\substack{\text{Feynman} \\ \text{diagrams}}} \left[\prod_{\substack{\text{independent} \\ \text{loops } \{\alpha\}}} \int \frac{dq^0_\alpha}{2\pi} \frac{1}{L^3} \sum_{\mathbf{q}_\alpha \in \frac{2\pi}{L} \mathbb{Z}^3} \right] \left[\prod_{\text{lines } \{\ell\}} \frac{1}{p_\ell^2 + m_\pi^2} \right] V(q, k)$$

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Poisson summation formula

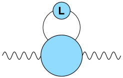
$$\frac{1}{L^3} \sum_{\mathbf{q}_\alpha \in \frac{2\pi}{L}\mathbb{Z}^3} \quad \to \quad \sum_{\mathbf{n}_\alpha \in \mathbb{Z}^3} \int \frac{d^3\mathbf{q}_\alpha}{(2\pi)^3} e^{iL\mathbf{n}_\alpha \mathbf{q}_\alpha}$$

$\mathbf{n}_\alpha = (n^1_\alpha, n^2_\alpha, n^3_\alpha)$ can be interpreted as the wrapping numbers of the loop α around the torus.

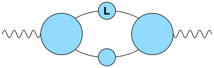
How to calculate finite-volume effects, $T = \infty$

$$\Delta a^{\mathrm{HVP}}(L) = a^{\mathrm{HVP}}(L) - a^{\mathrm{HVP}}(\infty) =$$

$$= \int_0^\infty dx_0 \, K(x_0) \int_0^L d^3\mathbf{x} \, \Big\{$$



$$+ \quad \frac{1}{2}$$



$$\Big\} + O\left(e^{-2m_\pi L}\right)$$

- Leading contributions: only one loop wraps around the torus.
- Feynman diagrams can be resummed in a skeleton expansion.
- Small blobs: infinite-volume dressed propagators. Big blobs: infinite-volume 1PI vertices.
- Small blob with **L**: finite-volume propagator

$$\frac{1}{p^2 + m^2 + \Sigma(p^2)} \to \sum_{\mathbf{n} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}} \frac{e^{iL\mathbf{n}\mathbf{p}}}{p^2 + m_\pi^2 + \Sigma(p^2)}$$

- Where is the exponential decay?

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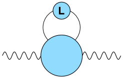

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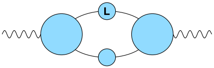

$$\Bigg\} + O\left(e^{-2m_\pi L}\right)$$

- Decompose $\mathbf{p} = p_n \dfrac{\mathbf{n}}{|\mathbf{n}|} + \mathbf{p}_\perp$
- Shift the p_n integral in the complex plane to $\mathbb{R} + 2im_\pi$

$$\sum_{\mathbf{n}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}} \int \frac{d^4p}{(2\pi)^4} e^{iL|\mathbf{n}|p_n} f(p_n, \mathbf{p}_\perp, k) = \int \frac{d^3p_\perp}{(2\pi)^3} e^{-L|\mathbf{n}|\sqrt{m_\pi^2+\mathbf{p}_\perp^2}} \times \text{(residue at the poles)} +$$

$$+ \int \frac{d^4p}{(2\pi)^4} e^{-2Lm_\pi|\mathbf{n}|} e^{iL|\mathbf{n}|p_n} f(p_n + 2im_\pi, \mathbf{p}_\perp, k)$$

Pole contributions = On-shell pions $\implies$ Compton scattering amplitude

Final formula, $T = \infty$

$$\Delta a^{\mathrm{HVP}}(L) = -\frac{\alpha^2}{2\pi L m_\mu^2} \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{|\mathbf{n}|} \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p_3^2}} \int_0^\infty dx_0 \, \widehat{\mathcal{K}}(m_\mu x_0) \times$$
$$\times \int \frac{dk_3}{2\pi} \cos(x_0 k_3) \, \mathrm{Re} \, T(-k_3^2, -k_3 p_3) + \mathcal{O}(e^{-2m_\pi L})$$

Forward Compton scattering amplitude $\pi\gamma^* \to \pi\gamma^*$

$$T(k^2, kp) = i \lim_{\mathbf{p}'\to\mathbf{p}} \sum_{q=0,\pm1} \int d^4x e^{ikx} \langle \mathbf{p}', q|\mathrm{T}\mathcal{J}_\rho(x)\mathcal{J}^\rho(0)|\mathbf{p}, q\rangle \ .$$

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Integration kernel

$$\widehat{\mathcal{K}}(t) = t^2 - 2\pi t + (8\gamma_E - 2) + \frac{4}{t^2} + 8\log(t) - \frac{8K_1(2t)}{t} - 8\int_0^\infty dv\, \frac{e^{-t\sqrt{v^2+4}}}{(v^2+4)^{3/2}}$$

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- Reference to the effective field theory has disappeared, everything is now defined in terms of the fundamental theory (QCD).
- Only space-like photons contribute!
- Can we use this formula for a numerical estimate of finite-L effects?

Estimates

$$\Delta a^{\mathrm{HVP}}(L) = -\frac{\alpha^2}{2\pi L m_\mu^2} \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{|\mathbf{n}|} \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p_3^2}} \int_0^\infty dx_0\, \widehat{\mathcal{K}}(m_\mu x_0) \int \frac{dk_3}{2\pi} \cos(x_0 k_3)\, \mathrm{Re}\; T(-k_3^2, -k_3 p_3)$$

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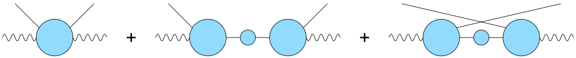

Compton scattering amplitude in the space-like region (it is a distribution!)

$$T(-\mathbf{k}^2, -\mathbf{kp}) = \frac{2(4m_\pi^2 + \mathbf{k}^2)\, F_\pi^2(\mathbf{k}^2)}{\mathbf{k}^2 + 2\mathbf{pk} - i\epsilon} + \frac{2(4m_\pi^2 + \mathbf{k}^2)\, F_\pi^2(\mathbf{k}^2)}{\mathbf{k}^2 - 2\mathbf{pk} - i\epsilon} + T^{\mathrm{reg}}(-\mathbf{k}^2, -\mathbf{kp})$$

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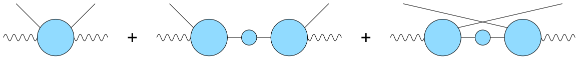

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Space-like electromagnetic form factor of the pion, phenomenological fit

D. Brmmel *et al.* [QCDSF/UKQCD Collaboration], *The Pion form-factor...*, Eur. Phys. J. C **51** (2007) 335.

$$F_\pi(\mathbf{k}^2) = \left(1 + \frac{\mathbf{k}^2}{M^2}\right)^{-1}, \qquad M = 727 \text{ MeV}$$

Estimates

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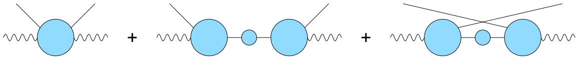

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Regular part of the Compton scattering amplitude, NLO chiral perturbation theory

$$T^{\rm reg}(-\mathbf{k}^2, -\mathbf{kp}) = c_0 + c_1 \mathbf{k}^2 - \frac{7m_\pi^2 + 4\mathbf{k}^2}{6\pi^2 f_\pi^2} \sqrt{1 + \frac{4m_\pi^2}{\mathbf{k}^2}}\, \mathrm{cotg}^{-1} \sqrt{1 + \frac{4m_\pi^2}{\mathbf{k}^2}}$$

Estimates

$m_\pi = 137$ Mev $\quad m_\mu = 106$ MeV $\quad M = 727$ MeV $\quad a = 700 \times 10^{-10}$

$m_\pi L$	$-10^2 \frac{\Delta a(L)}{a}$	form-factor contribution $e^{-m_\pi L}$	$e^{-\sqrt{2} m_\pi L}$	$e^{-\sqrt{3} m_\pi L}$	$e^{-2 m_\pi L}$	$e^{-\sqrt{5} m_\pi L}$	T^{reg} $e^{-m_\pi L}$
4	**3.15**	1.26	1.16	0.32	0.10	0.19	-0.019
5	**1.42**	0.852	0.428	0.081	0.020	0.029	-0.005
6	**0.629**	0.461	0.141	0.019	0.003	0.004	-0.001
7	**0.274**	0.226	0.043	0.004	0.001	0.001	–
8	**0.118**	0.104	0.013	0.001	–	–	–

Comparison with the free-pion approximation

$m_\pi L$	$F_\pi(\mathbf{k}^2) = \left(1 + \frac{\mathbf{k}^2}{M^2}\right)^{-1}$	$F_\pi(\mathbf{k}^2) = 1$
4	**3.15**	**2.12**
5	**1.42**	**1.05**
6	**0.629**	**0.491**
7	**0.274**	**0.223**
8	**0.118**	**0.099**

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arXiv:1904.10010 — next paper and this talk — next paper and maybe this talk

Finite-T corrections

$$\Delta a^{\mathrm{HVP}}(T) = \Delta a_0^{\mathrm{OB}}(T) + \Delta a_0^{\mathrm{BPS}}(T) + \Delta a_0^{\mathrm{FT}}(T) + O(e^{-\frac{3}{2}mT})$$

Out-of-the-Box contribution

$$\Delta a_0^{\mathrm{OB}}(T) = \int_{\frac{T}{2}}^{\infty} dx_0 \; K(x_0) \int d^3\mathbf{x} \sum_{\mu=1}^{3} \langle j_\mu(x_0,\mathbf{x}) j_\mu(0)\rangle_\infty$$

Backward-Propagating-States contribution

$$\Delta a_0^{\mathrm{BPS}}(T) = \int_0^{\frac{T}{2}} dx_0 \; K(x_0) \int d^3\mathbf{x} \sum_{\mu=1}^{3} \langle j_\mu(T-x_0,\mathbf{x}) j_\mu(0)\rangle_\infty$$

Proper Finite-Temperature contribution

$$\Delta a_0^{\mathrm{FT}}(T) = \int_0^{\frac{T}{2}} dx_0 \; K(x_0) \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{-T\sqrt{m_\pi^2+\mathbf{p}^2}}}{2\sqrt{m_\pi^2+\mathbf{p}^2}} \times$$

$$\times \lim_{\mathbf{p}'\to\mathbf{p}} \int d^3\mathbf{x} \sum_{q=0,\pm1} \sum_{\rho=1}^{3} \langle \mathbf{p}', q | j_\rho(x_0,\mathbf{x}) j_\rho(0) | \mathbf{p}, q\rangle$$

TO do: estimates!

Some comments – Conclusions

- The space-like form factor is the natural quantity that enters in the finite-L effects, not the time-like one (but one can always use dispersion relation).

- We studied the finite-volume effects of a particular estimator of a^{HVP}, i.e.

$$a^{\mathrm{HVP}}(T,L) \to \int_0^{T/2} dx_0\, K(x_0) \int_0^L d^3\mathbf{x} \sum_{\mu=1}^{3} \langle j_\mu(x_0,\mathbf{x}) j_\mu(0) \rangle_{T,L}$$

  Different estimators will have different finite-volume effects.

- Because of the noise at large distance, one may need to cut the integral over x_0, i.e.

$$a^{\mathrm{HVP}}(T,L) \to \int_0^{\bar{x}_0} dx_0\, K(x_0) \int_0^L d^3\mathbf{x} \sum_{\mu=1}^{3} \langle j_\mu(x_0,\mathbf{x}) j_\mu(0) \rangle_{T,L}$$

  The finite-volume corrections of this integral can be calculated with the same techniques.

- If the tail of the two-point functions is reconstructed with a subset of states e.g. extracted with a GEVP, power-law finite-L corrections are artificially introduced (people are doing more refined things).

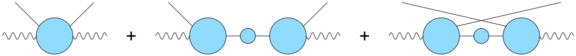
- NNLO chiral perturbation theory gives divergent integrals.

- Finite-T corrections can be estimated as well, but they are likely to be negligible in standard setups $T \geq 2L$. Choosing $T = L$ seems to be a bad idea.

Backup slides

A representation suitable for numerics

$$\Delta a^{\mathrm{HVP}}(L) = -\frac{\alpha^2}{2\pi L m_\mu^2} \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{|\mathbf{n}|} \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p_3^2}} \int_0^\infty dx_0\, \widehat{\mathcal{K}}(m_\mu x_0) \int \frac{dk_3}{2\pi} \cos(x_0 k_3)\, \mathrm{Re}\; T(-k_3^2, -k_3 p_3)$$



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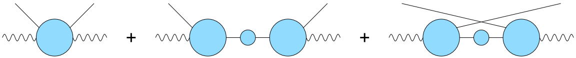


Compton scattering amplitude in the space-like region (it is a distribution!)

$$T(-\mathbf{k}^2, -\mathbf{kp}) = \frac{2(4m_\pi^2 + \mathbf{k}^2)F_\pi^2(\mathbf{k}^2)}{\mathbf{k}^2 + 2\mathbf{pk} - i\epsilon} + \frac{2(4m_\pi^2 + \mathbf{k}^2)F_\pi^2(\mathbf{k}^2)}{\mathbf{k}^2 - 2\mathbf{pk} - i\epsilon} + T^{\mathrm{reg}}(-\mathbf{k}^2, -\mathbf{kp})$$

Integrate over p_3 and isolate the distribution

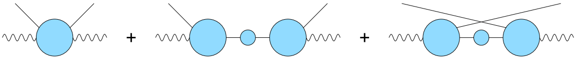
$$\mathcal{T}(k_3^2; |\mathbf{n}|L) = \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p_3^2}}\, T(-k_3^2, -k_3 p_3) = 2(4m_\pi^2 + k_3^2)F_\pi^2(k_3^2)\zeta(k_3^2; |\mathbf{n}|L) + \mathcal{T}^{\mathrm{reg}}(k_3^2; |\mathbf{n}|L)$$

$$\zeta(k_3^2; |\mathbf{n}|L) = \mathrm{Re} \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p_3^2}} \left\{ \frac{1}{k_3^2 + 2p_3 k_3 - i\epsilon} + \frac{1}{k_3^2 - 2p_3 k_3 - i\epsilon} \right\}$$

$$\mathcal{T}^{\mathrm{reg}}(k_3^2; |\mathbf{n}|L) = \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p_3^2}}\, T^{\mathrm{reg}}(-k_3^2, -k_3 p_3)$$

A representation suitable for numerics

$$\Delta a^{\mathrm{HVP}}(L) = -\frac{\alpha^2}{2\pi L m_\mu^2} \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{|\mathbf{n}|} \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p_3^2}} \int_0^\infty dx_0\, \widehat{\mathcal{K}}(m_\mu x_0) \int \frac{dk_3}{2\pi} \cos(x_0 k_3)\ \mathrm{Re}\ T(-k_3^2, -k_3 p_3)$$



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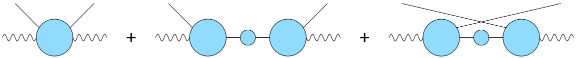

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$$\mathcal{T}^{\text{reg}}(k_3^2; |\mathbf{n}|L) = \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2 + p_3^2}} T^{\text{reg}}(-k_3^2, -k_3 p_3)$$

A representation suitable for numerics

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Fourier representation of the integration kernel

$$\widehat{\mathcal{K}}(t) = 2 \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} f(\omega^2) \left[\omega^2 t^2 - 2 + 2\cos(\omega t)\right]$$

$$f(\omega^2) = \frac{16}{\sqrt{4+\omega^2}\left[\sqrt{4+\omega^2} + \sqrt{\omega^2}\right]}$$

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Integrate over x_0

$$\int_0^\infty dx_0\, \cos(x_0 k_3)\widehat{\mathcal{K}}(m_\mu x_0) = -\pi m_\mu^2 \delta''(k_3) + 4\pi \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} f(\omega^2)\left[\delta(\omega m_\mu - k_3) - \delta(k_3)\right]$$

A representation suitable for numerics

$$\Delta a^{\mathrm{HVP}}(L) = -\frac{\alpha^2}{2\pi L m_\mu^2} \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{|\mathbf{n}|} \int_0^\infty dx_0\, \widehat{\mathcal{K}}(m_\mu x_0) \int \frac{dk_3}{2\pi} \cos(x_0 k_3) \mathcal{T}(k_3^2; |\mathbf{n}|L)$$

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Integrate over k_3

$$\int_0^\infty dx_0\, \widehat{\mathcal{K}}(m_\mu x_0) \int \frac{dk_3}{2\pi} \cos(x_0 k_3) \mathcal{T}(k_3^2; |\mathbf{n}|L) =$$

$$= -m_\mu^2 \mathcal{T}'(0; |\mathbf{n}|L) + 4 m_\mu \int_0^\infty dk_3 f(m_\mu^{-2} k_3^2) \frac{\mathcal{T}(k_3^2; |\mathbf{n}|L) - \mathcal{T}(0; |\mathbf{n}|L)}{k_3^2}$$

A representation suitable for numerics: summary

$$\Delta a^{\mathrm{HVP}}(L) = \frac{\alpha^2}{2\pi L m_\mu} \sum_{\mathbf{n}\neq\mathbf{0}} \frac{1}{|\mathbf{n}|} \left\{ m_\mu \mathcal{T}'(0;|\mathbf{n}|L) - 4\int_0^\infty dk_3 f(m_\mu^{-2}k_3^2) \frac{\mathcal{T}(k_3^2;|\mathbf{n}|L) - \mathcal{T}(0;|\mathbf{n}|L)}{k_3^2} \right\}$$

Forward Compton scattering amplitude in space-like region

$$T(-\mathbf{k}^2, -\mathbf{kp}) = \frac{2(4m_\pi^2 + \mathbf{k}^2)F_\pi^2(\mathbf{k}^2)}{\mathbf{k}^2 + 2\mathbf{pk} - i\epsilon} + \frac{2(4m_\pi^2 + \mathbf{k}^2)F_\pi^2(\mathbf{k}^2)}{\mathbf{k}^2 - 2\mathbf{pk} - i\epsilon} + T^{\mathrm{reg}}(-\mathbf{k}^2, -\mathbf{kp})$$

where $F_\pi^2(\mathbf{k}^2)$ is the pion electromagnetic form factor and T^{reg} is the regular part.

Intermediate quantities

$$\mathcal{T}(k_3^2;|\mathbf{n}|L) = 2(4m_\pi^2 + k_3^2)F_\pi^2(k_3^2)\zeta(k_3^2;|\mathbf{n}|L) + \mathcal{T}^{\mathrm{reg}}(k_3^2;|\mathbf{n}|L)$$

$$\zeta(k_3^2;|\mathbf{n}|L) = \int \frac{dp_3}{2\pi} \frac{e^{-|\mathbf{n}|L\sqrt{m_\pi^2 + \left(p_3 - \frac{k_3}{2}\right)^2}} - e^{-|\mathbf{n}|L\sqrt{m_\pi^2 + \left(p_3 + \frac{k_3}{2}\right)^2}}}{2p_3 k_3}$$

$$\mathcal{T}^{\mathrm{reg}}(k_3^2;|\mathbf{n}|L) = \int \frac{dp_3}{2\pi} e^{-|\mathbf{n}|L\sqrt{m_\pi^2 + p_3^2}}\, T^{\mathrm{reg}}(-k_3^2, -k_3 p_3)$$

Integration kernel in Fourier sapce

$$f(\omega^2) = \frac{16}{\sqrt{4+\omega^2}\left[\sqrt{4+\omega^2} + \sqrt{\omega^2}\right]}$$