Heavy quarks as a tool in non-perturbative renormalization

Alberto Ramos <alberto.ramos@maths.tcd.ie>

Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

ALPHA Collaboration

In collaboration with: M. Dalla Brida, T. Korzec, F. Knechtli, R. Höllwieser, S. Sint, R. Sommer

Overview

THE SCALE OF QCD: Λ-PARAMETER

$$\Lambda_s = \mu \left[b_0 \bar{g}_s^2(\mu) \right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0 \bar{g}_s^2(\mu)}} \underbrace{\exp\left\{ -\int_0^{\bar{g}_s(\mu)} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x} \right] \right\}}_{\mathcal{O}(\bar{g}_s{}^2(\mu))}$$

The intrinsic scale of QCD

- Λ_s same units as μ
- Λ_s is RGI: $d\Lambda_s / d\mu = 0$
- $\bar{g}_s^2(\mu)$ is a function of Λ_s/μ: Λ_s dictates what are "low" and "high" energies.
- (i.e. $\alpha_{\overline{\mathrm{MS}}}(M_Z)$ is trivial to compute if one knows $\Lambda^{(5)}_{\overline{\mathrm{MS}}}$)
- Scheme dependent, but if

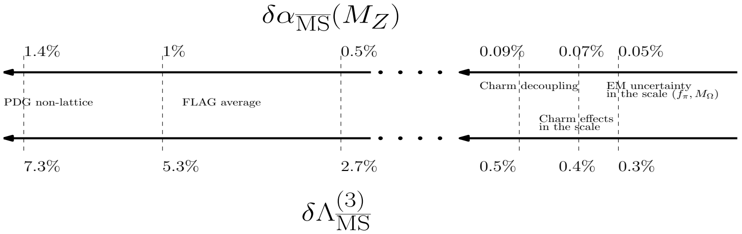
$$\bar{g}_{s'}^2(\mu) \overset{\bar{g}_s \to 0}{\sim} \bar{g}_s^2(\mu) + c_{ss'} \bar{g}_s^4(\mu) + \ldots$$

then

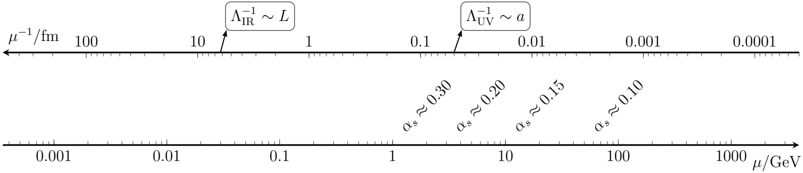
$$\frac{\Lambda_{s'}}{\Lambda_s} = \exp\left(\frac{-c_{ss'}}{2b_0} \right) .$$

- Defined non-perturbatively (even $\Lambda_{\overline{\mathrm{MS}}}$)

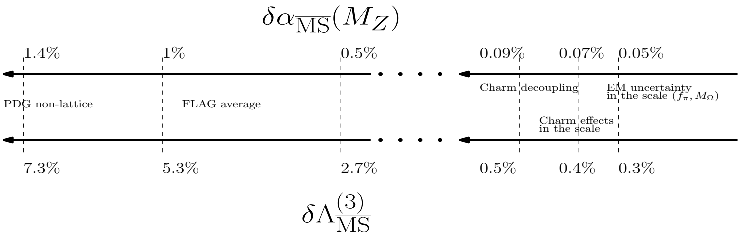
The challenge in determining Λ: Connecting quarks with hadrons



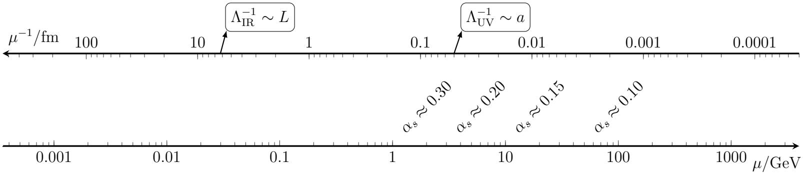
- 5.3% error in $\Lambda^{(3)} \leftrightarrow$ 1% error in $\alpha_{\overline{\mathrm{MS}}}(M_Z)$:
  - Electromagnetic corrections
  - Charm effects (i.e. $N_\mathrm{f} = 2+1+1$ vs. $N_\mathrm{f} = 2+1$)
  - Connecting hadronic physics with EW scale **without assumptions** on low scale physics (i.e. perturbation theory)



THE CHALLENGE IN DETERMINING Λ: CONNECTING QUARKS WITH HADRONS

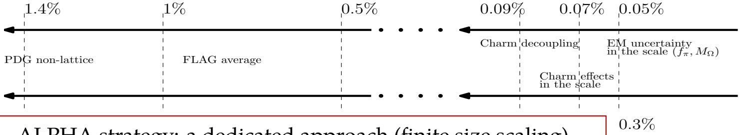


- 5.3% error in $\Lambda^{(3)} \leftrightarrow$ 1% error in $\alpha_{\overline{\rm MS}}(M_Z)$:
  - ~~Electromagnetic corrections~~
  - ~~Charm effects (i.e. $N_f = 2+1+1$ vs. $N_f = 2+1$)~~
  - Connecting hadronic physics with EW scale **without assumptions** on low scale physics (i.e. perturbation theory)



THE CHALLENGE IN DETERMINING Λ: CONNECTING QUARKS WITH HADRONS

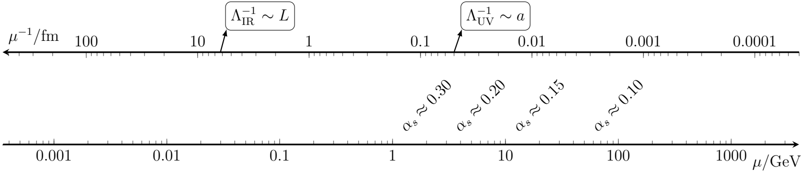
$$\delta\alpha_{\overline{\mathrm{MS}}}(M_Z)$$



ALPHA strategy: a dedicated approach (finite size scaling).

We aim at a 3% error in $\Lambda^{(3)}$. We do not care about EM, charm quarks, ...

- 5.3% error in $\Lambda^{(3)} \leftrightarrow$ 1% error in $\alpha_{\overline{\mathrm{MS}}}(M_Z)$:
  - Connecting hadronic physics with EW scale **without assumptions** on low scale physics (i.e. perturbation theory)



Overview

3M: A universe with three heavy degenerate quarks ($M \gg \Lambda$)

Alice uses fundamental theory

$$S_{\text{fund}}[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^4 x \left\{ -\frac{1}{2g^2} \mathrm{Tr}\,(F_{\mu\nu} F_{\mu\nu}) + \sum_{i=1}^{3} \bar{\psi}_i (\gamma_\mu D_\mu + M) \psi_i \right\}$$

Bob uses effective theory

$$S_{\text{eff}}[A_\mu] = -\frac{1}{2g_{\text{eff}}^2} \int \mathrm{d}^4 x \, \{\mathrm{Tr}\,(F_{\mu\nu} F_{\mu\nu})\} + \frac{1}{M^2} \sum_k \omega_k \int \mathrm{d}^4 x \, \mathcal{L}_k^{(6)} + \dots$$

3M: A universe with three heavy degenerate quarks ($M \gg \Lambda$)

Alice uses fundamental theory

$$S_{\text{fund}}[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^4x \left\{ -\frac{1}{2g^2} \mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right) + \sum_{i=1}^{3} \bar{\psi}_i (\gamma_\mu D_\mu + M)\psi_i \right\}$$

Bob uses effective theory

$$S_{\text{eff}}[A_\mu] = -\frac{1}{2g_{\text{eff}}^2} \int \mathrm{d}^4x \, \{\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right)\} \cancel{+ \frac{1}{M^2} \sum_k \omega_k \int \mathrm{d}^4x \, \mathcal{L}_k^{(6)} + \ldots}$$

Decoupling

- Dimensionless "low energy quantities" $\sqrt{t_0}/r_0, w_0/\sqrt{8t_0}, r_0/w_0, \ldots$ from effective theory

$$\frac{\mu_1^{\text{fund}}(M)}{\mu_2^{\text{fund}}(M)} = \frac{\mu_1^{\text{eff}}}{\mu_2^{\text{eff}}} + \mathcal{O}\left(\frac{\mu^2}{M^2}\right)$$

Renormalization in 3M: Alice determines the strong coupling

$$\frac{\Lambda}{\mu} = \left[b_0 \bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}.$$

- Determine non-perturbatively the β-function in the fundamental ($N_f = 3$) theory, mass-less scheme.
- Integral up to $\bar{g}^{(3)}(\mu_{\mathrm{ref}})$ (in a mass-less scheme!) gives:

$$\frac{\Lambda^{(3)}}{\mu_{\mathrm{ref}}}$$

- Only needs to compute a dimensionless ratio

$$\mu_{\mathrm{ref}}\sqrt{8t_0(M)}$$

- Result

$$\sqrt{8t_0(M)}\Lambda^{(3)} = \frac{\Lambda^{(3)}}{\mu_{\mathrm{ref}}} \times \mu_{\mathrm{ref}}\sqrt{8t_0(M)}$$

Renormalization in 3M: Bob determines the strong coupling

$$\frac{\Lambda}{\mu} = \left[b_0 \bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\} .$$

- Determine non-perturbatively the β-function in the effective ($N_f = 0$) theory.
- Integral up to $\bar{g}^{(0)}(\mu'_{\mathrm{ref}})$ gives:

$$\frac{\Lambda^{(0)}}{\mu'_{\mathrm{ref}}}$$

- Determine the dimensionless ratio

$$\mu'_{\mathrm{ref}}\sqrt{8t_0}$$

- Match across quark threshold to convert to $\Lambda^{(3)}$ (using perturbation theory)

$$\Lambda^{(3)}\sqrt{8t_0} = \Lambda^{(0)}\sqrt{8t_0} \times \frac{1}{P(\Lambda/M)} .$$

- Matching factor $P(\Lambda/M)$ **[ALPHA 1809.03383]**:
  - Known in perturbation theory up to three-loops. Power series in $\alpha(m^\star)$
  - “Good” PT: corrections very small even at $m_c^\star$.

Relation between Alice and Bob computation

Relation between Alice and Bob results:

$$\Lambda^{(3)}\sqrt{8t_0(M)} = \Lambda^{(0)}\sqrt{8t_0} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^3(m^\star)) + \mathcal{O}\left(\frac{1}{t_0 M^2}\right)$$

Bob is telling us that $\Lambda^{(3)}$ can be computed from $\Lambda^{(0)}$

$$\Lambda^{(3)} = \lim_{M\to\infty} \frac{1}{\sqrt{8t_0(M)}} \times \Lambda^{(0)}\sqrt{8t_0} \times \frac{1}{P(\Lambda/M)}$$

We need

- Running in pure gauge: $\Lambda^{(0)}\sqrt{8t_0}$
- A scale in a world with degenerate massive quarks: $\sqrt{8t_0(M)}$ in fm/MeV.

Lattice QCD can simulate unphysical worlds

$$\sqrt{8t_0(M)} = \sqrt{8t_0^{\text{phys}}} \times \frac{\sqrt{8t_0(M)}}{\sqrt{8t_0^{\text{phys}}}}$$

with $\sqrt{8t_0^{\text{phys}}} = 0.415(4)(2)$ fm from **[Bruno et al. '17]**

NON-PERTURBATIVE RENORMALIZATION BY DECOUPLING

Master relation

$$\frac{\Lambda^{(N_\mathrm{f})}}{\mu_\mathrm{dec}(M)} = \frac{\Lambda^{(0)}}{\mu_\mathrm{dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^3(m^\star)) + \mathcal{O}\left(\frac{\mu_\mathrm{dec}^2}{M^2}\right)$$

With a proper limit:

$$\Lambda^{(N_\mathrm{f})} = \lim_{M\to\infty} \mu_\mathrm{dec}(M) \times \frac{\Lambda^{(0)}}{\mu_\mathrm{dec}} \times \frac{1}{P(\Lambda/M)}$$

Where

- Pure gauge running: $\Lambda^{(0)}/\mu_\mathrm{dec}$
- A scale with N_f massive degenerate quarks: $\mu_\mathrm{dec}(M)$

NOTE: this is not completely trivial

$$\frac{\mu_\mathrm{dec}(M)}{\mu_\mathrm{dec}^\mathrm{phys}} = \lim_{a\to 0} \frac{a\mu_\mathrm{dec}(M)}{a\mu_\mathrm{dec}^\mathrm{phys}}$$

is a difficult extrapolation (M wants to be large, aM wants to be small).

Overview

Computing the Λ-parameter in units of μ_{ref}

$$\frac{\Lambda_s}{\mu_{\mathrm{ref}}} = \left[b_0 \bar{g}_s^2(\mu_{\mathrm{ref}})\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}_s^2(\mu_{\mathrm{ref}})}} \exp\left\{-\int_0^{\bar{g}_s(\mu_{\mathrm{ref}})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

The non-perturbative extrapolation

- We need $\beta_s(x)$ in the range $x \in [0, \bar{g}_s(\mu_{\mathrm{ref}})]$ (NOT possible!)
- Instead, $\beta_s(x)$ known[a] for $x \in [\bar{g}_s(\mu_{\mathrm{PT}}), \bar{g}_s(\mu_{\mathrm{ref}})]$ and perfom an extrapolation

$$\int_0^{\bar{g}_s(\mu_{\mathrm{ref}})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right] = \int_{\bar{g}(\mu_{\mathrm{PT}})}^{\bar{g}_s(\mu_{\mathrm{ref}})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right] + \mathcal{O}(\bar{g}_s{}^2(\mu_{\mathrm{PT}}))$$

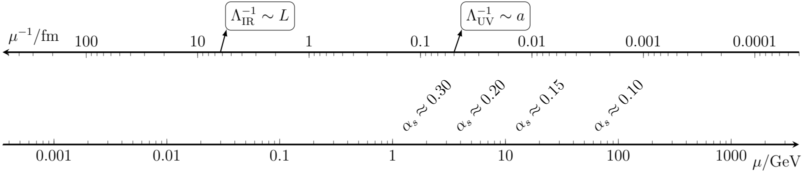
- Perturbation theory helps improving convergence

$$\begin{aligned}\int_0^{\bar{g}_s(\mu_{\mathrm{ref}})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right] &= \int_{\bar{g}(\mu_{\mathrm{PT}})}^{\bar{g}_s(\mu_{\mathrm{ref}})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right] \\ &+ \int_0^{\bar{g}(\mu_{\mathrm{PT}})} dx \left[\frac{1}{\beta_s{}^{(3l)}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right] \\ &+ \mathcal{O}(\bar{g}_s{}^4(\mu_{\mathrm{PT}}))\end{aligned}$$

[a] Sometimes β_s is not known but $\mu_{\mathrm{PT}}/\mu_{\mathrm{ref}} = \exp\left\{\int_{\bar{g}(\mu_{\mathrm{PT}})}^{\bar{g}_s(\mu_{\mathrm{ref}})} \mathrm{d}x/\beta_{s(x)}\right\}$. This is equivalent.

Continuum vs. non-perturbative extrapolation

	Continuum	Non-perturbative
Corrections:	Depend on action and observable $a, a\log a, a^2, a^2\log^k a, a^4\log^k a, \ldots$	Depend **only** on scheme $\bar{g}_s{}^2(\mu_{\mathrm{PT}}), \bar{g}_s{}^4(\mu_{\mathrm{PT}}), \bar{g}_s{}^6(\mu_{\mathrm{PT}}), \ldots$
Improvement:	Symanzik EFT NP $\mathcal{O}(a)$-improvement –	Perturbation Theory $\beta_s{}^{(\text{3-loops})}(x)$ $\beta_s{}^{(\text{4-loops})}(x)$
Quality:	$a_{\max} < 0.1\,\mathrm{fm}$ $a^2_{\max}/a^2_{\min} = 4$	$\alpha_s{}^2(\mu_{\mathrm{PT}}) = \bar{g}_s{}^2(\mu_{\mathrm{PT}})/4\pi < 0.25$ Change $\bar{g}_s{}^{2k}(\mu_{\mathrm{PT}})$ by a factor 4
Cost:	$a \to a/2 \implies \times 128$	$\alpha_s(\mu_{\mathrm{PT}}) = 0.25 \to 0.12 \implies \times 10^6$



Continuum vs. non-perturbative extrapolation

ative

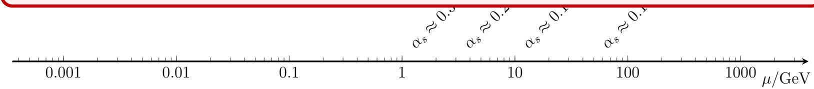
Exploring the non-perturbative extrapolation is challenging

- Very challenging to decrease $\alpha_s(\mu_{\mathrm{PT}})$ substantially
- Instead explore "truncation errors"
  - Change $\mu_{\mathrm{PT}} \to 2\mu_{\mathrm{PT}}$
  - Change order in PT
  - …
- Every lattice (and non-lattice) determination of $\Lambda, \alpha_s, m_q, B_K, \ldots$ is quoting the result of a non-perturbative extrapolation
- Strong opinion here: <u>solved</u> using finite size scaling **[Lüscher, Weisz, Wolff '91]**.
- Step scaling function: How much changes the coupling when we change the renormalization scale:

$$\sigma_2(u) = g^2(\mu/2)\Big|_{g^2(\mu)=u}$$

achieved by simple changing $L/a \to 2L/a$!

0^6



Simulation details [M. Dalla Brida, A. Ramos 1905.05147]

- Wilson Plaquette gauge action.
- Schrödinger Functional boundary conditions (w 2-loop c_t)
- Algorithm: Hybrid OR: $L/a \times (1\mathrm{HB} + L/a \times \mathrm{OR})$. Almost independent measurements of $\bar{g}^2_{\mathrm{GF}}$.
- We always work in the scheme with $c = 0.3$. Two discretizations of $\bar{g}^2_{\mathrm{GF}}$
  - Wilson Flow + Clover observable
  - Zeuthen Flow + Improved observable
- Measurements for $\Sigma_2(u, a/L)$ at $L/a = 8, 10, 12, 16, 24$. Factor 3 in a.
- Measurements for $\Sigma_{3/2}(u, a/L)$ at $L/a = 8, 16, 32$. Factor 4 in a.
- Topology freezing overcome by using measurements at fixed topology

$$\bar{g}^2_{\mathrm{m,e}}(\mu) = t^2 \hat{\mathcal{N}}^{-1}_{\mathrm{e,m}}(c, a/L) \frac{\langle E_{\mathrm{m,e}}(t,x)\hat{\delta}_Q\rangle}{\langle\, \hat{\delta}_Q\rangle}\bigg|_{\mu=1/\sqrt{8t},\ \sqrt{8t}=cL,\ x_0=T/2} \qquad (c = 0.3)\,.$$

with topology mesured on the flowed configuration

$$Q = -\frac{1}{16\pi^2}\sum_x \epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\{G^{\mathrm{cl}}_{\mu\nu}(t,x)G^{\mathrm{cl}}_{\rho\sigma}(t,x)\}\,, \quad \hat{\delta}_Q = \begin{cases} 1\,, & \text{if } |Q| < 0.5, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

DETERMINATION OF THE β-FUNCTION AT HIGH ENERGIES

- Basic relation: Raw measurements for $\Sigma_s(u, a/L)$, $(s = 3/2, 2)$ obey

$$\log s = \int_{\sqrt{u}}^{\sqrt{\Sigma_s(u,a/L)}} \frac{\mathrm{d}x}{\beta_{\mathrm{GF}}(x)} + \text{cutoff effects}(s)$$

- Parametrize β-function (b_0, b_1, b_2 known from PT)

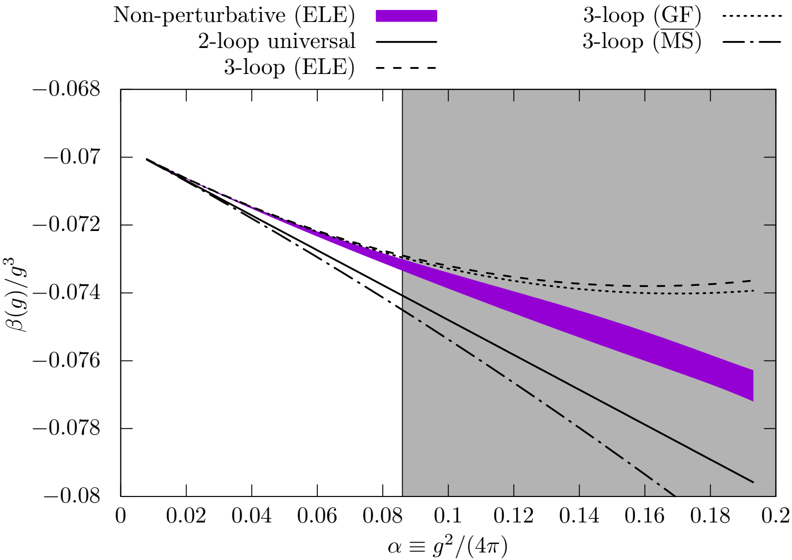
$$\beta_{\mathrm{GF}}(x) = -x^3 \left(b_0 + b_1 x^2 + b_2 x^4 + \sum_{k=3}^{n_b} p_k x^{2k} \right)$$

- Fit data to model (uncorrelated fit, but errors include correlations)

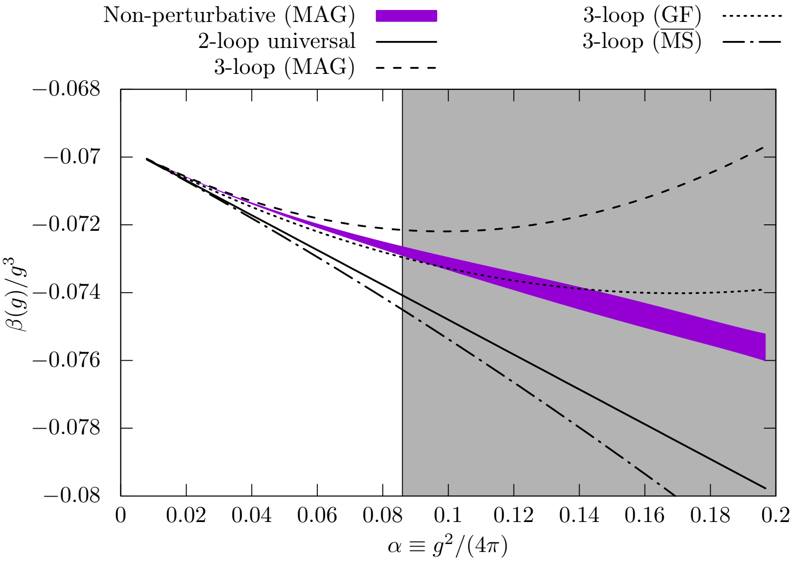
$$\chi^2 = \sum_{i \in \text{data}} \left[\frac{\log(s) + \rho^{(s)}(u_i)(a/L)^2 - F_i^{(s)}}{\delta F_i^{(s)}} \right]^2, \quad \rho^{(s)}(u) = \sum_{k=0}^{n_c} \rho_k^{(s)} u^k,$$

- Use $n_b = 4, 5$. Results insensitive to $n_c \geq 2$ (up to $n_c = 10$).
- Good fits ($\chi^2/\mathrm{dof} \sim 0.5 - 0.9$). Compared with many other analysis.

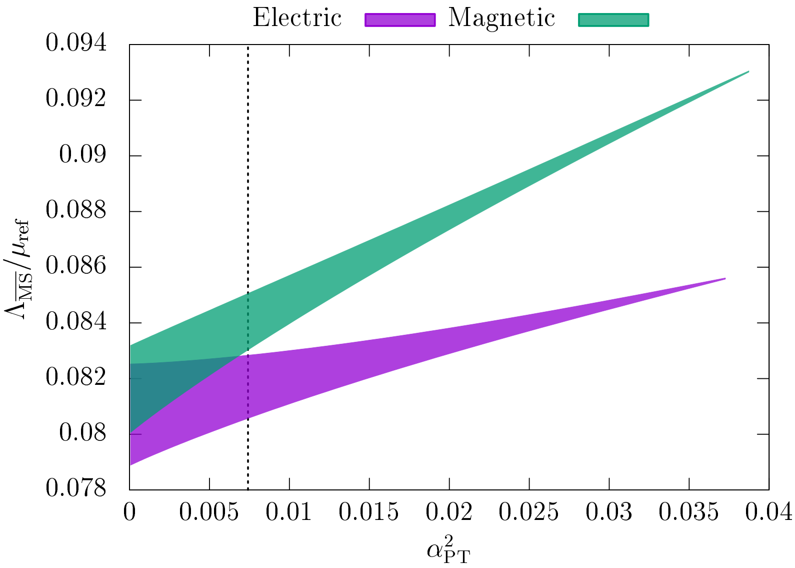
Determination of the β-function at high energies: Electric scheme



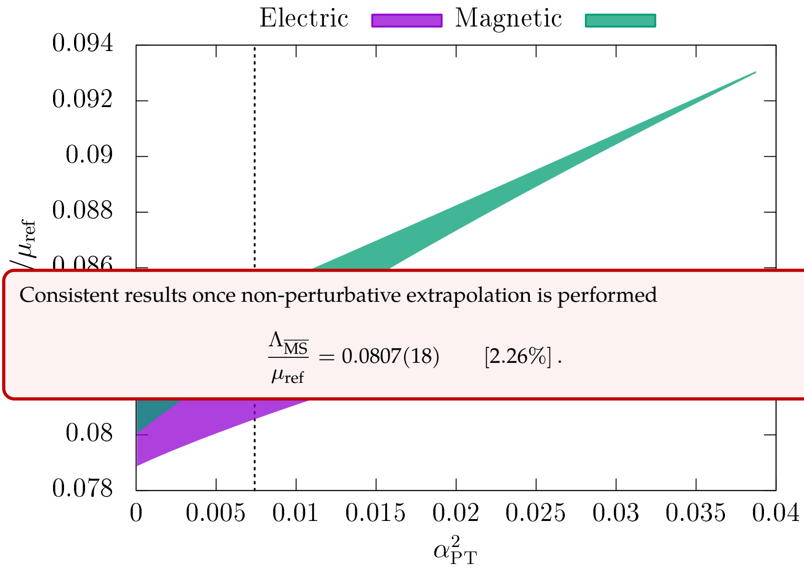
Determination of the β-function at high energies: Magnetic scheme



Approach $\alpha_{\mathrm{PT}} \to 0$: 6% (13%) deviation at $\alpha_{\mathrm{PT}} = 0.2$



Approach $\alpha_{\mathrm{PT}} \to 0$: 6% (13%) deviation at $\alpha_{\mathrm{PT}} = 0.2$



Consistent results once non-perturbative extrapolation is performed

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{ref}}} = 0.0807(18) \qquad [2.26\%]\,.$$

Observations

- 3-loop coefficients **[M. Dalla Brida, M. Lüscher '17]**:

$$(4\pi)^3 b_2^{\overline{\mathrm{MS}}} \approx 0.72\,, \quad (4\pi)^3 b_2^{\mathrm{GF,e}} \approx -2.0\,, \quad (4\pi)^3 b_2^{\mathrm{GF,m}} \approx -3.2\,.$$

- Electric: Contact with PT only at very high energies
- Magnetic: Significant deviation from PT even at $\bar{g}^2_{\mathrm{GF}} = 1$
- 4-loop coefficient probably large, and positive (alternating series?).
- Without finite size scaling, these schemes seem useless to extract Λ with high precision.
- Results are consistent **ONLY if NP extrapolation is performed**: Even at $\alpha_{\mathrm{PT}} = 0.09$ significant deviation.
- Extrapolation has a cost in precision: 2.6% is **not** great.

Matching with SF schemes

- $\bar{g}^2_{\mathrm{SF},\nu}$ have shown much better non-perturbative behavior[a]

$$\frac{1}{\bar{g}^2_{\mathrm{SF},\nu}(\mu)} = \frac{1}{k}\left\langle \frac{\partial S}{\partial \eta}\right\rangle\Big|_{\eta=\nu=0} - \frac{\nu}{k}\left\langle \frac{\partial^2 S}{\partial\eta\partial\nu}\right\rangle\Big|_{\eta=\nu=0},$$

$$(4\pi)^3 b_2^{\mathrm{SF},\nu} = 0.482(7) + \nu\times 0.7523(1)$$

- Non-perturbative matching GF $\to$ SF,ν:
  - At the same values of g_0^2 on $L/a = 12, 16, 20, 24, 32$
  - Measure $\bar{g}^2_{\mathrm{SF},\nu}$ on $L/a = 6, 8, 10, 12, 16$
  - Combine GF $\to$ SF,ν with $\mu \to (2c)\mu$
- Fit data

$$\frac{1}{\bar{g}^2_{\mathrm{SF}}(\mu)} - \frac{1}{\bar{g}^2_{\mathrm{GF}}(\mu/(2c))} = f(u) + \tilde{\rho}(u)\left(\frac{a}{L}\right)^2, \qquad (u = \bar{g}^2_{\mathrm{GF}}(\mu/(2c)))\,.$$

- Function

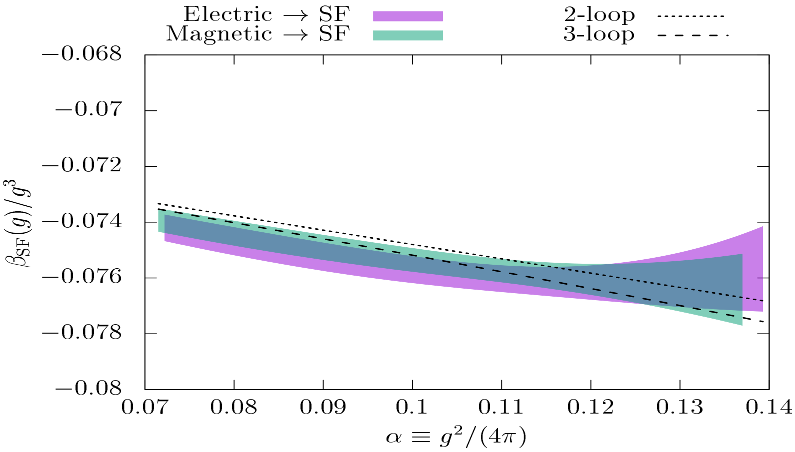
$$f(u) = \sum_{k=0}^{n_f} f_n u^n\,.$$

describe the non-perturbative matching between schemes (No PT imposed!)

[a] But even here there are surprises **[ALPHA '17]**. NP extrapolation has to be checked!.

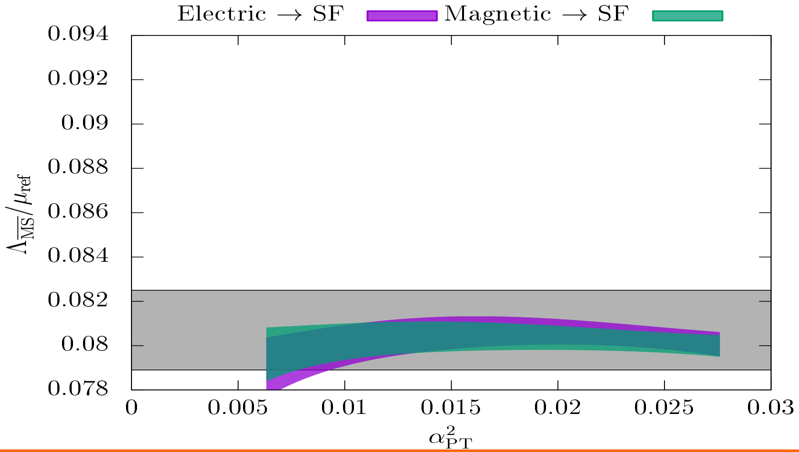
DETERMINATION OF $\beta_{\rm SF}$-FUNCTION: $u = \bar{g}^2_{\rm GF}(\mu/(2c))$

$$\mu \frac{\mathrm{d}\bar{g}_{\rm SF}(\mu)}{\mathrm{d}\mu} = \beta_{\rm SF}(\bar{g}_{\rm SF}(\mu)) = \sqrt{1+uf(u)}\left[-\frac{f'(u)-1/u^2}{(f(u)+1/u)^2}\right]\beta_{\rm GF}(\sqrt{u})$$



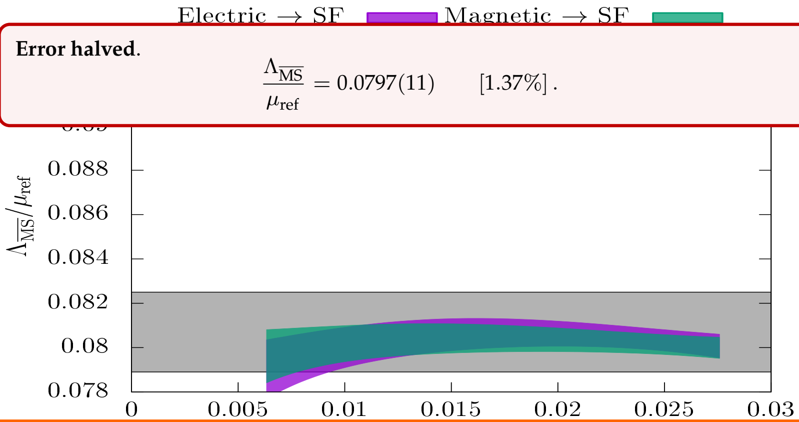
Determination of Λ: Easy non-perturbative extrapolation

$$
\begin{tikzcd}
\mu_{\mathrm{ref}} \arrow[r, "\beta_{\mathrm{GF}}"] & \mu_{\mathrm{PT}} \arrow[rr, "\beta^{3-\mathrm{loop}}_{\mathrm{GF}}"] \arrow[dr, "\mathrm{GF}\to\mathrm{SF}"'] & & \dfrac{\Lambda_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{ref}}} \\
 & & 2c\mu_{\mathrm{PT}} \arrow[ur, "\beta^{3-\mathrm{loop}}_{\mathrm{SF}}"'] &
\end{tikzcd}
$$



Determination of Λ: Easy non-perturbative extrapolation

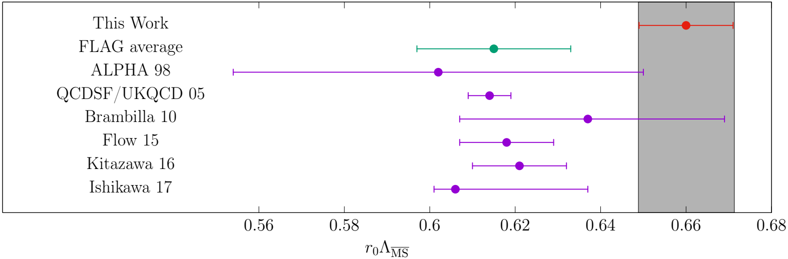
$$
\begin{tikzcd}
\mu_{\mathrm{ref}} \arrow[r, "\beta_{\mathrm{GF}}"] & \mu_{\mathrm{PT}} \arrow[rr, "\beta_{\mathrm{GF}}^{3-\mathrm{loop}}"] \arrow[dr, "\mathrm{GF}\to\mathrm{SF}"'] & & \dfrac{\Lambda_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{ref}}} \\
& & 2c\mu_{\mathrm{PT}} \arrow[ur, "\beta_{\mathrm{SF}}^{3-\mathrm{loop}}"'] &
\end{tikzcd}
$$



Error halved.

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{ref}}} = 0.0797(11) \qquad [1.37\%]\,.$$

But situation in pure gauge not clear.



Significant discrepancy with several works based on observables defined at the cutoff scale.

Overview

Our setup: Choices optimized to be able to simulate heavy quarks

$$\Lambda^{(3)} = \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^3(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

- Work in finite volume schemes with Schrödinger Functional boundary conditions: $T \times L^3$ with Dirichlet bcs. in time. ($\mu \sim 1/L$): "Only" two scales.
- Use Gradient Flow couplings

$$\bar{g}^2(\mu) = \mathcal{N}^{-1}(c, a/L)\, t^2 \langle E(t) \rangle \Big|_{\mu^{-1}=\sqrt{8t}=cL}.$$

- Matching condition ($\{N_{\rm f} = 3, M\} \leftrightarrow \{N_{\rm f} = 0\}$) between massive scheme and effective theory

$$\bar{g}^2(\mu_{\rm dec}(M))\Big|_{N_{\rm f}=3,M,T=2L} = \bar{g}^2(\mu_{\rm dec})\Big|_{N_{\rm f}=0,T=2L}.$$

Caveats: Schrödinger Functional boundary conditions break chiral symmetry

- $\mu_{\rm dec}/M$ corrections to decoupling
- Choose $T = 2L$. Boundary effects very suppressed
- Most probably completly negligible, but keep in mind…

OUR SETUP: CHOICES OPTIMIZED TO BE ABLE TO SIMULATE HEAVY QUARKS

$$\Lambda^{(3)} = \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^3(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

We only need to fill in a table!

M [GeV]	$\mu_{\rm dec}(M)$ [GeV]	$\bar{g}^2(\mu_{\rm dec}(M))\Big\rvert_{N_{\rm f}=3,M,T=2L}$	$\Lambda^{(0)}/\mu_{\rm dec}$	$1/P(\Lambda/M)$	$\Lambda^{(3)}$
-	-	-	-	-	-
-	-	-	-	-	-
-	-	-	-	-	-
-	-	-	-	-	-
...	...	...	...	...	...

Our setup: Most columns of the table already known

3-flavor renormalization program by ALPHA

- $\mu_{\mathrm{dec}}(M)$ [GeV]: Switch to mass-less scheme. Use ALPHA **[ALPHA 1706.03821]**

$$\bar{g}^2(\mu_{\mathrm{dec}}(M))\Big|_{N_\mathrm{f}=3,M,T=2L} \Longrightarrow \bar{g}^2(\mu_{\mathrm{dec}}(M))\Big|_{N_\mathrm{f}=3,M=0,T=L} \Longrightarrow \mu_{\mathrm{dec}}(M) \text{ in [fm]}\,.$$

- M [GeV]: NP-renormalization ALPHA **[ALPHA 1802.05243]**

$$LM = \frac{L}{a}\, Z_\mathrm{m}(\mu_{\mathrm{dec}}(M))(1+b_\mathrm{m} am_q)\, Z_{RGI}(\mu_{\mathrm{dec}}(M))\ (am_0 - am_c)$$

  - Z_m determined non-perturbatively ($\rightarrow$ no details here!)
  - Z_{RGI} Known non-perturbatively **[ALPHA 1802.05243]**
  - 1-loop value for b_m, b_g: Not fully $\mathcal{O}(a)$-improved.
- $\Lambda^{(0)}/\mu_{\mathrm{dec}}$: Known very precisely

OUR SETUP: MOST COLUMNS OF THE TABLE ALREADY KNOWN

3-flavor renormalization program by ALPHA

- $\mu_{\rm dec}(M)$ [GeV]: Switch to mass-less scheme. Use ALPHA **[ALPHA 1706.03821]**

$$\bar{g}^2(\mu_{\rm dec}(M))\Big|_{N_{\rm f}=3,M,T=2L} \Longrightarrow \bar{g}^2(\mu_{\rm dec}(M))\Big|_{N_{\rm f}=3,M=0,T=L} \Longrightarrow \mu_{\rm dec}(M) \text{ in [fm]}\,.$$

- M [GeV]: NP-renormalization ALPHA **[ALPHA 1802.05243]**

$$LM = \frac{L}{a} Z_{\rm m}(\mu_{\rm dec}(M))(1+b_{\rm m}am_q)\, Z_{RGI}(\mu_{\rm dec}(M))\ (am_0 - am_c)$$

Missing piece: massive $\leftrightarrow$ massless: LCP accureately known at $M=0$

L/a	β	$\bar{g}^2(\mu_{\rm dec}(M))\Big\vert_{N_{\rm f}=3,M=0,T=L}$	$\mu_{\rm dec}(M)$ [GeV]
12	4.3020	3.9533(59)	0.789(15)
16	4.4662	3.9496(77)	0.789(15)
20	4.5997	3.9648(97)	0.789(15)
24	4.7141	3.959(50)	0.789(15)
32	4.90	3.949(11)	0.789(15)

Determine massive coupling for matching (PRELIMINARY)

Example: $L/a = 20$

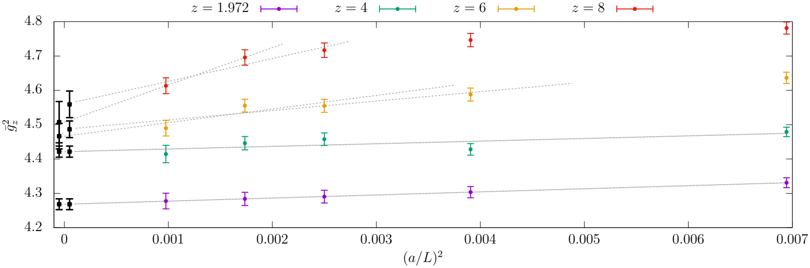
β	κ	$z = M/\mu_{\rm dec}(M)$	M [GeV]	$\bar{g}^2(\mu_{\rm dec}(M))\big\vert_{N_{\rm f}=3,M,T=2L}$
4.5997	0.1352889	0	0	3.9648(97)
4.6083	0.133831710060	1.972(18)	1.6	4.290(15)
4.6172	0.132345249425	4.000(37)	3.2	4.458(14)
4.6266	0.130827894135	6.000(58)	4.7	4.555(14)
4.6364	0.129273827559	8.000(85)	6.3	4.717(14)

Extrapolate results to the continuum: Example $z = 6$

We determine $\bar{g}^2(\mu_{\rm dec}(M))\big\vert_{N_{\rm f}=3,M,T=2L}$ change cuts $aM < 0.35, 0.5$ and flow

L/a	β	aM	ZFL	WFL
12	4.3499	0.50	4.636(17)	5.477(23)
16	4.5008	0.37	4.588(19)	5.023(22)
20	4.6266	0.30	4.555(19)	4.823(21)
24	4.7359	0.25	4.555(19)	4.738(20)
32	4.9159	0.18	4.490(23)	4.590(24)
∞	$aM < 0.50$		4.487(24)	4.470(26)
∞	$aM < 0.35$		4.466(37)	4.458(39)

Continuum extrapolations with two cuts: $aM < 0.50, 0.35$ (PRELIMINARY)



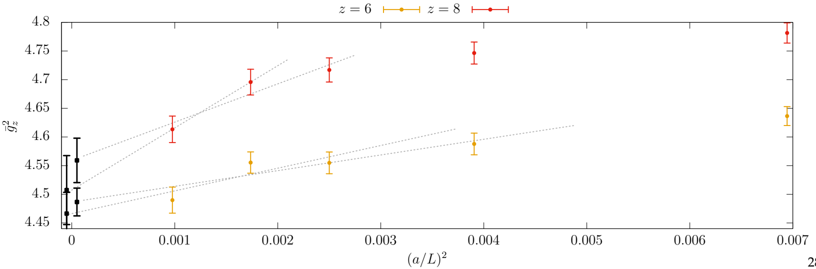
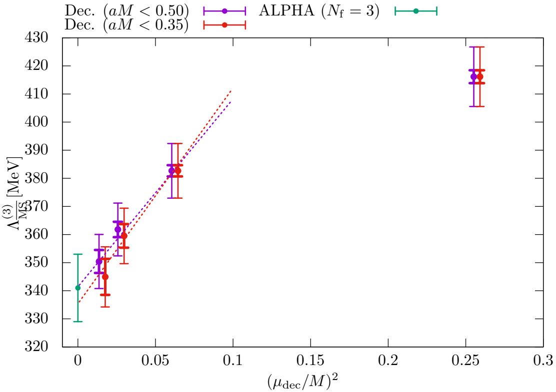


TABLE CAN BE FILLED (PRELIMINARY)

M [GeV]	$\mu_{\rm dec}(M)$ [GeV]	$\bar{g}^2(\mu_{\rm low}(M))\Big\vert_{N_{\rm f}=3,M,T=2L}$	$\Lambda^{(0)}/\mu_{\rm low}$	$\frac{1}{P(\Lambda/M)}$	$\Lambda^{(3)}$ [MeV]
1.6	0.789(15)	4.559(39)	0.689(11)	0.7662(44)	416(11)
3.2	0.789(15)	4.421(16)	0.725(11)	0.6693(37)	382.7(96)
4.7	0.789(15)	4.466(37)	0.741(12)	0.6198(34)	362.0(92)
6.3	0.789(15)	4.507(60)	0.757(13)	0.5871(32)	350.3(92)



Overview

CONCLUSIONS: NON-PERTURBATIVE RENORMALIZATION BY DECOUPLING

- Lattice QCD can determine scales at un-physical values of the parameters

$$\Lambda^{(N_\mathrm{f})} = \lim_{M\to\infty} \mu_\mathrm{dec}(M) \times \frac{\Lambda^{(0)}}{\mu_\mathrm{dec}} \times \frac{1}{P(\Lambda/M)}$$

with

  - $\mu_\mathrm{dec}(M)$: Scale with N_f heavy quarks ($M \gg \Lambda$)
  - $\Lambda^{(0)}/\mu_\mathrm{dec}$: Computed non-perturbatively in pure gauge
  - $P(\Lambda/M)$: Perturbative relation between fundamental and effective theories
- Method is generic
  - Similar expressions for other RGI invariants: $M, \hat{B}_K, \hat{B}_B, \ldots$
  - Valid in finite or infinite volume renormalization schemes
  - If you can, just compute $\sqrt{8t_0}, w_0$ with 3-4 quarks as heavy as possible!
- Still working (larger L/a) but It works!!
  - Finite volume setup: Small PT corrections $\mathcal{O}(\alpha^3(m^\star))$, window problem ameliorated
  - $\mu_\mathrm{dec}(M) = 789(15)$ MeV. Applied with $M = 1.6, \ldots, 6.3$ GeV
  - Non-perturbative running in pure gauge from $\mu = 789$ MeV to $\mu = \infty$
  - $\Lambda^{(3)}$ in agreement with current knowledge
- Probably best approach to reduce error in α_s substantially
  - Running done in pure gauge!
  - Better precision in pure gauge: Small lattice spacing, efficient algorithms.
  - Switch massive $\leftrightarrow$ massless schemes little effect in total error.
  - $\lim_{M\to\infty}$ can be controlled

Conclusions: Precision physics with lattice QCD

- Electromagnetic corrections
- Charm effects (i.e. $N_{\mathrm{f}} = 2 + 1 + 1$ vs. $N_{\mathrm{f}} = 2 + 1$)
- Statistical precision in difficult observables.
- Connecting hadronic physics with EW scale without assumptions on low scale physics (i.e. perturbation theory).
- "Non-perturbative extrapolation" is difficult.
    - Correction decreases very slowly $\alpha^n(\mu_{\mathrm{PT}})$
    - Even in pure gauge, with much more computer power, the situation is far from clear
    - Is this under control for M, B_K, . . . ?