

# QED corrections to leptonic meson decays

Vera Gülpers

School of Physics and Astronomy  
University of Edinburgh

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# RBC/UKQCD Collaboration

## BNL and BNL/RBRC

Yasumichi Aoki (KEK)  
Taku Izubuchi  
Yong-Chull Jang  
Chulwoo Jung  
Meifeng Lin  
Aaron Meyer  
Hiroshi Ohki  
Shigemi Ohta (KEK)  
Amarjit Soni

## UC Boulder

Oliver Witzel

## CERN

Mattia Bruno

## Columbia University

Ryan Abbot  
Norman Christ  
Duo Guo  
Christopher Kelly  
Bob Mawhinney  
Masaaki Tomii  
Jiqun Tu

Bigeng Wang  
Tianle Wang  
Yidi Zhao

## University of Connecticut

Tom Blum  
Dan Hoying (BNL)  
Luchang Jin (RBRC)  
Cheng Tu

## Edinburgh University

Peter Boyle  
Luigi Del Debbio  
Felix Erben  
Vera Gülpers  
Tadeusz Janowski  
Julia Kettle  
Michael Marshall  
Fionn Ó hÓgáin  
Antonin Portelli  
Tobias Tsang  
Andrew Yong  
Azusa Yamaguchi

## University of Liverpool

Nicolas Garron

## MIT

David Murphy

## Peking University

Xu Feng

## University of Regensburg

Christoph Lehner (BNL)

## University of Southampton

Nils Asmussen  
Jonathan Flynn  
Ryan Hill  
Andreas Jüttner  
James Richings  
Chris Sachrajda

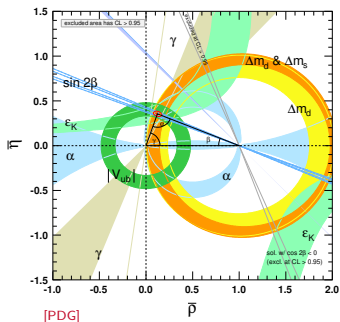
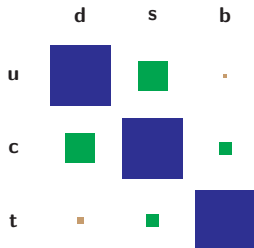
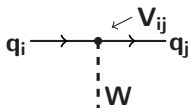
## Stony Brook University

Jun-Sik Yoo  
Sergey Syritsyn (RBRC)

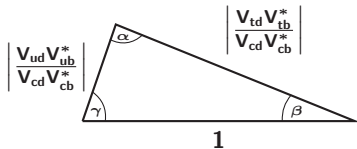
## Introduction

- ▶ quark-mixing Cabibbo–Kobayashi–Maskawa (CKM) matrix

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

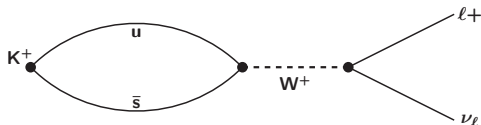


- ▶ Unitarity of the CKM matrix

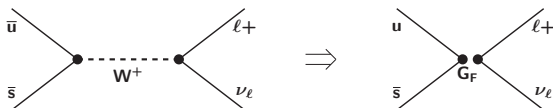


## $V_{us}$ from leptonic Kaon decays

- ▶ leptonic Kaon decay  $K^+ \rightarrow \ell^+ \nu_\ell$



- ▶ effective weak Hamiltonian



- ▶ decay rate (can be measured experimentally)

$$\Gamma(K^+ \rightarrow \ell^+ \nu_\ell) = \frac{G_F^2 |V_{us}|^2 f_K^2}{8\pi} M_K m_\ell^2 \left(1 - \frac{m_\ell^2}{M_K^2}\right)^2$$

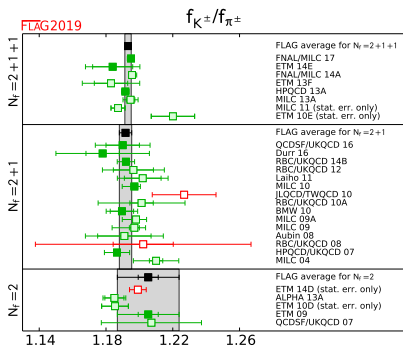
- ▶ known factors (Fermi constant  $G_F$ , masses  $m$ )
- ▶ kaon decay constant  $f_K$ , can be calculated on the lattice
- ▶ CKM matrix element  $V_{us}$

# $f_K/f_\pi$ from the lattice

- ▶ pseudoscalar meson decay constant from the lattice
- ▶ axial-vector matrix element

$$\mathcal{A}_K = \langle 0 | \bar{u} \gamma_0 \gamma_5 s | K \rangle = M_K f_K$$

- ▶ overview Kaon/Pion decay constants



- ▶ results with precision  $< 1\%$

## Isospin Breaking Corrections

- ▶ lattice calculations usually done in the isospin symmetric limit
- ▶ two sources of isospin breaking effects
  - ▶ different masses for up- and down quark (of  $\mathcal{O}((m_d - m_u)/\Lambda_{\text{QCD}})$ )
  - ▶ Quarks have electrical charge (of  $\mathcal{O}(\alpha)$ )
- ▶ lattice calculation aiming at **1%** precision requires to include isospin breaking
- ▶ separation of strong IB and QED effects requires renormalization scheme
- ▶ definition of “physical point” in a “QCD only world” also scheme dependent

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- ▶ Euclidean path integral including QED

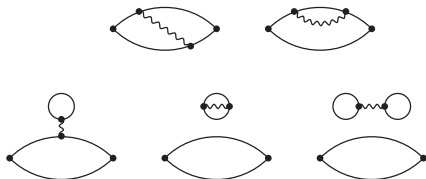
$$\langle \mathbf{O} \rangle = \frac{1}{Z} \int \mathcal{D}[\Psi, \bar{\Psi}] \mathcal{D}[U] \mathcal{D}[A] \mathbf{O} e^{-S_F[\Psi, \bar{\Psi}, U, A]} e^{-S_G[U]} e^{-S_\gamma[A]}$$

- ▶ photons in a box: finite volume corrections

## Expansion around IB symmetric (eg IB corrections to meson masses)

- perturbative expansion in  $\alpha$  [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]

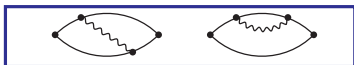
$$\langle \mathbf{O} \rangle = \langle \mathbf{O} \rangle_{e=0} + \frac{1}{2} e^2 \left. \frac{\partial^2}{\partial e^2} \langle \mathbf{O} \rangle \right|_{e=0} + \mathcal{O}(\alpha^2)$$



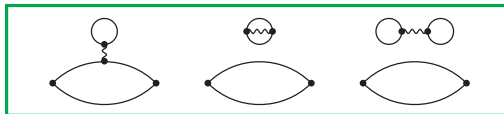
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electro-quenched approximation



sea-quark effects

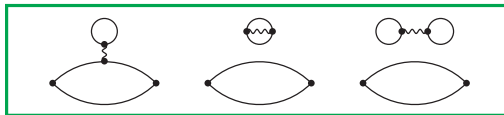
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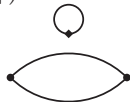
sea-quark effects

- perturbative expansion in  $\Delta m_f = (m_f^0 - m_f)$  [G.M. de Divitiis et al, JHEP 1204 (2012) 124]

$$\langle \mathbf{O} \rangle_{m_f} = \langle \mathbf{O} \rangle_{m_f^0} + \Delta m_f \left. \frac{\partial}{\partial m_f} \langle \mathbf{O} \rangle \right|_{m_f^0} + \mathcal{O}(\Delta m_f^2)$$



sea quark effects:  
quark-disconnected diagrams



## Tuning the quark masses

- ▶ isospin symmetric calculation using quark masses determined without QED
  - ▶ physical quark masses including QED:
- tune (**u,d,s**) masses to reproduce experimental  $\pi^+$ ,  $K^+$  and  $K_0$  mass (and check  $\pi^0$  mass)

$$am_{\pi^+}^{\text{exp}} = \left[ m_{\pi}^0 + \alpha m_{\pi^+}^{\text{QED}} + \Delta m_d m_{\pi^+}^{\Delta m_d} + \Delta m_u m_{\pi^+}^{\Delta m_u} \right]$$

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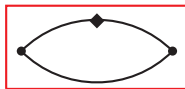


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- ▶ lattice spacing **a**: fix another mass including QED
- e.g. Omega-Baryon (**sss**)

$$a \rightarrow a(\Delta m_s) = \left( m_{\Omega}^0 + \alpha m_{\Omega}^{\text{QED}} + 3 \Delta m_s m_{\Omega}^{\Delta m_s} \right) / m_{\Omega}^{\text{exp}}$$

→ shift in **a** smaller than statistical error on lattice spacing

[T. Blum, VG, et al., Phys. Rev. Lett. 121, 022003 (2018)]

## Decay rate leptonic meson decays

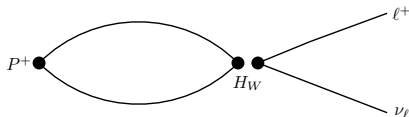
- ▶  $P^+$  decay rate in rest frame ( $\mathbf{P} = \{\pi, \mathbf{K}\}$ )

$$\Gamma(P^+ \rightarrow \ell^+ \nu_\ell) = \mathbf{K} \sum_{\mathbf{r}, \mathbf{s}} |\mathcal{M}^{\mathbf{r}, \mathbf{s}}|^2$$

summed over spins  $\mathbf{r}, \mathbf{s}$  of final state

- ▶ matrix element

$$\mathcal{M}^{\mathbf{r}, \mathbf{s}} = \langle \ell^+, \mathbf{r}; \nu_\ell, \mathbf{s} | \mathbf{H}_W | P^+ \rangle = \bar{u}_{\nu_\ell}^{\mathbf{r}} \tilde{\mathcal{M}} v_\ell^{\mathbf{s}}$$



- ▶ weak Hamiltonian  $\mathbf{H}_W$
- ▶ tree-level matrix element (hadronic and leptonic part factorisable)

$$\mathcal{M}_0^{\mathbf{r}, \mathbf{s}} = f_P \mathbf{M}_P (\bar{u}_{\nu_\ell}^{\mathbf{r}} \gamma_L^\mu v_\ell^{\mathbf{s}})$$

$$\gamma_L^\mu = \gamma_\mu (1 - \gamma_5)$$

## Decay rate leptonic meson decays

- ▶ tree-level decay rate

$$\Gamma^0(\mathbf{P}^+ \rightarrow \ell^+ \nu_\ell) = \frac{G_F^2 |V_{us}|^2 f_P^2}{8\pi} M_P m_\ell^2 \left(1 - \frac{m_\ell^2}{M_P^2}\right)^2$$

- ▶ full QCD+QED decay rate

$$\Gamma = \Gamma^0 + \delta\Gamma = \Gamma^0(1 + \delta R) \quad \delta R = \delta\Gamma/\Gamma_0$$

- ▶ first order  $\mathcal{O}(\alpha, \mathbf{m}_d - \mathbf{m}_u)$  in isospin breaking

$$\delta\Gamma = \delta K \sum_{r,s} |\mathcal{M}_0^{r,s}|^2 + 2K_0 \sum_{r,s} \Re(\mathcal{M}_0^{r,s} \delta\mathcal{M}^{r,s,*})$$

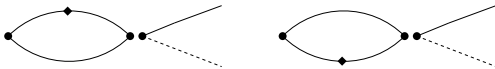
with

$$\mathcal{M}_0^{r,s} = f_P M_P (\bar{u}_{\nu_\ell}^r \gamma_L^\mu v_\ell^s) \quad \delta\mathcal{M}^{r,s} = \bar{u}_{\nu_\ell}^r \delta\widetilde{\mathcal{M}} v_\ell^s$$

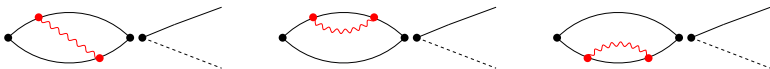
$$\Rightarrow \sum_{r,s} \Re(\mathcal{M}_0^{r,s} \delta\mathcal{M}^{r,s,*}) = f_P M_P \text{Tr}[\not{p}_\nu \delta\widetilde{\mathcal{M}} (-\not{p}_\ell + im_\ell) \gamma_L^\mu]$$

## perturbative expansion - leptonic meson decay

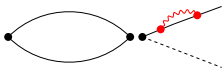
- ▶ strong IB corrections  $\mathcal{O}(m_d - m_u)$



- ▶ quark QED corrections  $\mathcal{O}(e_q^2)$



- ▶ lepton QED corrections  $\mathcal{O}(e_\ell^2)$

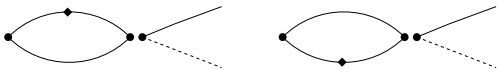


- ▶ quark-lepton QED correction  $\mathcal{O}(e_\ell e_q)$

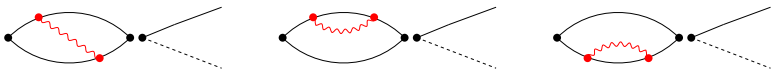


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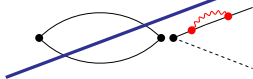
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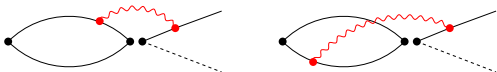


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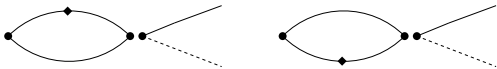
→ absorbed in renormalisation of lepton

- ▶ quark-lepton QED correction  $\mathcal{O}(e_\ell e_q)$

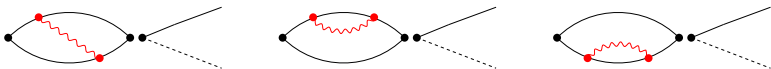


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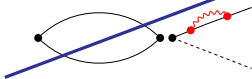
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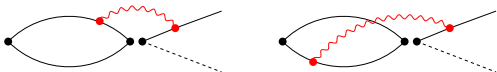


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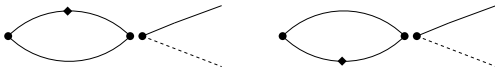
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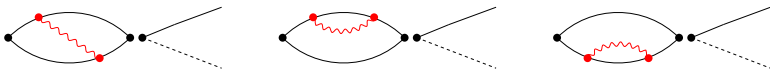
factorisable

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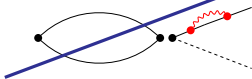
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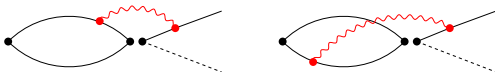


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- ▶ quark-lepton QED correction  $\mathcal{O}(e_\ell e_q)$



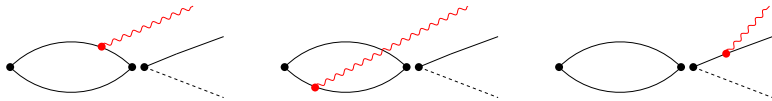
factorisable

non-factorisable

## IB corrections to leptonic meson decay

- ▶ Infrared divergencies canceled by diagrams with one final state photon

$$\Gamma(K^+ \rightarrow \ell^+ \nu_\ell, \alpha) + \Gamma(K^+ \rightarrow \ell^+ \nu_\ell \gamma)$$



- ▶ pioneering work to calculate IB correction to decay rate by RM123
  - formalism developed in [N. Carrasco *et al*, Phys.Rev. **D91**, 074506 (2015)]
  - finite volume effects [V. Lubicz *et al*, Phys. Rev. **D95**, 034504 (2017)]
  - first lattice results [M. Di Carlo *et al*, arXiv:1904.08731], [D. Giusti *et al*, Phys. Rev. Lett. **120**, 072001 (2018)]
- ▶ this work: calculation directly at the physical point

## Lattice Setup

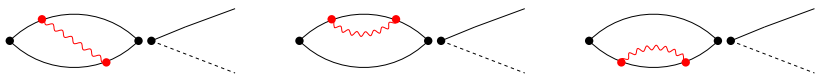
- ▶  $N_f = 2 + 1$  Möbius Domain Wall Fermions
- ▶ near physical quark masses
- ▶ inverse lattice spacing  $a^{-1} = 1.730(4)$  GeV
- ▶  $48^3 \times 96$  with  $L_s = 24$
- ▶ valence light quarks: physical mass z-Möbius DWF with  $L_s = 10$
- ▶ Feynman gauge and QED<sub>L</sub> for photon propagators

$$\Delta_{\mu\nu}(\mathbf{x} - \mathbf{y}) = \langle \mathbf{A}_\mu(\mathbf{x}) \mathbf{A}_\nu(\mathbf{y}) \rangle = \delta_{\mu\nu} \frac{1}{N} \sum_{\mathbf{k}, \vec{k} \neq 0} \frac{e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}}{\hat{k}^2}$$

- ▶ use stochastic photon fields  $\mathbf{A}_\mu(\mathbf{x})$  to estimate  $\Delta_{\mu\nu}(\mathbf{x} - \mathbf{y})$   
[D. Giusti et al. Phys.Rev. D95 (2017) 114504]
- ▶ em vertices using local vector currents  $\gamma_\mu \mathbf{A}_\mu = \cancel{A}$
- ▶ all results shown in this talk are **very preliminary**

## factorisable QED diagrams

- ▶ factorisable diagrams QED correction



→ hadronic and leptonic part can be factorised (as in tree-level)

- ▶ only need to calculate



- ▶ IB correction from factorisable diagrams

$$\delta^{\text{qq}} \mathcal{M}^{rs} = (\bar{u}_{\nu\ell}^r \gamma_L^\mu v_\ell^s) \delta \mathcal{A}$$

with

$$\delta \mathcal{A} = \delta \langle 0 | \bar{q}_1 \gamma_0 \gamma_5 q_2 | P^+ \rangle$$

## all-to-all propagators and meson fields

- ▶ all-to-all propagator [J. Foley *et al*, *Comput.Phys.Commun.* **172**, 145-162 (2005)]

$$D^{-1}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{N_l+N_h} \mathbf{v}_i(\mathbf{x}) \mathbf{w}_i^\dagger(\mathbf{y}) = \sum_{i=1}^{N_l} \mathbf{v}_i(\mathbf{x}) \mathbf{w}_i^\dagger(\mathbf{y}) + \sum_{i=N_l+1}^{N_l+N_h} \mathbf{v}_i(\mathbf{x}) \mathbf{w}_i^\dagger(\mathbf{y})$$

- ▶ all-to-all vectors  $\mathbf{v}_i(\mathbf{x})$ ,  $\mathbf{w}_i(\mathbf{x})$

low-modes (eigenvectors  $\phi(\mathbf{x})$ )

$$\mathbf{v}_i(\mathbf{x}) = \phi_i(\mathbf{x})$$

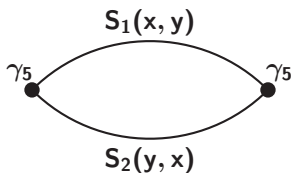
$$\mathbf{w}_i(\mathbf{x}) = \phi_i(\mathbf{x}) / \lambda_i$$

high-modes (from stochastic solves)

$$\mathbf{v}_i(\mathbf{x}) = D_{\text{defl}}^{-1}(\mathbf{x}, \mathbf{y}) \eta_i(\mathbf{y})$$

$$\mathbf{w}_i(\mathbf{x}) = \eta_i(\mathbf{x})$$

- ▶ e.g. pseudoscalar two-point function



$$C(\mathbf{y}_0 - \mathbf{x}_0) = \sum_{\vec{x}, \vec{y}} \text{Tr} [\gamma_5 S_1(\mathbf{x}, \mathbf{y}) \gamma_5 S_2(\mathbf{y}, \mathbf{x})]$$

## all-to-all propagators and meson fields

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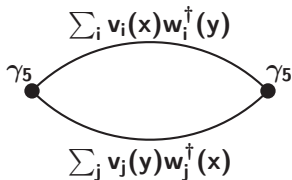
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$$C(\mathbf{y}_0 - \mathbf{x}_0) = \sum_{\vec{x}, \vec{y}} \text{Tr} \left[ \gamma_5 \sum_i \mathbf{v}_i(\mathbf{x}) \mathbf{w}_i^\dagger(\mathbf{y}) \gamma_5 \sum_j \mathbf{v}_j(\mathbf{y}) \mathbf{w}_j^\dagger(\mathbf{x}) \right]$$

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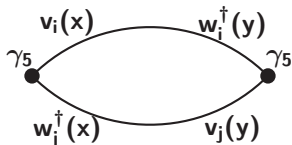
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- ▶ e.g. pseudoscalar two-point function



$$C(\mathbf{y}_0 - \mathbf{x}_0) = \sum_{i,j} \text{Tr} \left[ \sum_{\vec{y}} \mathbf{w}_i^\dagger(\mathbf{y}) \gamma_5 \mathbf{v}_j(\mathbf{y}) \sum_{\vec{x}} \mathbf{w}_j^\dagger(\mathbf{x}) \gamma_5 \mathbf{v}_i(\mathbf{x}) \right]$$

## all-to-all propagators and meson fields

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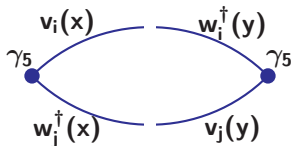
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- ▶ e.g. pseudoscalar two-point function



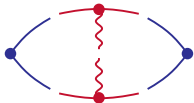
Meson field

$$\Pi_{ji}(\mathbf{x}_0; \Gamma) = \sum_{\vec{x}} \mathbf{w}_j^\dagger(\mathbf{x}) \Gamma \mathbf{v}_i(\mathbf{x})$$

$$C(\mathbf{y}_0 - \mathbf{x}_0) = \sum_{i,j} \text{Tr} [\Pi_{ij}(\mathbf{y}_0; \gamma_5) \Pi_{ji}(\mathbf{x}_0; \gamma_5)]$$

## QED meson fields

- ▶  $\cancel{A}$ -meson fields  $\Pi_{ji}(\mathbf{x}_0; \cancel{A}) = \sum_{\vec{x}} \mathbf{w}_j^\dagger(\mathbf{x}) i\gamma_\mu \mathbf{A}^\mu(\mathbf{x}) \mathbf{v}_i(\mathbf{x})$



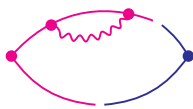
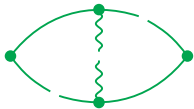
- ▶ sequential propagators on  $\mathbf{v}$ -vectors & sequential  $\cancel{A}$ -meson fields

$$\tilde{\mathbf{v}}_i(\mathbf{x}) = \sum_y \mathbf{S}(\mathbf{x}, y) i\gamma_\mu \mathbf{A}^\mu(y) \mathbf{v}_i(y)$$

$$\Pi_{ji}(\mathbf{x}_0; \Gamma \mathbf{S} \cancel{A}) = \sum_{\vec{x}} \mathbf{w}_j^\dagger(\mathbf{x}) \Gamma \tilde{\mathbf{v}}_i(\mathbf{x})$$

$$\tilde{\tilde{\mathbf{v}}}_i(\mathbf{x}) = \sum_y \mathbf{S}(\mathbf{x}, y) i\gamma_\mu \mathbf{A}^\mu(y) \tilde{\mathbf{v}}_i(y)$$

$$\Pi_{ji}(\mathbf{x}_0; \Gamma \mathbf{S} \cancel{A} \mathbf{S} \cancel{A}) = \sum_{\vec{x}} \mathbf{w}_j^\dagger(\mathbf{x}) \Gamma \tilde{\tilde{\mathbf{v}}}_i(\mathbf{x})$$

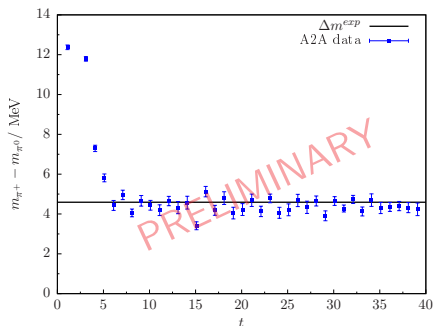


- ▶ sequential  $\cancel{A}$ -meson fields give smaller statistical errors
- ▶  $\cancel{A}$ -meson fields can be used for disconnected diagrams (sea-quark effects)

[see talk by J. Richings, Lattice 2019]

## QED corrections to meson masses

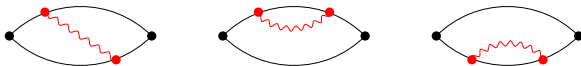
- ▶ **2000** low-modes for light quark, **96** × **12** time-diluted, spin-color diagonal stochastic sources for high-modes with sequential **A**-meson fields
- ▶ difference of charged and neutral pion mass



- ▶ Finite volume corrections  $\text{QED}_L$  [BMW Collaboration, Science **347** (2015) 1452–1455]

$$m^2(\mathbf{L}) \sim m^2 \left\{ 1 - q^2 \alpha \left[ \frac{\kappa}{m\mathbf{L}} \left( 1 + \frac{2}{m\mathbf{L}} \right) \right] \right\} \quad \text{with } \kappa = 2.837297$$

## IB correction from factorisable diagrams



- ▶ correlators w/o QED (example: Kaon)

$$C_{PP}^0(t) = \langle 0 | (\bar{s} \gamma_5 u) (\bar{u} \gamma_5 s) | 0 \rangle = A_0 e^{-m_0 t}$$

$$A_0 = \frac{\phi_0^2}{2m_0}$$

$$C_{AP}^0(t) = \langle 0 | (\bar{s} \gamma_0 \gamma_5 u) (\bar{u} \gamma_5 s) | 0 \rangle = B_0 e^{-m_0 t}$$

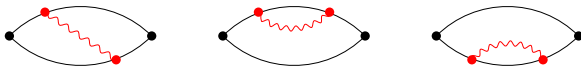
$$B_0 = \frac{\phi_0 \mathcal{A}_0}{2m_0}$$

- ▶  $\mathcal{O}(\alpha)$  QED corrections

$$\frac{\delta C_{PP}(t)}{C_{PP}^0(t)} = \frac{\delta A}{A_0} - \delta m t = 2 \frac{\delta \phi}{\phi_0} - \frac{\delta m}{m_0} - \delta m t$$

$$\frac{\delta C_{AP}(t)}{C_{AP}^0(t)} = \frac{\delta B}{B_0} - \delta m t = \frac{\delta \phi}{\phi_0} + \frac{\delta \mathcal{A}}{\mathcal{A}_0} - \frac{\delta m}{m_0} - \delta m t$$

## IB correction from factorisable diagrams



- ▶ correlators w/o QED (example: Kaon)

$$C_{PP}^0(t) = \langle 0 | (\bar{s} \gamma_5 u) (\bar{u} \gamma_5 s) | 0 \rangle = A_0 e^{-m_0 t}$$

$$A_0 = \frac{\phi_0^2}{2m_0}$$

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$$B_0 = \frac{\phi_0 \mathcal{A}_0}{2m_0}$$

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$$\frac{\delta C_{PP}(t)}{C_{PP}^0(t)} = \frac{\delta A}{A_0} - \delta m t = 2 \frac{\delta \phi}{\phi_0} - \frac{\delta m}{m_0} - \delta m t$$

$$\frac{\delta C_{AP}(t)}{C_{AP}^0(t)} = \frac{\delta B}{B_0} - \delta m t = \frac{\delta \phi}{\phi_0} + \frac{\delta \mathcal{A}}{\mathcal{A}_0} - \frac{\delta m}{m_0} - \delta m t$$

## IB correction from factorisable diagrams

- ▶ correlators w/o QED (including backward propagating signal)

$$C_{PP}^0(t) = 2 A_0 e^{-m_0 \frac{T}{2}} \cosh \left[ m_0 \left( \frac{T}{2} - t \right) \right]$$

$$C_{AP}^0(t) = 2 B_0 e^{-m_0 \frac{T}{2}} \sinh \left[ m_0 \left( \frac{T}{2} - t \right) \right]$$

- ▶  $\mathcal{O}(\alpha)$  QED corrections

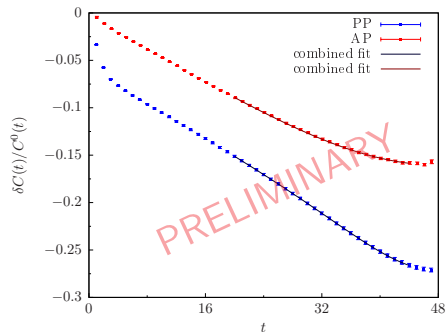
$$\frac{\delta C_{PP}(t)}{C_{PP}^0(t)} = \frac{\delta A}{A_0} - \delta m \frac{T}{2} + \delta m \left( \frac{T}{2} - t \right) \tanh \left[ m_0 \left( \frac{T}{2} - t \right) \right]$$

$$\frac{\delta C_{AP}(t)}{C_{AP}^0(t)} = \frac{\delta B}{B_0} - \delta m \frac{T}{2} + \delta m \left( \frac{T}{2} - t \right) \coth \left[ m_0 \left( \frac{T}{2} - t \right) \right]$$

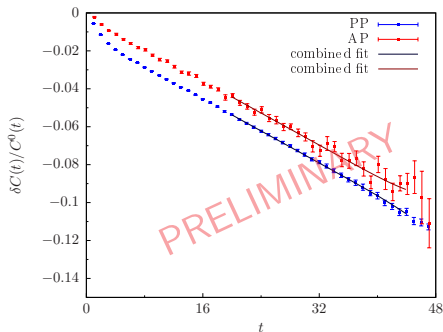
## results QED factorisable diagrams

- ▶ 20 configurations on the physical point ensemble
- ▶ combined fit to  $\delta C_{PP}(t)/C_{PP}^0(t)$  and  $\delta C_{AP}(t)/C_{AP}^0(t)$ 
  - three parameters  $\delta m$ ,  $\delta A/A_0$ ,  $\delta B/B_0$

pion



kaon



## non-factorisable diagrams

- ▶ non-factorisable diagrams



- ▶ include lepton in lattice calculation
- ▶ neutrino can be done analytically
- ▶ amputated weak Hamiltonian and matrix element

$$\bar{H}_W^\alpha = (\gamma_\mu^L \ell)^\alpha (\bar{q}_1 \gamma_\mu^L q_2) \quad \bar{\mathcal{M}}^{r,\alpha} = \langle \ell^+, r | \bar{H}_W^\alpha | P^+ \rangle = (\tilde{\mathcal{M}} \mathbf{v}_\ell^r)^\alpha$$

- ▶ Euclidean three-point function (in full QCD+QED)

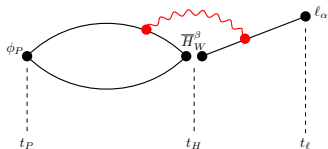
$$C^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_H, \mathbf{t}_P) = \langle \bar{\ell}^\alpha(\mathbf{t}_\ell) \bar{H}_W^\beta(\mathbf{t}_H) \phi_P^\dagger(\mathbf{t}_P) \rangle$$

- ▶ at  $\mathcal{O}(\alpha)$

$$C^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_H, \mathbf{t}_P) = C_0^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_H, \mathbf{t}_P) + \delta C^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_H, \mathbf{t}_P) + \mathcal{O}(\alpha^2)$$

# lattice setup non-factorisable diagrams

- ▶ lattice calculation



- ▶ correlation function

$$\delta^{\ell q} \mathbf{C}^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_H, \mathbf{t}_P) = \sum_{\mathbf{y}, \mathbf{z}} \sum_{\vec{\mathbf{x}}_P, \vec{\mathbf{x}}_H} \text{Tr} \left[ \gamma_5 \mathbf{S}^u(\mathbf{x}_P, \mathbf{y}) \mathbf{A}(\mathbf{y}) \mathbf{S}^u(\mathbf{y}, \mathbf{x}_H) \gamma_\mu^L \mathbf{S}^s(\mathbf{x}_H, \mathbf{x}_P) \right] \\ \times \left( \gamma_\mu^L \mathbf{S}^\ell(\mathbf{x}_H, \mathbf{z}) \mathbf{A}(\mathbf{z}) \mathbf{S}^\ell(\mathbf{z}, \mathbf{x}_\ell) \right)_{\alpha\beta}$$

- ▶ recap: QED correction to decay rate

$$\delta\Gamma \sim \sum_{\mathbf{r}, \mathbf{s}} \Re(\mathcal{M}_0^{\mathbf{r}, \mathbf{s}} \delta\mathcal{M}^{\mathbf{r}, \mathbf{s}, *}) \\ \sum_{\mathbf{r}, \mathbf{s}} \Re(\mathcal{M}_0^{\mathbf{r}, \mathbf{s}} \delta\mathcal{M}^{\mathbf{r}, \mathbf{s}, *}) = \mathbf{f}_P \mathbf{M}_P \text{Tr}[\not{\mathbf{p}}_\nu \delta\widetilde{\mathcal{M}}(-\not{\mathbf{p}}_\ell + i\mathbf{m}_\ell) \gamma_L^\mu]$$

## extract correction to decay rate

- ▶ spectral representation

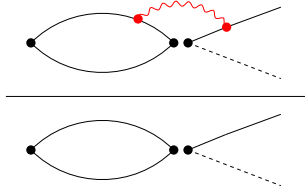
$$\delta^{\ell q} \mathbf{C}^{\alpha\beta}(\mathbf{t}_\ell, \mathbf{t}_H, \mathbf{t}_P) = \frac{\phi_0 \left[ \delta^{\ell q} \widetilde{\mathcal{M}}(-\not{\mathbf{p}}_\ell + i\mathbf{m}_\ell) \right]_{\alpha\beta}}{4E_\ell M_P} e^{-(t_H - t_P)M_P} e^{-(t_\ell - t_H)E_\ell}$$

- ▶ include the lepton trace

$$\text{Tr} \left[ \not{\mathbf{p}}_\nu \delta^{\ell q} \mathbf{C}(\mathbf{t}_H) \gamma_L^\mu \right] \propto \text{Tr} \left[ \not{\mathbf{p}}_\nu \delta^{\ell q} \widetilde{\mathcal{M}}(-\not{\mathbf{p}}_\ell + i\mathbf{m}_\ell) \gamma_L^\mu \right]$$

- ▶  $\delta^{\ell q} \Gamma / \Gamma_0$  can be obtained from long distance behaviour of

$$\mathbf{R}(\mathbf{t}_\ell, \mathbf{t}_H, \mathbf{t}_P) = \frac{\text{Tr} \left[ \not{\mathbf{p}}_\nu \delta^{\ell q} \mathbf{C}(\mathbf{t}_\ell, \mathbf{t}_H, \mathbf{t}_P) \gamma_L^\mu \right]}{\text{Tr} \left[ \not{\mathbf{p}}_\nu \mathbf{C}^0(\mathbf{t}_\ell, \mathbf{t}_H, \mathbf{t}_P) \gamma_L^\mu \right]}$$



## extract correction to decay rate

- ▶ spectral representation

$$\delta^{\ell q} \mathbf{C}^{\alpha\beta}(t_\ell, t_H, t_P) = \frac{\phi_0 \left[ \delta^{\ell q} \widetilde{\mathcal{M}}(-\not{p}_\ell + im_\ell) \right]_{\alpha\beta}}{4E_\ell M_P} e^{-(t_H - t_P)M_P} e^{-(t_\ell - t_H)E_\ell}$$

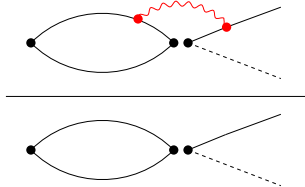
part of correction to decay rate

- ▶ include the lepton trace

$$\text{Tr} \left[ \not{p}_\nu \delta^{\ell q} \mathbf{C}(t_H) \gamma_L^\mu \right] \propto \text{Tr} \left[ \not{p}_\nu \delta^{\ell q} \widetilde{\mathcal{M}}(-\not{p}_\ell + im_\ell) \gamma_L^\mu \right]$$

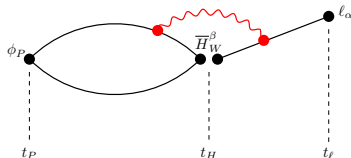
- ▶  $\delta^{\ell q} \Gamma / \Gamma_0$  can be obtained from long distance behaviour of

$$\mathbf{R}(t_\ell, t_H, t_P) = \frac{\text{Tr} \left[ \not{p}_\nu \delta^{\ell q} \mathbf{C}(t_\ell, t_H, t_P) \gamma_L^\mu \right]}{\text{Tr} \left[ \not{p}_\nu \mathbf{C}^0(t_\ell, t_H, t_P) \gamma_L^\mu \right]}$$



# lattice setup non-factorisable diagrams

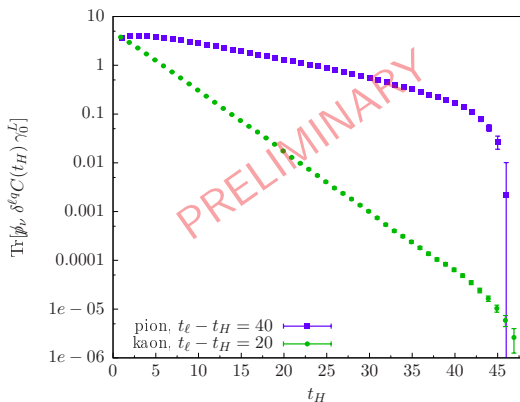
- ▶ lattice calculation



- ▶ 7 configurations on the physical point ensemble
- ▶ **96** source positions for  $\mathbf{t}_P$  (**Z2**-Wall sources)  
→ bin 8 consecutive source positions
- ▶  $\mathbf{t}_\ell - \mathbf{t}_H$  fixed (varied between **12** – **40** to check for systematic effects)
- ▶ lepton: free Domain Wall Fermion with muon mass as pole mass
- ▶ twisted boundary conditions for muon for energy/momentum conservation

## Results QED diagrams

$$\blacktriangleright \text{Tr} \left[ \not{p}_\nu \delta^{\ell q} \mathbf{C}(t_H) \gamma_0^L \right] \propto \text{Tr} \left[ \not{p}_\nu \delta^{\ell q} \widetilde{\mathcal{M}} (-\not{p}_\ell + i m_\ell) \gamma_0^L \right]$$



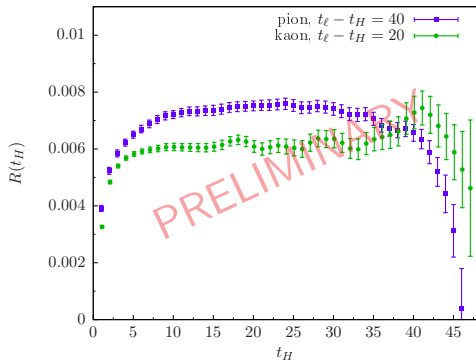
- ▶ decays exponentially with pion/kaon mass

## Results ratio

- ▶ ratio of QED diagram over tree-level diagram

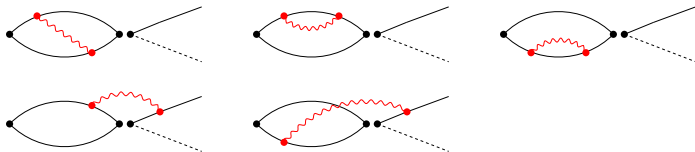
$$R(t_H) = \frac{\text{Tr} \left[ \not{p}_\nu \delta^{\ell q} \mathbf{C}(t_H) \gamma_0^L \right]}{\text{Tr} \left[ \not{p}_\nu \mathbf{C}^0(t_H) \gamma_0^L \right]} \longrightarrow \delta^{\ell q} \Gamma / \Gamma_0$$

- ▶ result physical point ensemble



## Summary

- ▶ lattice determinations of  $f_K$ ,  $f_\pi$  have reached precision of  $\lesssim 1\%$ 
  - isospin breaking correction become important
  - necessary to improve determination of CKM matrix elements
  
- ▶ preliminary results for QED corrections to leptonic meson decays
  - physical point ensemble
  - factorisable and non-factorisable QED corrections to decay rate



- low statistics, but results look encouraging

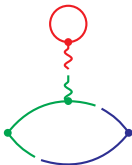
# Outlook

## ► work in progress

- increase statistics and complete analysis
- all-to-all setup for non-factorisable diagram



- disconnected diagrams & sea-quark effects, e.g.



## ► future work

- renormalisation of the weak Hamiltonian including QED
- diagrams with final state photon
- semi-leptonic meson decays  $\mathbf{K} \rightarrow \pi l \nu$

# Thank you



This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 757646.

# Backup

## The zero-mode of the photon field

- ▶ zero-mode of the photon field  
 shift symmetry of the of the photon action  $\mathbf{A}_\mu(\mathbf{x}) \rightarrow \mathbf{A}_\mu(\mathbf{x}) + \mathbf{c}_\mu$   
 → cannot be constrained by gauge fixing
- ▶ different prescriptions of QED:
- ▶ QED<sub>TL</sub>: remove the zero-mode of the photon field, i.e.  $\tilde{\mathbf{A}}_\mu(\mathbf{k} = \mathbf{0}) = \mathbf{0}$   
 [A. Duncan, E. Eichten, H. Thacker, Phys.Rev.Lett. **76**, 3894 (1996)]
- ▶ QED<sub>L</sub>: remove all the spatial zero-modes, i.e.  $\tilde{\mathbf{A}}_\mu(\mathbf{k}_0, \vec{\mathbf{k}} = \mathbf{0}) = \mathbf{0}$   
 [S. Uno and M. Hayakawa, Prog. Theor. Phys. **120**, 413 (2008)]
- ▶ QED<sub>m</sub>: use a massive photon and take  $\mathbf{m}_\gamma \rightarrow \mathbf{0}$   
 [M. Endres et al., Phys. Rev. Lett. **117** (2016) 072002]
- ▶ QED<sub>C</sub>:  $\mathbf{C}^*$  boundary conditions in spatial direction, i.e. fields are periodic up to charge conjugation [B. Luchini et al. JHEP **02** (2016) 076]