Hadron structure from the CLS $N_f = 2 + 1$ ensembles

S. Collins
University of Regensburg

RQCD Collaboration

Aim to determine nucleon structure observables with all the main systematics under control.

- $N_f = 2 + 1$ CLS ensembles.
- Excited state contamination.
- Octet baryon mass spectrum.
- Isovector baryon charges.
- Nucleon axial, induced pseudoscalar and pseudoscalar form factors.
- Use of a ChPT inspired fit to deal with large excited state contributions to the induced pseudoscalar and pseudoscalar form factors.

Preliminary!
Mass spectrum: Gunnar S. Bali, Wolfgang Söldner, ... 
Isovector charges: Wolfgang Söldner, Simon Weishäupl, Thomas Wurm, ... 
Form factors: Michael Gruber, Philipp Wein, Thomas Wurm, ...
Nucleon isovector charges and form factors

Charged currents ($n \rightarrow p$): $\bar{u} \Gamma d$

Neutral currents ($p \rightarrow p$): $\bar{u} \Gamma u, \bar{d} \Gamma d, \bar{s} \Gamma s$

Isospin limit: $\langle p | \bar{u} \Gamma d | n \rangle = \langle p | \bar{u} \Gamma u - \bar{d} \Gamma d | p \rangle = \langle n | \bar{d} \Gamma d - \bar{u} \Gamma u | n \rangle$

Also for $\Gamma = \gamma_\mu$: $\langle p | \bar{u} \gamma_\mu d | n \rangle = \langle p | J_{em} | p \rangle - \langle n | J_{em} | n \rangle$ etc.

Isovector form factors:

\[
\begin{align*}
\Gamma &= \gamma_\mu & F_1(Q^2) \gamma_\mu + \frac{F_2(Q^2)}{2m_N} \sigma_{\mu\nu} Q^\nu & \quad Q^2 \rightarrow 0 \\
\Gamma &= \gamma_\mu \gamma_5 & G_A(Q^2) \gamma_\mu \gamma_5 - i \gamma_5 \frac{\tilde{G}_P(Q^2)}{2m_N} Q_\mu & \quad \sim g_A \\
\Gamma &= \sigma_{\mu\nu} & G_T(Q^2) \sigma_{\mu\nu} & \quad \sim g_T \\
\Gamma &= 1 & G_S(Q^2) 1 & \quad \sim g_S \\
\Gamma &= \gamma_5 & G_P(Q^2) \gamma_5 & \quad \sim g_P
\end{align*}
\]
General considerations: $\langle N|\bar{q}\Gamma q|N\rangle$

Isovector: only connected quark line diagrams in the isosymmetric limit. Isoscalar combinations also contain disconnected contributions.

Systematics:

- Excited state pollution.
- Renormalisation\+ improvement: for $\bar{p} = \bar{p}' = 0$/some operators/actions $c_J = 0$ or $b_J = 0$, $J^{\overline{\text{MS}}} (\mu) = Z^{\overline{\text{MS}},\text{latt}} (a\mu, g^2) \left[ (1 + b_J a m_q) J^{\text{latt}} + c_J a J_1^{\text{latt}} \right]$
- Volume: exponentially suppressed $\sim e^{-LM_\pi}$, $LM_\pi > 4$.
- Discretisation effects: $O(a)$ or $O(a^2)$.
- Physical point extrapolation: ChPT (inspired) $M_\pi \rightarrow M_\pi^{\text{phys}}$. 
Excited states

Spectral decomposition (forward limit, zero momentum):

\[ C_{2pt}(t_f) = Z_N e^{-E_N t_f} + Z_{N_1} e^{-E_{N_1} t_f} + \ldots = Z_N e^{-E_N t_f} \left[ 1 + \frac{Z_{N_1}}{Z_N} e^{-\Delta E_{N_1} t_f} + \ldots \right] \]

\[ C_{3pt}(t_f, \tau) = Z_N \langle N(\vec{0})|J|N(\vec{0})\rangle e^{-E_N t_f} \times \]

\[ \left[ 1 + B_{01} (e^{-\Delta E_{N_1} (t_f - \tau)} + e^{-\Delta E_{N_1} \tau}) + B_{11} e^{-\Delta E_{N_1} t_f} + \ldots \right] \]

\[ Z_N = |\langle 0|\mathcal{N}|N\rangle|^2, \quad B_{10} = B_{01} \propto \sqrt{Z_{N_1}/Z_N} \langle N_1(\vec{0})|J|N(\vec{0})\rangle, \quad B_{11} \propto (Z_{N_1}/Z_N) \langle N_1(\vec{0})|J|N_1(\vec{0})\rangle \]

Ground state dominates for \( t_f \gg \tau \gg 0 \), noise increases with \( e^{[m_N - \frac{3}{2} M_\pi] t_f} \).

Lowest states are multi-particle states for \( M_\pi \sim M_\pi^{\text{phys}} \) and \( L M_\pi \sim 4 \).

2pt and 3pt (zero momentum): \( N_1 = N(\vec{0})\pi(\vec{0})\pi(\vec{0}) \) or \( N_1 = N(\vec{k})\pi(-\vec{k}) \), \ldots

3-momentum transfer \( \vec{q} \): \( N_1 = N(\vec{0})\pi(\vec{q}) \) or \( N_1 = N(\vec{q})\pi(\vec{0}) \), \ldots
Excited states

⋆ Optimise smearing of interpolator $\mathcal{N}$ to maximise $Z_N \gg Z_{N_i}$.

⋆ Use several $t_f \sim 0.7 - 1.4$ fm.

⋆ Include excited state terms in the fit.

⋆ Dominant excited state contribution also depends on $\langle N_i | J | N_j \rangle$.

⋆ More than one state may be significant.

⋆ ChPT has been used to provide some insight, e.g.

  — [Tiburzi,0901.0657,1503.06329] $N\pi$ excited state contribution to $G_A(0) = g_A$ (forward limit) in LO BChPT.

  — [Hansen,1610.03843] $N\pi$ excited state contribution to $g_A$, LO BChPT with finite volume interaction corrections a la Lellouch-Lüscher.

  — [Bär,1906.03652,1812.09191] $N\pi$ contributions to $G_A(Q^2)$, $\tilde{G}_P(Q^2)$ and $G_P(Q^2)$ computed in LO BChPT. Forward limit $g_A$, $g_S$, $g_T$ [Bär,1606.09385].

  — [Meyer,1811.03360] $N\pi$ contributions to $G_A(Q^2)$, $\tilde{G}_P(Q^2)$ and $G_P(Q^2)$ computed to tree-level in BChPT.
$N_f = 2 + 1$ CLS ensembles


★ Non-perturbatively improved clover fermion action and tree-level Lüscher-Weisz gauge action.

★ Six (five) lattice spacings: $a = 0.1 - 0.04$ fm.

★ $LM_\pi \gtrsim 4$ and multiple spatial volumes.

★ Mostly open boundary conditions in time.

Wilson flow action density, $t_0^2 E(t \approx t_0)$, $M_\pi \approx 340$ MeV, averaged over $\approx 1$ fm slice.
CLS ensembles: $M_\pi$ vs $a^2$

★ Three trajectories, physical point ensembles.
★ Typically 6000–10000 MDUs.

\[2m_\ell + m_s = \text{const.} \quad m_s = \text{const.} \quad m_\ell = m_s\]
CLS ensembles: $m_\ell-m_s$ plane

\[
m_s = \tilde{m}_{s}^{\text{phys}}
\]

$m_s = m_\ell$

\[2m_\ell + m_s = \text{const}\]

$\beta = 3.40$

$\beta = 3.46$

$\beta = 3.55$

$\beta = 3.70$

$\beta = 3.85$

physical point

\[s t_0 (2M_K^2 - M_\pi^2) \propto m_s\]

\[s t_0 M_\pi^2 \propto m_\ell\]
CLS ensembles: spatial volume

\[ 2m_\ell + m_s = \text{const.} \]

\[ m_s = \text{const.} \]

\[ \text{LM}_{\pi} < 4 \quad 4 \leq \text{LM}_{\pi} < 5 \quad \text{LM}_{\pi} \geq 5 \]
Octet baryon spectrum: $B \in \{N, \Lambda, \Sigma, \Xi\}$

Perform fits to 2pt functions.

⋆ 'Optimised' smeared interpolators $N$ with Wuppertal (Gaussian) smearing using APE smeared gauge transporters.

⋆ $\langle r^2 \rangle^1_{\psi^2} \sim 0.6 - 0.8$ fm for $M_\pi = 410 - 200$ MeV.

⋆ Multiple measurements per configuration (4–20).

$t_{\text{min}}$ for 1-state fit determined via 2-state fit. [ALPHA,1004.2661]

(Dependent on statistics)

$t_{\text{min}} \sim 0.8 - 1.0$ fm.

Right: $m_{\text{eff}}$ for nucleon with $M_\pi \sim 200$ MeV, $a = 0.064$ fm.
Mass extrapolations: \( B \in \{ N, \Lambda, \Sigma, \Xi \} \), Preliminary

Simultaneous fit to octet masses: (12 parameters)

\[
m_B(M_\pi, M_K, a) = m_B(M_\pi, M_K, 0) \left[ 1 + c_a^2 + \bar{c}a^2\bar{M}^2 + \delta c_B a^2\delta M^2 \right]
\]

\[
m_B = \sqrt{8t_0}m_B, \quad a = a/\sqrt{8t_0}, \quad \bar{M}^2 = (2M_K^2 + M_\pi^2)/3, \quad \delta M^2 = 2(M_K^2 - M_\pi^2)
\]

W. Söldner

Finite volume: \( LM_\pi > 3.4 \). \( O(a^2) \) effects removed using fit. Similarly, data shifted to \( \bar{M}^2 = M_\pi^2, \bar{M}^2_{ss} = M_{ss}^2, \bar{M}^2_{ss} = 2M_K^2 - M_\pi^2 \).
Mass extrapolations

$O(p^3)$ (NNLO) covariant SU(3) BChPT: ($\mathbb{M}_{\eta_8}^2 = \overline{M}^2 + \delta \overline{M}^2/3$)

$$m_B(\mathbb{M}_\pi, \mathbb{M}_K, 0) = m_0 + \overline{b}\overline{M}^2 + \delta \overline{b}_B \delta \overline{M}^2 +$$

$$g_{B,\pi} f_O \left( \frac{\mathbb{M}_\pi}{m_0} \right) + g_{B,K} f_O \left( \frac{\mathbb{M}_K}{m_0} \right) + g_{B,\eta_8} f_O \left( \frac{\mathbb{M}_{\eta_8}}{m_0} \right)$$

EOMS regularisation*: $f_O(x) = -2x^3[\sqrt{1 - x^2/4} \arccos(x/2) + x \ln(x)/2]$

SU(3) constraints:

* $\delta \overline{b}_B = \delta \overline{b}_{N,\Lambda,\Sigma,\Xi}$ determined by 2 parameters.

* $g_{B,\pi,K,\eta_8}$ known quadratic functions of the LECs $F$ and $D$.

$m_{\Xi}^{\text{latt}} = m_{\Xi}^{\text{ph}}$ imposed to fix the scale: $\sqrt{8} t_{0,\text{ph}} = 0.4128(22)\text{ fm}$. Compatible with $\sqrt{8} t_{0,\text{ph}} = 0.413(6)\text{ fm}$ from $F_\pi + 2F_K$ [ALPHA,1608.08900].

**Preliminary**. Still to try: SSE including decuplet baryons, Taylor expansion about the symmetric point, different mass/$a$ cuts.

* Extended on mass shell scheme.
Mass extrapolations: discretisation effects

Data shifted to physical quark mass using the fit.

\[ \sqrt{8t_0 m} a^2 / fm^2 \]

\( a^2 \) varied by a factor of 4.6.
\[ \sigma \text{ terms: } \sigma_{q,B} = m_q \langle B | q\bar{q} | B \rangle \]

Feynman-Hellmann theorem

\[ \sigma_{\pi B} = m_u \frac{\partial m_B}{\partial m_u} + m_d \frac{\partial m_B}{\partial m_d} \approx M^2_{\pi} \frac{\partial m_B}{\partial M^2_{\pi}}, \quad \sigma_{sB} = m_s \frac{\partial m_B}{\partial m_s} \approx \frac{1}{2} M^2_{s\bar{s}} \frac{\partial m_B}{\partial M^2_{s\bar{s}}} \]

\[ \sigma_{\pi B} = \sigma_{uB} + \sigma_{dB}. \]

\( \sigma_s \) is not well determined as \( m_s \) is not varied near the physical point.

Pion sigma terms: Preliminary

\[ \sigma_{\pi N} = 41(2)(2)(??) \text{ MeV} \quad \sigma_{\pi \Lambda} = 29(2)(1)(??) \text{ MeV} \]
\[ \sigma_{\pi \Sigma} = 23(1)(1)(??) \text{ MeV} \quad \sigma_{\pi \Xi} = 13(1)(0)(??) \text{ MeV} \]

Compatible with [FLAG,1902.08191] average (FH+direct) for \( N_f = 2 + 1 \) of
\[ \sigma_{\pi N} = 39.7(3.6) \text{ MeV}. \]

[BMW-c,1510.08013] \[ \sigma_{\pi N} = 38(3)(3) \text{ MeV} \]
Forward matrix elements: isovector charges $g_A$, $g_S$, $g_T$

★ Four source-sink separations $t_f \sim 0.7 - 1.2$ fm. $N_{\text{meas}}=1,2,3,4$.

★ Renormalisation: $\bar{J}^{\text{MS}}(\mu) = Z^{\text{MS,\,latt}}(a_\mu, g^2) \left[ (1 + b_J a m_q) J^{\text{latt}} + c_J a J_1^{\text{latt}} \right]$.

$c_S = 0$, $c_A$ [ALPHA,1502.04999] and $c_T$ only relevant for $\vec{q} \neq \vec{0}$.


★ Fit ratio $C_{3pt}(t_f, \tau)/C_{2pt}(t_f) \rightarrow \langle N|J|N \rangle$

★ 2 state fit. Fit range variation included in final error.
Forward matrix elements: isovector charges $g_A$, $g_S$, $g_T$

$M_{\pi} \sim 350$ MeV, $a = 0.064$ fm.
Axial charge: $g_A \equiv G_A(0)$

$g_A/g_V = 1.2724(23)$ PDG 2018 (assuming SM), $g_V = 1$ in the isospin limit.

Benchmark quantity: demonstration of lattice techniques, known to be sensitive to:

- excited state contamination,
- spatial volume,
- quark mass, ...

Right: [FLAG,1902.08191]
Axial charge: $g_A \equiv G_A(0)$

**Analysis in progress**: example of $a, m_q, V$ extrapolation for $2m_\ell + m_s = \text{const.}$ trajectory.

$$g_A(M_\pi, M_K, L, a) = g_A(M_\pi, M_K, L, 0) \left[ 1 + a^2 \left( c_0 + c_1 M_\pi^2 \right) \right]$$

$$g_A(M_\pi, M_K, L, 0) = g_0 + g_1 M_\pi^2 + g_2 M_\pi^2 \ln M_\pi + g_3 M_\pi^2 e^{-LM_\pi} / \sqrt{LM_\pi}$$

$$+ g_4 M_K^2 \ln M_K + g_5 M_K^2 e^{-LM_K} / \sqrt{LM_K}$$

$g_A$ vs $M_\pi^2$ (shifted in $a$ and $V$, $M_K$ depends on $M_\pi$): T. Wurm
Axial charge: $g_A \equiv G_A(0)$

$g_A = 1.27(3)(??)$ $\chi^2$/d.o.f. = 1.44

c.f. Mainz [Ottnad,1905.01291]: $g_A = 1.242(25)^{\text{stat}}(40)^{\text{sys}}$. 
SU(3) LECs $F$ and $D$

In SU(3) flavour limit: $g_A^N = F + D$, $g_A^\Sigma = 2F$ and $g_A^{\Xi} = F - D$.

Extrapolation along $m_\ell = m_s$ trajectory.

Preliminary S. Weishäupl

[Savanur,1901.00018] 'F' = 0.438(7)(6) and 'D' = 0.708(10)(6)

Expt. hyperon semi-leptonic decays + SU(3) symmetry [Cabbibo,hep-ph/0307298] $F = 0.463(8)$ and $D = 0.804(8)$. 
Isovector charges: $g_S$ and $g_T$

BSM contributions to $\beta$ decay.

$$\mathcal{L}_{CC} = -\frac{G_F^{(0)} V_{ud}}{\sqrt{2}} \left[ \epsilon_S \bar{e}(1 - \gamma_5)\nu_\ell \cdot \bar{u}d + \epsilon_T \bar{e}\sigma_{\mu\nu}(1 - \gamma_5)\nu_\ell \cdot \bar{u}\sigma^{\mu\nu}(1 - \gamma_5)d \right]$$

Studied in planned precision $\beta$ decay expts. + LHC $pp \rightarrow e\nu + X$

**Estimate of $g_T$:**

Transverse spin $g_T = \int_{-1}^{1} dx \left[ \delta u(x) - \delta d(x) \right]$

$\delta q(-x) = -\delta \bar{q}(x)$

Phenomenological estimates from fits to SIDIS data $g_T \approx 1$, $\Delta g_T / g_T \gtrsim 25%$.

**Estimate of $g_S$:**

CVC relation $\partial_\mu (\bar{u}\gamma_\mu d) = -i(m_u - m_d)\bar{u}d$ applied to $\langle p(p_f) \bar{u}_\gamma \gamma_\mu d | n(p_i) \rangle$

Forward limit: $(m_u - m_d)g_S = (m_p - m_n)^{\text{QCD}}$.

[Gonzalez-Alonso,1309.4434]:

Lattice estimates of $(m_u - m_d)$ and $(m_p - m_n)^{\text{QCD}} \rightarrow g_S = 1.02(11)$. 
Tensor charge well determined on the lattice.

**Preliminary**  $a \rightarrow 0, \ M_\pi \rightarrow M_\pi^{\text{phys}}, \ L \rightarrow \infty$ extrapolation:
\[
g_T^{\overline{\text{MS}}} (2 \text{ GeV}) = 1.01(3).
\]
c.f. Mainz [Ottnad,1905.01291]:  $g_T^{\overline{\text{MS}}} (2 \text{ GeV}) = 0.965(38)^{\text{stat}}_{-13}^{+10} \text{sys}.$

[FLAG,1902.08191]  $N_f = 2 + 1 + 1,$
\[
g_T^{\overline{\text{MS}}} (2 \text{ GeV}) = 0.989(32)(10) \text{ ([PNDME,1806.09006]).}
\]
Preliminary $a \rightarrow 0$, $M_\pi \rightarrow M_\pi^{\text{phys}}$, $L \rightarrow \infty$ extrapolation: $g_S^{\overline{\text{MS}}}(2 \text{ GeV}) = 0.97(14)$.

c.f. Mainz [Ottnad,1905.01291]: $g_S^{\overline{\text{MS}}}(2 \text{ GeV}) = 1.13(11)^{\text{stat}}(^{+07}_{-06})^{\text{sys}}$.

[FLAG,1902.08191] $N_f = 2 + 1 + 1$, $g_S^{\overline{\text{MS}}}(2 \text{ GeV}) = 1.02(8)(6)$ ([PNDME,1806.09006]).

Target of $10 - 15\%$ errors on $g_S$ and $g_T$ needed for constraining $\epsilon_S$ and $\epsilon_T$ [Bhattacharya,1110.6448] achieved.

Prediction for $(m_n - m_p)^{\text{QCD}}$ via CVC relation $(m_u - m_d)g_S = (m_p - m_n)^{\text{QCD}}$ [PNDME,1806.09006] find $(m_n - m_p)^{\text{QCD}} = 2.63(27)$ MeV using $(m_d - m_u) = 2.572(66)$ MeV from [MILC/Fermilab/TUMQCD,1802.04248].

c.f. $(m_n - m_p)^{\text{QCD}} = 2.52(29)$ MeV from $N_f = 1 + 1 + 1 + 1$ [BMW-c,1406.4088]
Global analysis of transversity with lattice constraints

Transverse spin: \( g_T = \langle 1 \rangle \delta q - \delta \bar{q} = \int_0^1 dx \left[ h^q_1(x, \mu^2) - h^{\bar{q}}_1(x, \mu^2) \right] \)

PDFs \( h^u_1(x) = \delta u(x), \ h^d_1(x) = \delta d(x), \) assuming \( \delta \bar{u} = \delta \bar{d} = \delta \bar{s} = \delta s \)

From JAM: [H-W Lin,1710.09858]: Fit SIDIS Collins asymmetry data \( (A_{UT}^{\sin(\phi_h+\phi_s)}) \) alone (yellow), and with \( \delta u - \delta \bar{u} - \delta d + \delta \bar{d} \) constrained to \( g_T = 1.01(6) \) (red+blue) (average of [LHPC,1206.4527], [PNDME,1606.07049], [RQCD,1412.7336])
Isovector form factors: $G_A(Q^2)$, $\tilde{G}_P(Q^2)$ and $G_P(Q^2)$

$$
\langle N(p_f)|A_{\mu}^{u-d}|N(p_i)\rangle = \bar{u}_N(p_f) \left[ G_A(Q^2)\gamma_\mu - i \frac{\tilde{G}_P(Q^2)}{2m_N} Q_\mu \right] \gamma_5 u_N(p_i)
$$

$$
\langle N(p_f)|P^{u-d}|N(p_i)\rangle = \bar{u}_N(p_f) G_P(Q^2) \gamma_5 u_N(p_i)
$$

$q_\mu = p_{f,\mu} - p_{i,\mu}$, $Q^2 = -q_\mu q^\mu \geq 0$.

Forward limit: $G_A(Q^2) \to g_A$, (by extrapolation) $\tilde{G}_P(Q^2) \to \tilde{g}_P$, $G_P(Q^2) \to g_P$.

Shape at low $Q^2$, $\langle r_X^2 \rangle = -6 \frac{dG_X(Q^2)}{dQ^2}$: different probe $\to$ different radius.

$$
G_X(Q^2) = G_X(0) \left[ 1 - \frac{1}{6} \langle r_X^2 \rangle Q^2 + \ldots \right]
$$

Parametrisation $\to$ dipole form, $z$ expansion etc $\to$ $\langle r_E^2 \rangle$, $\langle r_A^2 \rangle$. 

PCAC relation and pion pole dominance

PCAC relation for nucleon matrix elements:

\[
2 m_q \langle N(\vec{p}_f) | \mathbf{P} | N(\vec{p}_i) \rangle = \langle N(\vec{p}_f) | \partial_\mu A_\mu | N(\vec{p}_i) \rangle + \mathcal{O}(a^2)
\]

leads to

\[
m_q G_P(Q^2) = m_N G_A(Q^2) - \frac{Q^2}{4m_N} \tilde{G}_P(Q^2)
\]

Forward limit: \( m_q G_P(0) = m_q g_p = m_N g_A \)

Chiral limit: \( \tilde{G}_P(Q^2) = 4m_N^2 G_A(Q^2)/Q^2 \)

Pion pole dominance (LO ChPT):

\[
\tilde{G}_P(Q^2) = G_A(Q^2) \frac{4m_N^2}{Q^2 + M_\pi^2} + \text{corrections}
\]

PCAC + pion pole dominance (PPD) → one independent form factor but PPD is an approximation.
Nucleon axial form factor $G_A(Q^2)$

**Needed for neutrino oscillation experiments**: Charged current quasielastic (CCQE) neutrino-nucleus interaction must be known to high precision.

**Connecting quark - nucleon level**: $G_A(Q^2)$ form factor.

**nucleon - nucleus level**: nuclear model.

Pre 1990 information on $G_A(Q^2)$ extracted from $\bar{\nu}-p$ and $\nu-d$ scattering. [Mosel,1602.00696]

\[
G_A(Q^2) = \frac{g_A}{(1 + \frac{Q^2}{M_A^2})^2}
\]

Shown $M_A = 1.03(5) \text{ GeV}$

MiniBooNE: [Aguilar-Arevalo,1002.2680]

$M_A = 1.35(17) \text{ GeV}$

Systematics being explored including new analysis of old expt data:

$M_A = 1.01(24) \text{ GeV}$ from z-expansion [Meyer,1603.03048].
Nucleon induced pseudoscalar form factor $\tilde{G}_P(Q^2)$

Not relevant for neutrino oscillations: enters cross-section with $m_{\text{lep}}^2/m_N^2$.

Not well known from expt: muon capture $\mu^- p \rightarrow \nu_\mu n$ gives

$$g_P^* = \frac{m_\mu}{2m_N} \tilde{G}_P(Q^2 = 0.88 m_\mu^2) = 8.06(55) \quad [\text{MuCap,1210.6545}]$$

Compatible with pion pole dominance.
Extracting the form factors

Consider ratio:

$$R_{\Gamma_i, O} = \frac{C_{3pt}^O(\vec{p}_f, \vec{p}_i, t_f, \tau)}{C_{2pt}(\vec{p}_f, t_f)} = \sqrt{\frac{Z_{\vec{p}_f}}{Z_{\vec{p}_i}}} \frac{E_{\vec{p}_f}}{E_{\vec{p}_f} + m_N} B_{\Gamma_i, O}(\vec{p}_f, \vec{p}_i) e^{-(E_{\vec{p}_f} - E_{\vec{p}_i})\tau} + \ldots$$

Polarisation: $\Gamma_i = i\gamma_i\gamma_5$, overlap factor $Z_{\vec{p}} = |\langle 0|N|N(\vec{p})\rangle|^2$

$$B_{\Gamma_i, A\mu}(\vec{p}_f, \vec{p}_i) \propto G_A(Q^2)\gamma_\mu - i \frac{\tilde{G}_P(Q^2)}{2m_N} Q_\mu$$

Set-up: $\vec{p}_f = \vec{0}, \vec{q} = -\vec{p}_i$

$$R_{A_i \parallel \Gamma_i \perp \vec{q}} \propto G_A(Q^2)$$

$$R_{A_i \parallel \Gamma_i \parallel \vec{q}} \propto (M_N + E_{\vec{q}}) G_A(Q^2) - \frac{q_i^2}{2M_N} \tilde{G}_P(Q^2)$$

$$R_{A_4, \Gamma_i \parallel \vec{q}} \propto G_A(Q^2) + \frac{(M_N - E_{\vec{q}})}{2M_N} \tilde{G}_P(Q^2)$$

$$R_{P, \Gamma_i \parallel \vec{q}} \propto G_P(Q^2)$$
Extracting the form factors

2-state fit (grey curve) to $R_{\Gamma_i, A_\mu}$ and $R_{\Gamma_i, P}$ leads to large $\chi^2$/d.o.f. $M_\pi \sim 200$ MeV, $a = 0.064$ fm, $|\vec{q}| = 2\pi/(64a)$.

Well known problem: e.g. [RQCD, 1412.7336], [ETMC, 1705.03399], [PNDME, 1705.06834], [RQCD, 1810.05569], [PACS, 1811.07292].

Usually $C_{3pt}^A$ omitted, $\chi^2$ improves. Large excited state contributions to $\tilde{G}_P$, $G_P$. 

T. Wurm
Test PCAC relation and pion pole dominance (PPD) with:

\[
\begin{align*}
r_{PCAC} &= \frac{m_q G_P(Q^2) + \frac{Q^2}{4m_N} \tilde{G}_P(Q^2)}{m_N G_A(Q^2)} = 1 + O(a^2) \\
r_{PPD} &= \frac{M_\pi^2 + Q^2 \tilde{G}_P(Q^2)}{4m_N^2} \frac{G_A(Q^2)}{G_A(Q^2)} = 1 + \ldots
\end{align*}
\]

\(M_\pi\) variation, \(a = 0.064\) fm

\(a = 0.064\) fm, \(M_\pi\): Blue 410 MeV, Orange 350 MeV, Green 280 MeV, Red 200 MeV.
\[ a \text{ variation, } M_\pi \sim 280 \text{ MeV} \]

\[ M_\pi \sim 280 \text{ MeV, } a: \text{ Blue } 0.086 \text{ fm, Orange } 0.076 \text{ fm, Green } 0.064 \text{ fm, Red } 0.05 \text{ fm,} \]

[\text{RQCD,1810.05569}]: \, m_i^{PCAC} \text{ for } M_\pi \sim 290 \text{ MeV determined from applying the PCAC relation to the pion 2pt function and to the nucleon 3pt functions are consistent in the continuum limit.}
$N\pi$ excited state contributions

[Bär,1906.03652,1812.09191]: $N\pi$ contributions to $C_{2pt}$ and $C_{3pt}^J$ for $J = A_\mu, P$
computed in leading one-loop order of SU(2) covariant ChPT.

[Bär,1907.03284]

$R'_4 \text{ vs } \tau/a - t_f/(2a)$

$t_f \to \infty \quad R'_4 \to \text{const.}$

Data: [RQCD,1810.05569]:

$M_\pi \sim 150 \text{ MeV}, \quad a = 0.07 \text{ fm},$
$t_f = 1.06 \text{ fm}, \quad \vec{p}_f = \vec{0}, \quad |\vec{q}| = 2\pi/(64a)$

\[
C_{3pt}^{A_4}(\vec{0}, \vec{p}_i = -\vec{q}, t_f, \tau) = C_{3pt,N}^{A_4}(\vec{q}, t_f, \tau) + C_{3pt,N_\pi}^{A_4}(\vec{q}, t_f, \tau) = O \left( \frac{1}{m_N} \right) + O(1)
\]

In this channel $N(\vec{0})\pi(-\vec{q}) \to N(\vec{0})$ enhanced, relative to $N(-\vec{q}) \to N(\vec{0})$.

Suggestion: correct lattice data with factor e.g. $G_P(Q^2, t_f, \tau) = G_P(Q^2) \left[ 1 + \epsilon_P(Q^2, t_f, \tau) \right]$
ChPT inspired fit

[Bär,1906.03652,1812.09191]: loop diagrams (⇒ a whole tower of $N\pi$ states) contribute to $G_A$ while tree-level diagrams contribute to $\tilde{G}_P$ ($\Rightarrow N(0)\pi(-\bar{q})$).

Excited state contributions to $G_A(Q^2) \sim +5\%$ for $M_\pi = 140$ MeV, $M_\pi L = 3 - 6$, $t_f = 2$ fm, independent of $Q^2 \leq 0.25$ GeV$^2$. For $\tilde{G}_P(Q^2) = -10\%$ to $-40\%$ as $Q^2$ is decreased.

PCAC relation gives contributions to $G_P$.

Drawbacks:

☆ At LO wavefunction normalisation $Z_{N\pi} = Z_N$. Not expected with smearing unless $\langle r^2 \rangle^{1/2} \ll 1/m_\pi$.

☆ Applies to the plateau method.

Alternative:

☆ Use tree-level ChPT to determine form of $N\pi$ contributions.

☆ Allow for dependence on the smearing of the interpolators.

☆ Avoid any constraints on the ground state.
ChPT inspired fit

Tree-level diagrams:

Top diagram:

\[ \sim G_A \]
\[ = 0 \]

for \( O = A_\mu \)

for \( O = P \)

Bottom middle diagram

\[ \sim \tilde{G}_P + \text{excited states} \]

\[ \sim G_P + \text{excited states} \]

for \( O = A_\mu \)

for \( O = P \)

Other diagrams: only contribute to the excited states.
ChPT inspired fit
M. Gruber, P. Wein, T. Wurm

\[ C^{\Gamma_{\alpha}, A_{\mu}}_{3\text{pt}} (\vec{p}_f, \vec{p}_i, t_f, \tau) = \frac{\sqrt{Z_{\vec{p}_f} Z_{\vec{p}_i}}}{2E_{\vec{p}_f} E_{\vec{p}_i}} e^{-E_{\vec{p}_f} (t_f - \tau)} e^{-E_{\vec{p}_i} \tau} \left[ B_{\Gamma_i, A_{\mu}} (\vec{p}_f, \vec{p}_i) \times \right. \]

\[ \left( 1 + B_{10, A_{\mu}} e^{-\Delta E_{\vec{p}_f} (t-\tau)} + B_{01, A_{\mu}} e^{-\Delta E_{\vec{p}_i} \tau} + B_{11, A_{\mu}} e^{\Delta E_{\vec{p}_f} (t_f - \tau)} e^{-\Delta E_{\vec{p}_i} \tau} \right) \]

\[ + \frac{E_{\vec{p}_f}}{E_{\pi}} r_{+}^{\mu} c_{\vec{p}_f} q_{\alpha} e^{-\Delta E^{N\pi}_{\vec{p}_f} (t-\tau)} + \frac{E_{\vec{p}_i}}{E_{\pi}} r_{-}^{\mu} c_{\vec{p}_i} q_{\alpha} e^{-\Delta E^{N\pi}_{\vec{p}_i} (\tau)} \left] \right. \]

\[ C^{\Gamma_{\alpha}, P}_{3\text{pt}} (\vec{p}_f, \vec{p}_i, t_f, \tau) = \frac{\sqrt{Z_{\vec{p}_f} Z_{\vec{p}_i}}}{2E_{\vec{p}_f} E_{\vec{p}_i}} e^{-E_{\vec{p}_f} (t_f - \tau)} e^{-E_{\vec{p}_i} \tau} \left[ B_{\Gamma_i, P} (\vec{p}_f, \vec{p}_i) \times \right. \]

\[ \left( 1 + B_{10, P} e^{-\Delta E_{\vec{p}_f} (t-\tau)} + B_{01, P} e^{-\Delta E_{\vec{p}_i} \tau} + B_{11, P} e^{\Delta E_{\vec{p}_f} (t_f - \tau)} e^{-\Delta E_{\vec{p}_i} \tau} \right) \]

\[ + \frac{E_{\vec{p}_f}}{E_{\pi}} B_{0} c_{\vec{p}_f} q_{\alpha} e^{-\Delta E^{N\pi}_{\vec{p}_f} (t-\tau)} + \frac{E_{\vec{p}_i}}{E_{\pi}} B_{0} c_{\vec{p}_i} q_{\alpha} e^{-\Delta E^{N\pi}_{\vec{p}_i} (\tau)} \left] \right. \]

\[ \Delta E^{N\pi}_{\vec{p}_f} = E_{\pi} + E_{\vec{p}_f} - \vec{q} - E_{\vec{p}_f}, \Delta E^{N\pi}_{\vec{p}_i} = E_{\pi} + E_{\vec{p}_i} + \vec{q} - E_{\vec{p}_i}, \vec{p}_f = \vec{0} \]

\[ r_\pm = (E_{\pi}, \pm \vec{q}, E_{\vec{p}} = \sqrt{m^2_N + |\vec{p}|^2}, E_{\pi} = \sqrt{M^2_{\pi} + |\vec{q}|^2}, \]

\[ B_0 = -\langle \bar{u}u \rangle / F_0^2 \approx M_{\pi}/(2m_q). \]

\[ c_{\vec{p}_i} \text{ and } c_{\vec{p}_f} \text{ are the same for axial and pseudoscalar 3pt functions.} \]
Extracting the form factors

Performing new fit (yellow curve) to $R_{\Gamma, A\mu}$ and $R_{\Gamma, P}$ leads to reasonable $\chi^2$/d.o.f.

$M_\pi \sim 200$ MeV, $a = 0.064$ fm, $|\vec{q}| = 2\pi/(64a)$.

T. Wurm

$R_{A_i \parallel \Gamma_i \perp \vec{q}}$

$R_{A_i \parallel \Gamma_i \parallel \vec{q}}$

$R_{A_4, \Gamma_i \parallel \vec{q}}$

$R_{P, \Gamma_i \parallel \vec{q}}$

$\tau/a - t_f/2a$
Extracting the form factors

Modified $R_{\Gamma_i, A_\mu}$ and $R_{\Gamma_i, P}$, with $e^{-(E_{\bar{p}_i} - E_{\bar{p}_f})\tau}$ and other factors removed.

$M_\pi \sim 200$ MeV, $a = 0.064$ fm, $|\vec{q}| = 2\pi/(64a)$. T. Wurm
Testing the PCAC relation and pion pole dominance

\[ r_{\text{pcac}} \text{ Npi} \]

\[ r_{\text{ppd}} \text{ naive} \]

\[ Q^2 [\text{GeV}^2] \]

\[ a = 0.064 \text{ fm}, \ M_\pi: \text{ Blue} 410 \text{ MeV}, \text{ Orange} 350 \text{ MeV}, \text{ Green} 280 \text{ MeV}, \text{ Red} 200 \text{ MeV}. \]

\[ Q^2 [\text{GeV}^2] \]

\[ M_\pi \sim 280 \text{ MeV}, \ a: \text{ Blue} 0.076 \text{ fm}, \text{ Orange} 0.064 \text{ fm}, \text{ Green} 0.05 \text{ fm}. \]
Extrapolation in $a, m_q$ and $V$

Extrapolation in $a, m_q$ and $V$ as a function of $Q^2$.

Choose $Q^2$ parameterisation: in progress: $z$–expansion.

Dipole forms

$$G_A(Q^2) = \frac{g_A}{(1 + Q^2/M_A^2)^2}$$

$$\tilde{G}_P(Q^2) = \frac{1}{Q^2 + M_\pi^2} \frac{\tilde{g}_P}{(1 + Q^2/M_{\tilde{P}}^2)^2}$$

$$m_q G_P(Q^2) = \frac{1}{Q^2 + M_\pi^2} \frac{g'_P}{(1 + Q^2/M_{P^2})^2}$$

Using $m_q G_P$ means all form factors are renormalised with $Z_A$.

Consistent with the perturbative expectation at very large $Q^2$:

$$G_A(Q^2) \propto 1/Q^4$$

$$\tilde{G}_P(Q^2) \propto 1/Q^6$$

$$G_P(Q^2) \propto 1/Q^6$$

Fit form for $X \in \{M_A, g_A, M_P, g'_P, M_{\tilde{P}}, \tilde{g}'_P\}$:

$$X(M_\pi, M_K, L, a) = \left[ x_0 + x_1 M_\pi^2 + x_2 M_\pi^2 \ln M_\pi + x_3 M_\pi^2 e^{-LM_\pi} / \sqrt{LM_\pi} + x_4 M_K^2 + x_5 M_K^2 \ln M_K + x_6 M_K^2 e^{-LM_K} / \sqrt{LM_K} \right] \times \left[ 1 + x_7 a^2 + x_8 a^2 M_\pi^2 + x_9 a^2 M_K^2 \right]$$
Extrapolation of $G_A(Q^2)$

* Fit performed on a subset of the ensembles used for the charges.

* $\chi^2$/d.o.f. = 1.31, $G_A(0) = g_A \sim 1.2$, $\langle r_A^2 \rangle \sim 0.3$ fm$^2$, i.e. $\langle r_A^2 \rangle^{1/2} \sim 0.55$ fm.

T. Wurm

[ETMC,1705.03399], $\langle r_A^2 \rangle^{1/2} \sim 0.516(33)(14)$ fm, [PNDME,1705.06834], $\langle r_A^2 \rangle^{1/2} \sim 0.481(58)(62)$ fm, [PACS,1811.07292], $\langle r_A^2 \rangle^{1/2} \sim 0.647(22)(38)$ fm, [Mainz,1705.06186] $\langle r_A^2 \rangle^{1/2} \sim 0.600(60)(^{+133}_{-147})$ fm.
Extrapolation of $\tilde{G}_P(Q^2)$

$\chi^2$/d.o.f. = 1.37, $\tilde{G}_P(0) \sim 241$, $\langle r^2 \rangle \sim 12$ fm$^2$,

$g_P^* = \frac{m_\mu}{2m_N} \tilde{G}_P(Q^2 = 0.88 m_\mu^2) \sim 8 - 9$

Previously: $g_P^* = 2.8(7)$ [RQCD,1810.05569],
$g_P^* = 7.7(1.8)^{+0.8}_{-2.0}$ [Mainz,1705.06186], $g_P^* = 4.44(18)$ [PNDME,1705.06834],
$\tilde{G}_P(0) = 165.62(9.82)(18.46)$ [ETMC,1705.03399]
Extrapolation of $m_q G_P(Q^2)$

\[ \chi^2/d.o.f. = 1.07, \ m_q G_P(0) \sim 1.26, \ \langle r^2 \rangle \sim 12 \text{ fm}^2. \]

PCAC relation and expt.: \( m_q G_P = m_N g_A = 1.19. \)

T. Wurm
$r_{PCAC}$ and $r_{PPD}$ after extrapolation
Summary

⋆ Aim for determination of nucleon structure observables with all main systematics under control.

⋆ This is possible using the CLS ensembles with 5 lattice spacings, $M_\pi \sim 410 - 135$ MeV and $M_\pi L \gtrsim 4$.

⋆ Three trajectories allow the $m_l - m_s$ plane to be explored.

⋆ Excited state contributions to 2pt and 3pt functions in the forward limit are reasonably under control.

⋆ Reproduce octet baryon spectrum and results for the SU(3) LECs, $\sigma_{\pi N}$, charges and moments of PDFs: $g_A, S, T, \langle x \rangle_{u - d}, \langle x \rangle_{\Delta u - \Delta d}, \langle x \rangle_{\delta u - \delta d}$ to follow soon.

⋆ Form factors: well known problem of large excited state contributions to $\tilde{G}_P(Q^2), G_P(Q^2)$. Relatively small contributions to $G_A(Q^2)$.

⋆ Utilising a ChPT inspired fit to the axial and pseudoscalar 3pt function (with no constraints made on the ground state), then the PCAC relation is respected. Approximate pion pole dominance is also found.