Lattice QCD overview: form-factors

Advances in Lattice Gauge Theory
CERN, 22 July 2019 — 9 August 2019

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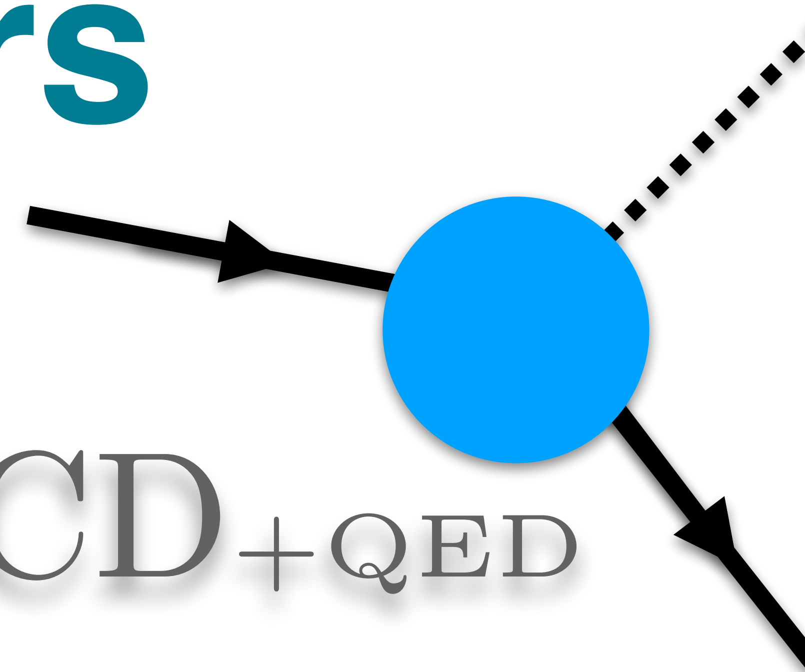
Outline

- Tree semileptonic decay — parametrisation
- Rare semileptonic decay — howto
- Radiative decay — new
- Leptonic decay — disconnected

Form factors



$$\langle P_f(p_f)|O|P_i(p_i)\rangle_{|\mathrm{QCD}+\mathrm{QED}}$$

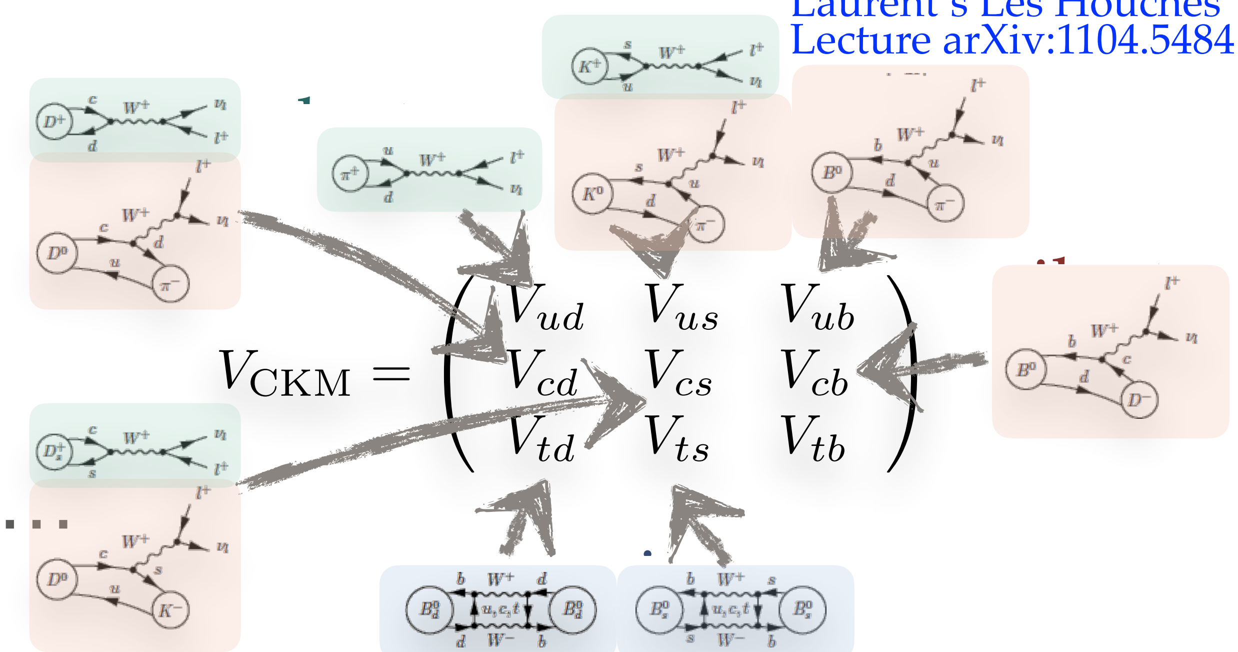
- initial and final states $P_{i,f}$ with momenta $p_{i,f}$
- current can be elm, weak, non-local, …
- can be single or multiple hadrons, e.g. $\langle P_{i,f}| = \langle 0|, \langle \pi|, \langle \pi\pi|, \ldots$
- states can be stable or unstable in QCD, e.g. $\langle P_{i,f}| = \langle \rho|, \langle K^*|, \ldots$
- **form factors** parametrise hadronic matrix elements

Form factors

Form factors are of crucial interest:

- For CKM: $\Gamma_{\text{exp.}} \overset{???}{=} V_{\text{CKM}}(\text{WEAK})(\text{EM})(\text{STRONG})$

illustrations from
Laurent's Les Houches
Lecture arXiv:1104.5484



- For finding the unknown:
  Flavour anomalies, light-by-light scattering, non-SM matrix elements, …

- For understanding structure:
  Parton-picture, momentum distribution, sea/valence effects, …

A form factor calculation

Generate ensembles (or use existing ones):

- good coverage of parameters: L, a
  we require continuum- and infinite-volume extrapolations
  note that $\overrightarrow{p} = \frac{2\pi}{L}\overrightarrow{n}$ with implications for accessible kinematics

- well tuned: $m_l, m_c\,, m_b, \ldots$ physical
  (not yet) possible in practice, requiring extrapolations or help from effective theory

- maybe include QED/strong IB

Assumptions

In QCD-only simulations we assume factorisation of SM:

$$\Gamma_{\text{exp.}} \overset{???}{=} V_{\text{CKM}}(\text{WEAK})(\text{EM})(\text{STRONG})$$

Strong contribution given in terms of hadronic form factors (lattice)

Weak & EM & strong treated separately — although in real world all three SM sectors talk to each other

In particular EM:
Note that $O(\alpha_{EM}) \approx 1\%$ — so OK as long as we keep it in mind

Challenges

A multi-scale problem

a^{-1} << physics of interest << L^{-1}

finite cutoff — finite box size

finite lattice spacing

- hard to discretise *b*-quarks (slowly getting there but need to play and control tricks like effective theory, improvement, extrapolation in m_h,…, which are not needed for light quarks)

finite lattice volume

- physical pion mass ‘expensive’ to reconcile with above bounds — we often dial heavier (cheaper) m_π and then extrapolate (model or EFT)

Challenges

a^{-1} << physics of interest << L^{-1}

finite cutoff finite box size

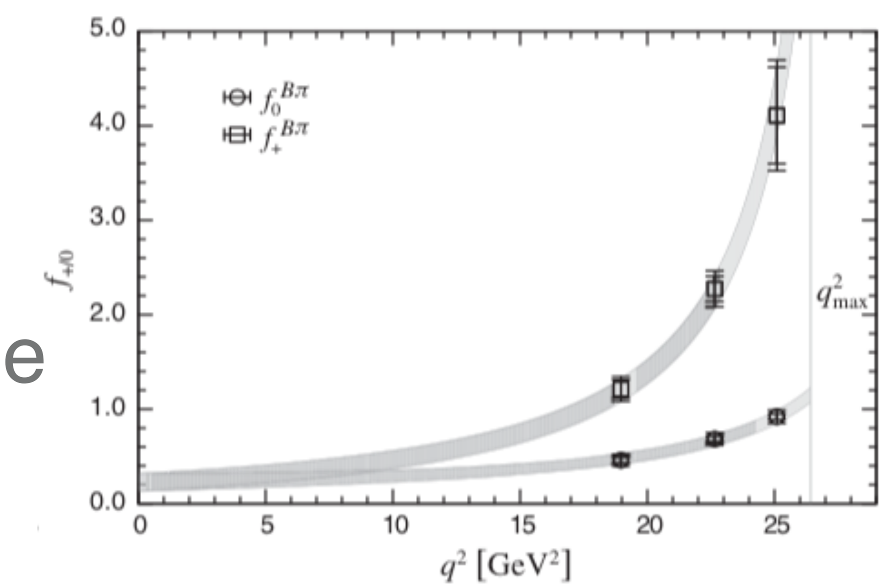
kinematics — e.g. semileptonic decay

$$q^2 = (E_i - E_f)^2 - (\vec{p}_i - \vec{p}_f)^2$$

lattice does best with mesons at rest
(statistical error and cutoff effects smaller)

E.g. for heavy-light SL this is at tension with the suppression of the decay rate at large q^2

Kinematical reach limited in lattice QCD →
extract value of V_{CKM} from
simultaneous analysis of exp. and lattice data



RBC/UKQCD PRD 91, 074510 (2015)2018

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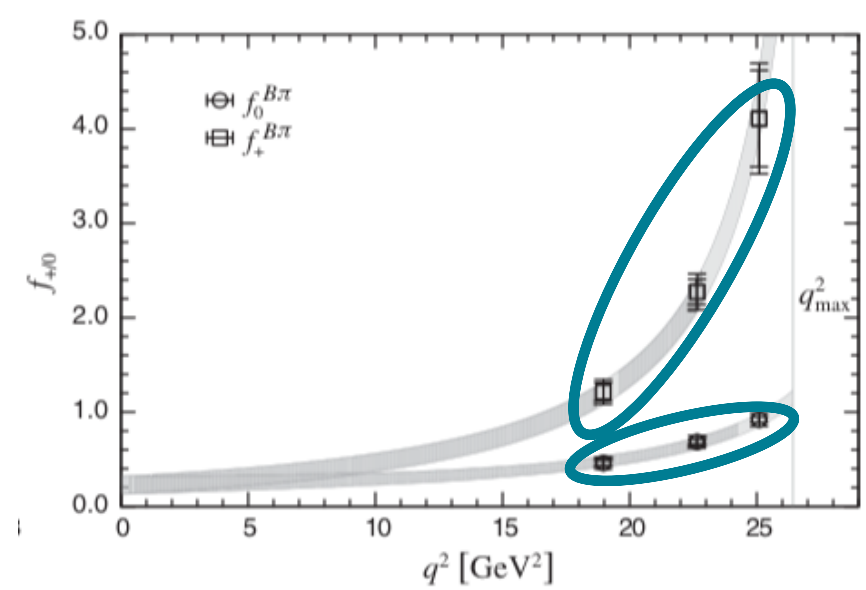
$$q^2 = (E_i - E_f)^2 - (\vec{p}_i - \vec{p}_f)^2$$

e.g. $B \to \pi l \nu$ decay $q^2_{\text{max}} = (m_B - m_\pi)^2 \approx 26.4 GeV^2$

E.g. on $L = 4fm$ lattice lowest Fourier modes lead to

$\lvert \vec{n} \rvert^2$	0	1	2	3	4
E_π/GeV	0.139	0.338	0.457	0.551	0.631
q^2/GeV^2	26.4	24.3	23.1	22.1	21.2



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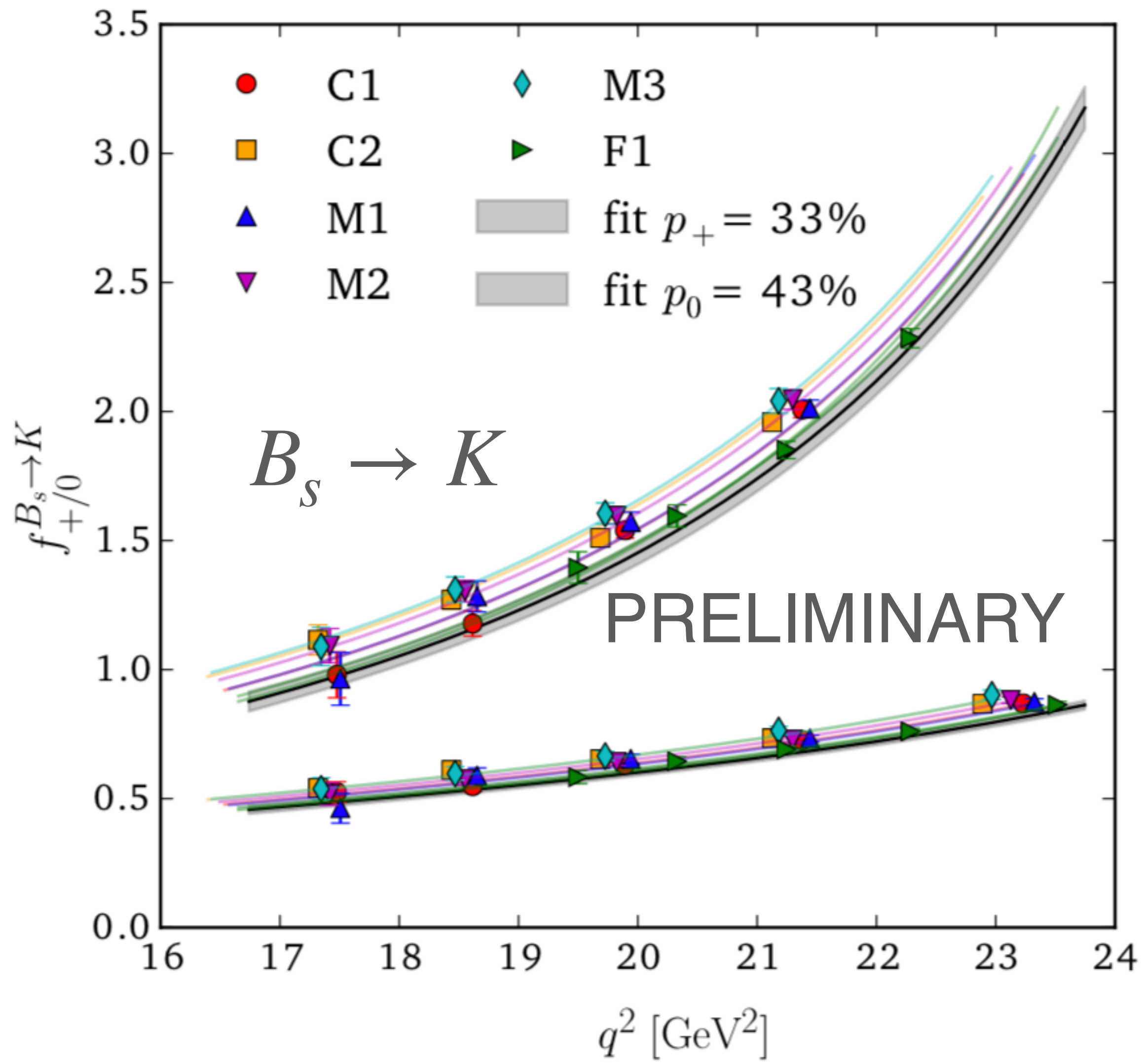
There is limited reach for small lattice momentum!

(Chris Bouchard et al. are pushing to large momenta Bouchard@Lattice2019)

Extrapolation of lattice data



a, L, m_l, m_h extrapolations

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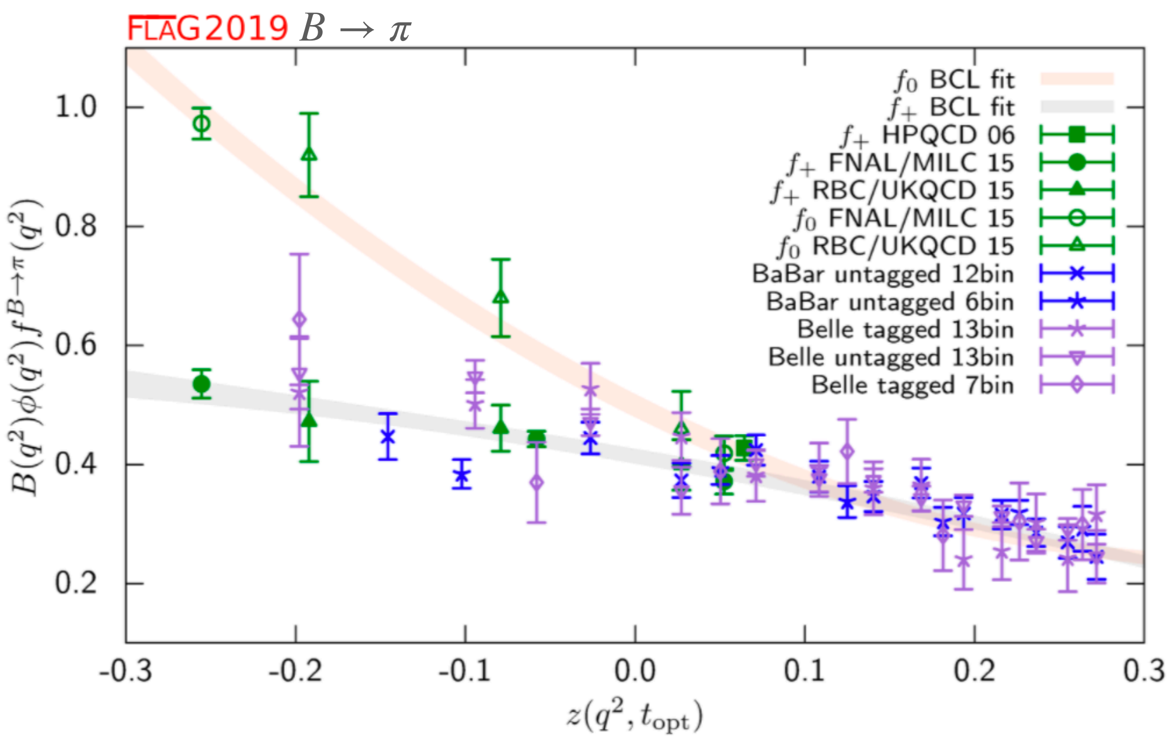


a, L, m_l, m_h extrapolations

extrapolate into into kinematic region inaccessible by lattice data (model/EFT/z-parametrisation)

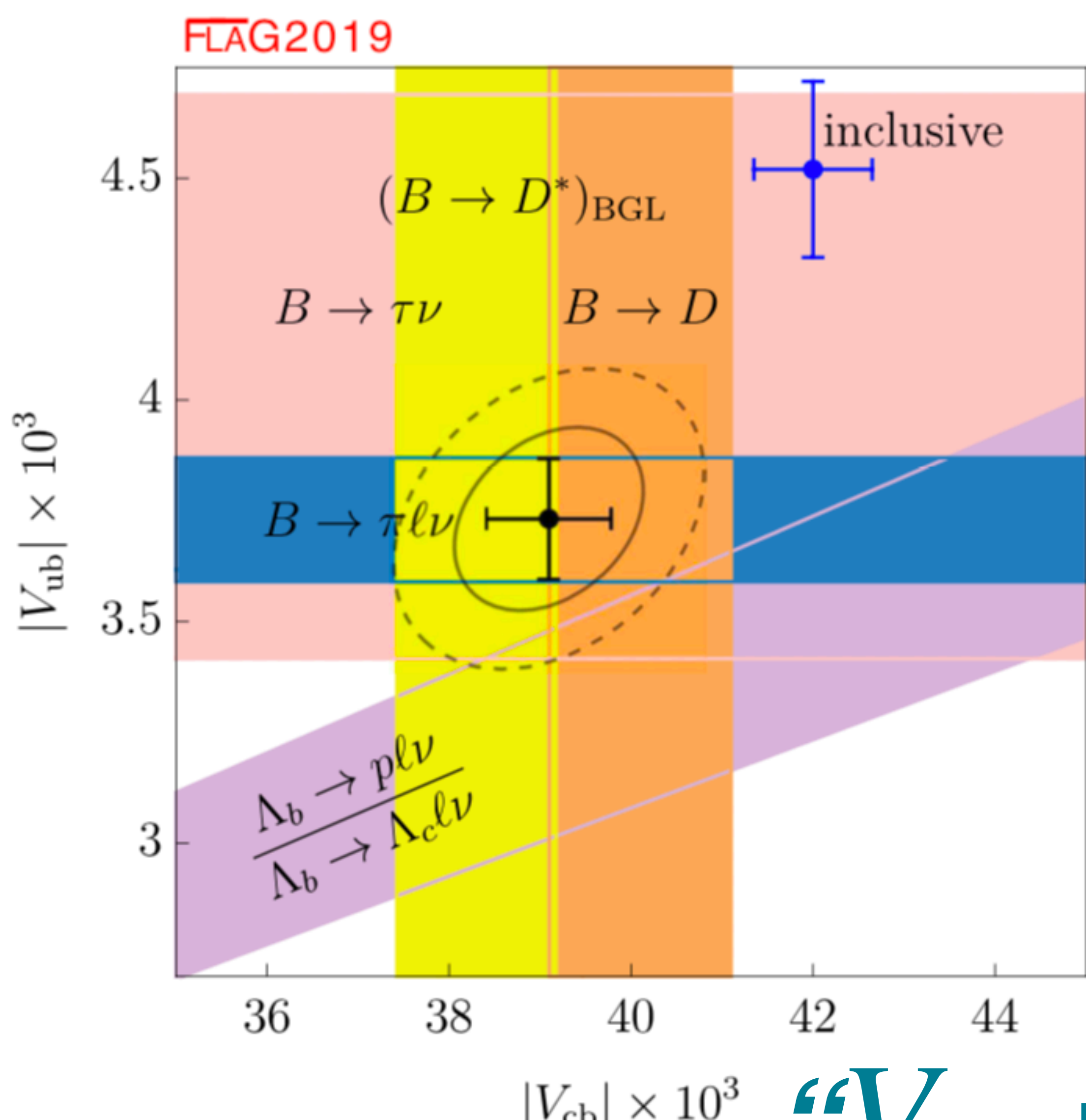
Ideally want model-independent parametrisation of form factor with QFT constraints

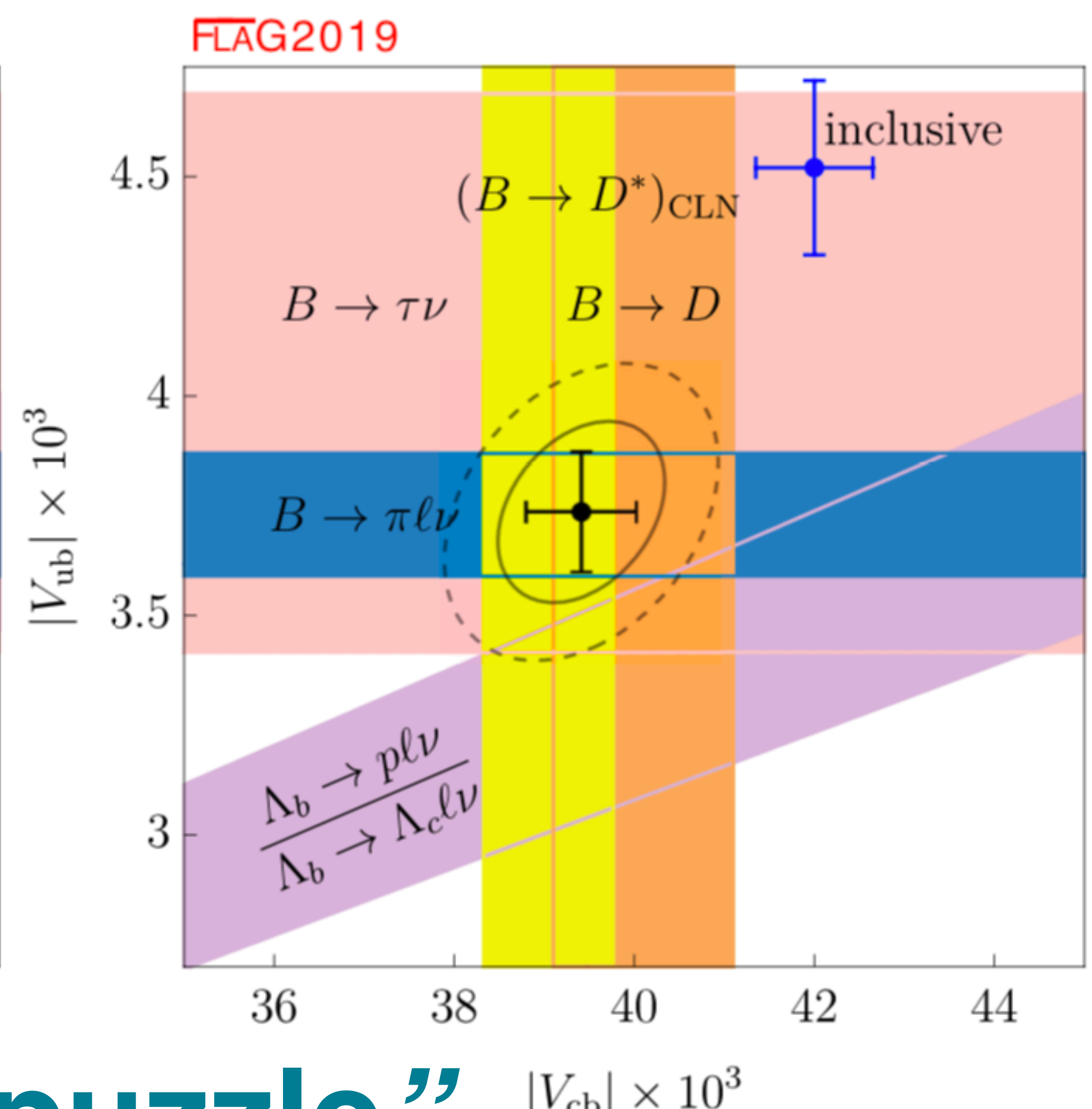
$B \to D^{(*)} l\nu$

an instructive example

- $|V_{cb}|$ — unitarity, indirect CP-violation in K mixing —
  CKM ME constitutes dominant error, not matrix element!

~3σ tension between inclusive and exclusive





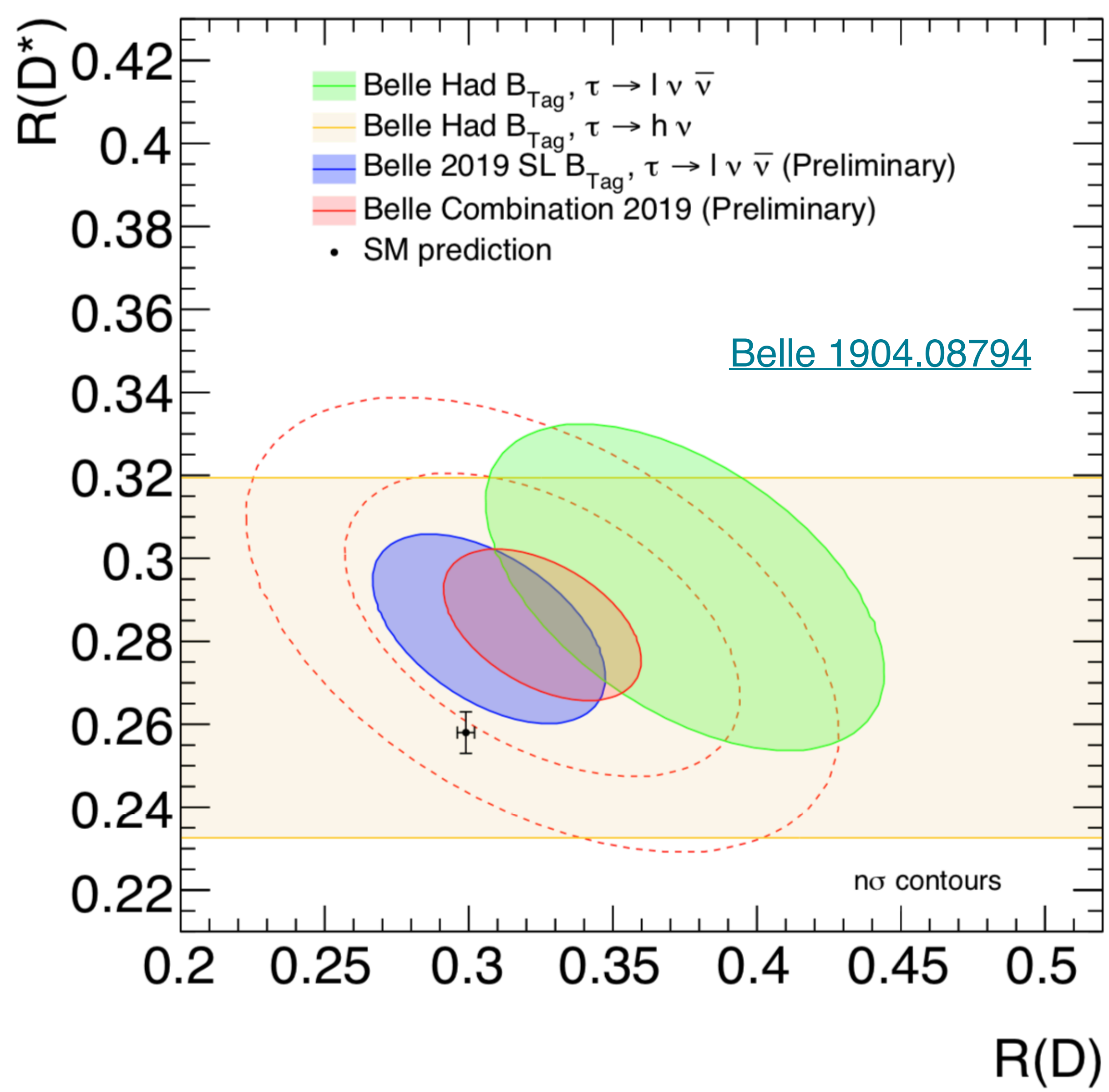
"V_{cb} puzzle"

$B \to D^{(*)} l\nu$
an instructive example

$$R(D^*) = \frac{\mathcal{B}(B \to D^* \tau \nu_\tau)}{\mathcal{B}(B \to D^* l \nu_l)}$$

- Tree-level decay
- Test of lepton-flavour universality
- Ratios are great
- Lepton-flavour ratios —
  2-3σ tension exp. vs. SM

Loads of speculation…..



Experimental prospects for $B \to D^* l \nu$

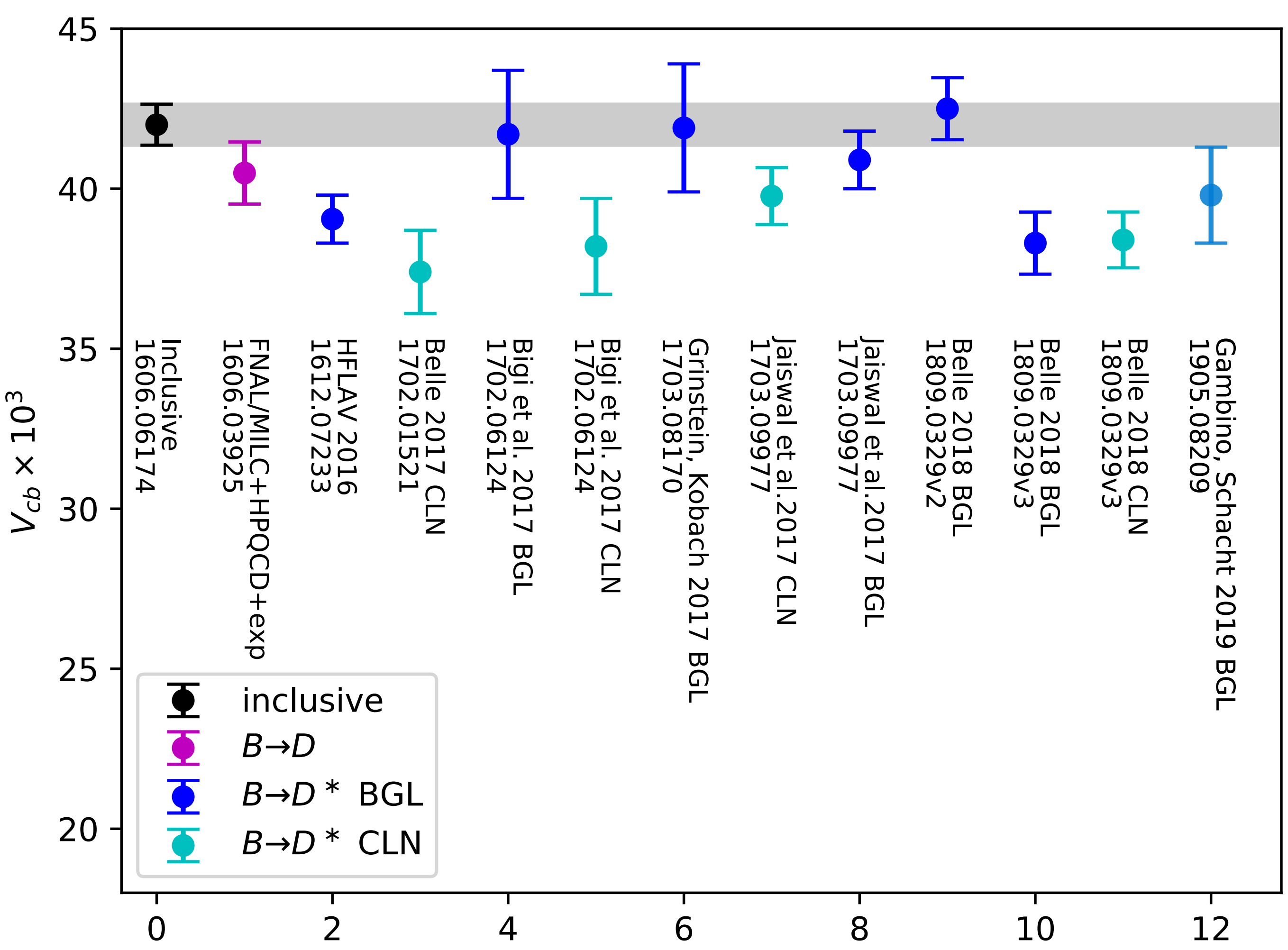
	Belle	BelleII ($5ab^{-1}$)	BelleII ($50ab^{-1}$)
Year		2021	2025
V_{cb} excl.	3.3%	1.8%	1.4%
V_{cb} incl.	1.8%	1.2%	
R(D)	16.5%	6%	3%
R(D*)	7.5%	3%	2%

Belle II Physics Book arXiv:1808.10567

LHCb $B \to D^{(*)}$ predictions aims at 2.5% for $R(D^*)$ with Upgrade II (2030ish)

$B \to D^{(*)} l\nu$
an instructive example

Recent analysis of BaBar/Belle + theory for V_{cb}:



Result seems unsettled - differences due to experimental analysis and form-factor parametrisation

Form factor parametrisation

Consider transition $Q \to q$ as mediated by current $J^\mu = \bar{Q}\Gamma q$

The corresponding vacuum polarisation tensor is

$$\Pi_J^{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle 0\,|\,TJ^\mu(x)J^\nu(0)^\dagger\,|\,0\rangle = \frac{1}{q^2}\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)\Pi_J^T(q^2) + \frac{q^\mu q^\nu}{q^2}\Pi_J^L(q^2)$$

And related subtracted dispersion relations

$$\chi_J^L(q^2) \equiv \frac{\partial \Pi_J^L}{\partial q^2} = \frac{1}{\pi}\int_0^\infty dt\,\frac{\mathrm{Im}\Pi_J^L(t)}{(t-q^2)^2} \qquad \chi_J^T(q^2) \equiv \frac{1}{2}\frac{\partial^2 \Pi_J^L}{\partial (q^2)^2} = \frac{1}{\pi}\int_0^\infty dt\,\frac{\mathrm{Im}\Pi_J^T(t)}{(t-q^2)^3}$$

χ can be evaluated in PT for suitable q^2

Form factor parametrisation

Spectral functions $\quad \mathrm{Im}\Pi_J^{T,L}(q^2) = \frac{1}{2}\sum_X (2\pi)^4\delta(q-p_X)\,|\langle 0|J|X\rangle|^2$

e.g. $X = BD^* \rightarrow \langle 0|J|BD^*\rangle \rightarrow \langle D^*|J|B\rangle$

cross. symm. $\rightarrow F^{B\rightarrow D^*}$

This allows us to constrain form factor:

$$\chi_J^T(q^2) \equiv \frac{1}{2}\frac{\partial^2\Pi_J^L}{\partial(q^2)^2} = \frac{1}{\pi}\int_0^\infty dt\,\frac{\mathrm{Im}\Pi_J^T(t)}{(t-q^2)^3} \longrightarrow \frac{1}{\pi\chi^T}\int_{t_+}^\infty dt\,\frac{W(t)\,|F(t)|^2}{(t-q^2)^3} \leq 1$$

$$t_\pm = (M \pm m)^2$$

Form factor parametrisation

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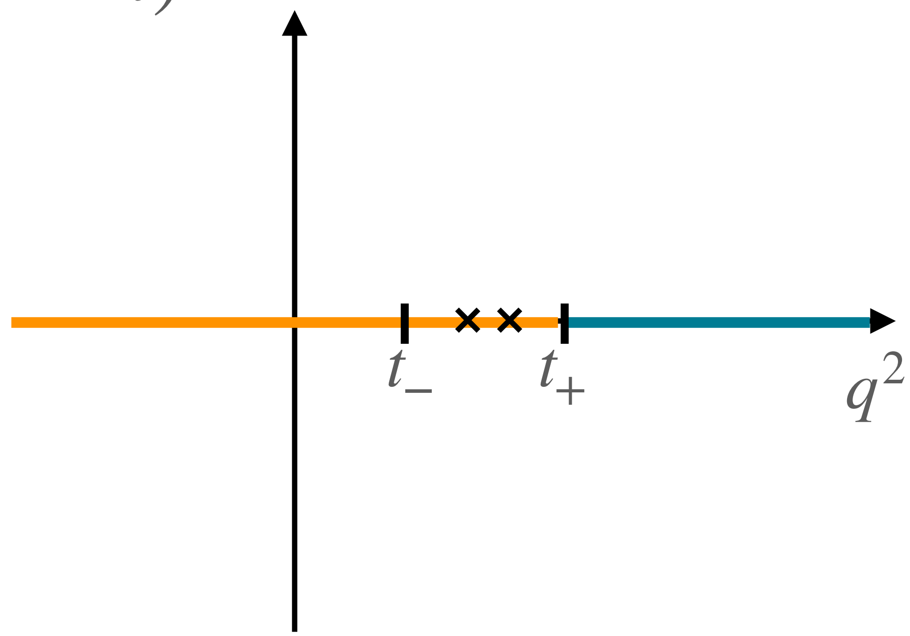

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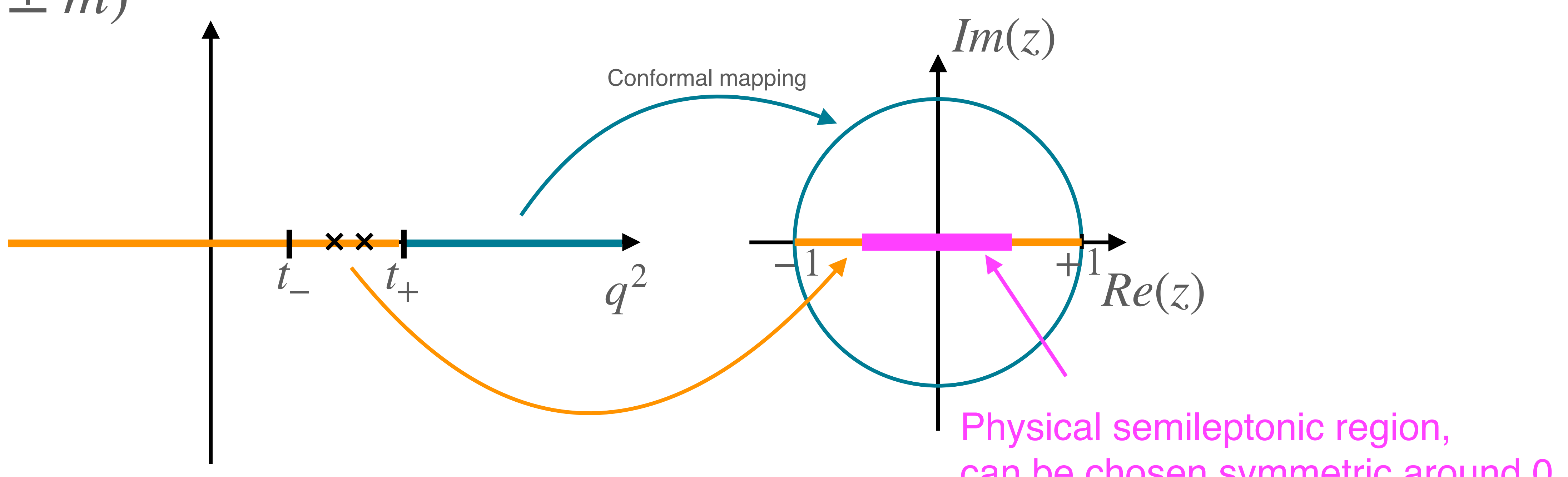

$$z(q^2, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$

Form factor parametrisation

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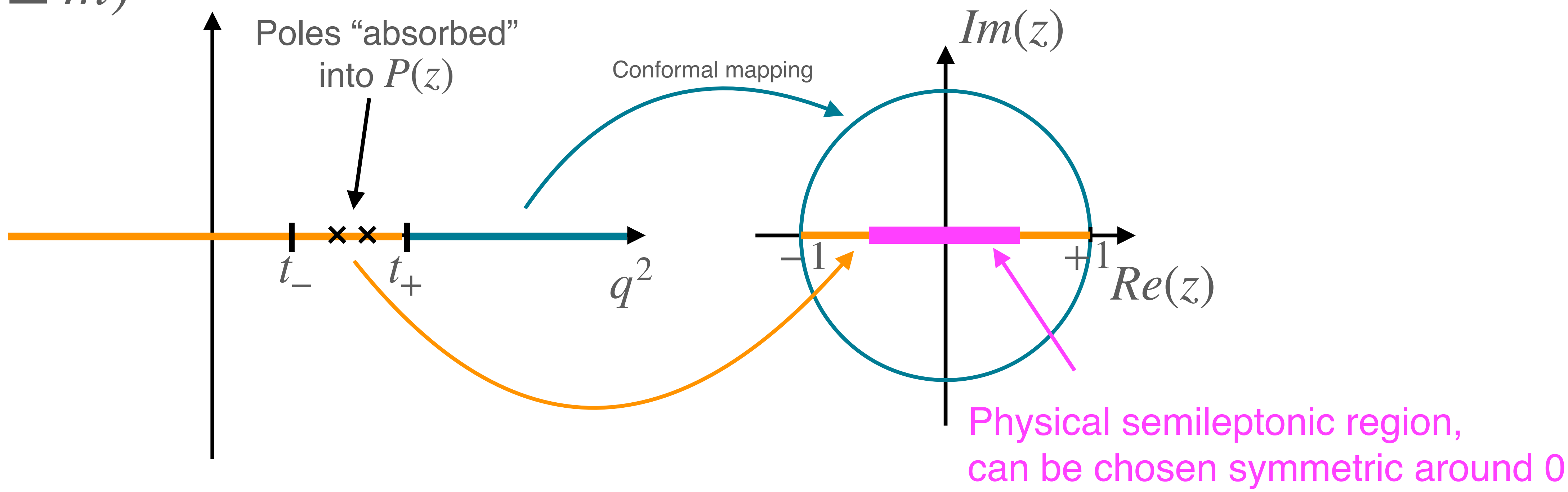
$$\frac{1}{2\pi i}\int_C \frac{dz}{z}\,|\phi(z)P(z)F(z)|^2 \leq 1$$

$$t_\pm = (M \pm m)^2$$



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Form factor parametrisation

z real in physical SL $B \to D^* l\nu$ region:

$q^2 \in \{0,10.7\}\text{GeV}^2 \to z \in \{-0.028,0.028\}$

Suited for polynomial expansion in z

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$F(t) = \frac{1}{|P(t)\phi(t;t_0)|}\sum_{n=0}^{\infty} a_n z(t;t_0)^n$ unitarity constraint $\sum_{n=0}^{\infty} a_n^2 \leq 1$

Boyd, Grinstein, Lebed (BGL) PRL 74 23 1995

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- **BGL** "original"

Boyd, Grinstein, Lebed (BGL) PRL 74 23 1995

- **CLN** with HQET constraints ($B^{(*)} \to D^{(*)}$) $O(\alpha_S, 1/m)$ Sum rules

Caprini, Lellouch, Neubert (CLN) NPB 530 1998

- **BCL** like BGL with fixes for finite truncation

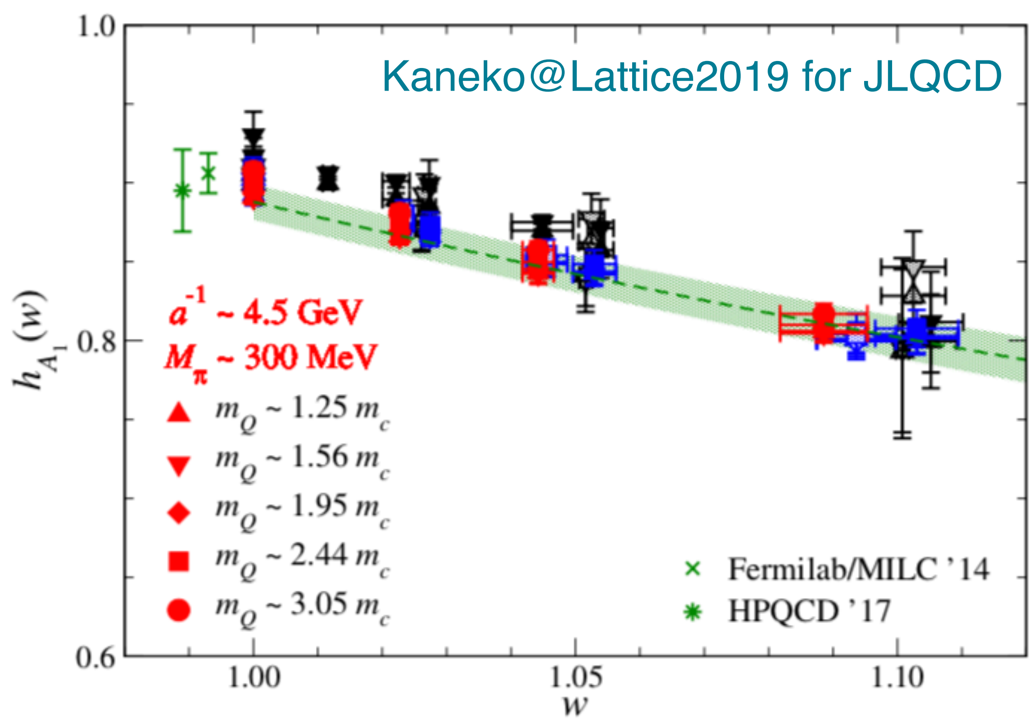
Bourrely, Caprini, Lellouch, (BCL) PRD 82 099902 2010

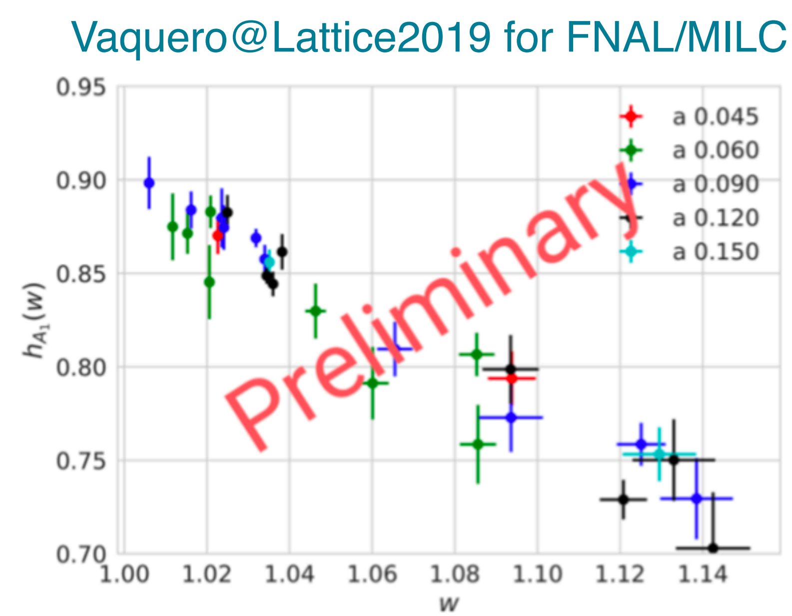
Two questions

1. **Are there still unresolved issues with BGL/CLN/BCL?**
   (Lattice simulations for $B \to D^* l\nu$ should come out shortly and help shed light on the slightly messy recent past of exclusive $|V_{cb}|$ results)

   So far only zero-recoil published FNAL/MILC PRD 89 2014, HPQCD 97 2018

   Results at non-zero recoil will shed light on the parametrisation puzzle
   JLQCD Kaneko@Lattice 2019, FNAL/MILC Vaquero@Lattice2019, LANL-SWME arXiv:1711.01786, 1812.07675





Two questions

2. Two extrapolation philosophies are being followed:

   A. First extrapolate lattice data to $a = 0$ and $L = \infty$, to physical m_π, etc.… and only then do z-fit (BGL, CLN, BCL, …)(

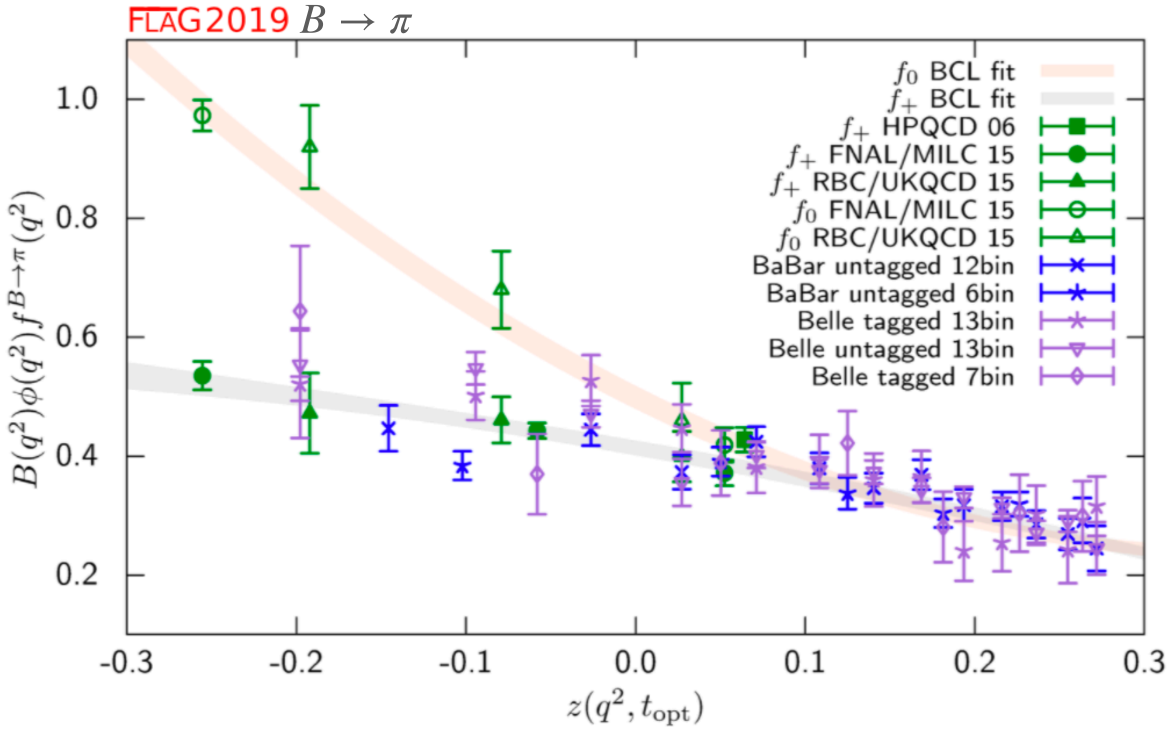
   B. Combine z-fit and above extrapolations in "modified z-expansion" (HPQCD):

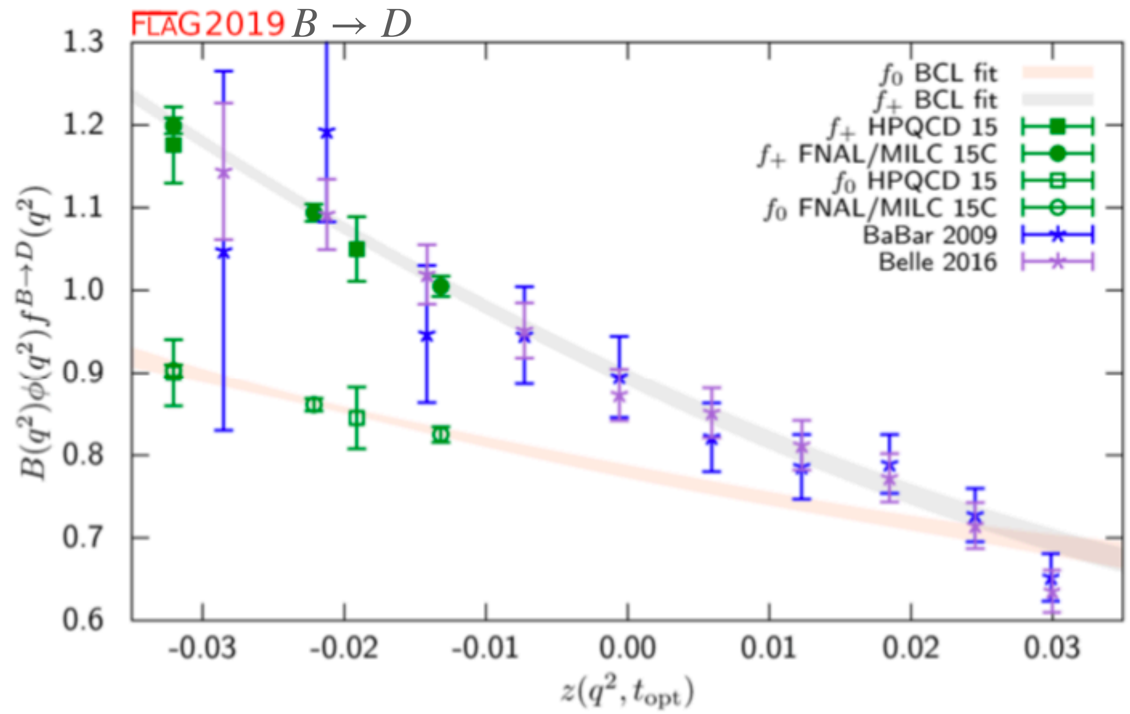
$$f(t) = \frac{1}{|P(t)\phi(t;t_0)|} \sum_{n=0}^{\infty} a_n(a, m_l, \ldots)\, z(t;t_0)^n$$

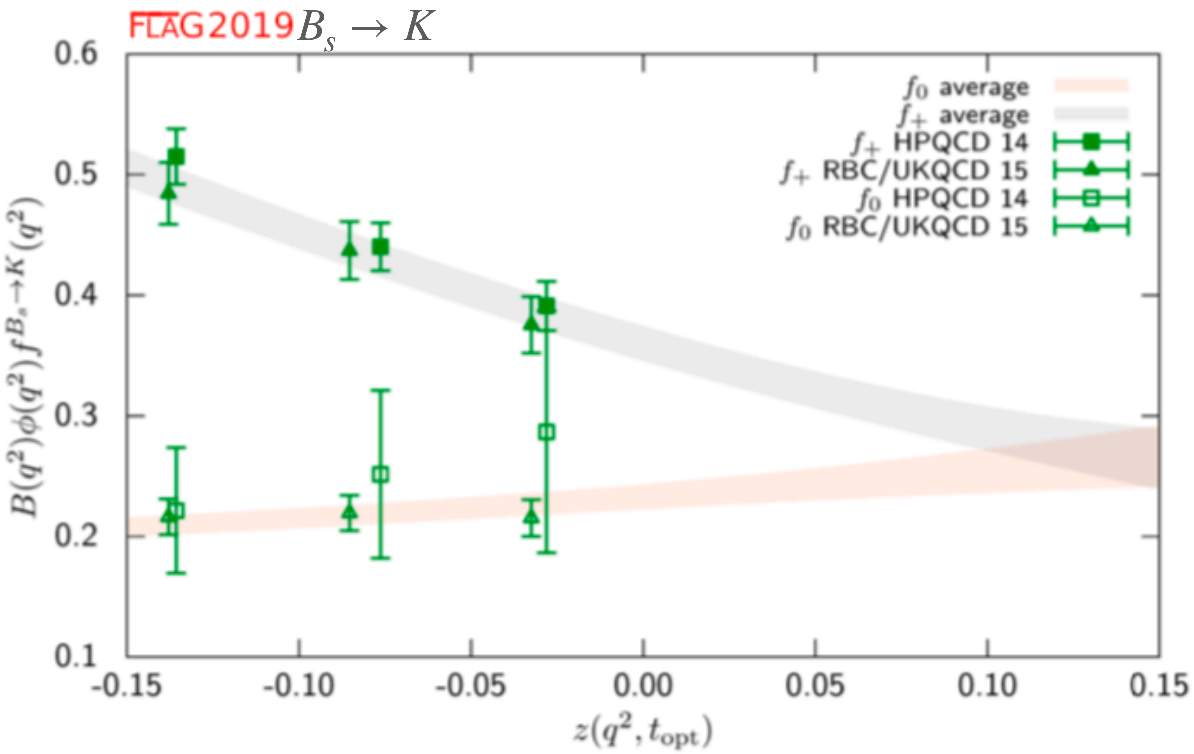
   Is it clear that it still works given conformal map, Blaschke-factor and outer function depend on QCD spectrum?

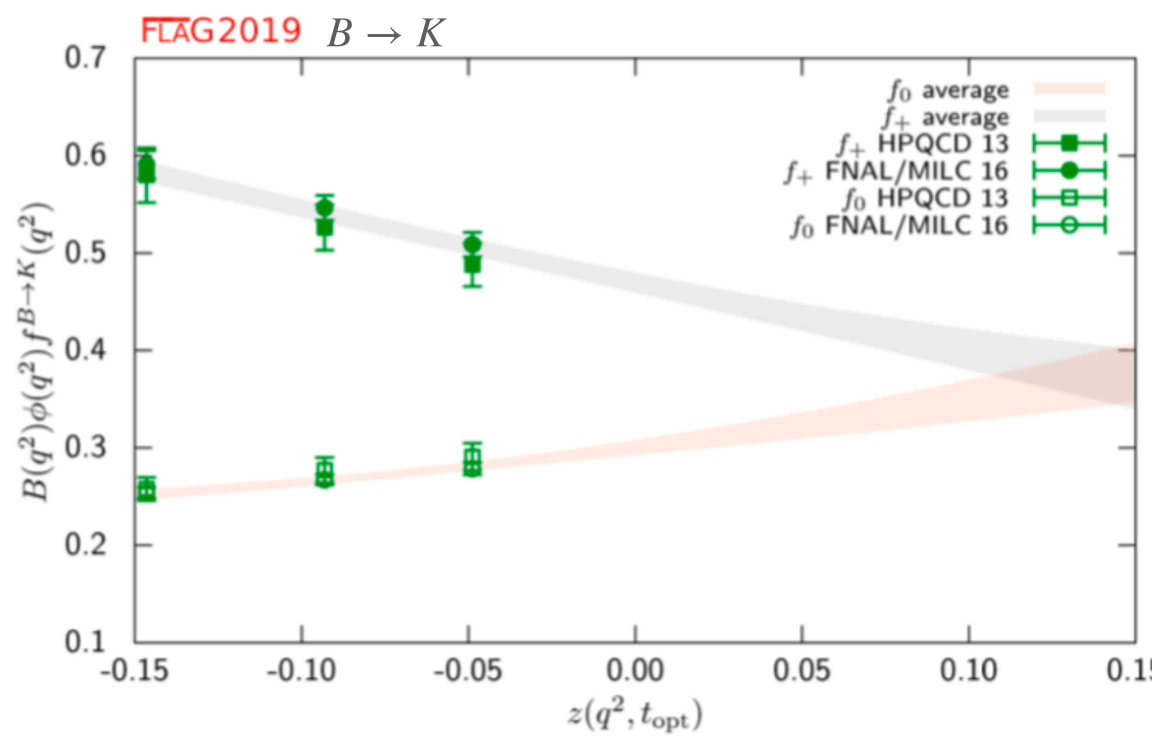
Quick summary of other CKM channels

Other FF calculations

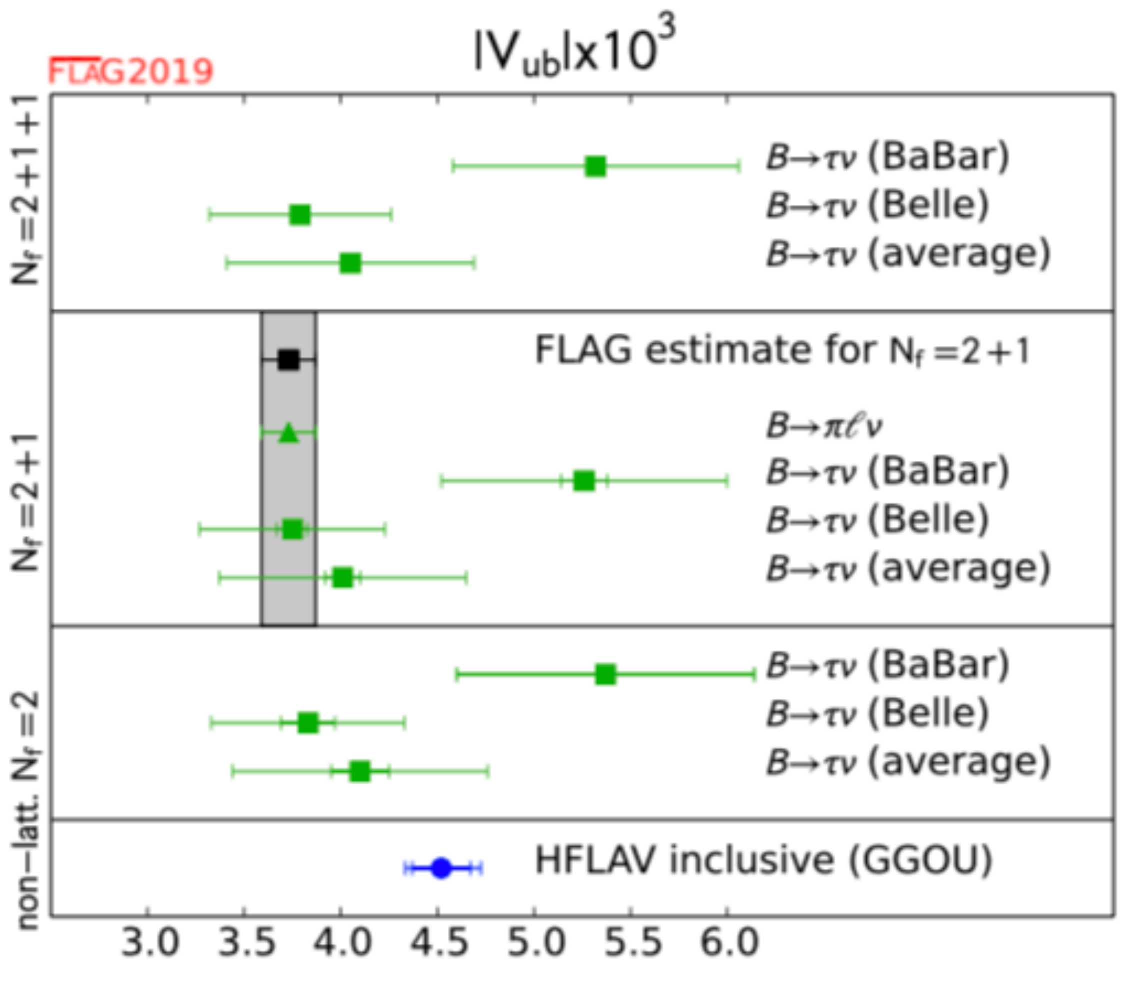


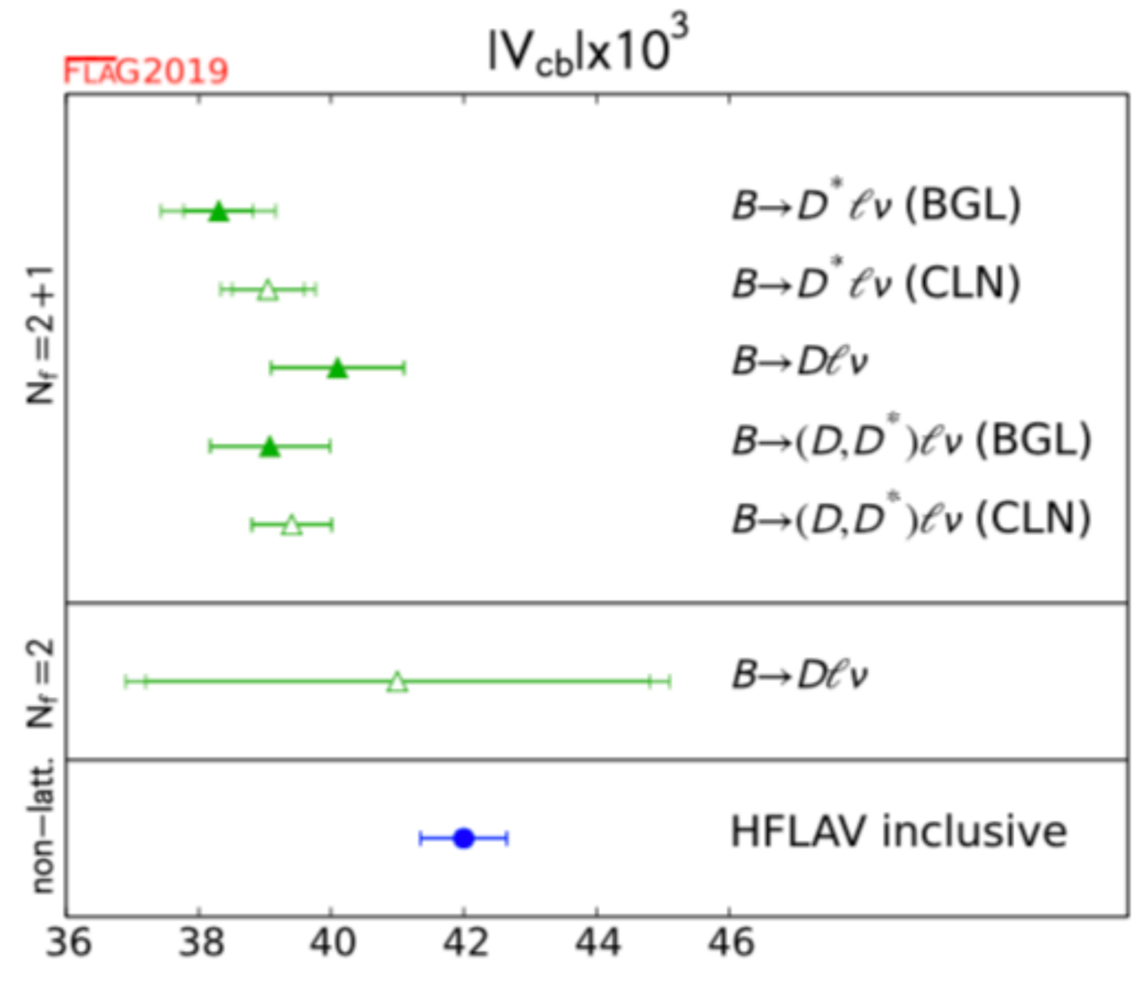






Lattice results for $|V_{ub}|, |V_{cb}|$





New directions

Rare Kaon decays

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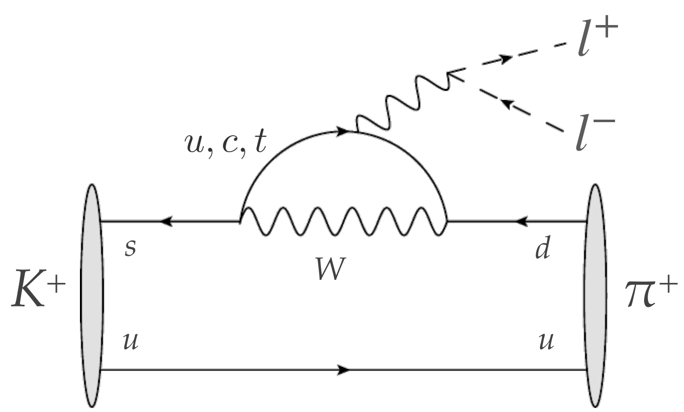
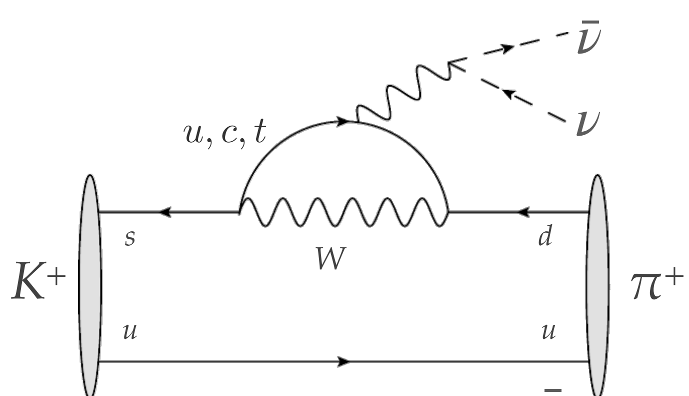
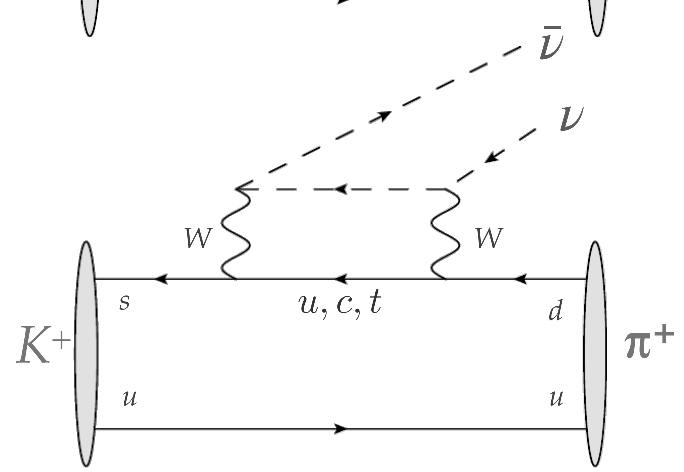






loop suppressed in the SM (FCNC via W-W or γ/Z-exchange diagrams)

hard to observe in nature deep probe into flavour mixing and SM/BSM

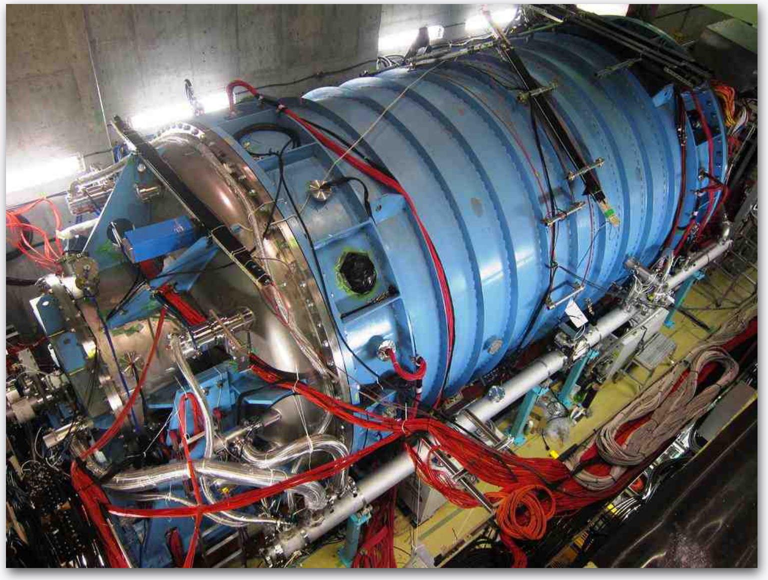
J-PARC's KOTO and CERN's NA62 are measuring these decays

results expected on the time scale of 5 years

Experiments

$K_L \to \pi^0 \nu \bar{\nu}$

- KOTO (J-PARC)
- direct CP violation
- GIM → top dominated and charm suppressed, pure SD
- phase 2 aims at 10% measurement of BR



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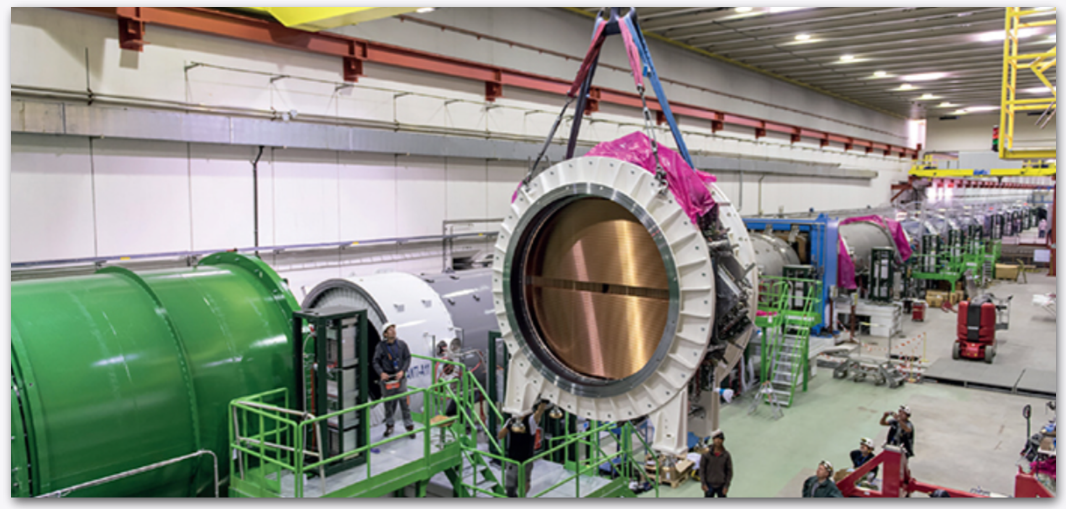

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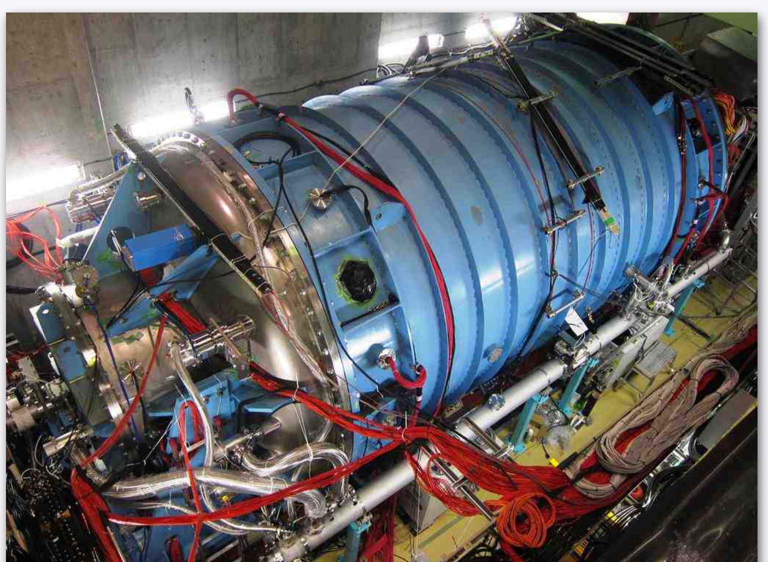

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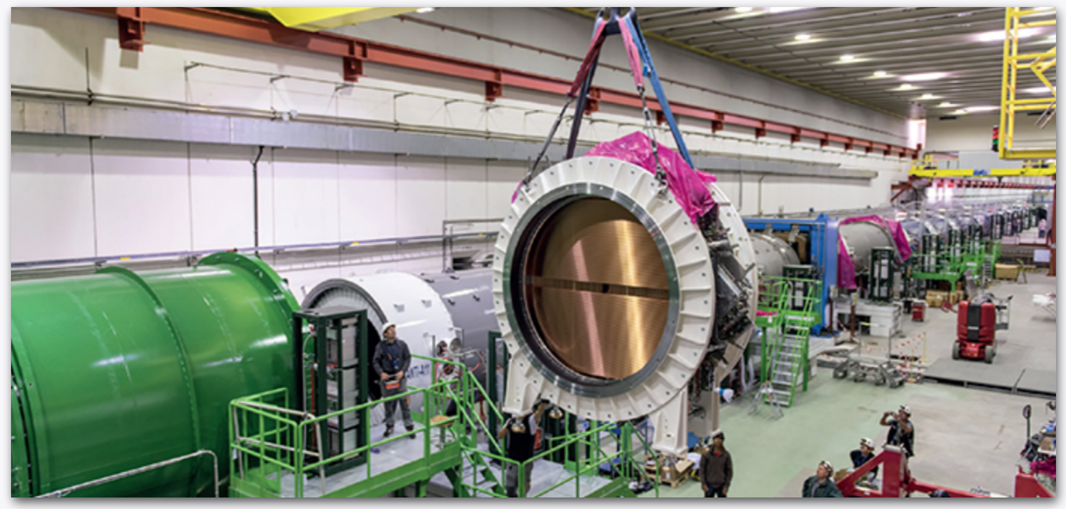

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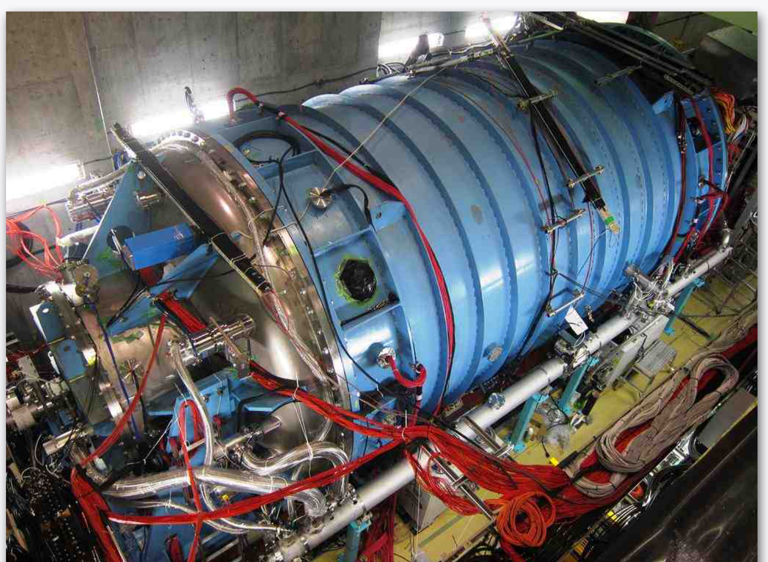

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- 1-photon exchange LD dom.
- SM prediction mainly ChPT
- lattice can predict ME and LECs
- well suited for experiment

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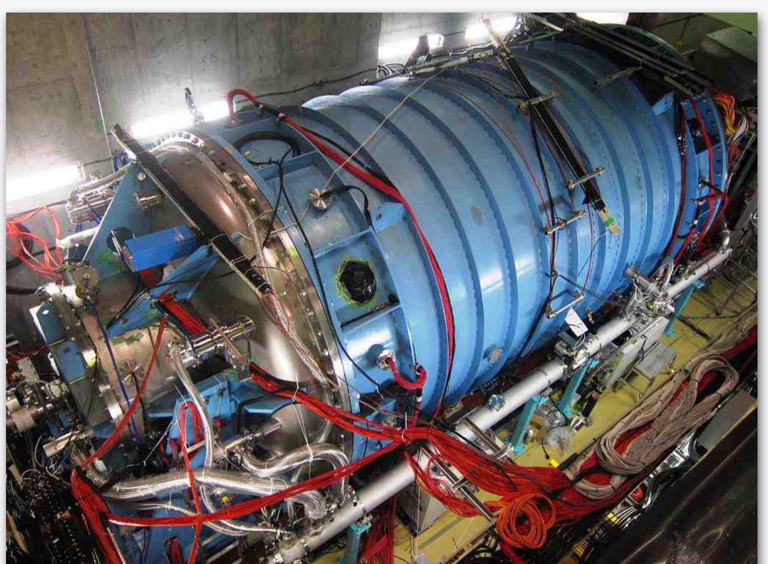

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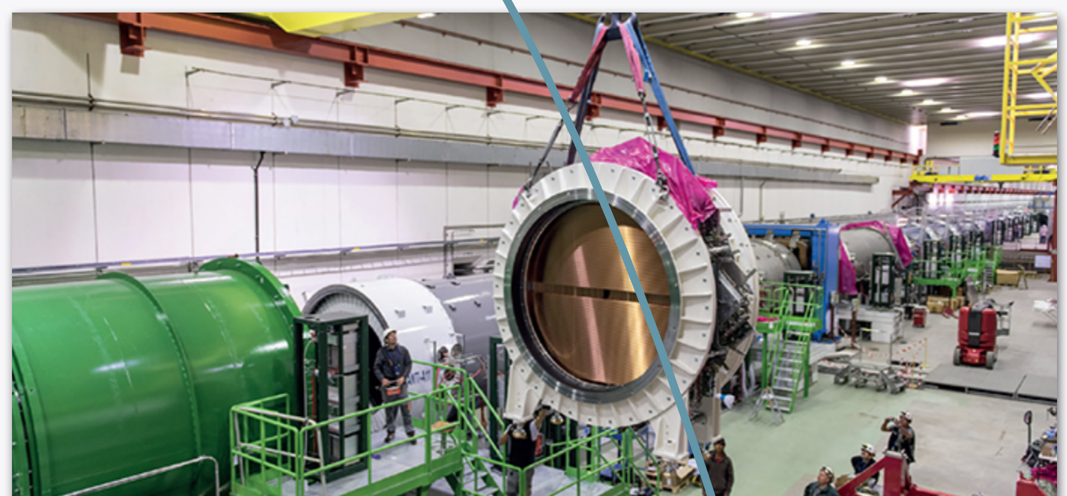

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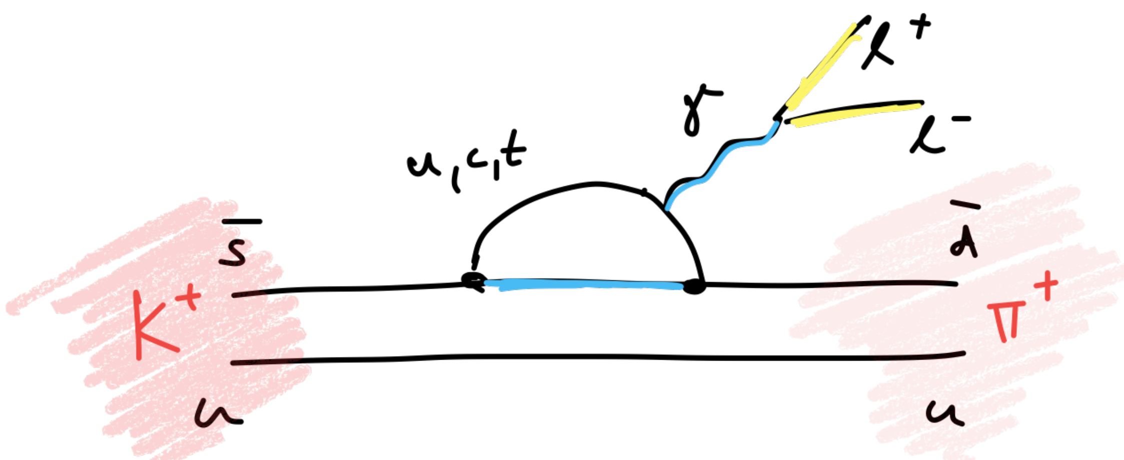
candidates for lattice computation

2nd order weak processes

consider $K^+ \to \pi^+ l^+ l^-$ with dominant 1-photon contribution:



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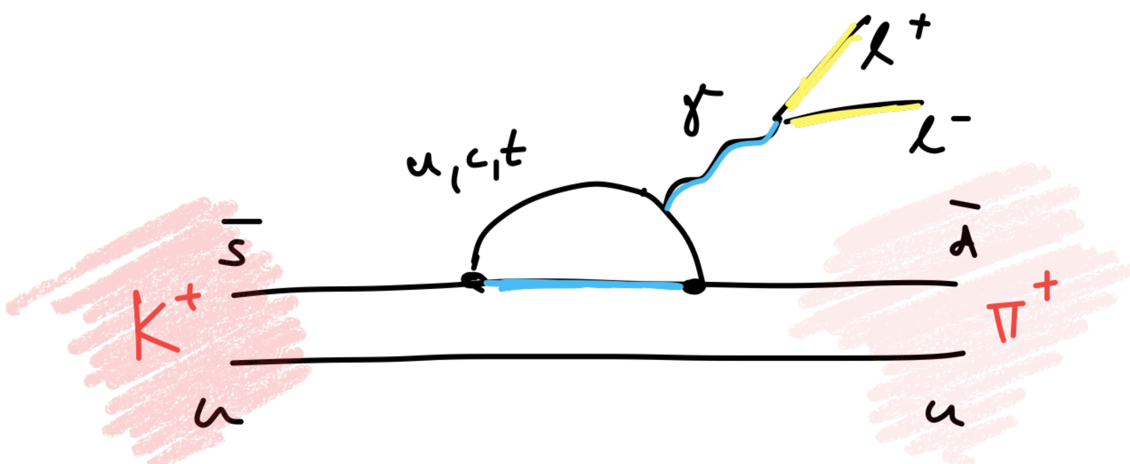

2nd order weak decay
→ 2 insertions of H_W/J_μ

$$\mathcal{A}_\mu = (q^2) \int d^4x \langle \pi(p) | T\left[J_\mu(0) H_W(x)\right] | K(k) \rangle$$

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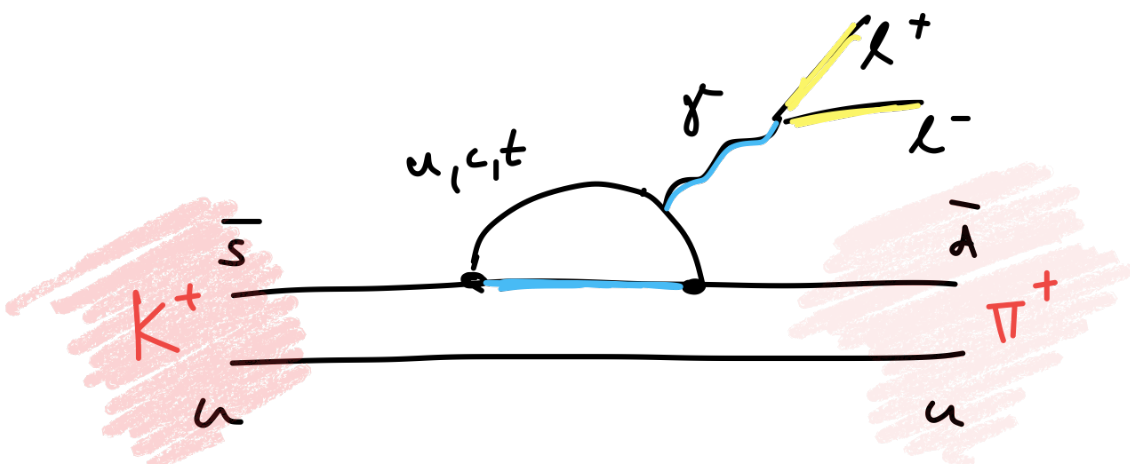

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This is not about precision — it's about being able to do it!

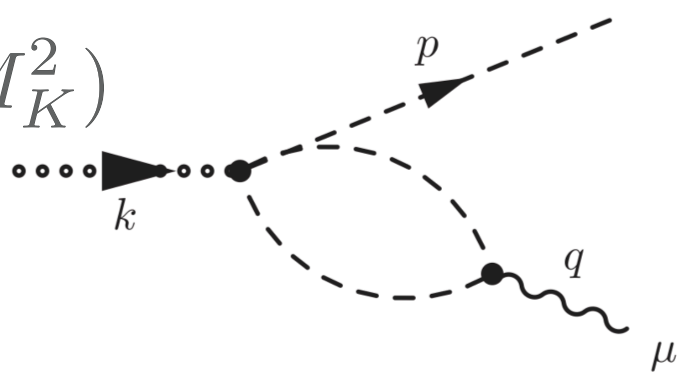
$K^+ \to \pi^+ l^+ l^-$ form factor

Decay amplitude in terms of elm. transition form factor:

$$\mathcal{A}_\mu^{c=+,S}(q^2) = -i\frac{G_F}{4\pi}^2 \left[q^2(k+p)_\mu - (M_K^2 - M_\pi^2)q_\mu\right] V_c(q^2/M_K^2)$$

D'Ambrosio et al., JHEP 9808, 004 (1998)

$$V_c(q^2/M_K^2) = a_c + b_c q^2/M_K^2 + V_c^{\pi\pi}(q^2/M_K^2)$$



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$c=+,S$

D'Ambrosio et al., JHEP 9808, 004 (1998)

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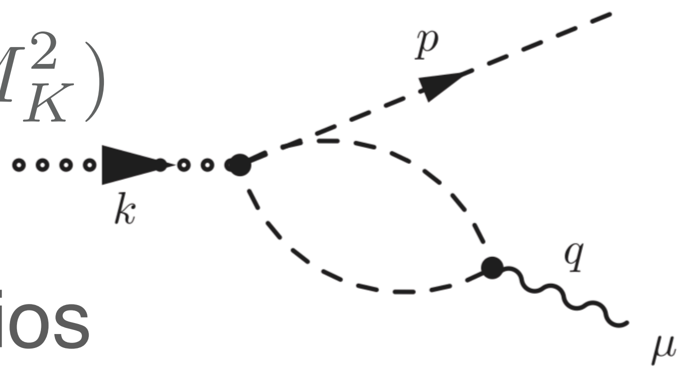


- the $|a_S|$ and $|a_+|$ can be extracted from branching ratios
- a_S parameterises also the CP-violating contribution to the K_L BR
- sign of a_S unknown - could be predicted by lattice — plays crucial role in BR prediction for $K_L \to \pi^0 e^+ e^- / \mu^+ \mu^-$

Difficulties

Difficulties

1. **Spectral representation:** Euclidean space intermediate states lead to artefacts that need to be controlled

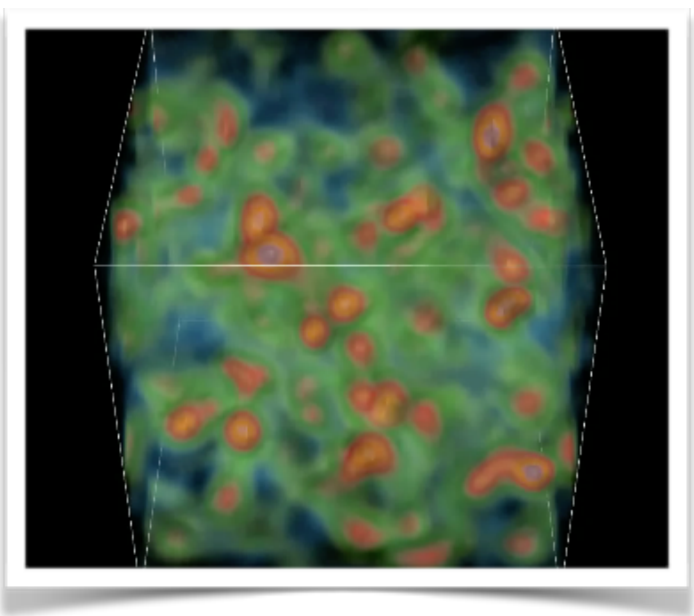
2. **Renormalisation:** EW operator contact terms lead to UV div.

3. **Finite volume effects:** The finite-volume corrections from intermediate on-shell states can be large

Isidori et al. PLBB 633 (2006) 75-83, **Christ et al.** PRD91 (2015), 114510
RBC/UKQCD PRD92 (2015) 094512, PRD94 (2016) 114516, PRD93 (2016) 114517, PRL118 (2017) 252001, arXiv:1806.11520

EXPLORATORY STUDY - Lattice setup

RBC/UKQCD exploratory study



- domain wall fermions (24^3, a~0.12fm)
- m_π~430MeV, m_K~625MeV
  $E_K(k)<2M_\pi \rightarrow$ only one-π intermediate state
- unphysically light charm quark mass
  m_c~533MeV
- no disconnected diagrams
- kaon at rest

Spectral representation - Minkowski

$$\mathcal{A}^c_\mu(q^2) = \int d^4x \langle \pi^c(p) | T\left[J_\mu(0) H_W(x)\right] | K^c(k) \rangle$$

Spectral representation - Minkowski

$$\mathcal{A}^c_\mu(q^2) = \int d^4x \langle \pi^c(p)| T\left[J_\mu(0) H_W(x)\right] |K^c(k)\rangle$$

$$\begin{aligned}\mathcal{A}^c_\mu(q^2) =& i\int_0^\infty dE \frac{\rho(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k}) - E + i\epsilon} \\ & -i\int_0^\infty dE \frac{\rho_S(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E - E_\pi(\mathbf{p}) + i\epsilon}\end{aligned}$$

Spectral representation - Minkowski

$$\mathcal{A}^c_\mu(q^2) = \int d^4x \langle \pi^c(p)| T\left[J_\mu(0) H_W(x)\right] |K^c(k)\rangle$$

non-strange intermediate states

$$\mathcal{A}^c_\mu(q^2) = i\int_0^\infty dE \frac{\rho(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k}) - E + i\epsilon}$$

$$-i\int_0^\infty dE \frac{\rho_S(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E - E_\pi(\mathbf{p}) + i\epsilon}$$

strange intermediate states

Spectral representation - Minkowski

$$\mathcal{A}^c_\mu(q^2) = \int d^4x \langle \pi^c(p)| T\left[J_\mu(0) H_W(x)\right] |K^c(k)\rangle$$

non-strange intermediate states

$$\mathcal{A}^c_\mu(q^2) = i\int_0^\infty dE \frac{\rho(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k}) - E + i\epsilon}$$

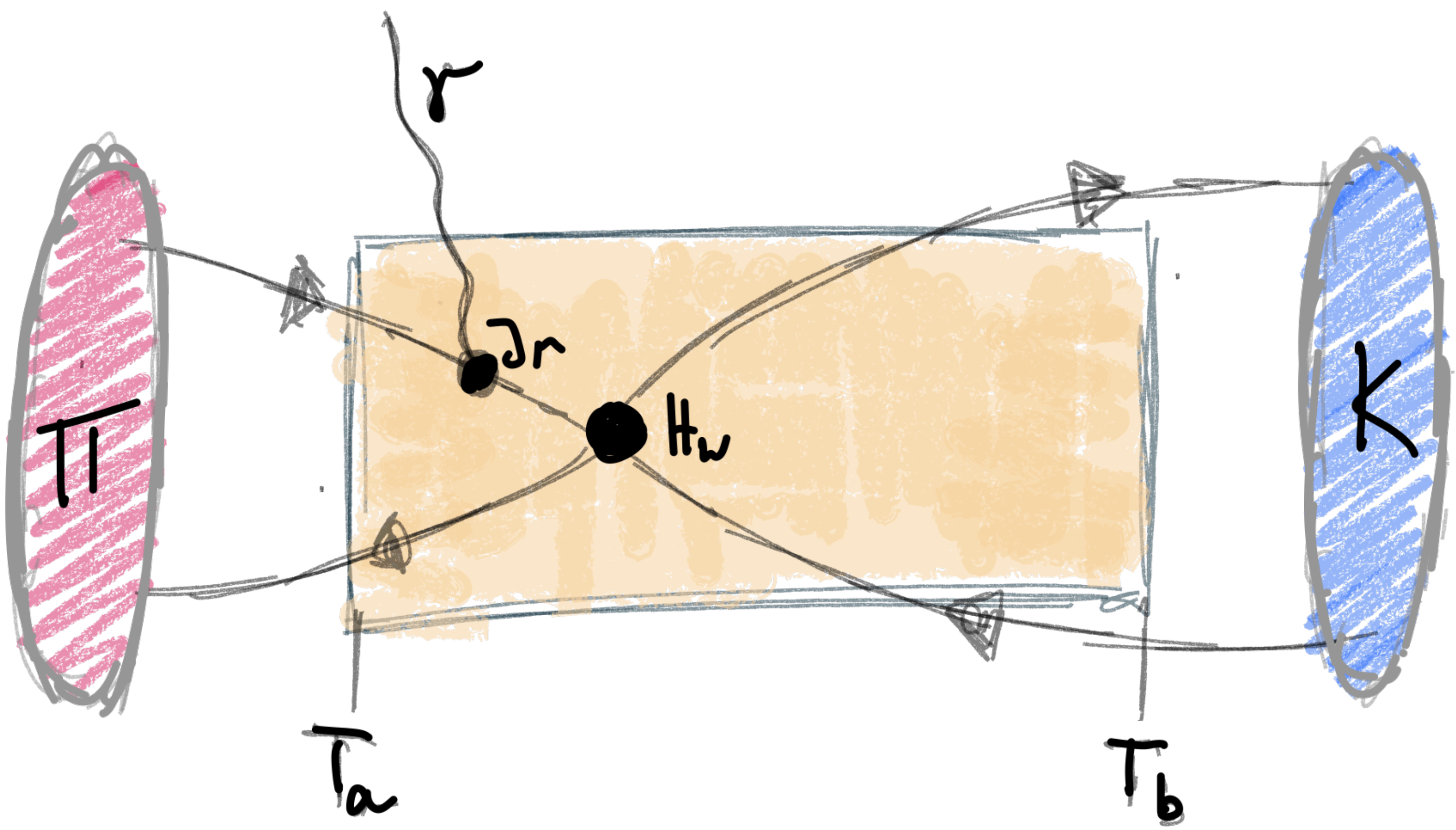
$$- i\int_0^\infty dE \frac{\rho_S(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E - E_\pi(\mathbf{p}) + i\epsilon}$$

strange intermediate states

complications arise when considering the amplitude in Euclidean space …

Spectral representation - Euclidean

$$\mathcal{A}^c_\mu(q^2) = \int d^4x \langle \pi^c(p)| T\left[J_\mu(0) H_W(x)\right] |K^c(k)\rangle$$



integrate EW operators over T_a-T_b

Spectral representation - Euclidean

$$A^c_\mu(T_a, T_b, q^2) = \int_0^\infty dE \frac{\rho(E)}{2E} \frac{\langle \pi^c(\mathbf{p}) | J_\mu(0) | E, \mathbf{k} \rangle \langle E, \mathbf{k} | H_W(0) | K^c(\mathbf{k}) \rangle}{E_K(\mathbf{k}) - E} \left(1 - e^{(E_K(\mathbf{k}) - E)T_a}\right)$$

$$+ \int_0^\infty dE \frac{\rho_S(E)}{2E} \frac{\langle \pi^c(\mathbf{p}) | H_W(0) | E, \mathbf{p} \rangle \langle E, \mathbf{p} | J_\mu(0) | K^c(k) \rangle}{E - E_\pi(\mathbf{p})} \left(1 - e^{-(E - E_\pi(\mathbf{k}))T_b}\right)$$

Spectral representation - Euclidean

$$A^c_\mu(T_a, T_b, q^2) = \int_0^\infty dE \frac{\rho(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k}) - E}\left(1 - e^{(E_K(\mathbf{k})-E)T_a}\right)$$

$$+ \int_0^\infty dE \frac{\rho_S(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E - E_\pi(\mathbf{p})}\left(1 - e^{-(E-E_\pi(\mathbf{k}))T_b}\right)$$

exponential in first terms on r.h.s.

- 1st line:
  - $E>E_K$: exponential term vanishes as $T_a \to \infty$
  - $E<E_K$: exponential term grows as $T_a \to \infty$, must be removed (possible intermediate states π, ππ, πππ)
- 2nd line: no problem, all intermediate states E larger E_π

Spectral representation - Euclidean

$$A^c_\mu(T_a,T_b,q^2) = \int_0^\infty dE \frac{\rho(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k})-E}\left(1-e^{(E_K(\mathbf{k})-E)T_a}\right)$$

$$+\int_0^\infty dE \frac{\rho_S(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E-E_\pi(\mathbf{p})}\left(1-e^{-(E-E_\pi(\mathbf{k}))T_b}\right)$$

subtraction of exponentially increasing states:

- π: either get amplitudes from 2pt and 3pt functions and subtract or replace

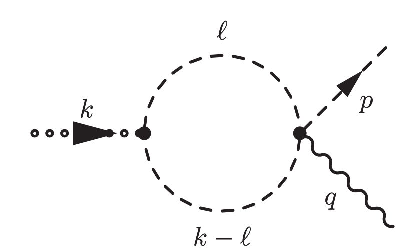
$$H_W(x) \to H'_W(x) = H_W(x) + c_S(\mathbf{k})\bar{s}(x)d(x)$$

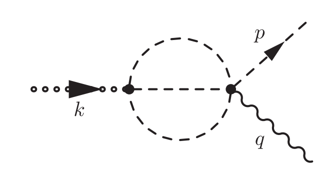
where c_S such that $\langle \pi^c(\mathbf{k})|H'_W(0,\mathbf{k})|K^c(\mathbf{k})\rangle = 0$ kills the unwanted divergent contribution and does not contribute to the amplitude itself

Spectral representation - Euclidean

subtraction of exponentially increasing states:

- $\pi\pi$: disallowed by O(4) invariance but can be present as discretisation effect — needs to be monitored
- $\pi\pi\pi$: comparison of experimental width (PDG) suggests
  - $\pi\pi\pi$ to be highly suppressed wt. respect to $\pi\pi$
  - techniques similar as for $\pi\pi$ possible but its own research topic ($K \rightarrow \pi\pi\pi$)



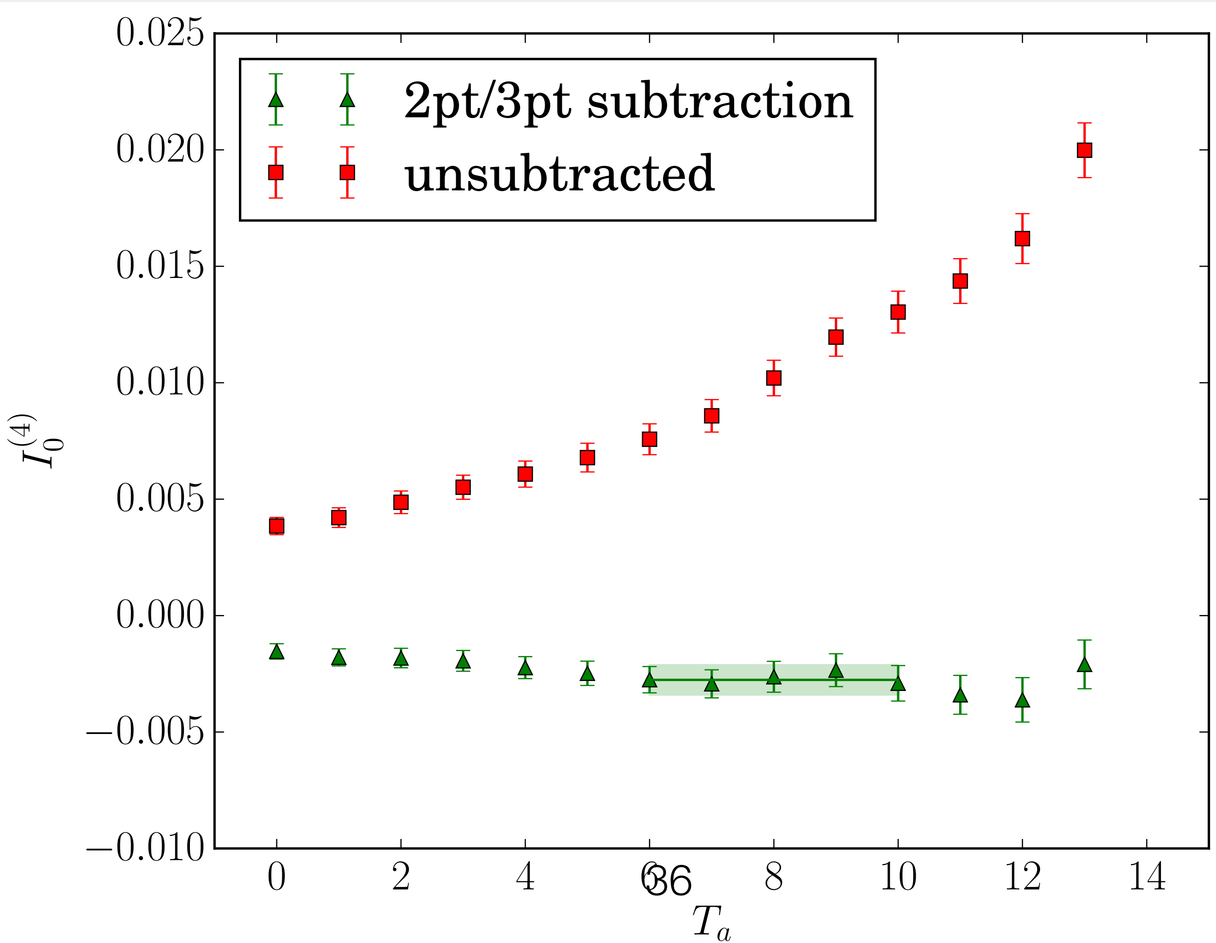


(b)

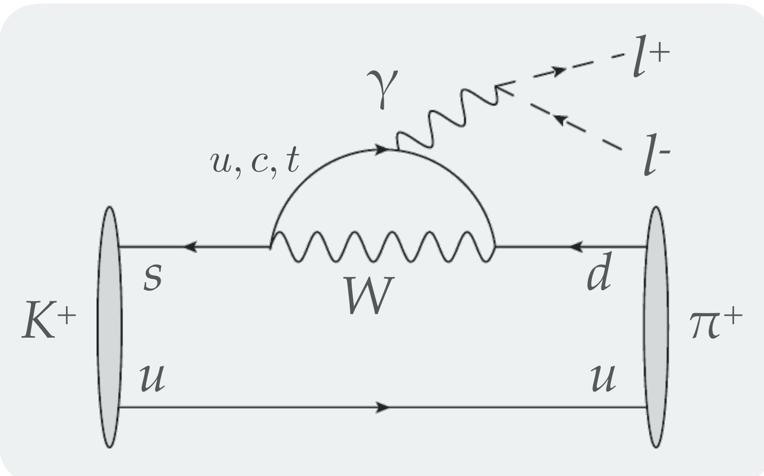
Removing the exponentially rising terms

$$A^c_\mu(T_a, T_b, q^2) = \int_0^\infty dE \frac{\rho(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k}) - E}\left(1 - e^{(E_K(\mathbf{k})-E)T_a}\right)$$

$$+ \int_0^\infty dE \frac{\rho_S(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E - E_\pi(\mathbf{p})}\left(1 - e^{-(E-E_\pi(\mathbf{k}))T_b}\right)$$



Renormalisation

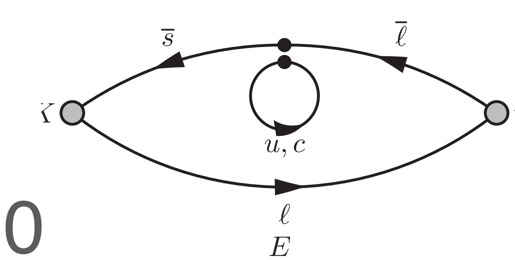


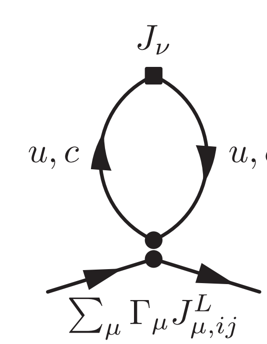
4-flavour

$$\mathcal{A}^c_\mu(q^2) = \int d^4x \langle \pi^c(p) | T\left[J_\mu(0) H_W(x)\right] | K^c(k) \rangle$$

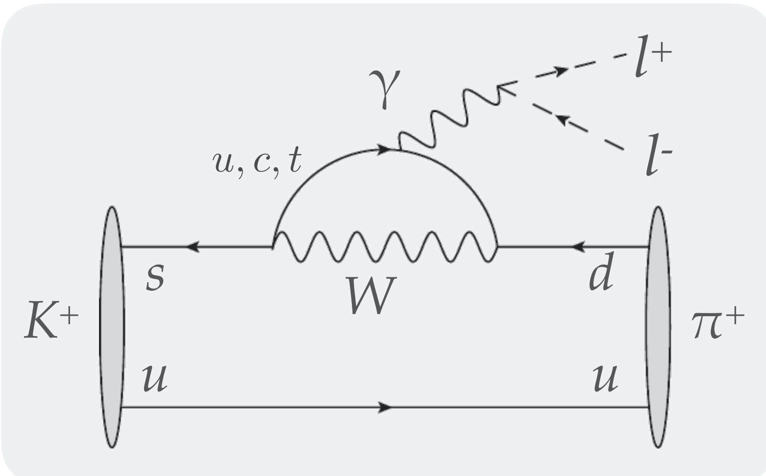
$$H_W(x) = \frac{G_F}{\sqrt{2}} V^*_{us} V_{ud} \left[C_1(Q^u_1 - Q^c_1) + C_2(Q^u_2 - Q^c_2)\right]$$

- Q_1 and Q_2 in H_W renormalise multiplicatively (chiral fermions)
- J_μ conserved
- divergences:
  - quadratic divergence can appear as x→0 but gauge invariance reduces it to a logarithmic one
  - remaining logarithmic divergence cancelled via GIM (→ need charm quark in lattice simulation)





Renormalisation



4-flavour

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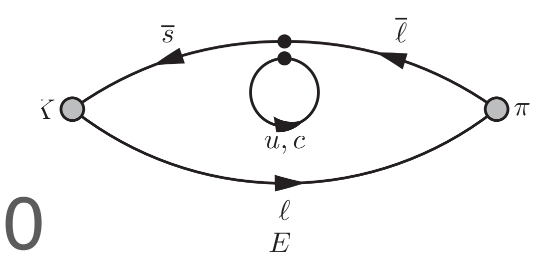

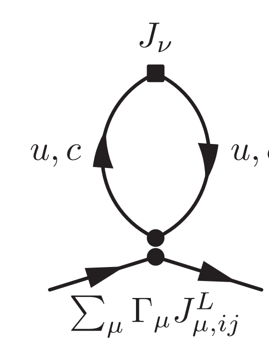

$K^+ \to \pi^+ \nu\bar{\nu}$ more involved due to axial current (also if local vector current)

Euclidean correlation functions

$$\mathcal{A}^c_\mu(q^2) = \int d^4x \langle \pi^c(p)| T\left[J_\mu(0) H_W(x)\right] |K^c(k)\rangle$$

$$\Gamma^{(4)\,c}_\mu(t_H, t_J, \mathbf{k}, \mathbf{p}) = \int d^3\mathbf{x} \int d^3\mathbf{y} e^{-i\mathbf{q}\cdot\mathbf{x}} \langle \phi_{\pi^c}(t_\pi, \mathbf{p}) T\left[J_\mu(t_j, \mathbf{x}) H_W(t_H, \mathbf{y})\right] \phi^\dagger_{K^c}(0, \mathbf{k})\rangle$$

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$$\mathcal{A}^c_\mu(q^2) = \int d^4x \langle \pi^c(p)| T\left[J_\mu(0) H_W(x)\right] |K^c(k)\rangle$$

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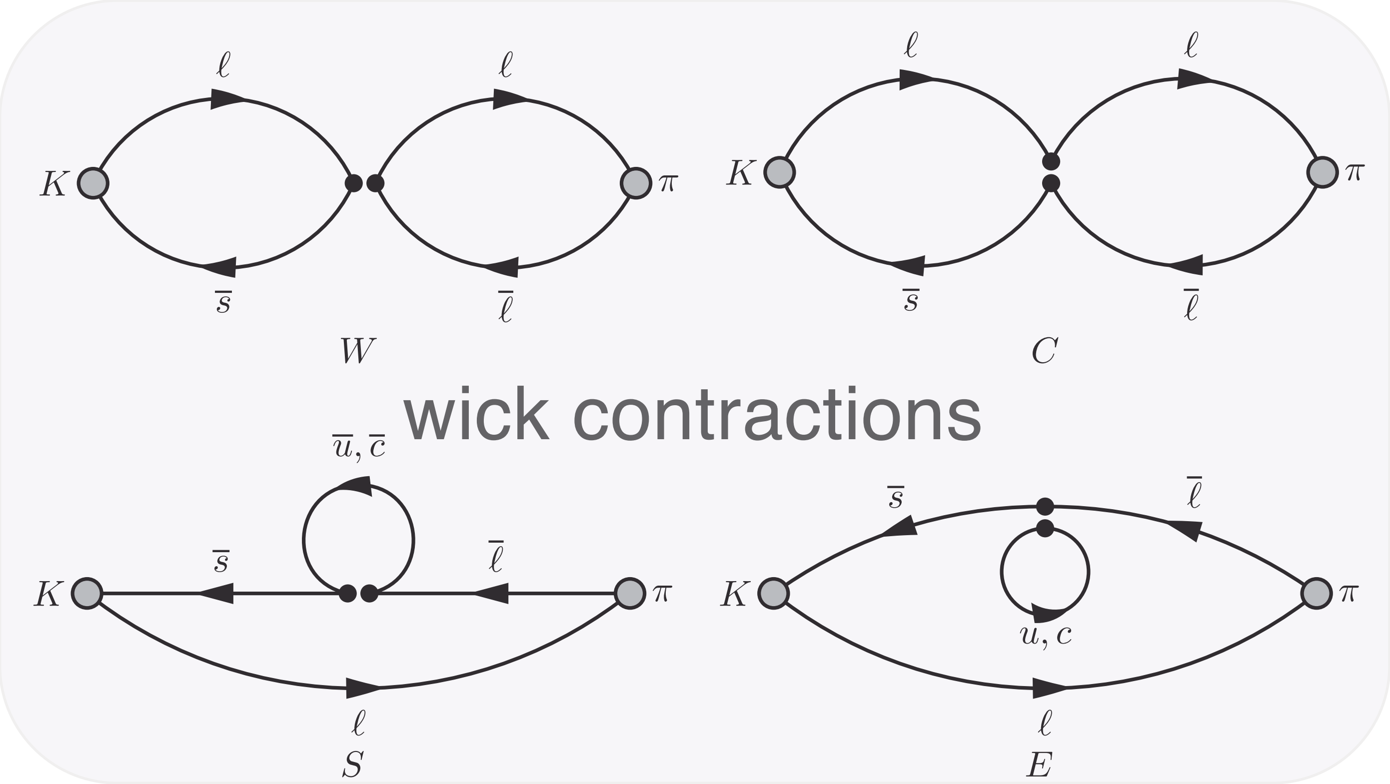
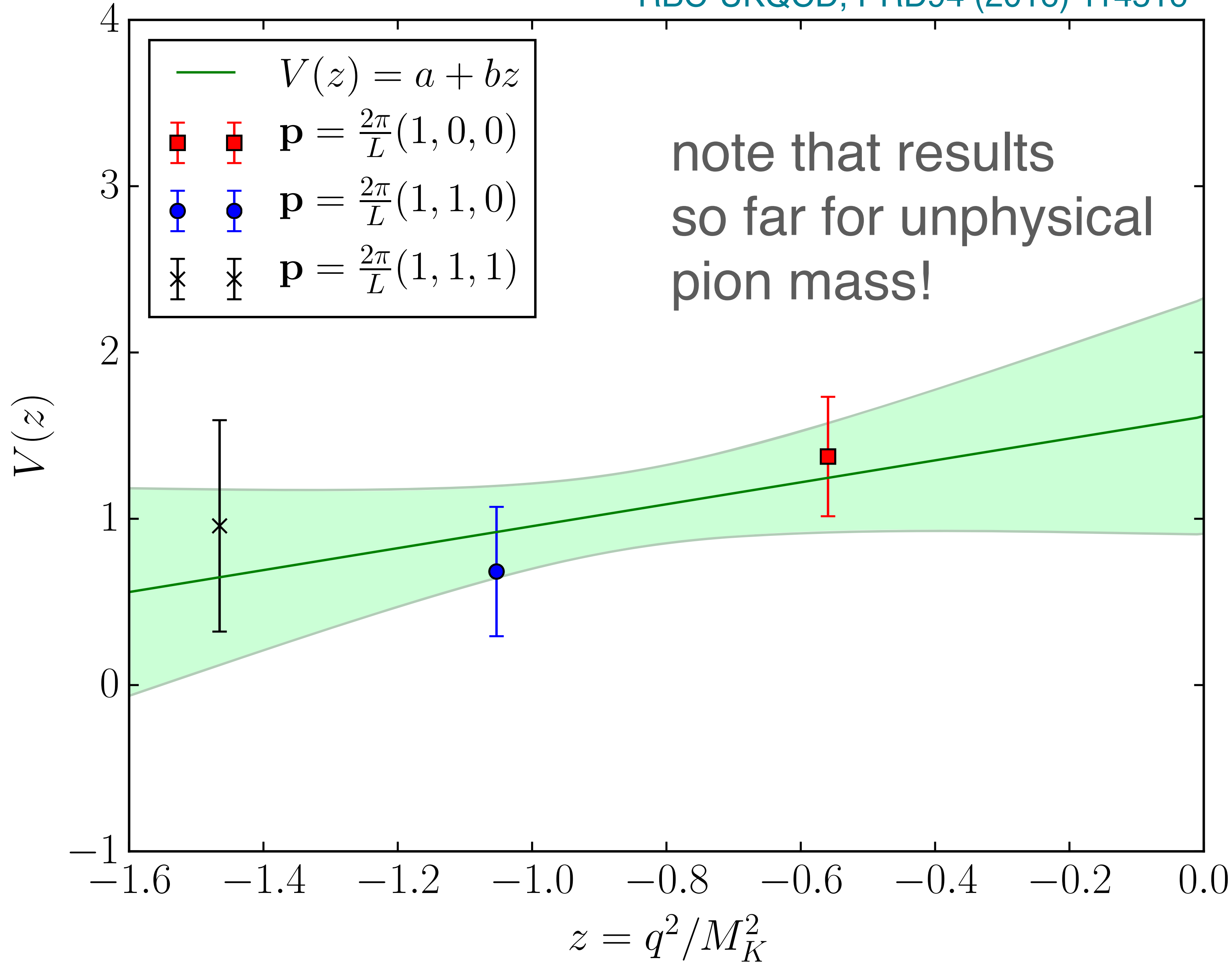


$K^+ \to \pi^+ l^+ l^-$ Results exploratory study

RBC-UKQCD, PRD94 (2016) 114516



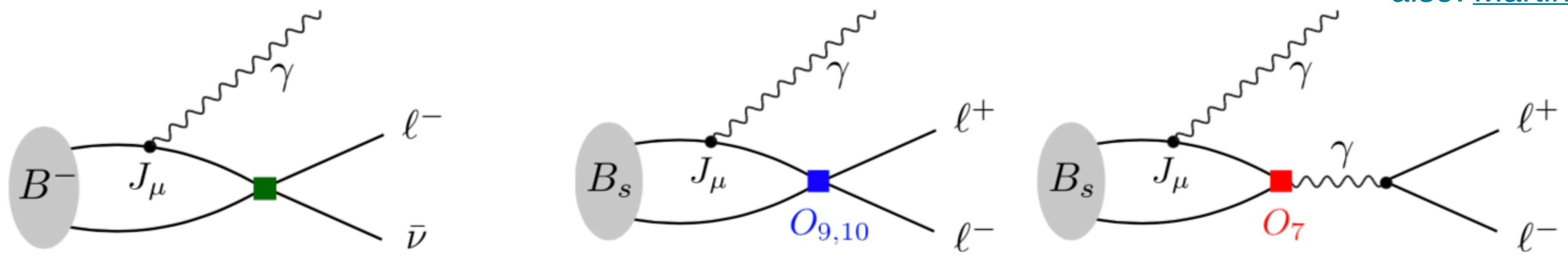
We are now running physical m_π — details in Fionn Ó hÓgáin's Lattice2019 talk

New directions

Radiative leptonic decays

Radiative leptonic decays

Kane et al. arXiv:1907.00279
also: Martinelli@Lattice2019



- Hard photon removes helicity suppression $(m_l/m_B)^2$
- Might allow constraining B-meson distribution amplitude
- Provide constraints for new physics searches
- Works also for D, K

Radiative leptonic decays

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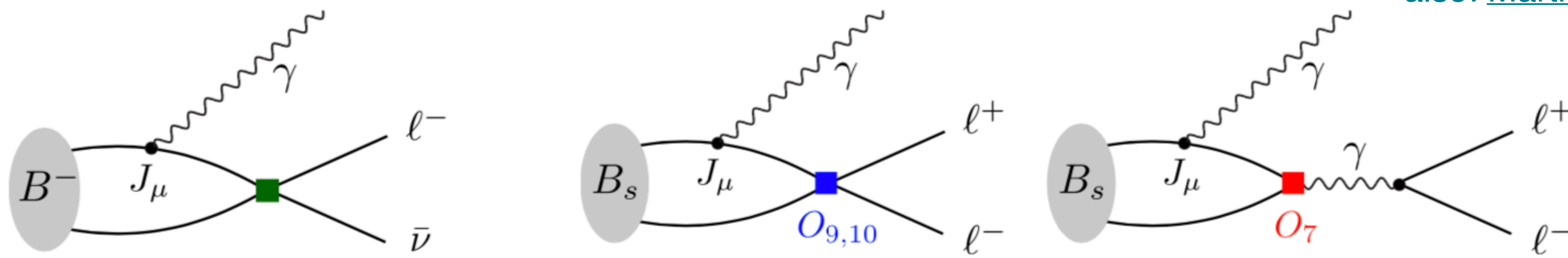


- Hard photon removes helicity suppression $(m_l/m_B)^2$
- Might allow constraining B-meson distribution amplitude
- Provide constraints for new physics searches
- Works also for D, K

$$T_{\mu\nu} = -i \int d^4x\, e^{ip_\gamma x} \langle 0 | T\left(J_\mu(x) J_\nu^{\text{weak}}(0)\right) | B^-(\vec{p}_B)\rangle$$

$$C_{\mu\nu}(t, t_B) = \int d^3x \int d^3y\, e^{-i\vec{p}_\gamma \vec{x}} \langle J_\mu(t,\vec{x}) J_\nu^{\text{weak}}(0,\vec{0}) \phi_B^\dagger(t_B, \vec{y})\rangle$$

Integrated over finite finite extent in Euclidean t

Radiative leptonic decays

Kane et al. arXiv:1907.00279
also: Martinelli@Lattice2019

Again we find exponential contaminations

$$I^{<}_{\mu\nu} = \langle B(\vec{p}_B)|\phi^{\dagger}_B(0)|0\rangle \frac{1}{2E_B} e^{E_B t_B} \sum_n \frac{1}{2E_{n,\vec{p}_B-\vec{p}_\gamma}}$$
$$\times \frac{\langle 0|J^{\text{weak}}_\nu(0)|n(\vec{p}_B-\vec{p}_\gamma)\rangle\langle n(\vec{p}_B-\vec{p}_\gamma)|J_\mu(0)|B(\vec{p}_B)\rangle}{E_\gamma + E_{n,\vec{p}_B-\vec{p}_\gamma} - E_B}$$
$$\times \left(1 - e^{-(E_\gamma + E_{n,\vec{p}_B-\vec{p}_\gamma}) - E_B)T}\right)$$

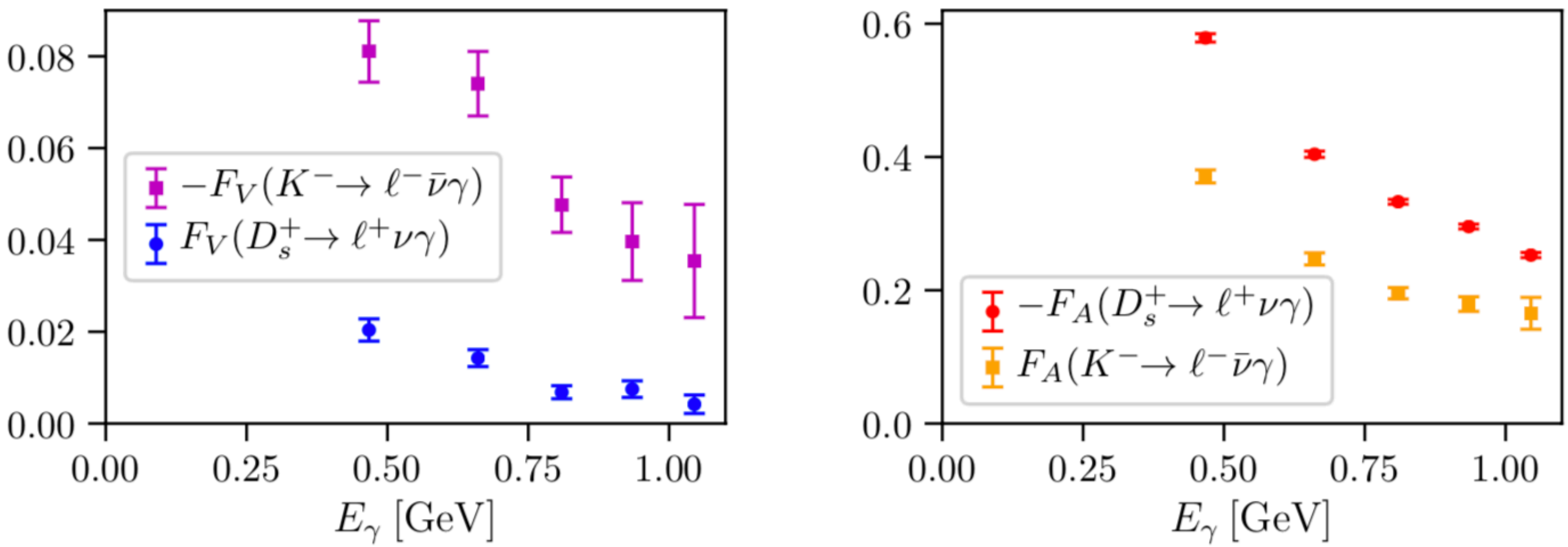
In $I^{<}_{\mu\nu}$ $e^{-(E_\gamma+E_{n,\vec{p}_B-\vec{p}_\gamma}-E_B)T}$ vanishes for non-zero photon momentum

In $I^{>}_{\mu\nu}$ $e^{(E_\gamma-E_{n,\vec{p}_\gamma})T}$ vanishes since hadronic state massive

Radiative leptonic decays

Kane et al. arXiv:1907.00279
also: Martinelli@Lattice2019

$$T_{\mu\nu} = \varepsilon_{\mu\nu\tau\rho} p_\gamma^\tau v^\rho F_V + i[-g_{\mu\nu}(p_\gamma \cdot v) + v_\mu (p_\gamma)_\nu] F_A - i \frac{v_\mu v_\nu}{p_\gamma \cdot v} m_B f_B + (p_\gamma)_\mu\text{-terms},$$

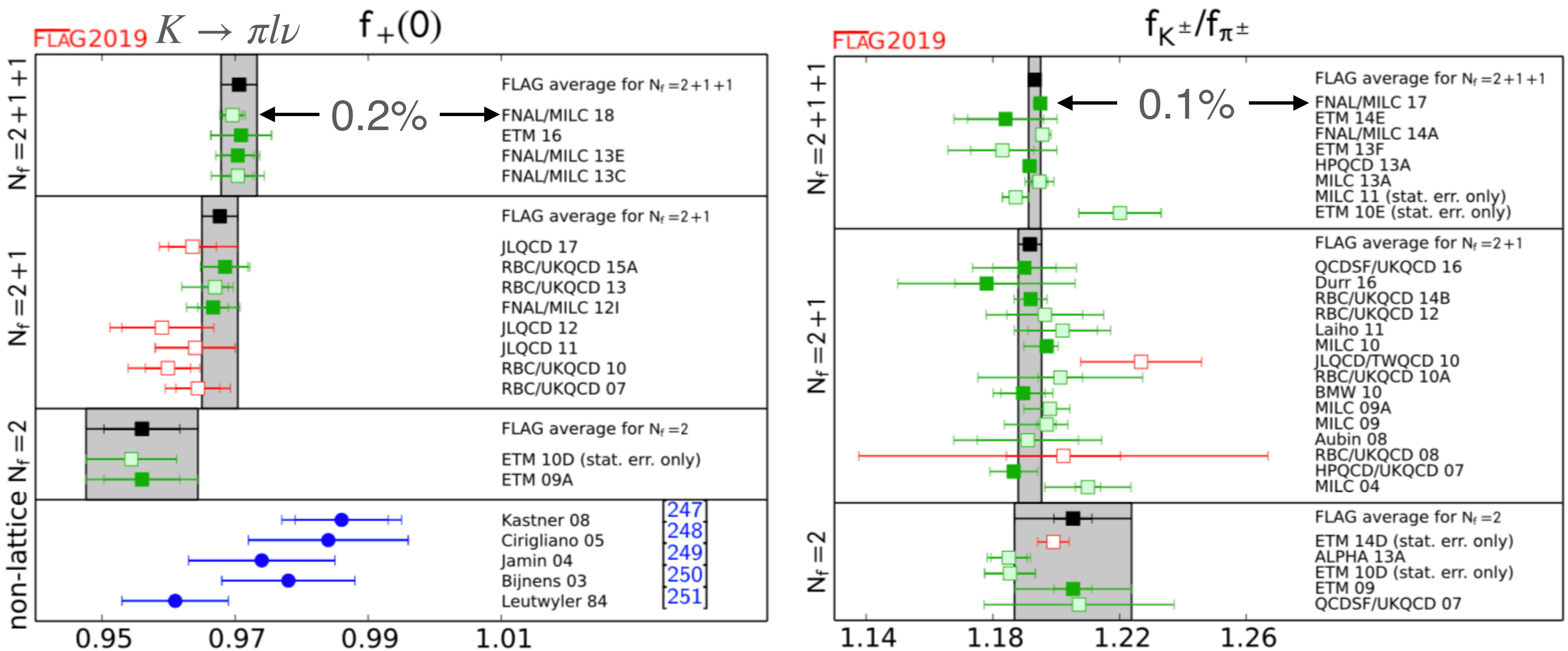


Kane et al. want to push forward and invest in simulations

New directions

Disconnected diagrams
for meson masses/decay

Beyond %-level precision



Beyond %-level precision

With a sub-percent precision goal we can't ignore isospin breaking effects:

- strong isospin breaking

$$m_u = 2.5\text{MeV} \qquad m_d = 5.0MeV \qquad \overline{\text{MS}}(2\text{GeV})$$

$$\frac{m_u - m_d}{\Lambda_{\text{QCD}}} \sim O(1\%)$$

- QED

$$\alpha \approx \frac{1}{137} \sim O(1\%)$$

Isospin breaking: EM effects

Factorisation Γ = Weak x EM x Strong

Many questions:

- Photon is massless and induces power-suppressed FSE
  BMWc Science 347 2015, Lubicz et al. PRD 95 2017, Davoudi, Savage, PRD 90 2014, Endres et al. PRL 117 2016, Lucini et al. JHEP 02 2016, Davoudi et al. PRD 99 2019
- How to formulate QED in finite volume
  Duncan et al. PRL 76 1996, Hayakawa, Uno Prog.Th.Ph. 120 2008, Endres et al. PRL 117 2016, Lucini et al. JHEP 02 2016
- Renormalising QCD+QED
- IR singularities (Bloch-Nordsiek) need to be dealt with Carrasco et al. PRD 91 2016
- Disconnected diagrams
- …

IB brk. in leptonic decay

QED

$$\langle O \rangle = \langle O \rangle_0 + \frac{e^2}{2} \frac{\partial^2}{\partial e^2} \langle O \rangle |_{e=0}$$

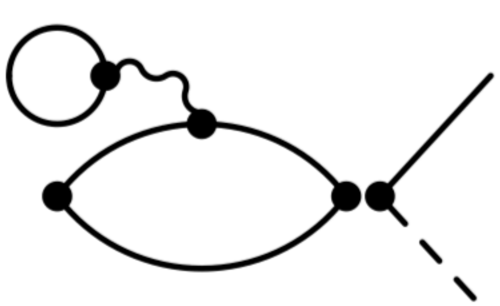
RM123 PRD 87 6 2013

strong IB

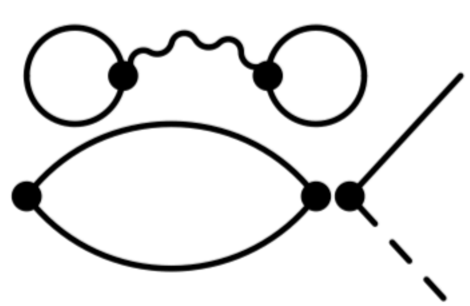
$$\langle O \rangle = \langle O \rangle_0 + \frac{(m_d - m_u)}{2} \langle OS \rangle$$

Divitiis et al. JHEP 1204 2012

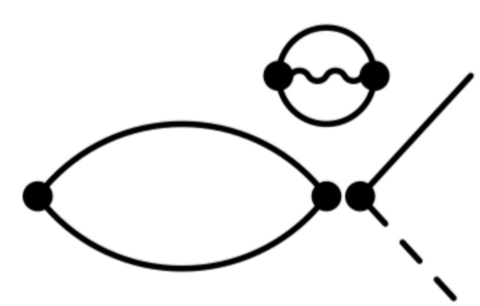
Quark-disconnected contributions to masses and decay



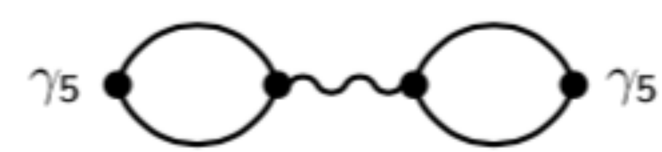
tadpole



spectacles diagram



burger diagram



neutr. pion. exch.

IB brk. in leptonic decay

Foley et al. Comm.Phys.Commun. 172 2005

$$D^{-1}(x,y) = \underbrace{\sum_{l=1}^{N_l} v_l(x) w_l^\dagger(y)}_{\text{Exact low-modes}} + \underbrace{\sum_{l=N_l+1}^{N_l+N_h} v_h(x) w_h^\dagger(y)}_{\text{Stochastic high-modes}}$$

Low modes:

$$v_l(x) = \phi_l(x)$$
$$w_l(y) = \phi_l(y)/\lambda_l$$

High modes:

$$v_h(x) = D^{-1}_{\text{defl}}\eta_h(x)$$
$$w_h(y) = \eta_h(y)$$

IB brk. in leptonic decay

Foley et al. Comm.Phys.Commun. 172 2005

$$D^{-1}(x,y) = \sum_{l=1}^{N_l} v_l(x) w_l^\dagger(y) + \sum_{l=N_l+1}^{N_l+N_h} v_h(x) w_h^\dagger(y)$$

Exact low-modes

Stochastic high-modes

$$C(t) = \sum_{\vec{x},\vec{y}} \mathrm{Tr}\left[\gamma_5 S_u y(x,y) \gamma_5 S_d(y,x)\right]$$

$$= \sum_{i,j} \mathrm{Tr}\left[\sum_{\vec{y}} w_i^\dagger(y)\gamma_5 v_j(y) \sum_{\vec{x}} w_j^\dagger(x)\gamma_5 w_i(x)\right]$$

$$= \sum_{i,j} \mathrm{Tr}\left[\Pi_{ij}(t,\gamma_5)\Pi_{ji}(t,\gamma_5)\right]$$

Meson fields Π_{ij} stored on disk — versatile, can be used for other offline contractions

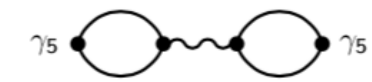
Richings@Lattice 2019

Our study

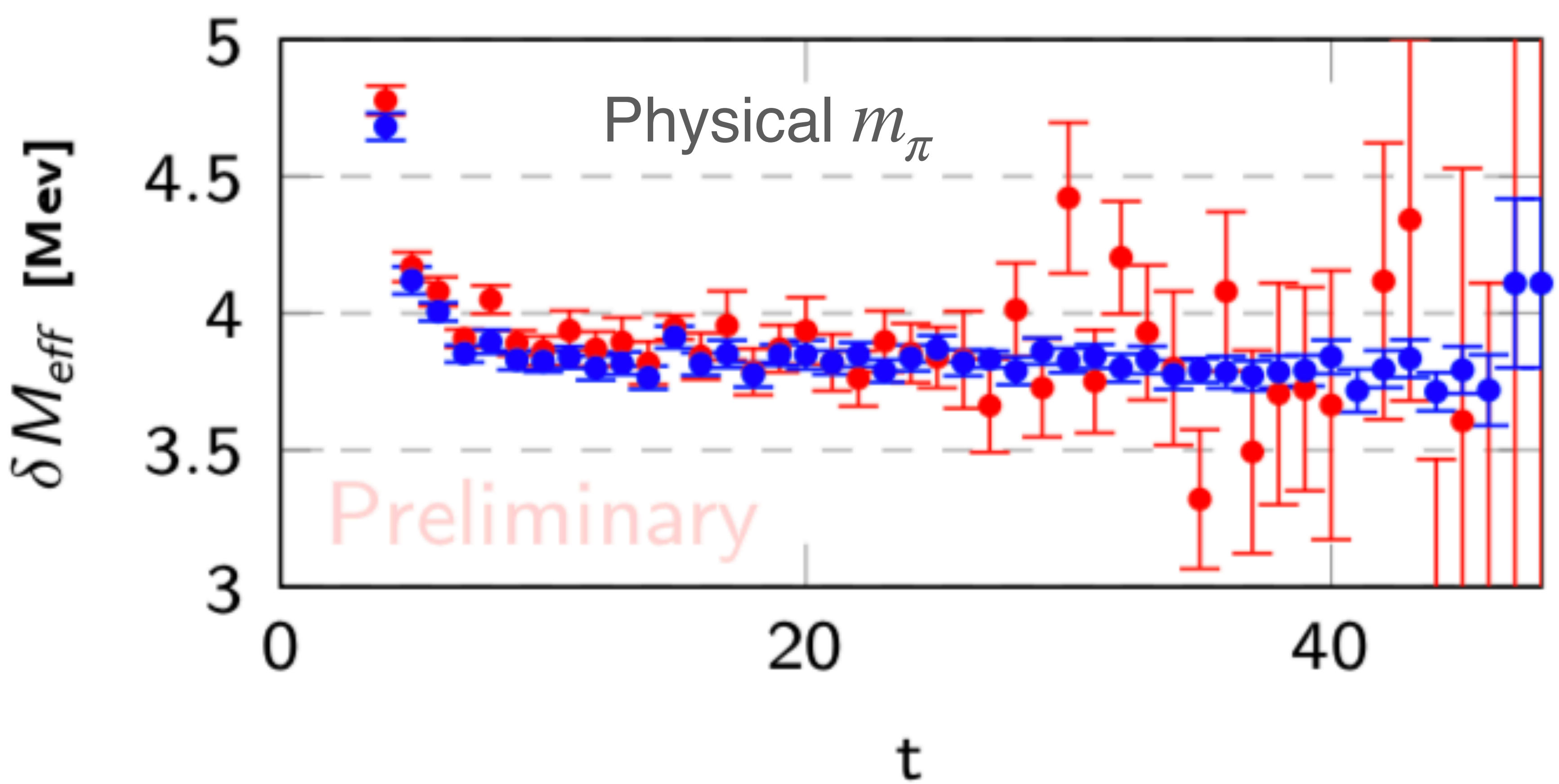
- Local vector currents
- Feynman gauge
- QED_L
- Stochastic photons $\Delta_{\mu\nu}(x-y) = \langle A_\mu(x) A_\nu(y) \rangle$

vol.	a^{-1}	m_π	N	N_l
48^3x96	1.73GeV	140MeV	19	2000
24^3x64	1.78GeV	340MeV	25	600

Disconected in pion mass-splitting



$$M_{\pi^+} - M_{\pi^0} = \frac{(Q_u - Q_d)^2}{2} e^2 \partial_t \left[\frac{C^{\pi}_{\rm exch} - C^{\pi}_{\rm neutr.\,exch.}}{C^{\pi}_0} \right]$$



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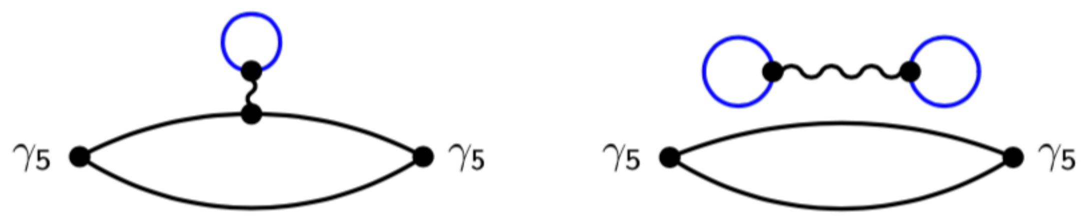
IB brk. in leptonic decay

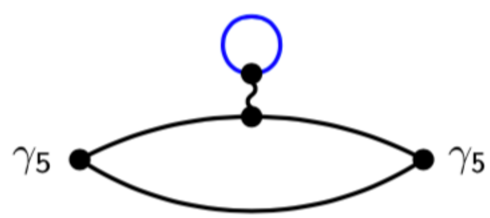
Meson fields including elm. Field $\Pi_{ij}(t; \not{A}) = \sum_{\vec{x}} w_i^\dagger(x) \not{A} v_j(x)$



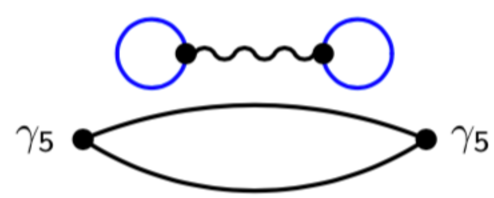
Tadpole diagram   Specs diagram

IB brk. in leptonic decay

Meson fields including elm. Field $\Pi_{ij}(t; \not{A}) = \sum_{\vec{x}} w_i^\dagger(x) \not{A} v_j(x)$



Tadpole diagram

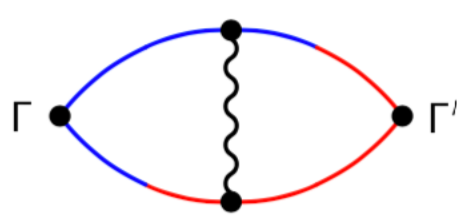


Specs diagram

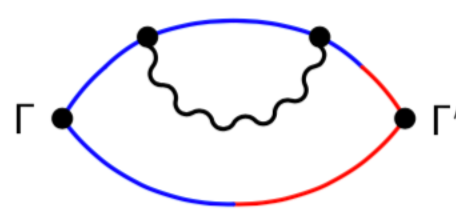
Or as *sequential meson field*

$$v_i'(x) = \sum_x S(x,z) i \not{A}(z) v_i(z)$$

$$\Pi_{ij}'(t; \Gamma) = \sum_{\vec{x}} w_i^\dagger(x) \Gamma v_j'(x)$$

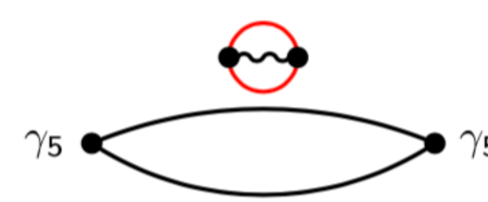


Exchange diagram



Self Energy diagram

Neutral pion exchange diagram



Burger diagram

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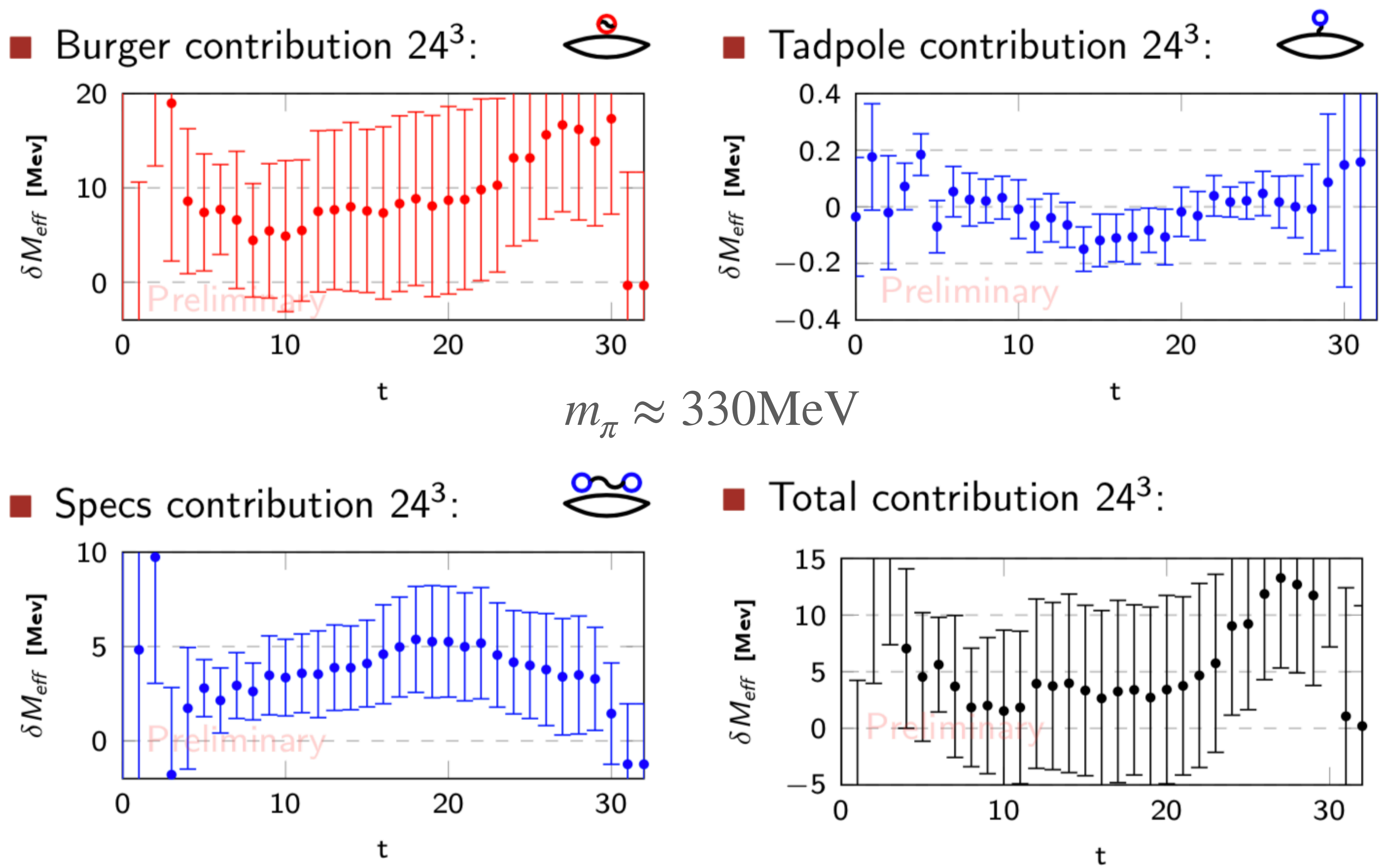

It works but signal needs improving — we are working on it.
More on out QCD+QED efforts next week …

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Summary

- Tree semileptonic decay — parametrisation
- Rare semileptonic decay — howto
- Radiative decay — new
- Leptonic decay — disconnected

Not covered:

- Heavy-quark discretisation
- Unstable states/multi-hadron states
- Baryon form factors
- Signal-to-noise
- Excited-state contaminations
- Renormalisation
- Critical slowing down
- …