1. Nucleon form factors
2. Lattice QCD: standard methods
3. Direct methods for radii and magnetic moment
4. Outlook
Elastic $ep$ scattering cross section depends on the strength of the coupling of a proton to a current.

\[ \langle p' | V^q_\mu | p \rangle = \bar{u}(p') \left[ \gamma^\mu F^q_1(Q^2) + \frac{i\sigma^{\mu\nu}(p' - p)^\nu}{2m_p} F^q_2(Q^2) \right] u(p), \]

where $V^q_\mu = \bar{q} \gamma^\mu q$.

Electric and magnetic form factors:

\[ G^q_E(Q^2) = F^q_1(Q^2) - \frac{Q^2}{(2m_p)^2} F^q_2(Q^2), \quad G^q_M(Q^2) = F^q_1(Q^2) + F^q_2(Q^2). \]

For a photon, weight quarks with their charges:

\[ G^\gamma_{E,M} \equiv \frac{2}{3} G^u_{E,M} - \frac{1}{3} G^d_{E,M} - \frac{1}{3} G^s_{E,M} + \ldots \]
Electromagnetic form factors: nonrelativistic

In the Breit frame ($\vec{p}' = -\vec{p} = \vec{q}/2$):

$$G_E(Q^2) \sim \rho_{em}, \quad G_M(Q^2) \sim \vec{J}_{em}.\,$$

Non-relativistically, this motivates the definition of a charge density:

$$\rho_{NR}(\vec{r}^2) = \int \frac{d^3 \vec{q}}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} G_E(\vec{q}^2).$$

Integrating the density yields

$$\int d^3 \vec{r} \rho_{NR}(\vec{r}^2) = G_E(0), \quad \int d^3 \vec{r} \vec{r}^2 \rho_{NR}(\vec{r}^2) = -6G'_E(0).$$

At $Q^2 = 0$, we get the charge and magnetic moment of the proton, and the slopes define the mean-squared electric and magnetic radii:

$$G_E(Q^2) = 1 - \frac{1}{6} (r_E^2)^p Q^2 + O(Q^4)$$

$$G_M(Q^2) = \frac{\mu_p}{\mu_N} (1 - \frac{1}{6} (r_M^2)^p Q^2 + O(Q^4))$$
“Earthrise” photo taken by Apollo 8 astronauts.

- Photo has 4 ms exposure time.
- Image of Earth taken $\sim 1$ s before Moon.
- Farthest point of Earth viewed $\sim 20$ ms before nearest point.

We actually see things along the light cone rather than at fixed time.

Apply same light-front approach to the proton. Then we get a 2d transverse charge density

$$ \rho(\vec{b}^2) = \int \frac{d^2 \vec{q}}{(2\pi)^2} e^{i\vec{q} \cdot \vec{b}} F_1(\vec{q}^2). $$


Electron-proton scattering

Electromagnetic form factors are studied using elastic scattering of an electron off a fixed proton target.

The differential scattering cross section behaves like

$$\frac{d\sigma}{d\Omega} \propto G_E(Q^2)^2 + \frac{\tau}{\epsilon} G_M(Q^2)^2,$$

$$\tau = \frac{Q^2}{4m_p^2}, \quad \epsilon^{-1} = 1 + 2(2 + \tau) \tan^2 \frac{\theta}{2},$$

so that $G_E$ and $G_M$ can be measured in experiments (Rosenbluth separation).

- First experiments in 1950s (R. Hofstadter).
- Recent years: experiments in Mainz and JLab.
- Ongoing work studying low and high $Q^2$ regions.
Proton radius

How to measure $r_E$:

- From scattering experiments: measure $G_E(Q^2)$, then do curve fitting to find slope at $Q^2 = 0$.
- From atomic spectroscopy: the 2S–2P Lamb shift is sensitive to $r_E$. \[ \Delta E_{\text{finite size}} \propto r_E^2 m_e^3 \]

Muonic hydrogen spectrum is much more sensitive to $r_E$. This experiment led to proton radius puzzle.
Proton radius: ongoing experiments

- **New $ep$ scattering experiments at low $Q^2$**
  - Mainz ISR: $r_E = 0.870(28)$ fm \cite{Mihovilovic:2019}
  - JLab PRad: $r_E \sim 0.830(20)$ fm \cite{JLabPRad}
  - Tohoku: planned ULQ2 experiment using low beam energy (20–60 MeV).

- **Muon-proton scattering**
  - **MUSE**: $\mu^\pm p$, $e^\pm p$ scattering at PSI
  - **COMPASS**: proposed $\mu^\pm p$ experiment

- **New hydrogen spectroscopy experiments**: Garching, Paris, Toronto

Also note: analyses of scattering data based on dispersion relations yield small radius. \cite{Belushkin:2007}

Jeremy Green | DESY, Zeuthen | July 23, 2019 | Page 8
An older puzzle: $G_E/G_M$ at high $Q^2$

Polarization transfer, $\bar{e}p \to e\bar{p}$, gives a direct measurement of $G_E/G_M$.

Result disagreed with Rosenbluth separation.

Can be explained by contributions from two-photon exchange.

Explanation was tested via $\frac{\sigma(e^+p)}{\sigma(e^-p)}$, but not at high $Q^2$. 
Flavour separation

Elastic $ep$ scattering gives one flavour combination:

$$G^p_{E,M} = \frac{2}{3} G^u_{E,M} - \frac{1}{3} G^d_{E,M} - \frac{1}{3} G^s_{E,M} - \ldots$$

Separating out $u$, $d$, and $s$ contributions requires two more independent combinations

1. Neutron electromagnetic form factors (assuming isospin): swap role of $u$ and $d$. Obtained using $^2$H or $^3$He targets.

2. Contribution from $Z$ exchange. Obtained from parity-violating asymmetry in elastic $\bar{e}p$ scattering.

Neutron-electron scattering length yields neutron charge radius:

$$b_{ne} = \frac{\alpha}{3} m_n r^2_{En}.$$

PDG average: $r^2_{En} = -0.1161(22)$. 
Magnetic moment

Good benchmark observable: experimental situation is solid.

\[ \mu^p = 2.792\,847\,344\,62(82)\mu_N \ (0.3 \ \text{ppb}) \quad \text{PDG; G. Schneider et al., Science 358, 1081–1084 (2017)} \]

\[ \mu^n = -1.913\,042\,73(45)\mu_N \ (0.2 \ \text{ppm}) \quad \text{CODATA; G. L. Greene et al., Phys. Rev. D 20, 2139 (1979)} \]

Nuclear magneton: \[ \mu_N = \frac{e}{2m_p} \]
Magnetic moment

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\[ \mu^s \approx -0.02\mu_N \text{ (LQCD)} \]

Nuclear magneton: \( \mu_N = \frac{e}{2m_p} \)
Magnetic radius

No independent measurements of $r_M$: only elastic scattering data.

Proton: reanalysis using $z$ expansion


$$r_{Mp} = 0.776(34)(17) \text{ Mainz data}$$

$$r_{Mp} = 0.914(35) \text{ world data excluding Mainz}$$

Neutron: PDG 2019 cites two analyses

$$r_{Mn} = 0.89(3) \quad z \text{ expansion} \quad Z. \text{ Epstein et al., Phys. Rev. D 90, 074027 (2014) [1407.5687]}$$

$$r_{Mn} = 0.862^{+9}_{-8} \quad \text{disp. rel.} \quad M. \text{ A. Belushkin et al., Phys. Rev. C 75, 035202 (2007) [hep-ph/0608337]}$$
Need for physical pion mass

Pion cloud makes large contribution to isovector radii.


\[
(r_1^2)_{p-n} = -\frac{1}{(4\pi F_\pi)^2} \left[ 1 + 7g_A^2 + (2 + 10g_A^2) \log \left( \frac{m_\pi}{\Lambda} \right) + 12B_{10}^r(\Lambda) \right]
\]

\[
\kappa_{p-n} = \kappa_{0}^{p-n} - \frac{g_A^2 m_\pi m_N}{4\pi F_\pi^2}
\]

\[
\kappa_{p-n} (r_2^2)_{p-n} = \frac{g_A^2 m_N}{8\pi F_\pi^2 m_\pi}
\]

Isovector \( r_1^2 \) and \( r_2^2 \) diverge as \( m_\pi \to 0 \).
Hadron correlation functions

Compute two-point and three-point functions, using interpolator $\chi$ and operator insertion $O$. In simplest case:

$$C_{2\text{pt}}(t) \equiv \langle \chi(t)\chi^\dagger(0) \rangle$$

$$= \sum_n |Z_n|^2 e^{-E_n t}$$

$$\to |Z_0|^2 e^{-E_0 t} \left( 1 + O(e^{-\Delta E t}) \right),$$

where $Z_n = \langle \Omega | \chi | n \rangle$,

$$C_{3\text{pt}}(\tau, T) \equiv \langle \chi(T)O(\tau)\chi^\dagger(0) \rangle$$

$$= \sum_{n,n'} Z_n Z_n^* \langle n'|O|n \rangle e^{-E_n \tau} e^{-E_{n'}(T-\tau)}$$

$$\to |Z_0|^2 \langle 0|O|0 \rangle e^{-E_0 T} \left( 1 + O(e^{-\Delta E \tau}) + O(e^{-\Delta E(T-\tau)}) \right)$$
Hadron matrix elements

**Ratio method**

\[ R(\tau, T) \equiv \frac{C_{3\text{pt}}(\tau, T)}{C_{2\text{pt}}(T)} = \langle 0|\mathcal{O}|0 \rangle + O(e^{-\Delta E \tau}) + O(e^{-\Delta E (T-\tau)}) \]

Midpoint yields \( R\left(\frac{T}{2}, T\right) = \langle 0|\mathcal{O}|0 \rangle + O(e^{-\Delta E T/2}) \).

**Summation method**

\[ S(T) \equiv \sum_{\tau} R(\tau, T), \quad \frac{d}{dT} S(T) = \langle 0|\mathcal{O}|0 \rangle + O(Te^{-\Delta E T}) \]

Sum can be over all timeslices or from \( \tau_0 \) to \( T - \tau_0 \).

Improved asymptotic behaviour noted in talks at Lattice 2010.

S. Capitani et al., PoS LATTICE2010 147 [1011.1358]; J. Bulava et al., *ibid.* 303 [1011.4393]

In practice noisier than ratio method at same \( T \).
Two-point function: excited states

\[ C_{2pt}(t) = \langle \chi(t) \chi^\dagger(0) \rangle = \sum_n e^{-E_n t} \left| \langle n | \chi^\dagger | 0 \rangle \right|^2 \]

For a nucleon, the signal-to-noise asymptotically decays as \( e^{-\left( m_N - \frac{3}{2} m_\pi \right) t} \).
Two-point function: excited states

\[ C_{2pt}(t) = \langle \chi(t) \chi^+(0) \rangle = \sum_n e^{-E_n t} \left| \langle n | \chi \rvert 0 \rangle \right|^2 \]

For a nucleon, the signal-to-noise asymptotically decays as \( e^{-\left( m_N - \frac{3}{2} m_\pi \right) t} \).
Three-point function: excited states

\[ R(T, \tau) = \frac{C_{3pt}(T, \tau)}{C_{2pt}(T)} \rightarrow \langle N|O|N \rangle \]

\[ \chi \chi J \]

Three-point function: excited states

\[ \text{ratio: } R(T, \tau) = \frac{C_{3\text{pt}}(T, \tau)}{C_{2\text{pt}}(T)} \rightarrow \langle N|O|N \rangle \]
Off-forward matrix elements

\[ C_{3\text{pt}}^O(\tau, T; \vec{p}, \vec{p}') \]

\[ = \int d^3\vec{x} d^3\vec{y} e^{-i\vec{p} \cdot \vec{x}} e^{i(\vec{p}' - \vec{p}) \cdot \vec{y}} \text{Tr} \left[ \Gamma_{\text{pol}} \langle \chi(\vec{x}, T)O(\vec{y}, \tau)\bar{\chi}(0) \rangle \right] \]

\[ \rightarrow \frac{Z(\vec{p})Z(\vec{p}')}{{4E(\vec{p})E(\vec{p}')}} e^{-E(\vec{p})\tau} e^{-E(\vec{p}')(T-\tau)} \sum_{s, s'} \bar{u}(p, s) \Gamma_{\text{pol}} u(p', s') \langle p', s' | O | p, s \rangle. \]

Cancel overlaps and exponents using ratio:

\[ R^O(\tau, T; \vec{p}, \vec{p}') = \frac{C_{3\text{pt}}^O(\tau, T; \vec{p}, \vec{p}')}{{\sqrt{C_{2\text{pt}}(T, \vec{p})C_{2\text{pt}}(T, \vec{p}')}}} \frac{\sqrt{C_{2\text{pt}}(\vec{p}, T-\tau)C_{2\text{pt}}(\vec{p}', \tau)}}{{\sqrt{C_{2\text{pt}}(\vec{p}', T-\tau)C_{2\text{pt}}(\vec{p}, \tau)}}}. \]
Matrix elements are linear combinations of form factors:

\[
\langle p', s' | V_\mu | p, s \rangle = \bar{u}(p', s') \left[ \gamma_\mu F_1(Q^2) + \frac{i\sigma_{\mu\nu}(p' - p)^\nu}{2m_p} F_2(Q^2) \right] u(p, s).
\]

Gather matrix elements with the same \( Q^2 \):
- different \( \mu \)
- different \( \vec{p}, \vec{p}' \)
- different polarization (i.e. \( s, s' \)).

This gives a linear system of equations with two unknowns: \( F_1(Q^2) \) and \( F_2(Q^2) \).
Consider states with $\vec{p'} = 0$ and spin in the $+\hat{z}$ direction.

\[
\Re (R^V_i) \sim \epsilon_{ij3} p_j G_M(Q^2)
\]
\[
\Im (R^V_i) \sim -p_i G_E(Q^2)
\]
\[
\Re (R^V_4) \sim (m_N + E_N(\vec{p})) G_E(Q^2)
\]

Most common approach: take $\Re (R^V_i)$ and $\Re (R^V_4)$. 
Lattice results

Largest volume: $L = 10.8$ fm $\rightarrow Q_{\text{min}}^2 = 0.013$ GeV$^2$

E. Shintani et al. (PACS), Phys. Rev. D 99, 014510 (2019) [1811.07292]
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Minimum momentum transfer $Q_{\text{min}}^2 \approx \left( \frac{2\pi}{L} \right)^2$.

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Background field methods

Alternative approach with different systematics.
E.g. neutral hadron in uniform magnetic field: $E = m - \mu \cdot B + O(B^2)$. 

Z. Davoudi and W. Detmold, Phys. Rev. D 93, 014509 (2016) [1510.02444]
Alternative approach with different systematics.
E.g. neutral hadron in uniform magnetic field: \( E = m - \mu \cdot \vec{B} + O(\vec{B}^2) \).

Implement by modifying gauge links: \( U_\mu(x) \rightarrow e^{ie_q \alpha A_\mu(x)} U_\mu(x) \).
For constant \( \vec{B} = B\hat{z} \), can choose \( A_\mu(x) \) such that with periodic boundary conditions,
\[
e_q B = \frac{2\pi}{L^2} n, \quad n \in \mathbb{Z}.
\]

Can study radii using spatially-varying \( \vec{E} \) fields.
Z. Davoudi and W. Detmold, Phys. Rev. D 93, 014509 (2016) [1510.02444]
Direct calculation of $\mu$ (naïve attempt)

For spin $+\hat{z}$, we have $\Re [RV_x(p\hat{y}, \vec{0})] \sim pG_M(Q^2)$. Therefore

$$
\frac{\mu}{\mu_N} = G_M(0) \sim \lim_{p \to 0} \frac{1}{p} \Re [RV_x(p\hat{y}, \vec{0})] = \left. \frac{\partial}{\partial p_y} \Re [RV_x(\vec{p}, \vec{0})] \right|_{\vec{p}=0} = \frac{1}{C_{2pt}(T, \vec{0})} \left. \frac{\partial}{\partial p_y} \Re \left[ C_{3pt}^V(\tau, T; \vec{p}, \vec{0}) \right] \right|_{\vec{p}=0},
$$

where

$$
\left. \frac{\partial}{\partial p_y} C_{3pt}^V(\tau, T; \vec{p}, \vec{0}) \right|_{\vec{p}=0} = \int d^3\vec{x}_1 d^3\vec{x}_2 (-ix_{2y}) \text{Tr} \left[ \Gamma_{\text{pol}} \langle \chi(\vec{x}_1, T)V_x(\vec{x}_2, \tau)\bar{\chi}(0) \rangle \right].
$$
Failure of naïve approach


Need to choose finite-volume expression for the moment in $y$:

$$y \rightarrow f_L(y) = \begin{cases} 
  y & y < L/2 \\
  0 & y = L/2 \\
  y - L & y > L/2 
\end{cases}$$

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In finite volume, this can always be decomposed into Fourier modes:

\[
f_L(y) = \sum_{n \in \mathbb{Z}} c_n e^{-ip_n y}, \quad p_n = \frac{2\pi}{L} n \quad \Rightarrow \quad c_n = \begin{cases} 
0 & n = 0 \\
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At large $\tau$, $C_{3pt}$ will be dominated by the lowest-lying initial states with energies $E_N(\vec{p} = p_n \hat{y})$, i.e. with $n = \pm 1$. 
Failure of naïve approach

[hep-lat/0204024]

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At large $\tau$, $C_{3pt}$ will be dominated by the lowest-lying initial states with energies $E_N(\vec{p} = p_n \hat{y})$, i.e. with $n = \pm 1$. Two problems:

1. The denominator $C_{2pt}(T, \vec{0})$ doesn’t cancel $Z(\vec{p}) e^{-E_N(\vec{p}) \tau}$.

2. We get matrix elements proportional to $G_M(Q^2_{\text{min}})$. Nothing new over standard approach.
Need to take $L \to \infty$ before $p \to 0$.
Correct order of limits for standard calculation:

$$\frac{\mu}{\mu_N} \propto \lim_{p \to 0} \lim_{L \to \infty} \lim_{\tau, T-\tau \to \infty} \frac{1}{p} \Re \left[ R^V_x(\tau, T; p\hat{y}, \vec{0}) \right].$$
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Can postpone ground-state isolation:

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\]

We can take finite-volume moments of a correlator:

\[
\frac{\mu}{\mu_N} \propto \lim_{\tau, T-\tau \to \infty} \lim_{L \to \infty} \frac{\partial}{\partial p_y} \mathcal{R} \left[ R_x^V (\tau, T; \bar{p}, \bar{0}) \right] \bigg|_{\bar{p}=0}.
\]

This approach not currently being taken, but discussed in

C. Alexandrou et al. (ETMC), Phys. Rev. D 94, 074508 (2016) [1605.07327].
Twisted boundary conditions

For $p_\mu = \frac{2\pi}{L_\mu} n_\mu$, $n_\mu \in \mathbb{Z}$, can absorb momentum into quark propagator:

$$e^{-ip \cdot (x-y)} D^{-1}(x, y; U) = D^{-1}(x, y; U^{(p)}), \quad U^{(p)}_\mu(x) = e^{iap_\mu} U_\mu(x).$$
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For non-Fourier modes $p_\mu = \frac{1}{L_\mu} (2\pi n_\mu + \theta_\mu)$, using $U^{(p)}$ implies twisted boundary conditions

$$D^{-1}(x + L_\mu \hat{\mu}, y; U) = e^{i\theta_\mu} D^{-1}(x, y; U).$$
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\[
D^{-1}(x + L_\mu \hat{\mu}, y; U) = e^{i\theta_\mu} D^{-1}(x, y; U).
\]

Derivatives with respect to \( p \) can be evaluated exactly


\[
\frac{\partial D^{-1}}{\partial p_\mu} = -D^{-1} \frac{\partial D}{\partial p_\mu} D^{-1}
\]

For Wilson-type fermions, \( \frac{\partial D}{\partial p_\mu} \) amounts to an insertion of the conserved vector current.
Twisted boundary conditions: form factors

Need to absorb momentum into \( D^{-1}(x, y) \) with \( x \neq y \)

\( \implies \) can’t use for disconnected diagrams. Equivalently, need
flavor-changing vector current \( \bar{q}_{\theta'} \gamma_{\mu} q_{\theta} \) with different twist angles \( \theta' \neq \theta \).
Twisted boundary conditions: form factors

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\[ \Rightarrow \text{can’t use for disconnected diagrams. Equivalently, need flavor-changing vector current } \bar{q}_{\theta'} \gamma_{\mu} q_{\theta} \text{ with different twist angles } \theta' \neq \theta. \]

Use partially twisted boundary conditions, i.e. only on valence quarks connected to current. Nonunitarity vanishes as $L \to \infty$. 
Twisted boundary conditions: form factors

Need to absorb momentum into $D^{-1}(x, y)$ with $x \neq y$

$\implies$ can’t use for disconnected diagrams. Equivalently, need flavor-changing vector current $\bar{q}_{\theta'}\gamma_{\mu}q_{\theta}$ with different twist angles $\theta' \neq \theta$. Use partially twisted boundary conditions, i.e. only on valence quarks connected to current. Nonunitarity vanishes as $L \to \infty$.

Two applications:

1. Studying finite-volume effects. Twisted BC allows for the same momentum on different volumes.
Twisted boundary conditions: form factors

Need to absorb momentum into $D^{-1}(x, y)$ with $x \neq y$

$\implies$ can’t use for disconnected diagrams. Equivalently, need
flavor-changing vector current $\bar{q}_{\theta'} \gamma_\mu q_\theta$ with different twist angles $\theta' \neq \theta$.

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Two applications:

1. Studying finite-volume effects. Twisted BC allows for the same
momentum on different volumes.

2. Momentum derivatives at $\vec{p} = 0$.

$$\frac{\partial}{\partial p_j} \to \mu,$$

$$\frac{\partial^2}{\partial p_j^2} \text{ or } \frac{\partial^2}{\partial p_j' \partial p_j} \to r_E^2.$$
Calculations at physical $m_\pi$ using BMW 2HEX-clover action. First version used $\frac{\partial}{\partial p_j}$ and $\frac{\partial^2}{\partial p_j^2}$ → extra propagators shared for all source-sink separations.


Revised approach also uses $\frac{\partial^2}{\partial p_j \partial p_j}$ → reduced statistical errors for $r_E^2$.

N. Hasan, talk at Lattice 2017
Heavy baryon partially quenched ChPT.

Two sea quarks plus additional valence and ghost quarks with twisted BC.

New unphysical baryon LEC $g_1$: equals $2g_A^{d,\text{conn}}$ in chiral limit.

E.g. for $\vec{p} = 0$, $\vec{p}' = p\hat{y}$:

$$
\delta_L[G_M^{u-d}(Q^2)] = \frac{-2m_N}{f^2p}(g_A^2 + g_A g_1)K^2(m_\pi, \vec{p}, 0)
$$

$$
+ \frac{3m_N}{f^2} \int_0^1 dx \left[ g_A^2 L^{33}(m_\pi P_\pi, \vec{0}, \vec{p}, x\vec{p}, 0)
$$

$$
+ \frac{2}{9} g_{\Delta N}^2 L^{33}(m_\pi P_\pi, \vec{0}, \vec{p}, x\vec{p}, \Delta) \right],
$$

where $P_\pi = \sqrt{1 + x(1-x)p^2/m_\pi^2}$ and $K^i$, $L^{ij}$ are finite-volume functions.
Derivative method:
\[
\delta_L[\mu/\mu_N] \rightarrow \frac{-m_N}{\pi f^2}(2g_A^2 + g_A g_1)m_\pi e^{-m_\pi L}.
\]

Compare with
\[
\delta_L[g_A] \sim m_\pi^2 e^{-m_\pi L}/\sqrt{m_\pi L}.
\]
Finite-volume effects from ChPT: isovector $G_E$

**Derivative method:**

$$\delta_L[r_E^2] \sim (m_\pi L)^{3/2} e^{-m_\pi L}.$$  
Slow approach to asymptote; better approx.: $(2/f^2)e^{-m_\pi L}$.

**Compare with**

$$\delta_L[g_A] \sim m_\pi^2 e^{-m_\pi L}/\sqrt{m_\pi L}.$$
Two ensembles with $m_\pi \approx 250$ MeV, $a = 0.116$ fm:

1. $32^3 \times 48$, $m_\pi L = 4.8$
2. $24^3 \times 48$, $m_\pi L = 3.6$, plus twisted BC to match momenta

Results are PRELIMINARY.
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Excited-state effects are significant!

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open symbols: 24³
filled symbols: 32³

\[ r_E^2 \sim -2\% \text{ effect for } r_E^2; \text{ ChPT: } (r_E^2)^{24} - (r_E^2)^{32} = +0.076 \text{ fm}^2 \]

\[ \mu \sim -5\% \text{ effect for } \mu; \text{ ChPT: } \mu^{24} - \mu^{32} = -0.28 \]
Excited-state effects are significant!
Disconnected diagrams?

![Graph showing disconnected diagrams](image)

Extreme case: isoscalar induced pseudoscalar form factor.

Connected diagrams have pole from partially quenched pion.

Disconnected diagrams must cancel it.

\[ m_\pi = 317 \text{ MeV}. \]

JG et al., Phys. Rev. D 95, 114502 (2017) [1703.06703]
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Likewise, disconnected diagrams must cancel \( \log(m_\pi) \) term in isoscalar \( r^2_E \).

In practice: contribution is less than 5%. e.g. \( \chi^QCD \), PRD 96, 114504 [1705.05849]

Large fitting uncertainty is acceptable.
Understanding excited-state effects

Two main models used:

2. ChPT: nucleon-pion states. Predictive at leading order but not consistent with data.
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Special case: ChPT provides good description for (induced) pseudoscalar form factor. Tree-level diagram dominates.

FIG. 7: PACS data for the momentum transfer dependence of the renormalized induced pseudo scalar form factor. Black symbols are the original plateau estimate data given in Ref. [15]. Red symbols correspond to the data corrected according to eq. (6.3). The prediction of the ppd model is given by the red dashed line, while the corrected ppd model is shown by the dashed brown line. The pole-ansatz description according to (6.2) of the original PACS data is given by the black dashed line.

B. Induced pseudo scalar form factor

In Ref. [15] the PACS collaboration reports plateau estimate data for the two nucleon form factors. The results were obtained on a 96^4 lattice with lattice spacing \( a \sim 0.085 \) fm and a pion mass \( M_\pi = 146 \) MeV. The spatial lattice extent \( L_\pi = 8.1 \) fm is rather large, corresponding to \( M_\pi L_\pi \sim 6.0 \). The source-sink separation equals 15 time slices, i.e. \( t = 1.3 \) fm, and the central four time slices with \( 6 \leq t/a \leq 9 \) were used to obtain the plateau estimates. For more simulation details see [15].

Figure 7 shows essentially fig. 16 of Ref. [15]. It displays the numerical PACS results for the renormalized induced pseudo scalar form factor (black data points) together with existing experimental results (blue and green data points) and the analytic expectation by the pion-pole-dominance (ppd) model (red dashed line). In this model the two form factors are given by

\[
\tilde{G}_P(Q^2) = \frac{4}{\pi} M^2 N G_A(Q^2) Q^2 + M^2 \tilde{G}_A(Q^2) = \frac{G_A(0)}{1 + Q^2/M_A^2}.
\]

(6.1)

In Ref. [15] the value \( M_A^2 = 1.04 \) GeV was chosen, stemming from \( r_A^2 = 12/\pi \) with \( r_A = 0.67 \) fm.

For small momentum transfers the lattice data are incompatible with the ppd model and the experimental data. The PACS collaboration found that the data are well described by a ppd-inspired ansatz (black dashed line in fig. 7) with \( M_{\text{pole}} = 256(17) \) MeV determined by a fit of (6.2) to the data. This mass is about twice as large as the pion mass in the simulation.
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Two main models used:


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Speculation:
Could resonance models provide good description of excited-state effects? Are $N\rho$-type states the dominant contribution for $G_E$ and $G_M$?

Outlook

Derivative method makes it possible to avoid uncertainty in $r_E$ from fitting $G_E(Q^2)$.

- Applies to connected diagrams: largest contribution to $r_E$.
- Exponentially suppressed finite-volume effects.

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▶ Magnetic moment: same sign, similar size.
▶ Charge radius: opposite sign, observed effect much smaller.

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Excited-state effects remain a significant challenge.