Three-body scattering
Unitarity-based approach

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May 9th, 2019
Content

1. Unitarity constraint
2. Final-state interaction
3. Three-particle scattering
4. Analysis of the $1^{++}$ sector
Hadronic excitations from Lattice QCD

[bottom citation]

Meanwhile, the first $\rho\pi$ scattering --


$m_\pi = 392$ MeV
$24^3 \times 128$

isoscalar

isovector

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Hadronic excitations from Lattice QCD

[Dudek et al., PRD 88, 094505 (2013)]

Meanwhile, the first $\rho\pi$ scattering –

I = 2


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Three-body scattering

May 9th, 2019
Hadronic excitations from Lattice QCD

$[\text{Dudek et al., PRD 88, 094505 (2013)}]$ 

Meanwhile, the first $\rho\pi$ scattering – $I = 2$ $[\text{A.Woss, et al. JHEP 1807 (2018) 043}]$
Unitarity of the scattering amplitude

unitarity cut, poles of resonances, dispersive relations

[books by Martin-Spearman, Collins, Gribov]
Resonances = Poles at the Complex plane

Example: Breit-Wigner amplitude

Features of the complex $s$ plane:
- $s = E^2$ – the total inv.mass squared
- The Real axis $\rightarrow$ physical world
- The Imaginary axis $\rightarrow$ analytical continuation
Resonances = Poles at the Complex plane

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Unitarity constraints for the two-body scattering
- $\hat{S}^\dagger \hat{S} = \hat{I}$, $\hat{S} = \hat{I} + i\hat{T}$
- $T(s, t) = \langle p'_1 p'_2 | \hat{T} | p_1 p_2 \rangle$
- Partial-wave expansion . . .
- The final form

$t_l - t_l^\dagger = t_l^\dagger \rho(s) t_l$
Resonances = Poles at the Complex plane

Example: Breit-Wigner amplitude

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- Partial-wave expansion . . .
- The final form

$$\hat{T} - \hat{T}^\dagger = i\hat{T}^\dagger \hat{T}.$$  

$$T(s, t) - T^\dagger(s, t) = \int d\Phi_2 T^\dagger(s, t') T(s, t'')$$

$$t_l - t_l^\dagger = t_l^\dagger \rho(s) t_l$$
Three-particle interaction: resonances are everywhere

COMPASS experiment

$[\text{MM, PhD thesis}]$

Intensity, $\frac{d^2I}{d\sqrt{s} d\sqrt{\sigma}}$ ($\sigma, s$ fixed)

$0.35$
$0.6$
$0.85$
$1.1$
$1.35$
$1.6$
$1.85$
$2.1$

Intensity, $\frac{d^2I}{d\sqrt{s} d\sqrt{\sigma}}$ ($\sigma$ fixed)

$0.35$
$0.6$
$0.85$
$1.1$
$1.35$
$1.6$
$1.85$
$2.1$

$M_{3\pi}$(GeV) $\equiv m_{\pi^+\pi^-}$ (GeV)

$M_{2\pi}$(GeV)

$\sqrt{s} \equiv m_{3\pi}$(GeV)

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Three-particle interaction: resonances are everywhere

Controlled by three-body unitarity

COMPASS experiment [MM, PhD thesis]

Controlled by two-body unitarity

Intensity, $d^2I/d\sqrt{s} d\sqrt{\sigma} (s, \text{fixed } \sigma)$

Intensity, $d^2I/d\sqrt{s} d\sqrt{\sigma} (\sigma, \text{fixed } s)$

$\rho(770)$ $f_2(1270)$

$\pi^+(\text{GeV})$ $\pi^-(\text{GeV})$

$\pi^-(\text{GeV})$ $\pi^+(\text{GeV})$

$p$ $p_r$

$\sqrt{s} \equiv m_{3\pi} (\text{GeV})$

$\sqrt{s} \equiv m_{3\pi} (\text{GeV})$

$\rho(770)$ $f_2(1270)$

$a_1(1260)$ $a_2(1320)$ $a_2(1670)$

Intensity, $d^2I/d\sqrt{s} d\sqrt{\sigma} (s, \text{fixed } \sigma)$

Intensity, $d^2I/d\sqrt{s} d\sqrt{\sigma} (\sigma, \text{fixed } s)$

$\rho(770)$ $f_2(1270)$

$p$ $p_r$

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$\sqrt{s} \equiv m_{3\pi} (\text{GeV})$
Three-body decay
Final-state interaction

isobar model, rescattering, ladder of exchanges
Three-body decay

Decay amplitude – \( \langle p_1 p_2 p_3 | \hat{T} | p_0 \rangle \)

\[ \tilde{\sigma}_3 \equiv m_{23}^2 (\text{GeV}^2) \]

\[ \sigma_1 \equiv m_{12}^2 (\text{GeV}^2) \]

Dalitz plot variables

- Subchannel resonances are bands.
- Angular distribution along the bands determined by angular momenta.
Partial-waves vs Isobar representation

Isobar representation

\[ F(\sigma_1, \sigma_2) = F^{(1)}(\sigma_1, \sigma_2) + F^{(2)}(\sigma_1, \sigma_2) + F^{(3)}(\sigma_1, \sigma_2) \]

\[ = \sum_{l} \sqrt{2l + 1} P_l(z_1) a^{(1)}_l(\sigma_1) + \sum_{l} \sqrt{2l + 1} P_l(z_2) a^{(2)}_l(\sigma_2) + \sum_{l} \sqrt{2l + 1} P_l(z_3) a^{(3)}_l(\sigma_3). \]

Simple model: \[ \sim a^{(i)}_l(\sigma_1) \rightarrow c^{(i)} BW(\sigma_1) = \sim \]
Partial-waves vs Isobar representation

Isobar representation

\[ F(\sigma_1, \sigma_2) = F^{(1)}(\sigma_1, \sigma_2) + F^{(2)}(\sigma_1, \sigma_2) + F^{(3)}(\sigma_1, \sigma_2) \]

\[ = \sum_l \text{few} \sqrt{2l + 1} P_l(z_1) a_l^{(1)}(\sigma_1) + \sum_l \text{few} \sqrt{2l + 1} P_l(z_2) a_l^{(2)}(\sigma_2) + \sum_l \text{few} \sqrt{2l + 1} P_l(z_3) a_l^{(3)}(\sigma_3). \]

Simple model: \( \sim = a_l^{(i)}(\sigma_1) \rightarrow c^{(i)} \text{BW}(\sigma_1) = \sim \).

Partial-wave representation

\[ \sim = F(\sigma_1, \sigma_2) = \sum_l \sqrt{2l + 1} P_l(z_1) f_l^{(1)}(\sigma_1) \]

Why would someone do this? – theoretical constant to \( f^{(1)}(\sigma_1) \) is straightforward.
Two-body unitarity and Khuri-Trieman model

Example of $f_0^{(1)}(\sigma_1)$ constraints:

$$f_0^{(1)}(\sigma_1) = \underbrace{a_0^{(1)}(\sigma_1)}_{\text{same channel}} + \int_{-1}^{1} \frac{dz_1}{2} \left( \sum_l \sqrt{2l+1} P_l(z_2) a_l^{(2)}(\sigma_2) + \sum_l \sqrt{2l+1} P_l(z_3) a_l^{(3)}(\sigma_3) \right) \underbrace{\text{cross-channel (c.-c.) projections}}_{\text{BW}}$$

Unitarity of $f_0^{(1)}(\sigma_1)$ – same RHC as $2 \rightarrow 2$ scattering amplitude, $BW_0^{(1)}(\sigma_1)$

⇒ consistency relation the direct term and the cross-channel projections

⇒ $a_l^{(1)}(\sigma_1)$ obtains corrections from one seen in $2 \rightarrow 2$.

KT model: analytic continuation of two-body unitarity

(source + unitarization of c.-c. projections) (the loop is a dispersive integral)
Diagramatic representation

Isobar representation with \( a_i^{(i)}(\sigma_i) = \hat{a}_i^{(i)}(\sigma_i) \text{BW}_i^{(i)}(\sigma_i) \)

\[
\begin{array}{c}
\text{Diagramatic representation} \\
\text{Isobar representation with } a_i^{(i)}(\sigma_i) = \hat{a}_i^{(i)}(\sigma_i) \text{BW}_i^{(i)}(\sigma_i) \\
\end{array}
\]

\[
\begin{array}{c}
\text{The amplitude prefactor is not constant: } a_i^{(i)}(\sigma_i) = c_i \text{BW}_i^{(i)}(\sigma_i) + \ldots \\
\end{array}
\]

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\end{array}
\]
Khuri-Treiman model in practice

Light-meson decays

- $\eta \rightarrow 3\pi$ [Sebastian et al. (2011)], [P.Guo et al., JPAC, 2015], [Albaladejo, Moussallam (2017)]
- $\eta' \rightarrow \eta\pi\pi$ [Isken, Kubis (2017)]
- $\omega/\phi \rightarrow 3\pi$ [Niecknig, Kubis (2012)], [Danilkin et al., JPAC (2012)]
- $a_1 \rightarrow 3\pi$ [JPAC (in progress)]

Open charm, $D \rightarrow K\pi\pi$, [Niecknig, Kubis (2015)], [Moussallam, in progress (2017)]

$\rho$-meson lineshape: direct and induced
Three-particle scattering

Three-body unitarity, Ladders and Resonances, short-range factorization

[arXiv:1904.11894]
Decomposition of the $3 \rightarrow 3$ scattering

Particle pairing (symmetrization or isobar decomposition), e.g. for the state of identical particles:

$$|p_1 p_2 p_3\rangle = \frac{1}{3!} (|p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle + \text{symm.})$$

$$= \frac{1}{3} \sum_{a=1}^{3} |p_a 1\rangle |p_a 2\rangle \otimes |p_a 3\rangle + |p_a 3\rangle \otimes |p_a 2\rangle \equiv t(\sigma)$$

Separation of connected and disconnected terms (LSZ reduction):

Partial wave expansion (spin $l$ in subchannels, spin $j$ overall)

Amputation of the last scattering bit

$T(\sigma', s, \sigma)$
Decomposition of the $3 \rightarrow 3$ scattering

- **Particle pairing** (symmetrization or isobar decomposition), e.g. for the state of identical particles:

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$$= \frac{1}{3} \sum_{a=1}^{3} |p_{a_1}\rangle \otimes |p_{a_2}\rangle + |p_{a_3}\rangle \otimes |p_{a_2}\rangle = \frac{1}{3} \sum_{a=1}^{3} |a\rangle \quad - \text{isobar-spectator states},$$

- Separation of connected and disconnected terms (LSZ reduction):

$$= \sum_{9} \left( \frac{1}{3} \times \right)$$

- Partial wave expansion (spin $l$ in subchannels, spin $j$ overall)

- Amputation of the last scattering bit

$$t(\sigma) \times \equiv \frac{t(\sigma') T(\sigma', s, \sigma)}{t(\sigma)}$$
Decomposition of the $3 \rightarrow 3$ scattering

- **Particle pairing** (symmetrization or isobar decomposition), e.g. for the state of identical particles:

$$|p_1 p_2 p_3\rangle = \frac{1}{3!} \left( |p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle + \text{symm.} \right)$$

$$= \frac{1}{3} \sum_{a=1}^{3} \frac{|p_{a_1}\rangle \otimes |p_{a_2}\rangle + |p_{a_3}\rangle \otimes |p_{a_2}\rangle}{2} = \frac{1}{3} \sum_{a=1}^{3} |a\rangle \ - \text{isobar-spectator states,}$$

- Separation of **connected** and **disconnected** terms (LSZ reduction):

$$\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = \sum_{9} \left( 3 \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \right)$$
Decomposition of the $3 \rightarrow 3$ scattering

- **Particle pairing** (symmetrization or isobar decomposition), e.g. for the state of identical particles:

\[
|p_1 p_2 p_3\rangle = \frac{1}{3!} (|p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle + \text{symm.})
\]

\[
= \frac{1}{3} \sum_{a=1}^{3} \left( \frac{|p_{a_2}\rangle \otimes |p_{a_3}\rangle + |p_{a_3}\rangle \otimes |p_{a_2}\rangle}{2} \right) = \frac{1}{3} \sum_{a=1}^{3} |a\rangle \quad - \text{isobar-spectator states,}
\]

- Separation of **connected** and **disconnected** terms (LSZ reduction):

\[
\begin{align*}
\begin{array}{c}
\text{connected terms}\end{array} & = \sum_{\sigma, j} \left( 3^{\sigma} + 1 \right) \\
\begin{array}{c}
\text{disconnected terms}\end{array} & = \sum_{\sigma, j} \left( 1 \right)
\end{align*}
\]

- **Partial wave expansion** (spin $l$ in subchannels, spin $j$ overall)
Decomposition of the $3 \rightarrow 3$ scattering

- **Particle pairing** (symmetrization or isobar decomposition), e.g. for the state of identical particles:

$$|p_1p_2p_3\rangle = \frac{1}{3!} (|p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle + \text{symm.})$$

$$= \frac{1}{3} \sum_{a=1}^{3} |p_{a_1}\rangle \otimes |p_{a_2}\rangle + \frac{|p_{a_3}\rangle \otimes |p_{a_2}\rangle}{2} = \frac{1}{3} \sum_{a=1}^{3} |a\rangle \quad \text{isobar-spectator states,}$$

- Separation of **connected** and **disconnected** terms (LSZ reduction):

$$\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array} = \sum_{i} \left( 3 \begin{array}{c}
\bigcirc
\end{array} + \begin{array}{c}
\bigtimes
\end{array} \right)$$

- Partial wave **expansion** (spin $l$ in subchannels, spin $j$ overall)
- **Amputation** of the last scattering bit

$$\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array} = t(\sigma),$$

$$\begin{array}{c}
\begin{array}{c}
\bigtimes
\end{array}
\end{array} \equiv \begin{array}{c}
\big\bigcirc
\end{array} = t(\sigma') T(\sigma', s, \sigma) t(\sigma),$$
Three-body-unitarity constraint

Three-body scattering amplitude must satisfy the integral equation

\[
\mathcal{T}(\sigma', s, \sigma) - \mathcal{T}^\dagger(\sigma', s, \sigma) = \\
2i \frac{1}{\lambda_s^{1/2}(\sigma')} \frac{1}{8\pi} \int_{\sigma'-(\sigma', s)}^{\sigma'+(\sigma', s)} d\sigma_3' t(\sigma_3') \mathcal{T}(\sigma_3', s, \sigma) \\
+ i \int_{4m^2_\pi} \frac{d\sigma''}{2\pi} \mathcal{T}^\dagger(\sigma', s, \sigma'') t(\sigma'') t^\dagger(\sigma'') \rho(\sigma'') \rho_s(\sigma'') \mathcal{T}(\sigma'', s, \sigma) \\
+ \frac{2i}{3} \frac{1}{(8\pi)^2} \int_{\phi(\sigma_2', s, \sigma''_3)>0} \frac{d\sigma'' d\sigma''_3}{2\pi s} \mathcal{T}^\dagger(\sigma', s, \sigma''_2) t^\dagger(\sigma''_2) t(\sigma''_3) \mathcal{T}(\sigma''_3, s, \sigma) \\
+ 2i \frac{1}{\lambda_s^{1/2}(\sigma')} \frac{1}{8\pi} \int_{\sigma'-(\sigma', s)}^{\sigma'+(\sigma', s)} d\sigma_2 \mathcal{T}^\dagger(\sigma', s, \sigma_2) t^\dagger(\sigma_2) \\
+ 6i \frac{2\pi s}{\lambda_s^{1/2}(\sigma') \lambda_s^{1/2}(\sigma)} \theta^+(\phi(\sigma', s, \sigma)).
\]


\[
\mathcal{T} - \mathcal{T}^\dagger = D\mathcal{T} \mathcal{T} + \mathcal{T}^\dagger(\mathcal{T} - \mathcal{T}^\dagger) \mathcal{T} + \mathcal{T}^\dagger \mathcal{T} D\mathcal{T} + \mathcal{T}^\dagger \mathcal{T} D + D,
\]
Splitting amplitude by the interaction range

\[ T(\sigma', s, \sigma) = \mathcal{L}(\sigma', s, \sigma) + \mathcal{R}(\sigma', s, \sigma). \]

Long-range part: exchange processes (on-shell) [Mai et al.(EPJ A53 (2017))]

- Infinite sum of the one-particle-exchange process

\[ \mathcal{L} = \mathcal{B} + \mathcal{L}_\tau \mathcal{B} = \mathcal{B} + \mathcal{B}_\tau \mathcal{L}. \]

- \( T = \mathcal{L}, \mathcal{R} = 0 \) already satisfy unitarity. Can it have resonances?

Short-range part: resonances [MM, Y.Wunderlich, et al. (JPAC) 1904.11894]

- Condition for \( \mathcal{R}(\sigma', s, \sigma) \) is complicated
- However, simplified significantly if FSI is factorized from both sides:

\[ \mathcal{R} \equiv (1 + \mathcal{L}_\tau) \hat{\mathcal{R}}(\tau \mathcal{L} + 1). \]
Unitarity constraint for the resonance kernel

Familiar form of the constraint (see two-body constraint at slide 4):

\[
\hat{R} - \hat{R}^\dagger = \hat{R}^\dagger (1 + \tau^\dagger \mathcal{L}^\dagger) \left[ \tau - \tau^\dagger + \tau^\dagger \mathcal{D} \tau \right] (1 + \mathcal{L} \tau) \hat{R}
\]

\[
= \hat{R}^\dagger \left[ \tau - \tau^\dagger + \tau \mathcal{L} \tau - \tau^\dagger \mathcal{L}^\dagger \tau^\dagger \right] \hat{R}.
\]
Unitarity constraint for the resonance kernel

Familiar form of the constraint (see two-body constraint at slide 4):

\[
\hat{R} - \hat{R}^\dagger = \hat{R}^\dagger (1 + \tau^\dagger L^\dagger) \left[ \tau - \tau^\dagger + \tau^\dagger D \tau \right] (1 + L \tau) \hat{R}
\]

\[
= \hat{R}^\dagger \left[ \tau - \tau^\dagger + \tau L \tau - \tau^\dagger L^\dagger \tau^\dagger \right] \hat{R}.
\]

- **K-matrix-like solution** (cf. \(K_d.f.\) [M.Hansen, S.Sharpe, PRD90 (2014), 116003])

\[
\hat{R} = \mathcal{X} + \mathcal{X}(\tau + \tau L \tau) \hat{R}
\]

\[
= \mathcal{X} + \mathcal{X}(\tau + \tau L \tau) \mathcal{X} + \mathcal{X}(\tau + \tau L \tau) \mathcal{X}(\tau + \tau L \tau) \mathcal{X} + \ldots
\]

\[
= \mathcal{X} + \ldots
\]
Unitarity constraint for the resonance kernel

Familiar form of the constraint (see two-body constraint at slide 4):

\[
\hat{R} - \hat{R}^\dagger = \hat{R}^\dagger (1 + \tau^\dagger L^\dagger) \left[ \tau - \tau^\dagger + \tau^\dagger D \tau \right] (1 + L \tau) \hat{R}
\]

\[
= \hat{R}^\dagger \left[ \tau - \tau^\dagger + \tau L \tau - \tau^\dagger L^\dagger \tau^\dagger \right] \hat{R}.
\]

- **K-matrix-like solution** (cf. $K_{d.f.}$ [M.Hansen, S.Sharpe, PRD90 (2014), 116003]):

\[
\hat{R} = \mathcal{X} + \mathcal{X}(\tau + \tau L \tau) \hat{R}
\]

\[
= \mathcal{X} + \mathcal{X}(\tau + \tau L \tau) \mathcal{X} + \mathcal{X}(\tau + \tau L \tau) \mathcal{X}(\tau + \tau L \tau) \mathcal{X} + \ldots
\]

\[
= \mathcal{X} + \ldots
\]

- **Rescattering interpretation** [MM, Y.Wunderlich, et al. (JPAC) 1904.11894]:

\[
\hat{R} - \hat{R}^\dagger = \hat{R}^\dagger (1 + \tau^\dagger L^\dagger) \left[ \tau - \tau^\dagger + \tau^\dagger D \tau \right] (1 + L \tau) \hat{R} \overset{\text{rescattering}}{\longrightarrow} \text{sort-of}
\]

\[
\begin{array}{c}
\text{direct + crossed coupling} \\
\text{rescattering} \\
\text{rescattering}
\end{array}
\]

Mikhail Mikhasenko (CERN)

Three-body scattering

May 9th, 2019
Factorization of the resonance kernel

- Strictly, one more (weak) assumption – **Factorization**

\[
\hat{R}(\sigma', s, \sigma) = k_f(\sigma')\hat{R}(s) k_i(\sigma) \quad \text{OR}
\]
\[
= \hat{R}_{00}(s) + \sigma'\hat{R}_{10}(s) + \hat{R}_{01}(s)\sigma + \sigma'\hat{R}_{11}(s)\sigma + \ldots,
\]

⇒ Unitarity requirement is algebraic!

\[
\hat{R}(s) - \hat{R}^\dagger(s) = i \hat{R}^\dagger(s)\Sigma(s) \hat{R}(s),
\]

with \( \Sigma \equiv \mathcal{K}^\dagger(\tau - \tau^\dagger)\mathcal{K} + \mathcal{K}^\dagger\tau^\dagger D\tau\mathcal{K}, \)

\( \mathcal{K} \) is the modification of the isobar lineshape due to the rescattering

\[
\mathcal{K} \equiv K^\dagger(\tau - \tau^\dagger)K + K^\dagger\tau^\dagger D\tau K. \]
Factorization of the resonance kernel

- Strictly, one more (weak) assumption – **Factorization**

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\hat{R}(\sigma', s, \sigma) = k_f(\sigma')\hat{R}(s) k_i(\sigma) \quad \text{OR} \quad \\
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\( \mathcal{K} \) is the modification of the isobar lineshape due to the rescattering

**An approximate-three-body unitarity**

\[
\hat{R}(s) = \frac{g^2}{m^2 - s - ig^2/2}
\]

contains effect of the subchannel-resonances **interference**

**The quasi-two-body approximation**

\[
\hat{R}(s) = \frac{g^2}{m^2 - s - ig^2/2}
\]

naively accounts for the subchannel-resonance decay

[J.Basdevant, Ed Berger, PRD19 (1979) 239]
Analysis of $a_1(1260)$

subchannel-resonance interference, analytic continuation

[PRD98 (2018), 096021]
$a_1(1260)$ state – ground axial vector – isospin partner of $\rho$

**$a_1(1260)$ WIDTH**

<table>
<thead>
<tr>
<th>VALUE (MeV)</th>
<th>EVTS</th>
<th>DOCUMENT ID</th>
<th>TECN</th>
<th>COMMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>250 to 600</td>
<td>OUR ESTIMATE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>389 ± 29</td>
<td>OUR AVERAGE</td>
<td></td>
<td></td>
<td>Error includes scale factor of 1.3.</td>
</tr>
<tr>
<td>430 ± 24 ± 31</td>
<td>DARGENT</td>
<td>2017</td>
<td>RVUE</td>
<td>$D^0 \rightarrow \pi^+ \pi^- \pi^+$</td>
</tr>
<tr>
<td>367 ± 128 ± 128</td>
<td>ALEKSEEV</td>
<td>2010</td>
<td>COMP</td>
<td>$190 \pi^- \rightarrow \pi^- \pi^- \pi^- Pb^+$</td>
</tr>
</tbody>
</table>

... We do not use the following data for averages, fits, limits, etc. ...

<table>
<thead>
<tr>
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<th>EVTS</th>
<th>DOCUMENT ID</th>
<th>TECN</th>
<th>COMMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>410 ± 31 ± 30</td>
<td>1 AUBERT</td>
<td>2007AU</td>
<td>BABR</td>
<td>$10.6 e^+ e^- \rightarrow \rho^0 \rho^+ \pi^+ \gamma$</td>
</tr>
<tr>
<td>520 - 680</td>
<td>6360</td>
<td>2 LINK</td>
<td>2007A</td>
<td>FOCS</td>
</tr>
<tr>
<td>480 ± 20</td>
<td>3 GOMEZ-DUMM</td>
<td>2004</td>
<td>RVUE</td>
<td>$\tau^+ \rightarrow \pi^+ \pi^- \pi^-$</td>
</tr>
<tr>
<td>580 ± 41</td>
<td>90k</td>
<td>SALVINI</td>
<td>2004</td>
<td>OBLX</td>
</tr>
<tr>
<td>460 ± 85</td>
<td>205</td>
<td>DRUTSKOY</td>
<td>2002</td>
<td>BELL</td>
</tr>
<tr>
<td>814 ± 36 ± 13</td>
<td>37k</td>
<td>ASNER</td>
<td>2000</td>
<td>CLE2</td>
</tr>
</tbody>
</table>
The $a_1(1260)$ state is a ground axial vector meson with isospin partner of $\rho$. The width of $a_1(1260)$ is given by PDG (2018)

<table>
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<tbody>
<tr>
<td>250 to 600</td>
<td>OUR ESTIMATE</td>
<td>RVUE</td>
<td></td>
<td>$D^0 \to \pi^+ \pi^- \pi^+$</td>
</tr>
<tr>
<td>380 ± 20</td>
<td>OUR AVERAGE</td>
<td>ALEKSEEV</td>
<td>2010</td>
<td>COMP $190 \pi^- \to \pi^- \pi^- \pi^+ Pb^+$</td>
</tr>
<tr>
<td>430 ± 24 ± 31</td>
<td>DARGENT</td>
<td>2017</td>
<td></td>
<td></td>
</tr>
<tr>
<td>367 ± 9 ± 28</td>
<td>420k</td>
<td>2010</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We do not use the following data for averages, fits, limits, etc.

410 ± 31 ± 30 | 1 | AUBERT | 2007AU | BABR | $10.6 e^+ e^- \to \rho^0 \rho^+ \pi^+ \gamma$

520 - 660 | 2 | LINK | 2007A | FOCS | $D^0 \to \pi^+ \pi^- \pi^+$

480 ± 20 | 3 | GOMEZ-DUMM | 2004 | RVUE | $\tau^+ \to \pi^+ \pi^- \pi^+ \nu_{\tau}$

580 ± 41 | 90k | 2004 | OBLX | $\bar{p} p \to 2 \pi^+ 2 \pi^-$

460 ± 88 | 205 | 2002 | BELL | $B (s) K^- K^0$

814 ± 36 ± 13 | 37k | 2000 | CLE2 | $10.6 e^+ e^- \to \tau^+ \tau^- \to \pi^- \pi^0 \pi^0 \nu_{\tau}$
$a_1(1260)$ state – ground axial vector – isospin partner of $\rho$

$J^{PC} = 1^{++}$

$\pi^+ \pi^- \pi^+ \pi^-$

$V-A$: Vector $(1^{--})$ or Axial $(1^{++})$

Isospin 1 due to the charge

Negative $G$-parity $\Rightarrow$ positive $C$-parity
Analysis of experimental (ALEPH measurements) [data from Phys.Rept.421 (2005)]

Two models of $\rho\pi$ scattering:
- SYMM-DISP: Approximate three-body unitarity (includes interference)
- QTB-DISP: Quasi-two-body unitarity both neglect rescattering, $K\rightarrow 1.$

Fit to the public data
Stat. cov. matrix is used in the fit
Syst. cov. matrix – in the bootstrap

Mikhail Mikhasenko (CERN)
Analysis of experimental (ALEPH measurements) [data from Phys.Rept.421 (2005)]

Two models of $\rho\pi$ scattering:

- SYMM-DISP: Approximate three-body unitarity (includes interference)
  \[ \Sigma(s) = \left[ \begin{array}{c} \text{\ldots} \end{array} \right] \]

- QTB-DISP: Quasi-two-body unitarity
  \[ \Sigma(s) = \left[ \begin{array}{c} \text{\ldots} \end{array} \right] \]

both neglect rescattering, $\mathcal{K} \to 1$.

Fit to the public data

- Stat. cov. matrix is used in the fit
- Syst. cov. matrix – in the bootstrap
Determination of the $a_1(1260)$ pole position

Non-trivial analytic continuation [MM et al. (JPAC), PRD98 (2018), 096021]

$$m(a_1) = (1209 \pm 4^{+12}_{-9}) \text{ MeV}$$
$$\Gamma(a_1) = (576 \pm 11^{+80}_{-20}) \text{ MeV}$$

- Large systematic uncertainties due to disregard of rescattering effects
- Effect of the subchannel-resonances interference is very important
Summary

- **Unitarity** is an important constraint
  - that guides the amplitude construction
    [Mai et al. (EPJ A53 (2017)), Jackura et al. (EPJ C79 (2019)), MM et al. (JPAC) 1904.11894]
  - is satisfied in a good FT consideration [R.Briceno, 1905.11188]
  - Separation between the short-range and the long-range is not unique [M.Doering et al., PLB681 (2009) 26-31]
  - Decomposition of the short-range is not unique [A.Jackura et al., 1905.12007]

- The **ladder** is a new phenomenon of the three-particle physics
  - sum of particle exchange diagrams
  - left-hand singularity in the physical region
  - genuine non-factorizable component

- The **resonance** part admits Factorization:
  - Effect of the **Ladder** is the common final-state interaction
  - Unitarity requirement casts to the familiar two-body-like form.
Summary

- **Unitarity** is an important constraint
  - that guides the amplitude construction
  - is satisfied in a good FT consideration [R.Briceno, 1905.11188]
  - Separation between the short-range and the long-range is not unique [M.Doering et al., PLB681 (2009) 26-31]
  - Decomposition of the short-range is not unique [A.Jackura et al., 1905.12007]

- The **ladder** is a new phenomenon of the three-particle physics

Outlook

Better understanding of the exchange processes is needed:

⇒ studies of the final-state interaction in the decays
⇒ studies of the fix-target production data (COMPASS, GlueX)
⇒ studies of the nice, clean scattering data from the lattice
Thank you for attention

Thanks to my collaborators (JPAC group):

Yannick Wundelich  Andrew Jakura  Alessandro Pilloni  Vincent Mathieu  Miguel Albaladejo  Cesar Fernandez  Bernhard Ketzer  Adam Szczepaniak
Backup
m(a_1) = (1209 ± 4^{+12}_{-9}) \text{ MeV}
\Gamma(a_1) = (576 ± 11^{+80}_{-20}) \text{ MeV}

\begin{align*}
\text{Pole mass } & \equiv \text{Re} \sqrt{s_\rho} \text{ (GeV)} \\
\text{Pole width } & \equiv 2 \text{Im} s_\rho \text{ (GeV)} \\
\end{align*}

# | Fit studies
--- | ---
1 | s < 2 \text{ GeV}^2
2 | R' = 3 \text{ GeV}^{-1}
3 | \rho'_m = \rho_m + 10 \text{ MeV}
4 | \rho'_m = \rho_m - 10 \text{ MeV}
5 | \rho'_m = \rho_m - 20 \text{ MeV}
6 | \Gamma'_\rho = \Gamma_\rho + 5 \text{ MeV}
7 | \Gamma'_\rho = \Gamma_\rho - 30 \text{ MeV}
Analytical continuation

\[ |t_{\parallel}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i \left( \frac{\bar{\rho}(s)}{2} + \rho(s) \right) \right|. \]

- Analytical continuation of \( \rho(s) \): integral over the Dalitz plot for the complex energy

\[ \rho(s) = \sum_\lambda \int d\Phi_3 \left| f_\rho(\sigma_1)d\lambda_0(\theta_{23}) - f_\rho(\sigma_3)d\lambda_0(\hat{\theta}_3 + \theta_{12}) \right|^2 \]

- Analytic continuation of \( \rho \)-meson decay amplitude \( f_\rho \)
Analytical continuation of $\rho$-meson decay amplitude $f_{\rho}$
Analytical continuation

\[ |t_{\parallel}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i \left( \frac{\tilde{\rho}(s)}{2} + \rho(s) \right) \right|. \]

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\]

- Analytic continuation of \( \rho \)-meson decay amplitude \( f_{\rho} \)
Analytical continuation

\[ |t^{-1}_I(s)| = \left| \frac{m^2 - s}{g^2} - i \left( \frac{\bar{\rho}(s)}{2} + \rho(s) \right) \right|. \]

- Analytical continuation of \( \rho(s) \): integral over the Dalitz plot for the complex energy

\[ \rho(s) = \sum_\lambda \int d\Phi_3 \left| f_\rho(\sigma_1)d_{\chi_0}(\theta_{23}) - f_\rho(\sigma_3)d_{\chi_0}(\hat{\theta}_3 + \theta_{12}) \right|^2 \]

- Analytic continuation of \( \rho \)-meson decay amplitude \( f_\rho \)
Analytical continuation

\[ |t_{ll}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i \left( \frac{\bar{\rho}(s)}{2} + \rho(s) \right) \right|. \]

- Analytical continuation of \( \rho(s) \): integral over the Dalitz plot for the complex energy

\[ \rho(s) = \sum_{\lambda} \int d\phi_3 \left| f_{\rho}(\sigma_1)d_{\lambda 0}(\theta_{23}) - f_{\rho}(\sigma_3)d_{\lambda 0}(\hat{\theta}_3 + \theta_{12}) \right|^2 \]

- Analytic continuation of \( \rho \)-meson decay amplitude \( f_{\rho} \)
Analytical continuation

\[ |t_{II}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i \left( \frac{\bar{\rho}(s)}{2} + \rho(s) \right) \right|. \]

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- Analytic continuation of \( \rho \)-meson decay amplitude \( f_{\rho} \)
The spurious pole in the Breit-Wigner model

Energy dependent width, stable particles

\[ t(s) = \frac{1}{m^2 - s - im\Gamma(s)}, \quad \Gamma(s) = \Gamma_0 \frac{p(s)}{p(m^2)} \frac{m}{\sqrt{s}}, \quad p(s) = \frac{\sqrt{(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)}}{2\sqrt{s}}. \]

Example: \( m_1 = 140 \text{ MeV}, \ m_2 = 770 \text{ MeV}, \ m = 1.26 \text{ GeV}, \ \Gamma_0 = 0.5 \text{ GeV} \)