ISOSPIN BREAKING IN TAU DECAYS AND

\((g - 2)_{\mu}\)

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$(g - 2)_\mu$ RECAP

$(g - 2)_\mu$: discrepancy between exp vs theory ($\gtrsim 3\sigma$) hadronic contributions dominate the error

**HLbL**: models, lattice QCD, dispersive method [A. Gerardin’s talk]

**HVP LO**: dispersive approach vs lattice QCD [C. Lehner’s talk]

Let’s focus on Hadronic Vacuum Polarization

1. dispersive approach more precise than lattice
2. alternative data set for dispersive approach: $\tau$
3. isospin-breaking corrections: unde venis?
4. isospin-breaking corrections: quo vadis?
Dispersive integral

\[ a_\mu = \frac{\alpha}{\pi} \int \frac{ds}{s} K(s, m_\mu) \frac{\text{Im}\Pi(s)}{\pi} \] [Brodsky, de Rafael '68]

analyticity \( \hat{\Pi}(s) = \Pi(s) - \Pi(0) = \frac{s}{\pi} \int_{4m^2_\pi}^{\infty} dx \frac{\text{Im}\Pi(x)}{x(x - s - i\varepsilon)} \)

unitarity

\[ \text{Im} = \sum X x \left| x \right|^2 \quad \frac{4\pi^2\alpha \text{Im}\Pi(s)}{s} \pi = \sigma_{e^+e^- \rightarrow \gamma^* \rightarrow \text{had}} \]

At present \( O(30) \) channels: \( \pi^0\gamma, \pi^+\pi^-, 3\pi, 4\pi, K^+K^-, \cdots \)

\( K(s, m_\mu) \rightarrow \pi^+\pi^- \) dominates due to \( \rho \) resonance

\( \pi\pi \) channel is \( \sim 70\% \) of signal and \( \sim 70\% \) of error
Some problems

[Davier '18]

BABAR and KLOE measurements most precise to date, but in poor agreement

- Others are in between, but not precise enough to decide
- No progress achieved in understanding the reason(s) of the discrepancy
- Conclusion: accuracy of combined results degraded
- Imperative to improve accuracy of prediction (forthcoming $g-2$ results at FNAL, J-PARC)
- Other efforts at VEPP-2000 underway
- Design a new independent BABAR analysis

The BABAR/KLOE discrepancy for $ppg(g)$

KLOE vs Babar

most precise exp. disagree on cross-sections in $\pi\pi$ channel

- averaging of cross-sections before dispersive integral $\rightarrow$ error of $3 \times 10^{-10}$
- difference of $a_\mu$ after dispersive integral as systematic error $\rightarrow 10 \times 10^{-10}$

open opportunity: lattice QCD can be arbiter!
Radiative events

per-mille accuracy goal:
\[ \sigma_{\pi^+\pi^-}(\gamma) : \text{contains } \pi\pi \text{ and } \pi\pi\gamma \]

- remove Initial state radiation (ISR)
- undress photon (remove VP)
+ leave final photon (FSR)

\[ \sigma_{\pi^+\pi^-}(\gamma) = \sigma_{\pi^+\pi^-}^{\text{bare}} \]

(C invariance, ISR FSR factorize)

experiments do (most of) it for us

We introduce spectral function \( v_0(s) = \frac{s}{4\pi\alpha^2} \sigma_{\pi^+\pi^-}^{\text{bare}}(s) \)

\( v_0(s) \) used in dispersive integral for \( a_\mu \)

define pion form factor \( v_0 = e_{\text{FSR}}\beta_0^3|F_\pi^0|^2 \)
Motivations for $\tau$

Final states $I = 1$ charged

$\tau$ data can improve $a_\mu [\pi \pi]$

$\rightarrow 72\%$ of total Hadronic LO

or $a_\mu^{ee} \neq a^\tau \rightarrow$ NP  [Cirigliano et al '18]
Isospin Corrections

Restriction to $e^+e^- \rightarrow \pi^+\pi^-$ and $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$

$$v_0(s) = \frac{s}{4\pi\alpha^2} \sigma_{\pi^+\pi^-}(s)$$

$$v_-(s) = \frac{m_\tau^2}{6|V_{ud}|^2} \frac{\mathcal{B}_{\pi\pi^0}}{\mathcal{B}_e} \frac{1}{N_{\pi\pi^0}} \frac{dN_{\pi\pi^0}}{ds} \left(1 - \frac{s}{m_\tau^2}\right)^{-1} \left(1 + \frac{2s}{m_\tau^2}\right)^{-1} \frac{1}{S_{EW}}$$

Isospin correction $v_0 = R_{IB}v_-$

$$R_{IB} = \frac{\text{FSR}}{G_{EM}} \frac{\beta_0^3 |F^0_\pi|^2}{\beta_-^3 |F^-_\pi|^2}$$ [Alemani et al. '98]

0. $S_{EW}$ electro-weak radiative correction. [Marciano, Sirlin '88][Braaten, Li '90]

1. Final State Radiation of $\pi^+\pi^-$ system [Schwinger '89][Drees, Hikasa '90]

2. $G_{EM}$ (long distance) radiative corrections in $\tau$ decays
   Chiral Resonance Theory [Cirigliano et al. '01, '02]
   Meson Dominance [Flores-Talpa et al. '06, '07]

3. Phase Space $(\beta_0, -)$ due to $(m_{\pi\pm} - m_{\pi^0})$
LONG DISTANCE QED - I

At low energies relevant degrees of freedom are mesons

Chiral Perturbation Theory

Meson dominance model

Corrections casted in one function $\nu_-(s) \rightarrow \nu_-(s)G_{EM}(s)$

Real photon corrections

Virtual photon corrections

Real + virtual

$\rightarrow$ IR divergences cancel
Pion form factors

\[ F_0^0(s) \propto \frac{m_\rho^2}{D_\rho(s)} \]

\[ \times \left[ 1 + \delta_\rho \frac{s}{D_\omega(s)} \right] \]

\[ + \frac{m_X^2}{D_X(s)} \quad X = \rho', \rho'' \]

\[ F_{\pi^-}^-(s) \propto \frac{m_{\rho^-}^2}{D_{\rho^-}(s)} + (\rho', \rho'') \]

Sources of IB breaking in phenomenological models

\[ m_{\rho^0} \neq m_{\rho^\pm}, \quad \Gamma_{\rho^0} \neq \Gamma_{\rho^\pm}, \quad m_{\pi^0} \neq m_{\pi^\pm} \]

\[ \rho - \omega \text{ mixing } \delta_\rho \omega \simeq O(m_u - m_d) + O(e^2) \]
**Status**

\[ a_{\mu}^{\text{HVP,LO}}[\pi\pi, ee] = 503.51(3.5) \times 10^{-10} \]  
with \( E \in [2m_\pi, 1.8 \text{ GeV}] \)

\[ a_{\mu}^{\text{HVP,LO}}[\pi\pi, \tau] = 531.3(3.3) \times 10^{-10} \]

\[ a_\mu[\pi\pi, ee] - a_\mu[\pi\pi, \tau] = -12.0(2.6) \quad \text{[Cirigliano et al.]} \]

\[ a_\mu[\pi\pi, ee] - a_\mu[\pi\pi, \tau] = -16.1(1.8) \quad \text{[Davier et al. '09]} \]

\( \approx -10 \) due to \( S_{\text{EW}} \), rest \( R_{IB} \)

\[ a_\mu[\tau] : \left\{ \begin{array}{l}
\text{model dependence} \\
\text{model dependence} \\
\text{data more precise}
\end{array} \right. = \text{abandoned} \]

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Additional \( \rho \gamma \) mixing correction \[ \text{[Jegerlehner, Szafron '11]} \]

partly accounted in \( m_{\rho^0} - m_{\rho^-} \) in \[ \text{[Davier et al. '09]} \]

\[ a_\mu[\pi\pi, ee] = 385.2(1.5) \text{ with } E \in [0.582 - 0.975] \text{ GeV} \]

\[ a_\mu[\pi\pi, \tau] = 386.0(2.4) \text{ after } R_{IB} \]
Details of calculation

Our calculation: Domain Wall Fermion ensemble $N_f = 2 + 1$

- $a^{-1} \simeq 1.73$ GeV $\simeq 0.11$ fm, $L \approx 5.4$ fm
- $a^{-1} \simeq 1.01$ GeV $\simeq 0.19$ fm, $L \approx 4.6, 6.1, 9.12$ fm
- $a^{-1} \simeq 1.43$ GeV $\simeq 0.14$ fm, $L \approx 4.5$ fm

Diagrammatic expansion to $O(\alpha)$ and $O(m_u - m_d)$ [RM123]

E.g. $\langle O \rangle_{QCD+QED} = \langle O_0 \rangle_{QCD} + \alpha \langle O_1 \rangle_{QCD} + O(\alpha^2)$

$QED_L$ and $QED_\infty$: remove zero-modes of photon [Hayakawa, Uno ’08]

Hadronic scheme at $O(\alpha)$ and $O(m_u - m_d)$: [Blum et al. '18]

- $\Omega^-$ mass $\rightarrow$ a latt.spacing
- $m_{\pi^\pm} - m_{\pi^0}$ and $m_{\pi^\pm} \rightarrow m_u, m_d$
- $m_{K^\pm} \rightarrow m_s$

Local vector current $\rightarrow Z_V$
\[ a_\mu = 4\alpha^2 \int dQ^2 K(Q^2)[\Pi(Q^2) - \Pi(0)] \quad (Q^2 \text{ euclidean}) \quad \text{[Blum '03]} \]

\[ \Pi_{\mu\nu}(Q^2) = \int d^4 x e^{iQ \cdot x} \langle j^\gamma_\mu(x) j^\gamma_\nu(0) \rangle \text{ on the lattice} \]

small \( Q^2 \lesssim m^2_\mu \) very difficult

Time-momentum representation \[ \text{[Bernecker, Meyer, '11]} \]

\[ G^\gamma(t) = \frac{1}{3} \sum_k \int d\vec{x} \langle j^\gamma_k(x) j^\gamma_k(0) \rangle, \quad [\Pi(Q^2) - \Pi(0)] = \int dt G^\gamma(t) f(t, Q^2) \]

\[ a_\mu = 4\alpha^2 \int dt w(t) G^\gamma(t), \quad w(t) \text{ muon kernel (weights)} \]

more natural to study \( G^\gamma \) in euclidean time

spectral decomposition (reconstruction)
Contribution to $a_\mu$

Time-momentum representation

$$G_\gamma(t) = \frac{1}{3} \sum_k \int d\vec{x} \langle \tilde{j}_k^\gamma(x) j_k^\gamma(0) \rangle \rightarrow a_\mu = 4\alpha^2 \sum_t w_t G_\gamma(t)$$

Isospin decomposition of $u, d$ current

$$j_\mu^\gamma = \frac{i}{6} (\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d) + \frac{i}{2} (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d) = j_\mu^{(0)} + j_\mu^{(1)}$$

$$G_{00} \leftarrow \langle j_k^{(0)}(x) j_k^{(0)}(0) \rangle = \quad \ldots$$

$$G_{01} \leftarrow \langle j_k^{(0)}(x) j_k^{(1)}(0) \rangle = \quad \ldots$$

$$G_{11} \leftarrow \langle j_k^{(1)}(x) j_k^{(1)}(0) \rangle = \quad \ldots$$

Decompose $a_\mu = a_\mu^{(0,0)} + a_\mu^{(0,1)} + a_\mu^{(1,1)}$
\[ \frac{i}{2} (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d), \left[ \begin{array}{c} I = 1 \\ I_3 = 0 \end{array} \right] \rightarrow j^{(1,-)}_\mu = \frac{i}{\sqrt{2}} (\bar{u} \gamma_\mu d), \left[ \begin{array}{c} I = 1 \\ I_3 = -1 \end{array} \right] \]

Isospin 1 charged correlator $G_{11}^{W} = \frac{1}{3} \sum_k \int d\vec{x} \langle j^{(1,+)}_k(x) j^{(1,-)}_k(0) \rangle$

\[ \delta G^{(1,1)} \equiv G_{11}^{\gamma} - G_{11}^{W} \]

\[ = Z_V^4 (4\pi\alpha)^4 \frac{(Q_u - Q_d)^4}{4} \left[ \begin{array}{c} \text{subleading diagrams currently not included} \end{array} \right] \]

\[ G_{01}^{\gamma} = Z_V^4 \frac{(Q_u^2 - Q_d^2)^2}{2} (4\pi\alpha) \left[ \begin{array}{c} \text{subleading diagrams currently not included} \end{array} \right] \]

\[ + Z_V^2 \frac{Q_u^2 - Q_d^2}{2} (m_u - m_d) \left[ \begin{array}{c} \text{subleading diagrams currently not included} \end{array} \right] \]
from QCD we need a 4-point function \( f(x, y, z, t) \):

- known kernel with details of photons and muon line
- 1 pair of point sources \((x, y)\), sum over \(z, t\) exact at sink
- stochastic sampling over \((x, y)\) (based on \(|x - y|\))

**Successfull strategy:** \(\times 10\) error reduction

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from QCD we need a 4-point function \( f(x, y, z, t) \):

- \((g - 2)\mu\) kernel + photon propagator

**Similar problem** → re-use HLbL point sources!
Synergy - II

contribution of diagram $F$ to pure $I = 1$ part of $\Delta a_\mu$

$\Delta a_\mu^{(I=1)}[F \text{ only}]$

$O(1000)$ point-src per conf.
$5 \cdot 10^5$ combinations
80 configurations
$\times 4$ reduction in error
finite volume errs relevant → dedicated study

data from [Blum et al. '18]: $O(500)$ point-src per conf.
76 configurations
Synergy - III


contribution of diagrams $V, S$ to $a_\mu$

$O(2000)$ point-src per conf.
$\sim 3000$ combinations
$O(10)$ configurations
$\times 4$ reduction in error

expected QED conn. error $\leq 3 \times 10^{-10}$ $\rightarrow$ matches target
Presently only leading diagrams are computed $V, F, S, M$ [Blum et al. '18]

same diagrams for isospin-breaking in $\tau$ spectral functions

improvement in precision beneficial to both $(g - 2)_{\mu}$ and $\tau$

preliminary numbers for $SU(3)$ and $1/N_c$ suppressed diagrams

[Blum et. al. '18]
Restriction to $2\pi \rightarrow$ neglect pure $I = 0$ part $a_\mu^{(0,0)}[\pi^0 \gamma, 3\pi, \ldots]$

Lattice: $\Delta a_\mu[\pi\pi, \tau] = 4\alpha^2 \sum_{t} w_t \times \left[ G^\gamma_{01}(t) + G^\gamma_{11}(t) - G^W_{11}(t) \right]$  

Pheno: $\Delta a_\mu[\pi\pi, \tau] = \int_{4m^2_\pi}^{m^2_\tau} ds K(s) \left[ v_0(s) - v_-(s) \right]$  

Conversion to Euclidean time for direct comparison  

$Lattice \text{ fully inclusive}$  

$Lattice$ manipulate $G(t)$ (e.g. Backus-Gilbert) to implement cut $E < m_\tau$  

include additional channels in $v_0/v_-$  

$effects$ above $\sim 1 \text{ GeV}$ suppressed by (muon) kernel  

preliminary: smaller than current precision for $\Delta a_\mu$  

additional investigations on the way
LATTICE: PRELIMINARY RESULTS - I

$\Delta a_\mu \to G_{01} + \delta G_{11}$:

$\Delta a_\mu(t) \times 10^{-10}$

$V = \bullet$  $F = \bullet$  $S = \bullet$

$M = \bullet$  $O = \bullet$  relevant, negative, neglected

Pure $I = 1$ only $O(\alpha)$ terms:
Systematic errors

\[ a_{\mu}^{\text{QED,conn}} = V + 2S \]

FV study at coarse \( a^{-1} \sim 1 \text{ GeV} \)

Finite volume errors

empirical observation: diagrams may have largish FV errors

cancellation of FV effects in physical combinations

similar observation in ChPT, e.g. [Bijnens, Portelli ’19]
Dilemma

I am interested in comparing integrands beyond integrals
I have computed correlation functions in Euclidean time

To be or not to be Euclidean

1. leave lattice as it is, convert experiment to Euclidean time
   well-posed problem, simple Laplace trafo

2. spectral reconstruction of lattice data  [talks by Tantalo, Bulava, Portelli]
   ill-posed problem, not needed for integrals like $a_\mu$

let's do the comparison it in Euclidean time

Calculation incomplete, what follows mostly qualitative!
Lattice: Preliminary results - II

Study integrand in euclidean time → as important as integral

direct comparison
Lattice vs. EFT+Pheno

1. validate previous estimates of $R_{IB}$
2. study neutral/charged $\rho$ and $\omega$ properties

Preliminary lattice (full) calculation: $G_{01}^\gamma + \delta G$

Not included:
1. relevant
2. sub-leading $1/N_c$, $SU(N_f)$
3. finite-volume errors
4. discretization errors
**MODEL CALCULATIONS**

Preliminary (using $G_{EM}^\pi$ and without $S_{EW}$)

\[ \Delta a_\mu(t) \times 10^{-10} = \text{[Jegelehner, Szafron '11]} \]

depends on $\rho^0$ and $\rho^-$ masses/widths

requires $G_{EM}^\pi$ to compare with lattice

resembles lattice results qualitative agreement

Data from private comm. with F. Jegelehner
Experimental results

\[ \Delta a_\mu(t) = 4\alpha^2 \sum_t w_t \left\{ \int ds h(s, t) \left[ v_0(s) - \frac{v_1(s)}{G_{EM}(s)} \right] \right\} \]

\( v_0 \) BaBar, \( v_1 \) Aleph

preliminary GEM\( ^\pi \)

\( v_1 \rightarrow kv_1 \)
\( k = 1 \) Standard Model
\( k \neq 1 \) BSM (SMEFT)

[Cirigliano et al. '18]

lattice suggests a different answer
Towards a comparison

Lattice contains $\pi^0\pi^-\gamma$ states →

Re-evaluation of $G_{EM} \rightarrow G_{EM}^{\pi}$ [in collab. with Cirigliano]

Real photon corrections

Virtual photon corrections

$G_{EM}^{\pi}$ w/o $\pi^0\pi^-\gamma$ FSR

$\frac{v_-}{G_{EM}^{\pi}}$ w $\pi^0\pi^-\gamma$ FSR
Outlook

use arbitrary kernels with desired properties [with M. Gonzales-Alonso]

even stronger suppression of neglected channels at high energies

suppression of short distances (cutoff effects)

suppression of long distances (noise)

map other spectral functions to the corresponding correlators

  e.g. $K^*$ channel in vector-vector correlator

Eventually proper calculation is isospin-breaking corrections of $\pi\pi$ form factors
Conclusions

These are exciting times for $(g - 2)_\mu$:

1% goal for lattice results to be expected soon

QED+SIB crucial to reach target uncertainty

As a bi-product we get $\Delta a_\mu[\tau]$:

1. **first lattice calculation** of $\Delta a_\mu[\tau]$ almost complete
2. tests/checks previous calculations
   - comparing $v_-$ with experiment requires $G^{\pi}_{\text{EM}}$
   - study $G^{\gamma}_{01}$ alone $\rightarrow \rho \omega$ mixing; $\delta G^{(1,1)}$ alone $\rightarrow \rho^0 \text{ vs } \rho^-$
3. possibly sensitive to new physics

Thanks for your attention
Gauss law + periodic BC: no states with total electric charge

Solution: remove zero-spatial mode $\tilde{A}_\mu(p_0, \vec{0}) = 0 \ \forall p_0$

local in time: Hamiltonian and transfer matrix
non-local in space: renormalizability? OPE?

Do we really need it?

at $O(\alpha)$ I can factorize my observable

$$\sum_{x,y} \langle O(0, x, y, z) \rangle_{QCD} \Delta_\gamma(x, y)$$

freedom on photon propagator $\rightarrow$ analytic (infinite volume)

my view on Lattice QCD+QED (at $O(\alpha)$):

$\langle n\text{-point functions} \rangle_{QCD} \times$ analytic QED kernels

provided QED kernels fall-off sufficiently fast
Gounaris-Sakurai based on VMD model w/o EM gauge invariance
- generation of a photon mass
  + based on phase shift (proper pion rescattering behavior)
widely used: e.g. PDG estimates of $m_\rho$, $\Gamma_\rho$

VMD model with gauge-invariance
at 1-loop $s$-dependent mass matrix

limits of validity pion-loop? high enough energy must break down
$\rho\gamma$ MIXING - II

[Jefferson, Szafron '11]

Fig. 6. a) Ratio of the full $|F_\pi(E)|^2$ in units of the same quantity omitting the mixing term together with a standard GS fit with PDG parameters. b) The same mechanism scaled up by the branching fraction $\Gamma_V/\Gamma_V(V\to\pi\pi)$ for $V=\omega$ and $\phi$.

In the $\pi\pi$ channel the effects for resonances $V\neq\rho$ are tiny if not very close to resonance.

Fig. 7. CMD-2 data for $|F_\pi|^2$ in $\rho-\omega$ region together with Gounaris-Sakurai fit. Left before subtraction right after subtraction of the $\omega$.

has to be applied in the relation between the spectral functions. Finally state radiation correction FSR(s) and vacuum polarization effects we have been subtracted from all $e^+e^-$-data.

In Fig. 8 we illustrate the consequence of $\rho-\gamma$ mixing. After applying the correction (for our set of parameters, which is not far from standard GS fit parameters) the consistency of $\tau$ and $e^+e^-$ data is

30% correction at 1 GeV

$\delta_1^{\rho}$ in good agreement $E < 800$ MeV

perhaps restrict the $\rho\gamma$ below 800 MeV?
$[1] = [\text{Jegelehner, Szafron '17}]$

Modified $\rho\gamma$ coupling

large negative $\Delta a_\mu$

Modified $\rho\gamma$ suggests different behavior from lattice data

direct comparison with lattice not possible $\rightarrow$ hard cut at 1 GeV
Some QED corrections computed in Chiral PT

\[ \text{e.g. photon exchange between } \tau \text{ and hadrons} \]

\[ \text{relevant to compare lattice data vs } v_- \]

\[ \text{is current precision enough?} \]

\[ \text{alternative calculation from lattice possible} \]

[Giusti et al. ’17]