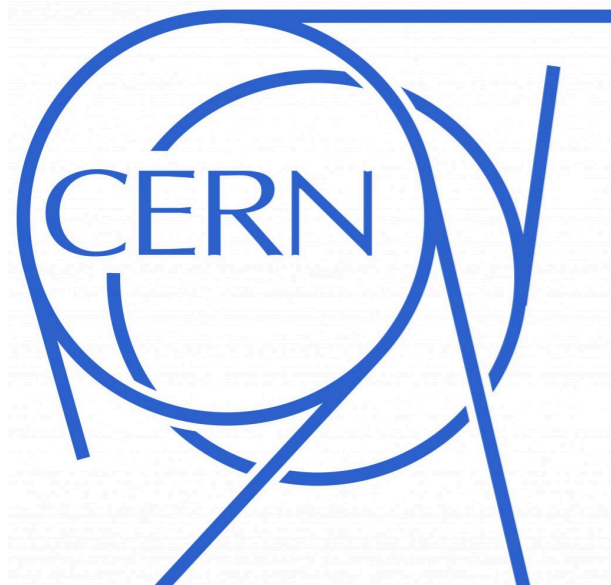


Introduction to scattering on the lattice

Maxwell T. Hansen

July 22nd, 2019

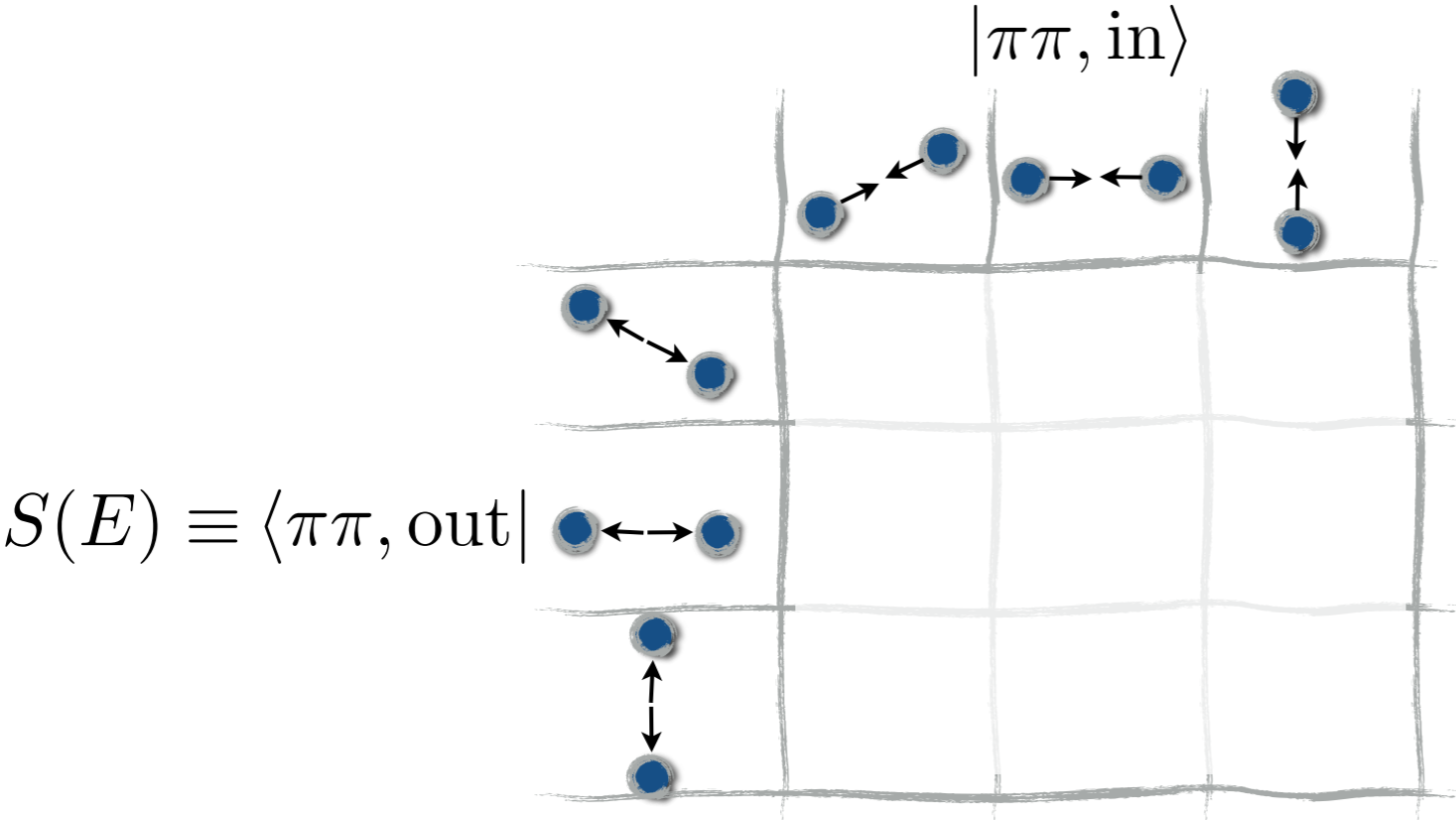


QCD Fock space

□ At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d$, $K \sim \bar{s}u$, $p \sim uud$

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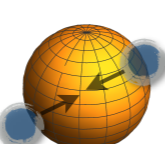
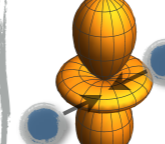
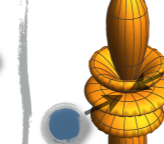
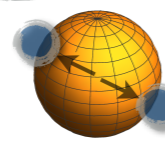
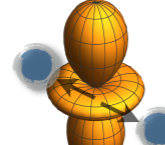



□ Overlaps of multi-hadron **asymptotic states** → S matrix

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d$, $K \sim \bar{s}u$, $p \sim uud$

$|\pi\pi, \text{in}\rangle$

			
	$S_0(E)$	0	0
	0	$S_1(E)$	0
	0	0	$S_2(E)$

$S(E) \equiv \langle \pi\pi, \text{out} |$

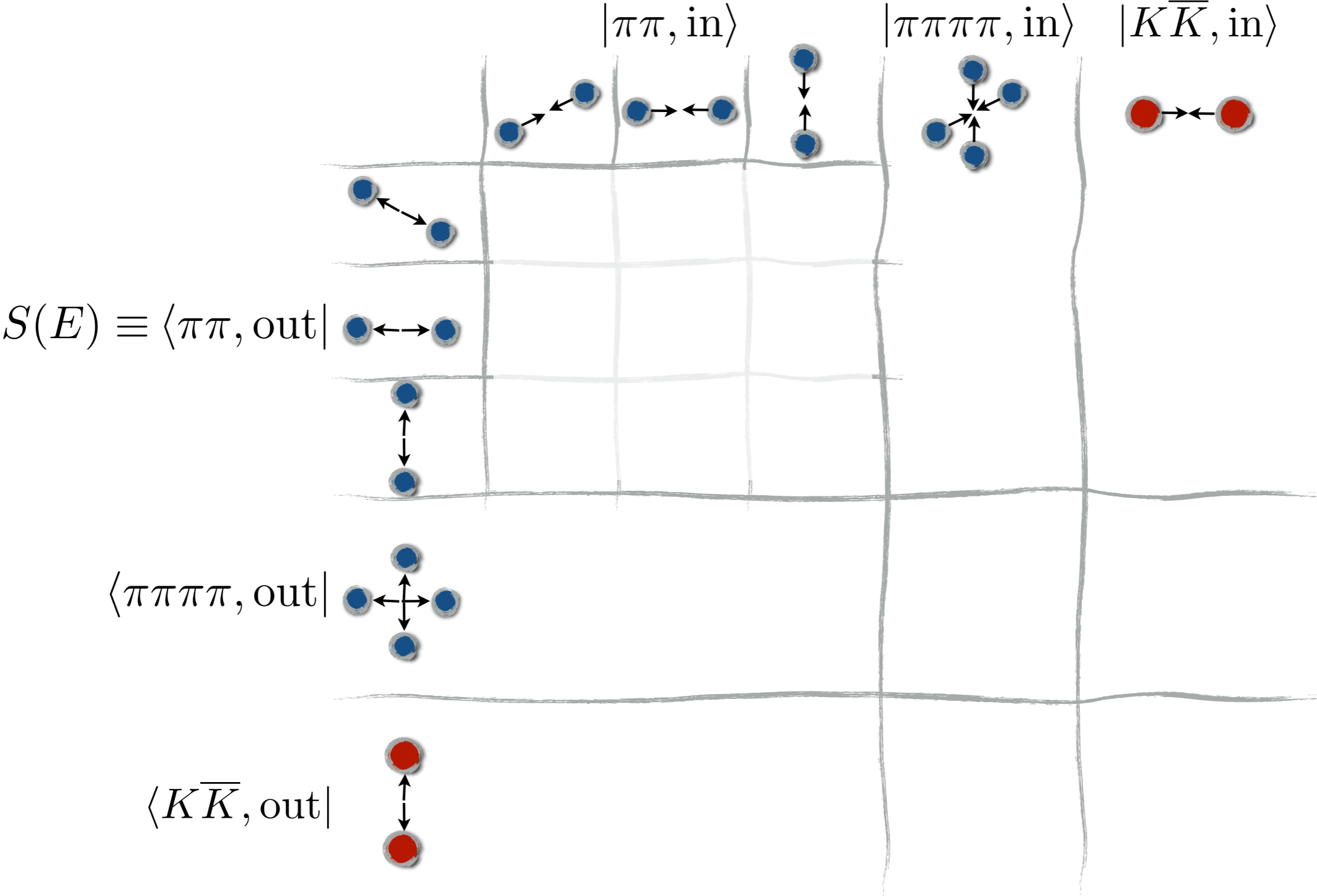
- Overlaps of multi-hadron **asymptotic states** \rightarrow S matrix
- Diagonalized in angular-momentum basis

$$S_0(E) = e^{2i\delta_0(E)}$$

$$\mathcal{M}_0(E) \propto e^{2i\delta_0(E)} - 1$$

QCD Fock space

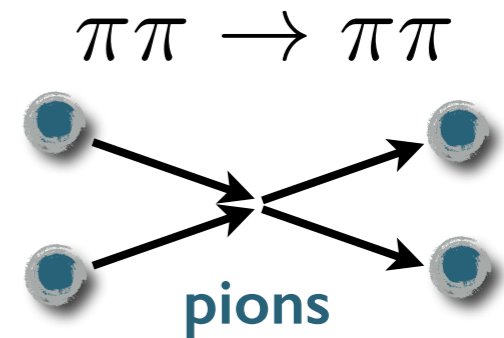
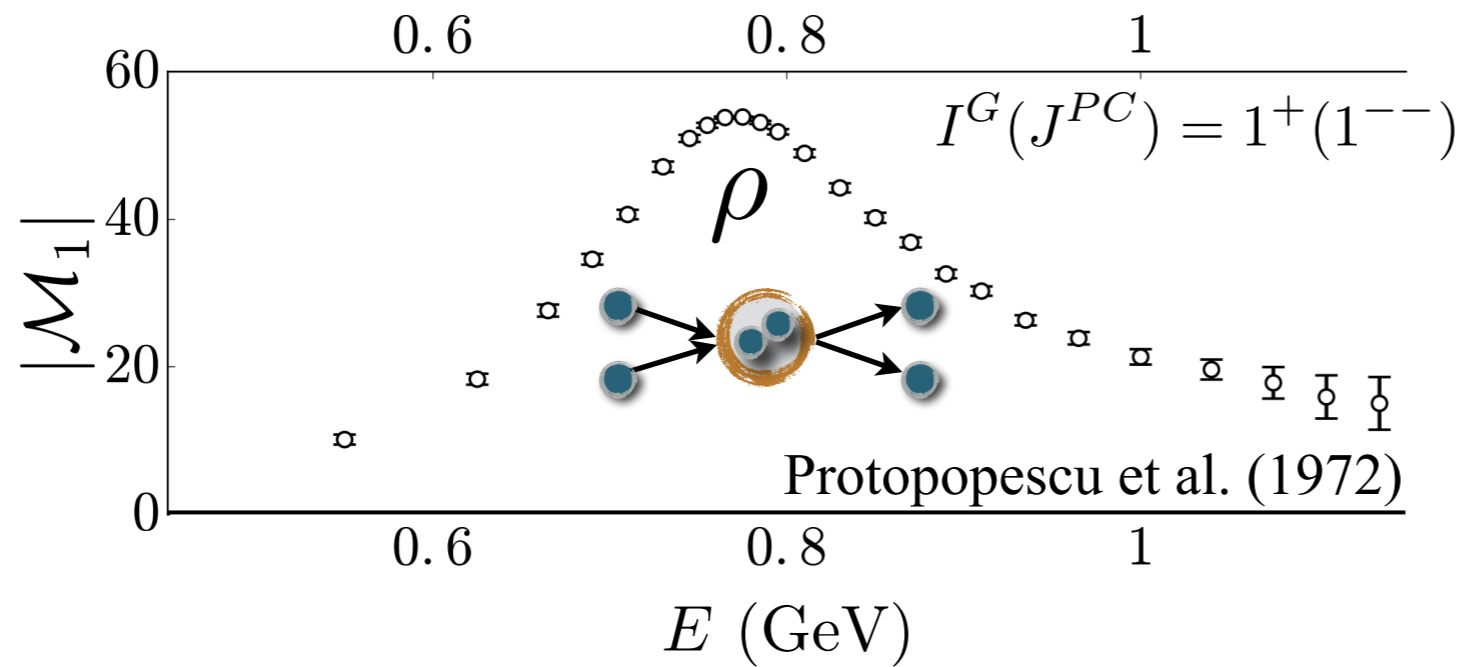
□ At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d$, $K \sim \bar{s}u$, $p \sim uud$



Infinite-dimensional matrix for each set of quantum numbers

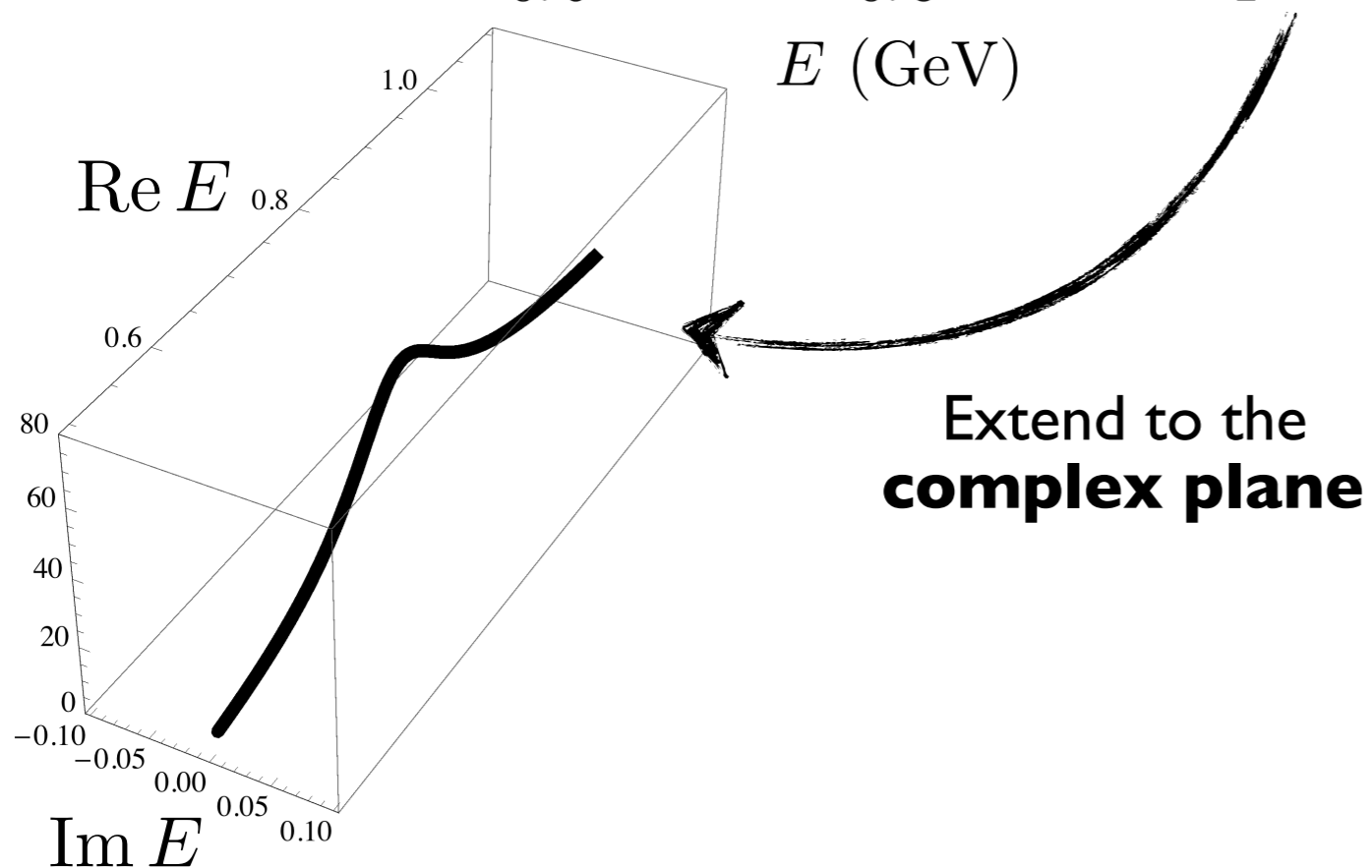
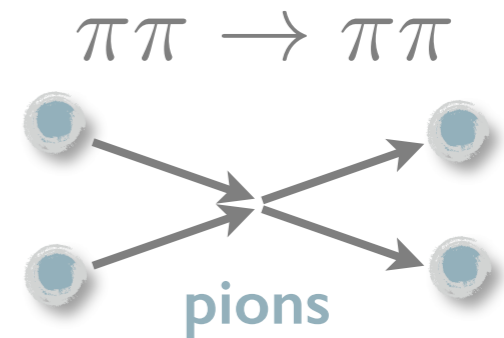
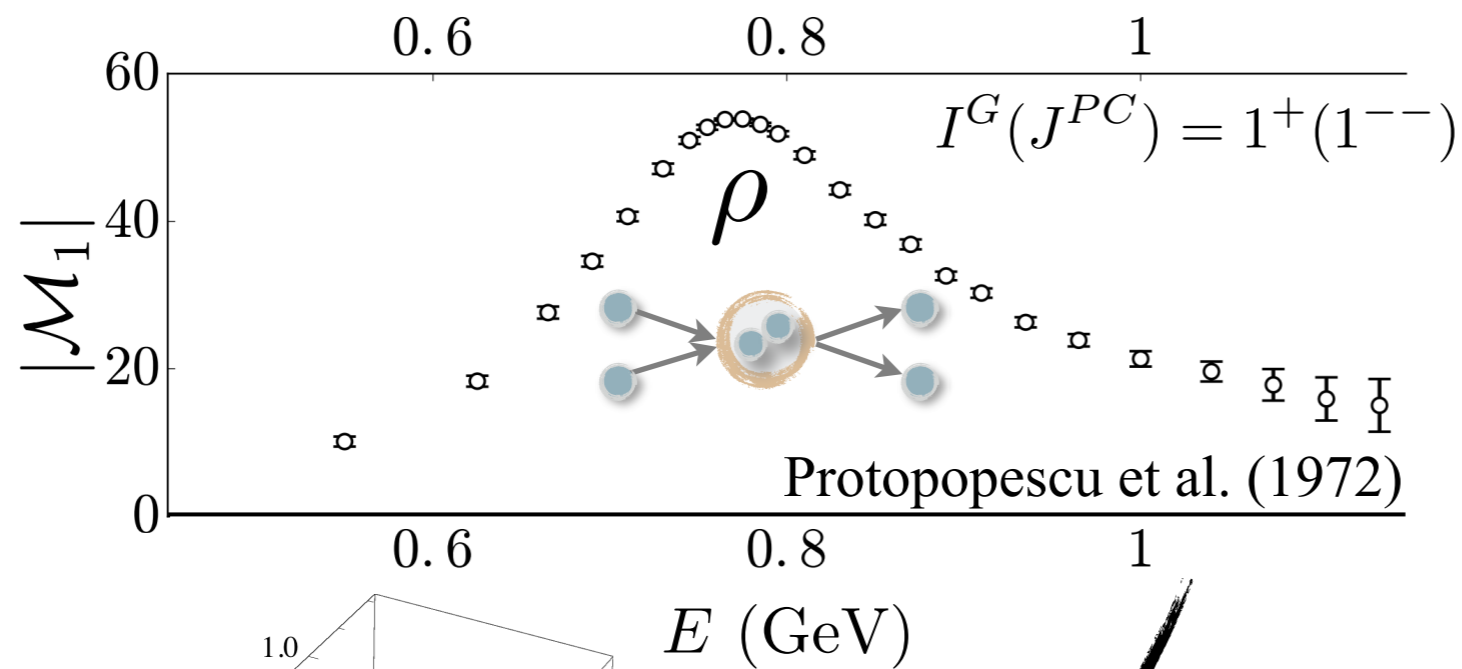
Extracting resonance properties

- Roughly speaking, a bump in: $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$
- scattering rate unitarity relation



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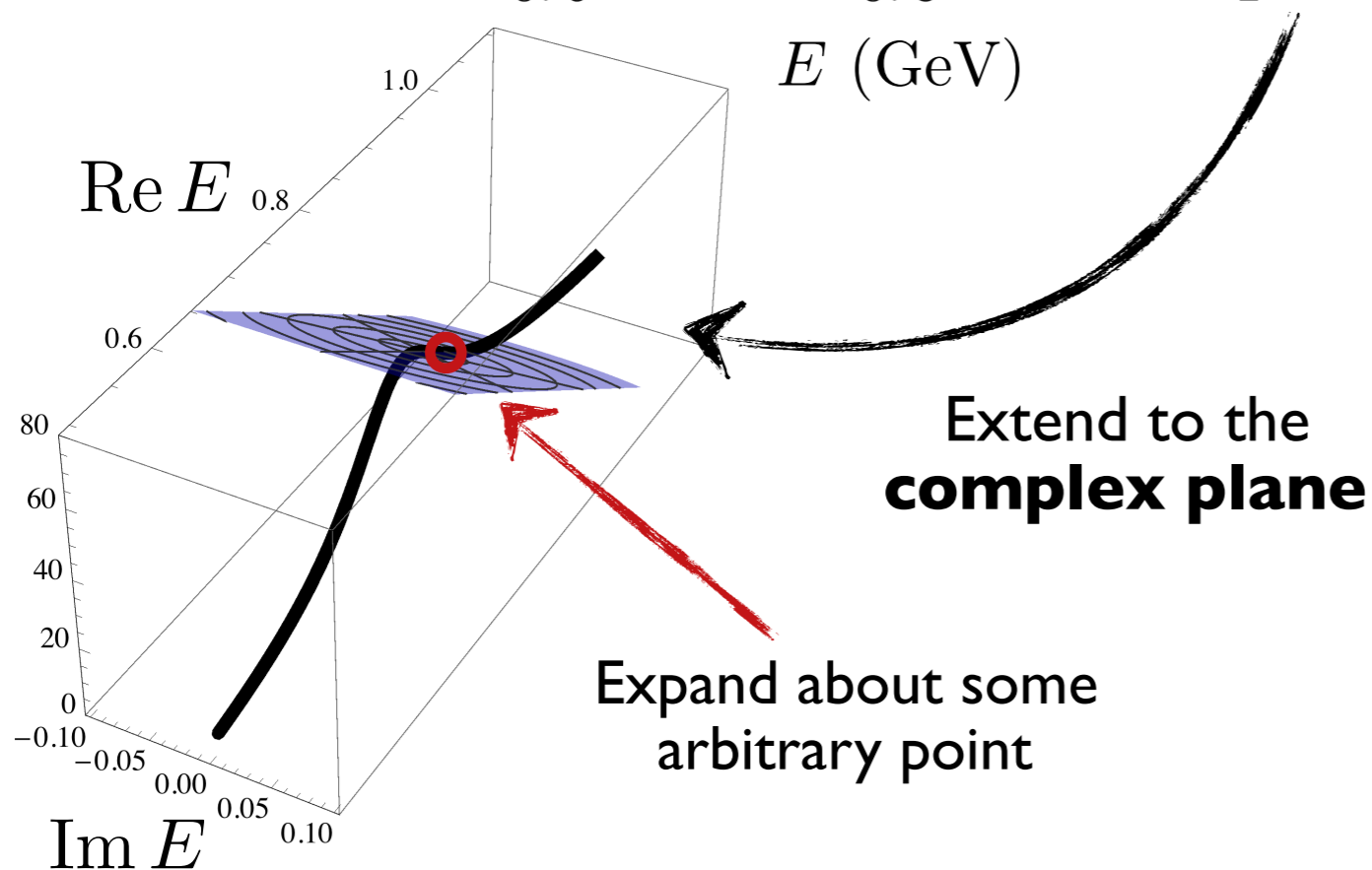
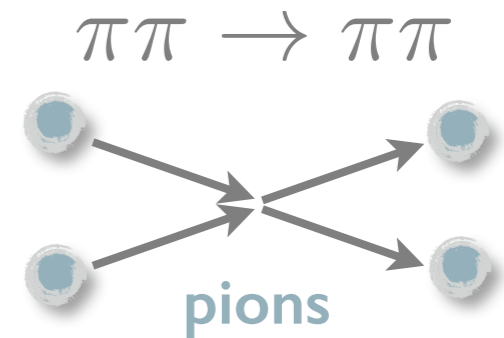
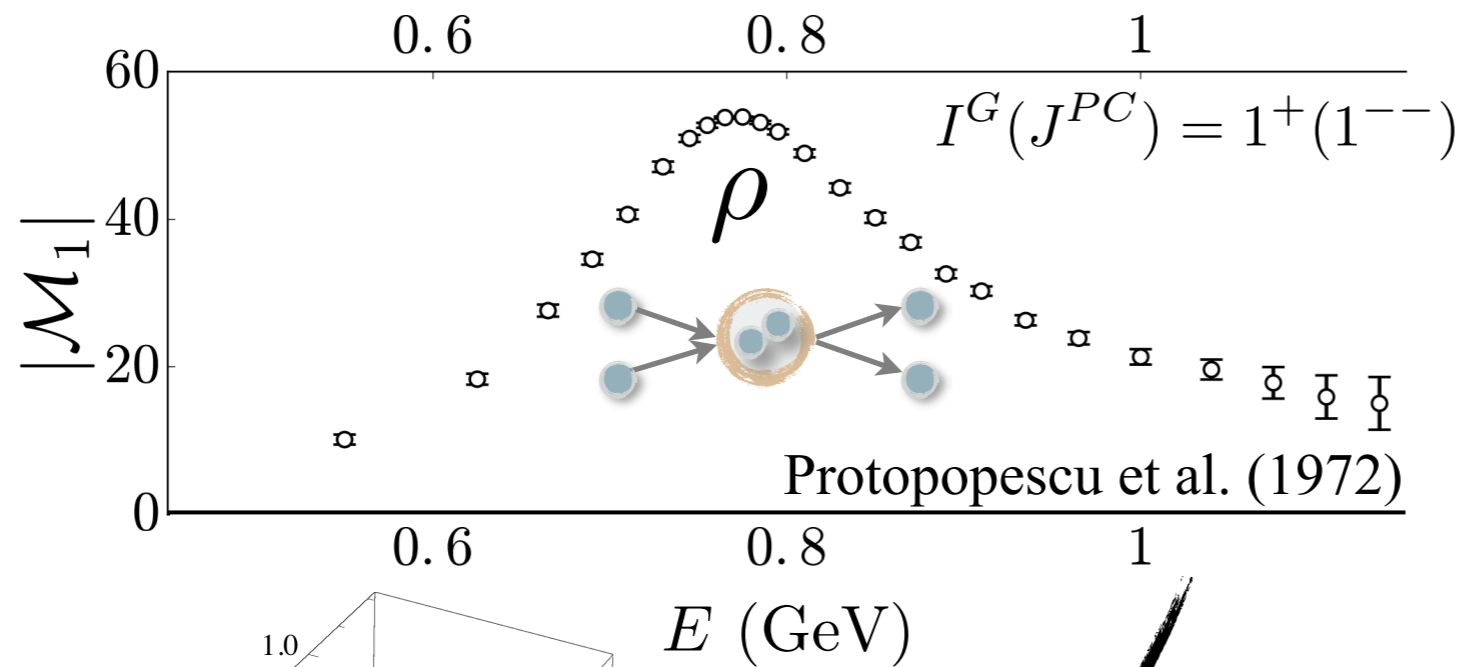


Extracting resonance properties

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scattering rate
unitarity relation

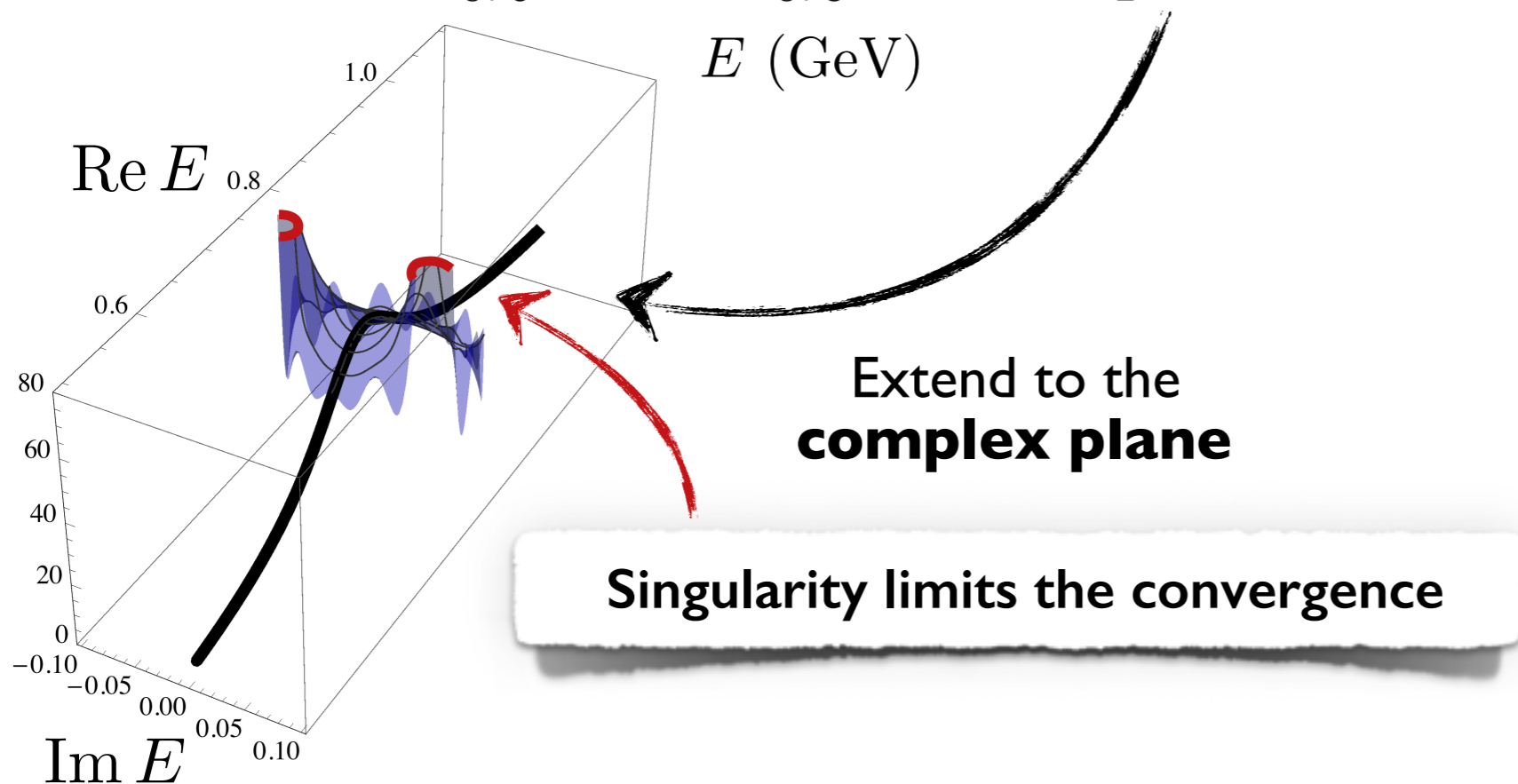
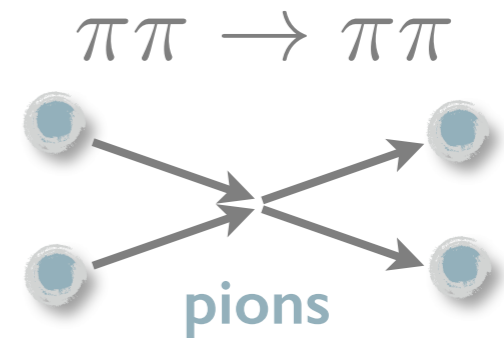
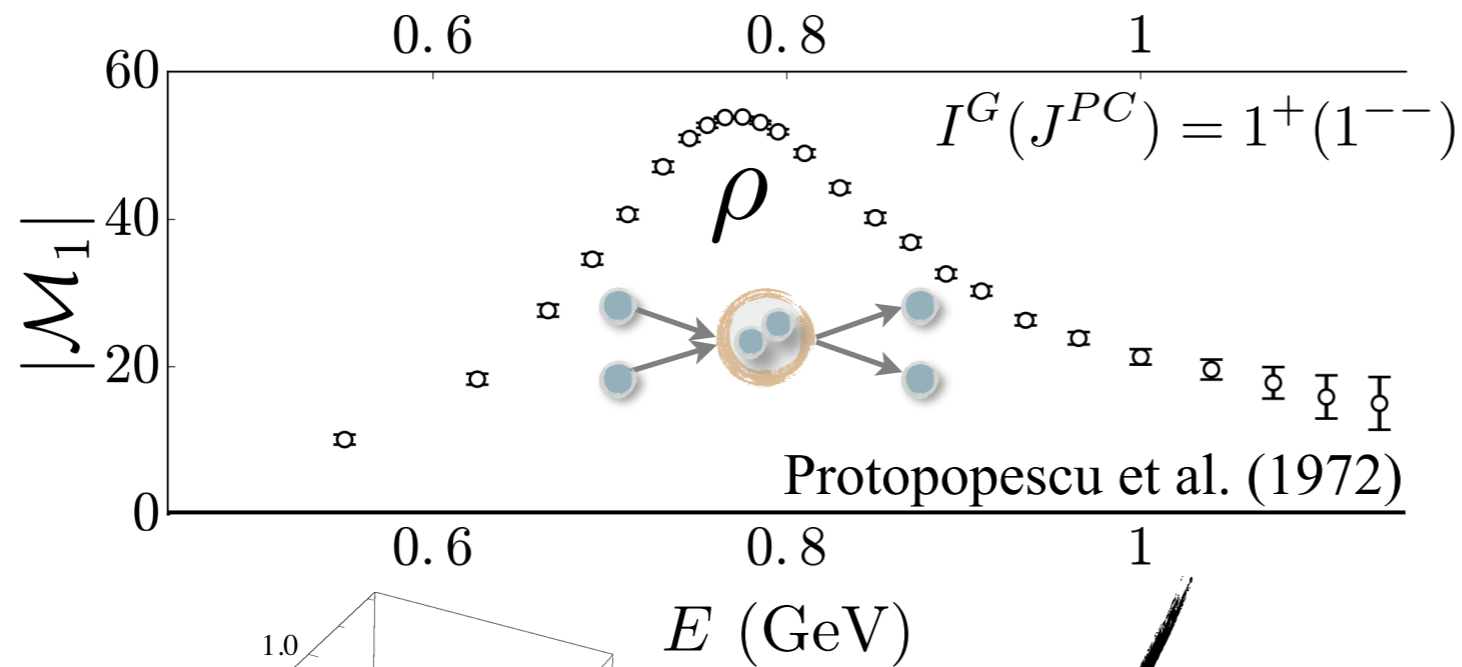


Extracting resonance properties

- Roughly speaking, a bump in:

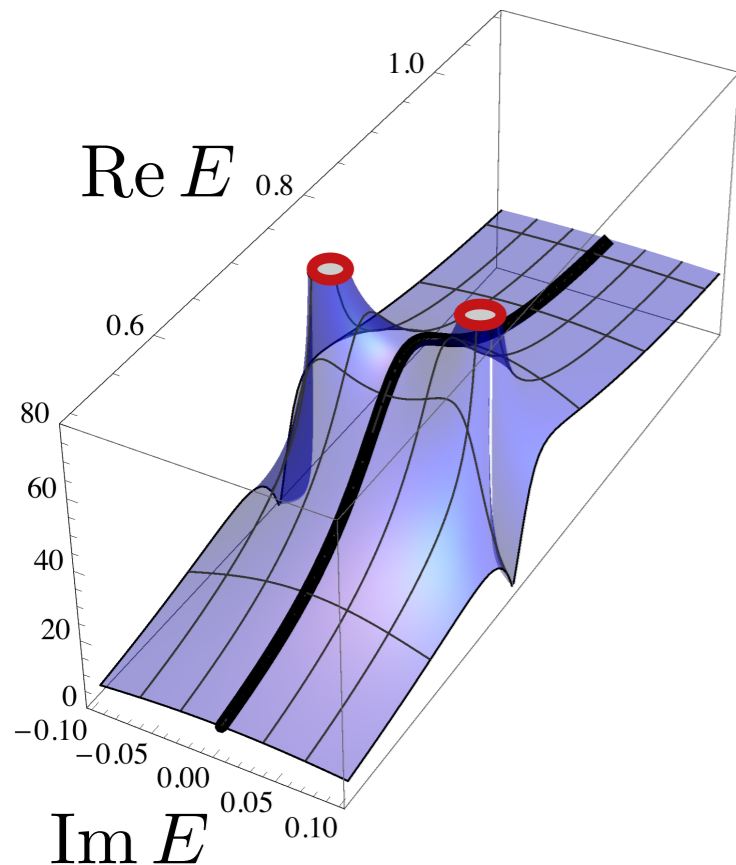
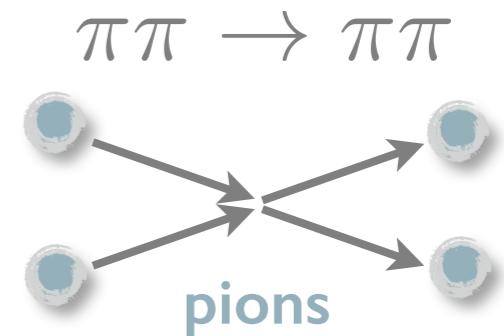
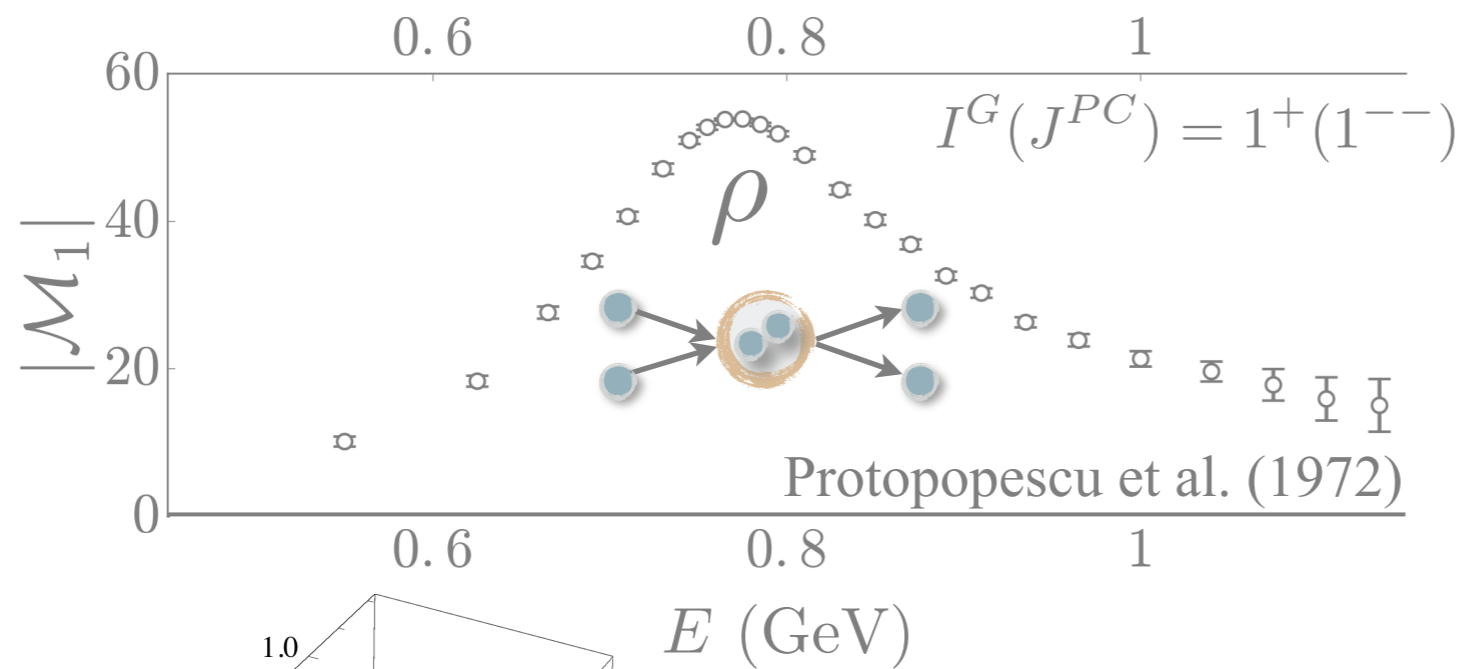
$$|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$$

scattering rate
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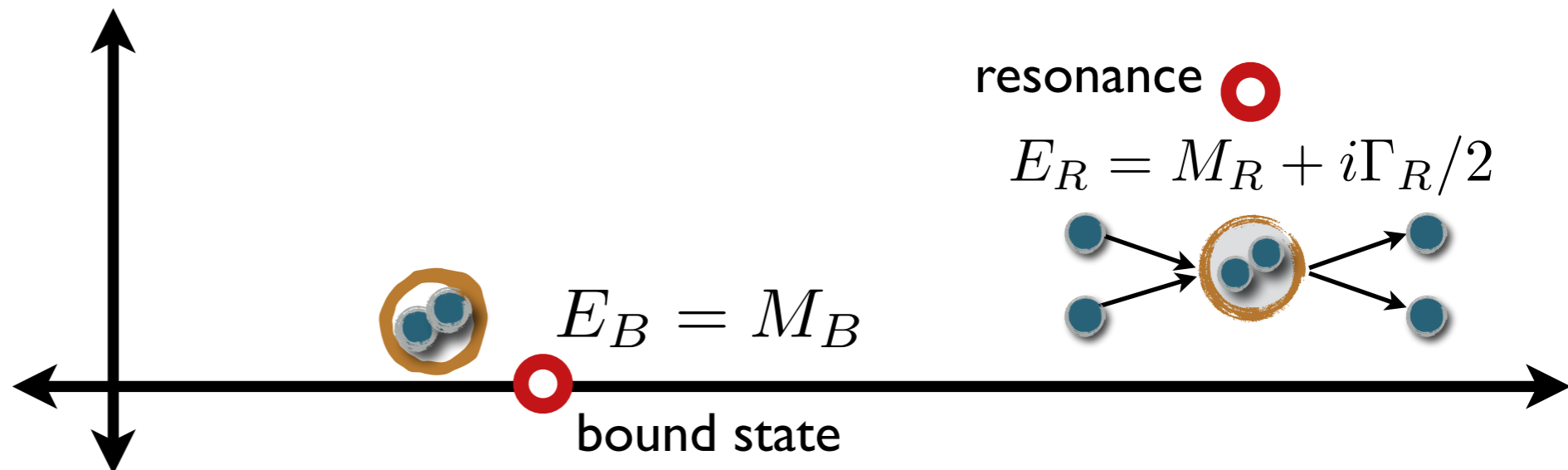


Extracting resonance properties

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Analytic continuation reveals a **complex pole**



Riemann sheets

- Most useful to analytically continue the scattering **amplitude**

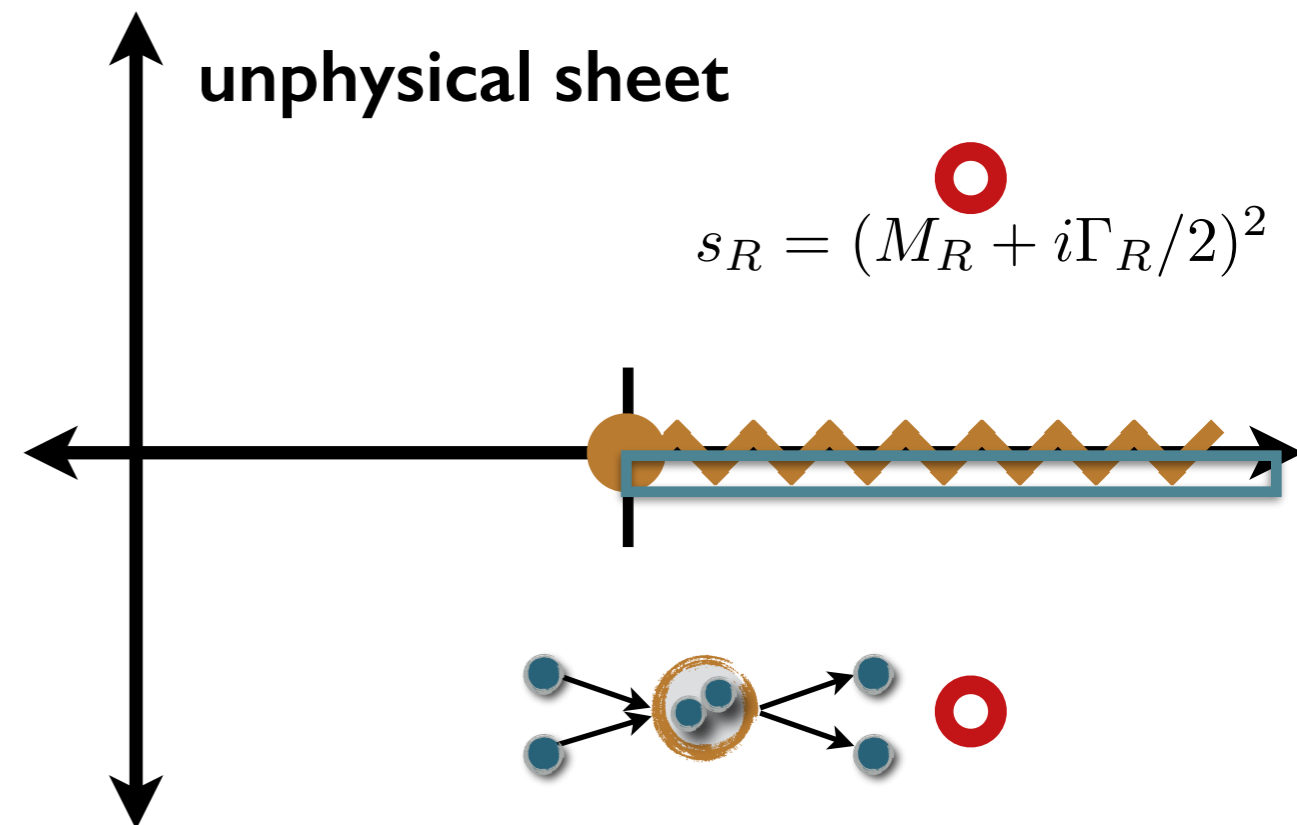
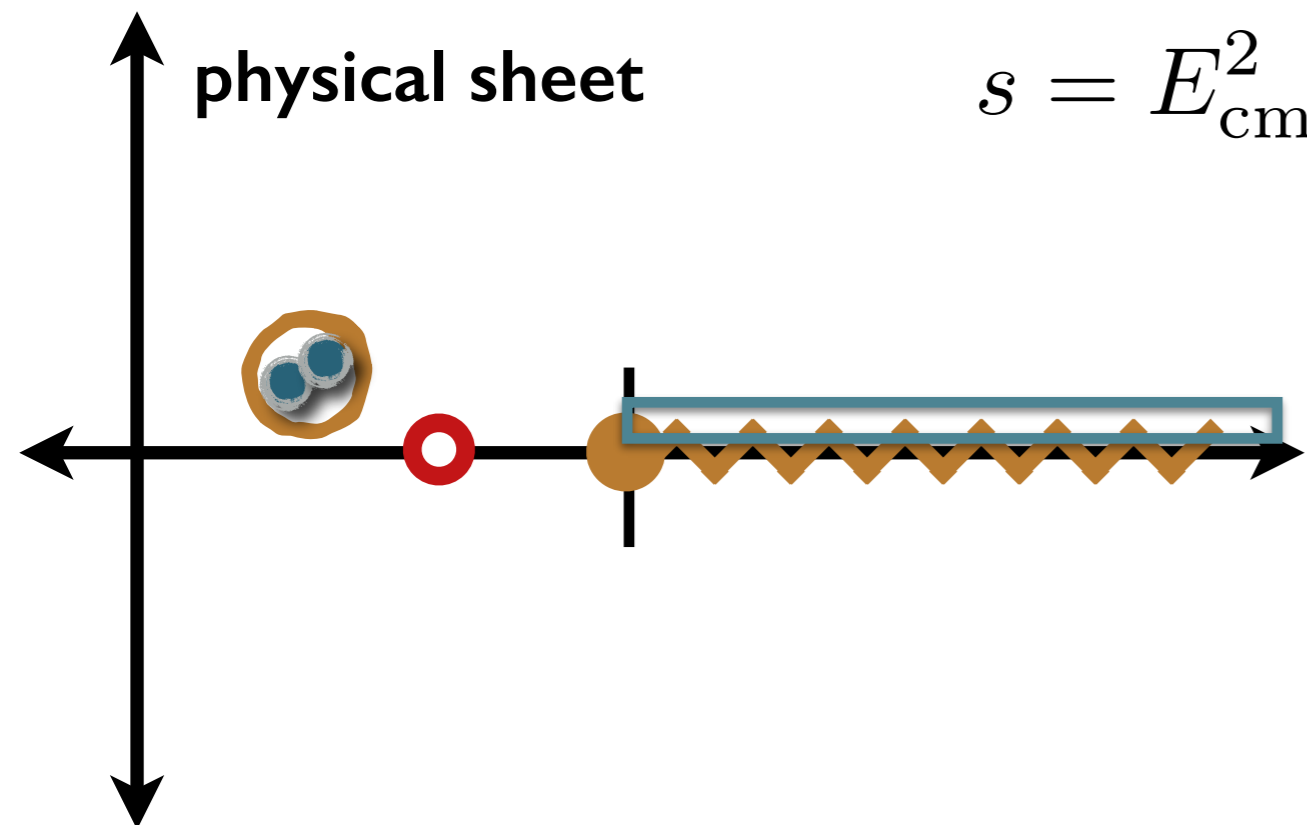
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} + \rho(s)} \quad \rho(s) \propto -i\sqrt{s - (2M_\pi)^2}$$

Riemann sheets

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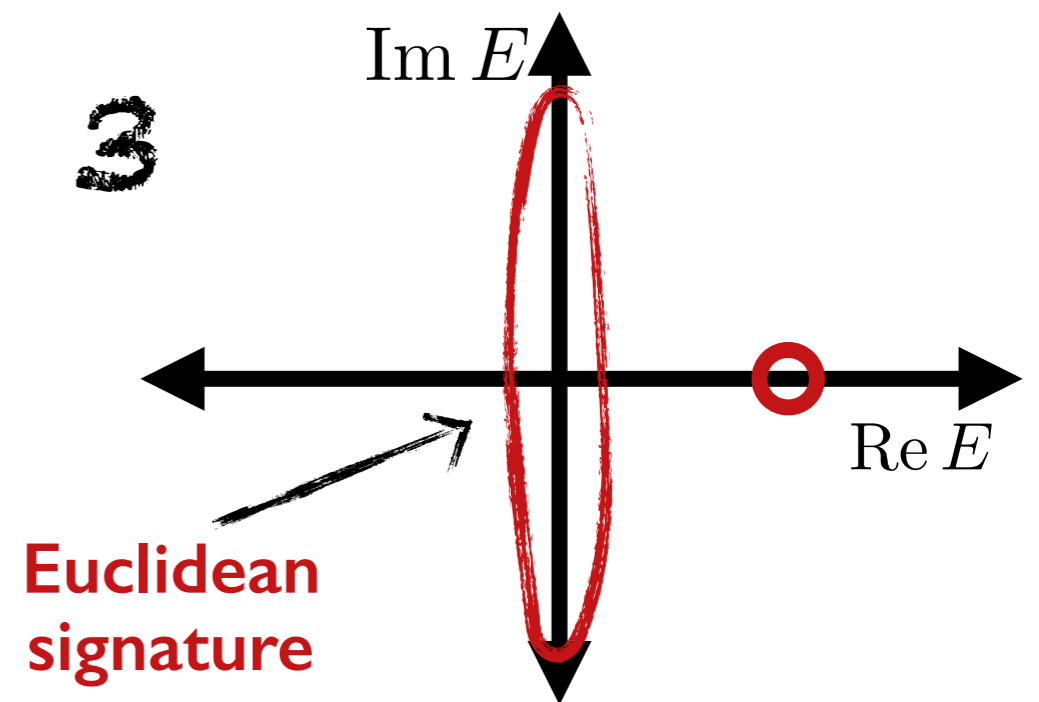
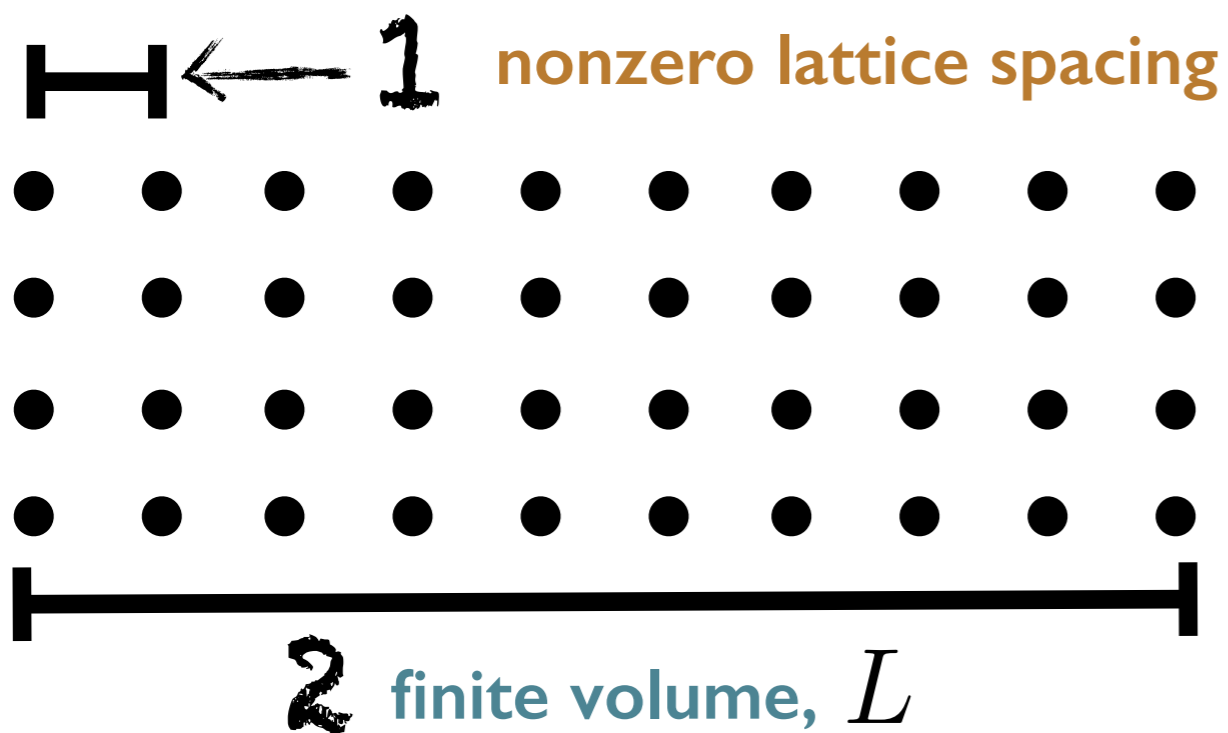
- Each channel generates a *square-root cut* → **doubles** the number of sheets



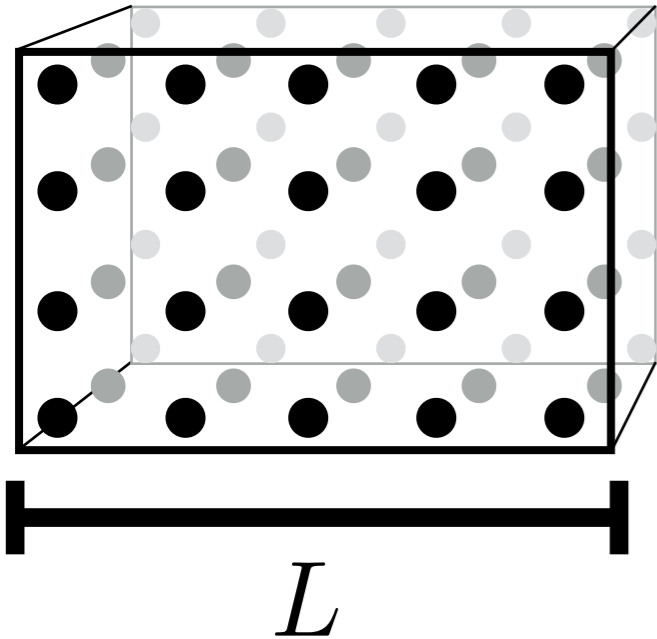
Lattice QCD

$$\text{observable?} = \int d^N \phi e^{-S} \left[\begin{array}{l} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

To proceed we have to make **three modifications**

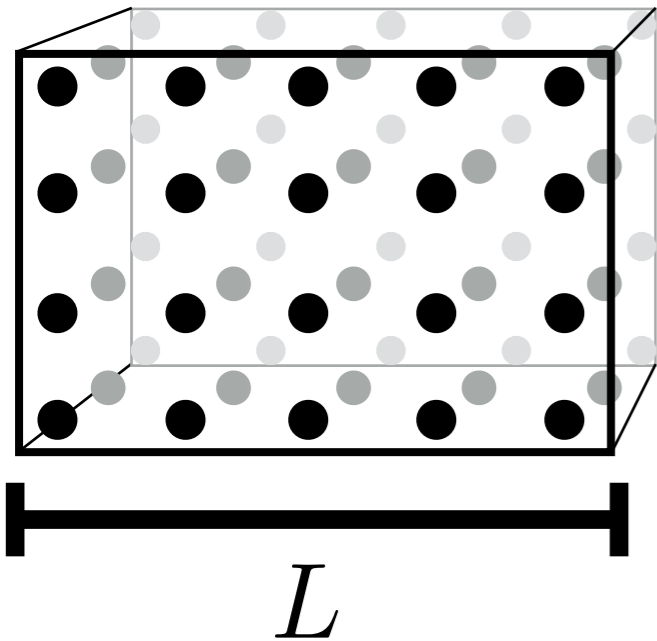


Difficulties for multi-hadron observables

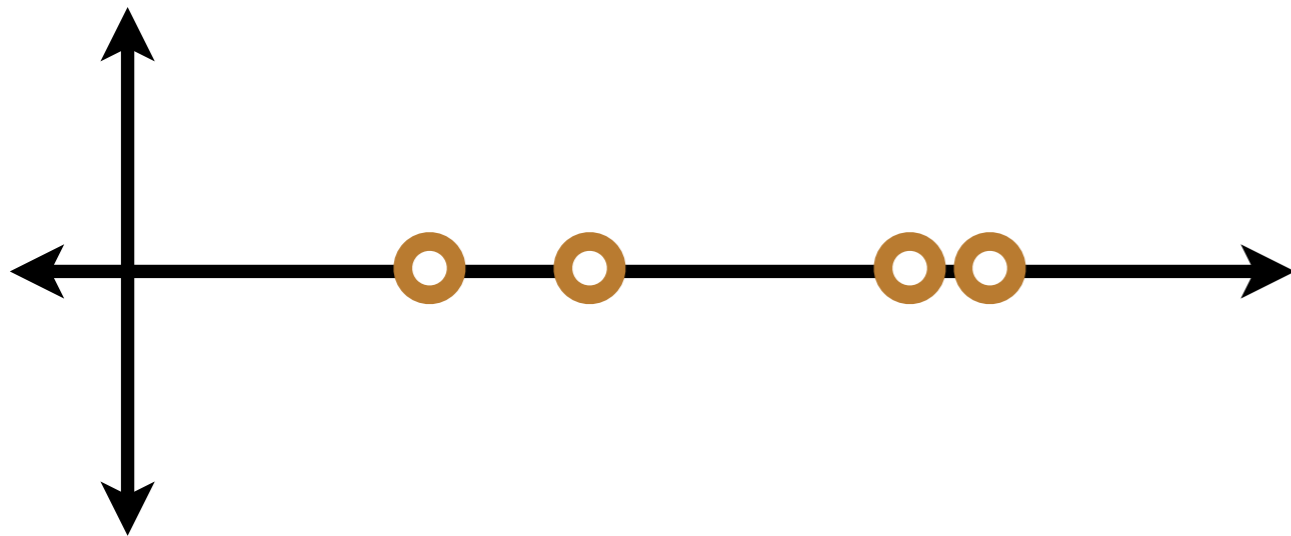


- The **finite volume**...
 - **Discretizes** the spectrum
 - **Eliminates** the branch cuts and extra sheets
 - **Hides** the resonance poles

Difficulties for multi-hadron observables



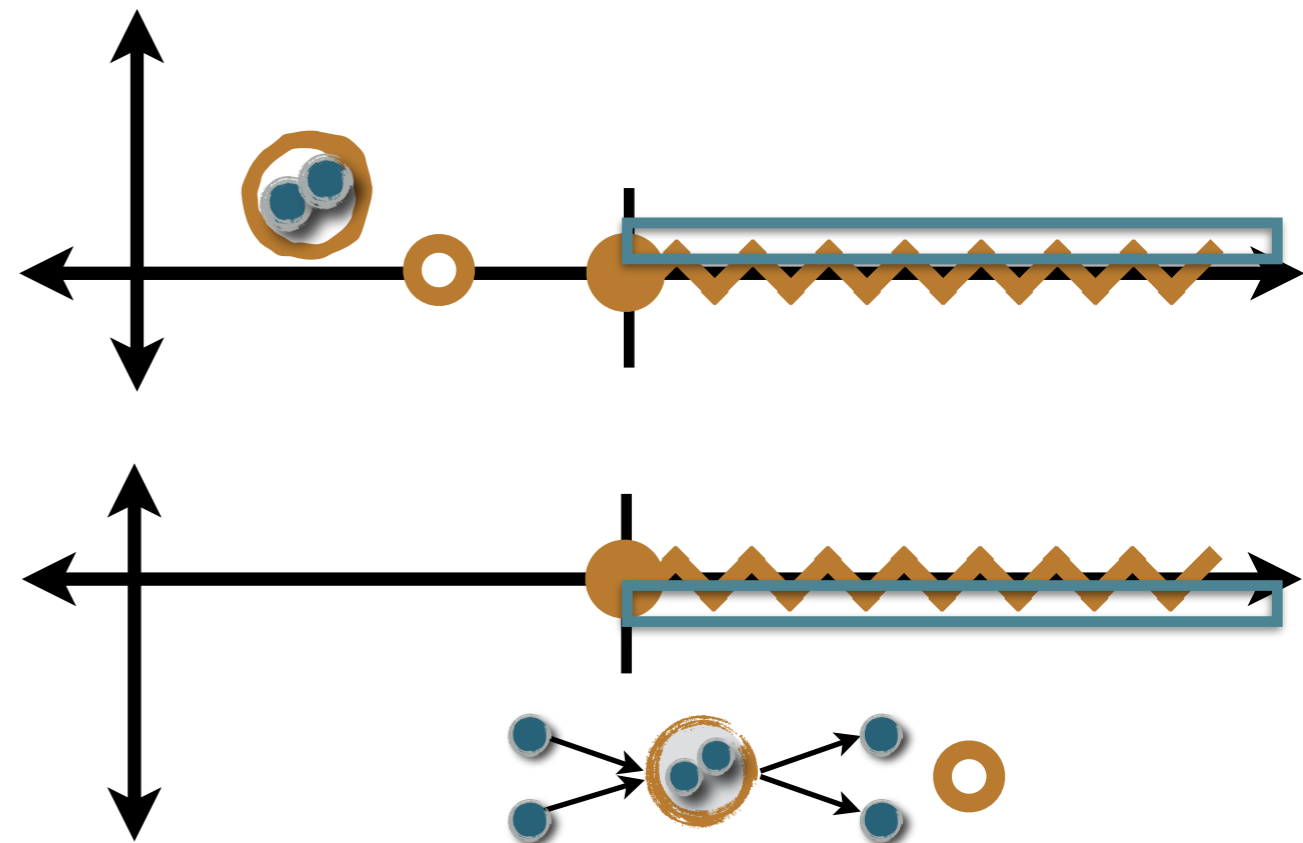
Finite-volume analytic structure



□ The **finite volume**...

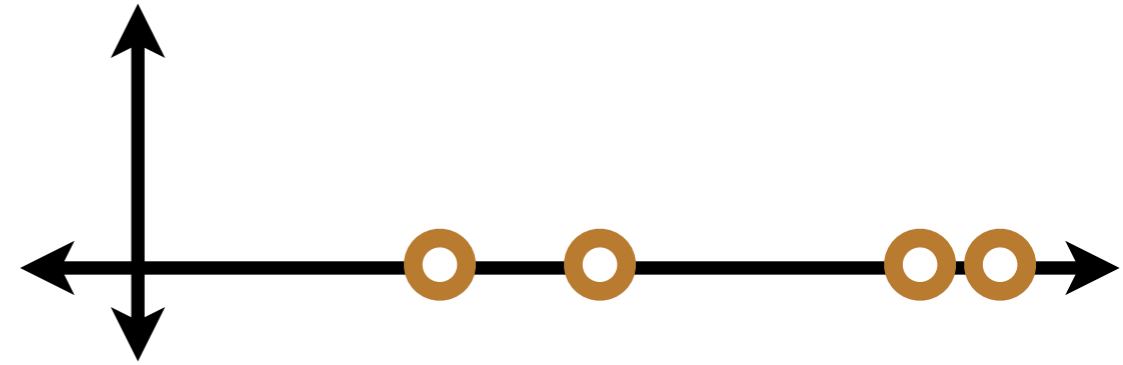
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Infinite-volume analytic structure



Observables available in LQCD

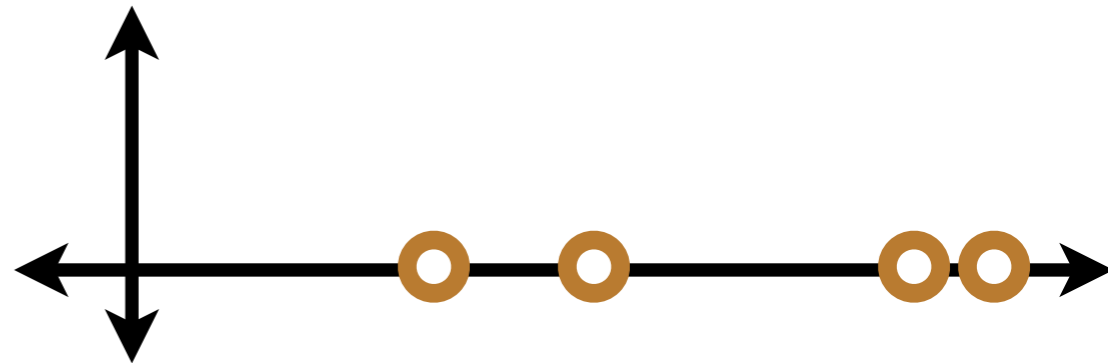
□ LQCD → **Energies** and **matrix elements**



$$\langle \mathcal{O}_j(\tau) \mathcal{O}_i^\dagger(0) \rangle = \sum_n \langle 0 | \mathcal{O}_j(\tau) | E_n \rangle \langle E_n | \mathcal{O}_i^\dagger(0) | 0 \rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

Observables available in LQCD

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□ Determine optimized operators by *diagonalizing* (GEVP) - requires **distillation**

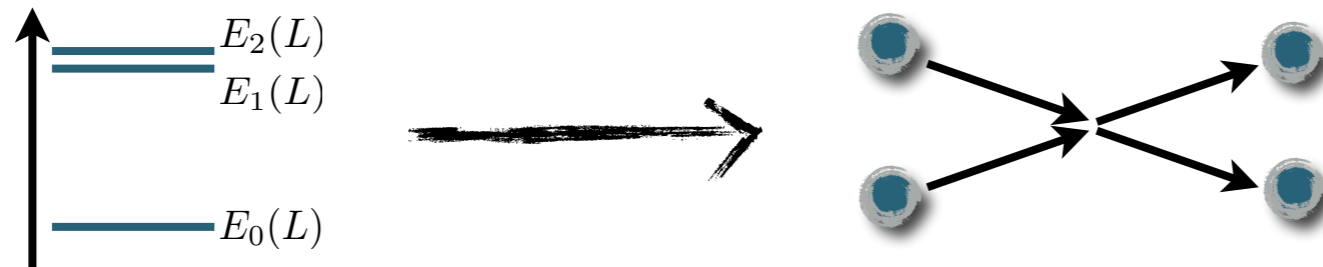
$$\begin{aligned} \langle \Omega_m(\tau) \Omega_m^\dagger(0) \rangle &\sim e^{-E_m(L)\tau} + \dots \\ \langle \Omega_{m'}(\tau) \mathcal{J}(0) \Omega_m^\dagger(-\tau) \rangle &\sim e^{-E_{m'}\tau} e^{-E_m\tau} \langle E_{m'} | \mathcal{J}(0) | E_m \rangle + \dots \end{aligned}$$

□ Our task is relate $E_n(L)$ and $\langle E_{m'} | \mathcal{J}(0) | E_m \rangle$ to **experimental observables**

Multi-hadron processes from LQCD

KEY IDEA: We can use the finite volume as a **tool** to extract multi-hadron observables

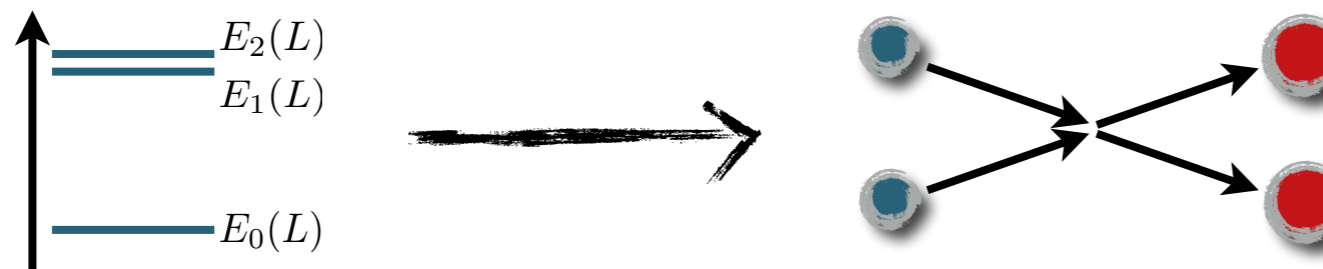
□ Single-channel two-to-two scattering



W2 - Fri - *Aaron*

W3 - Thurs - *Antonio, Dave*

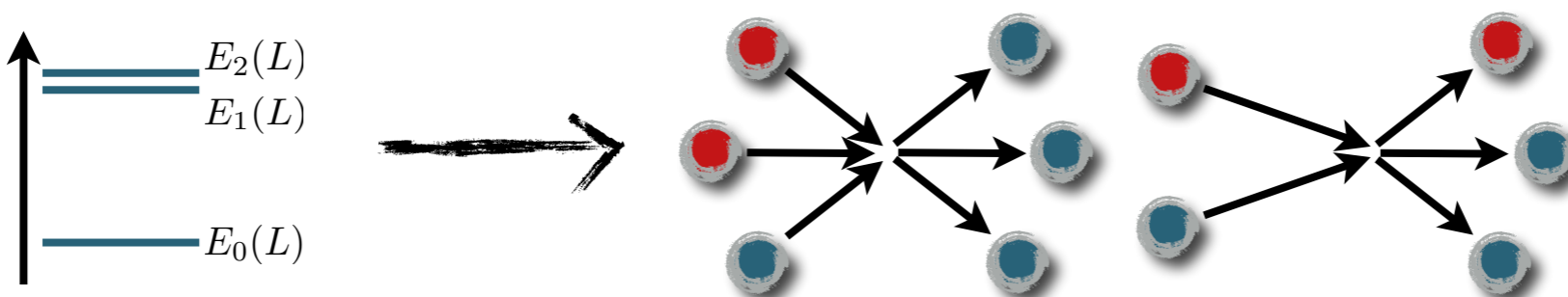
□ Coupled-channel two-to-two scattering + spin



Wed - *Jo*

Thurs - *Christopher, Antoni*

□ Two-to-three and three-to-three scattering

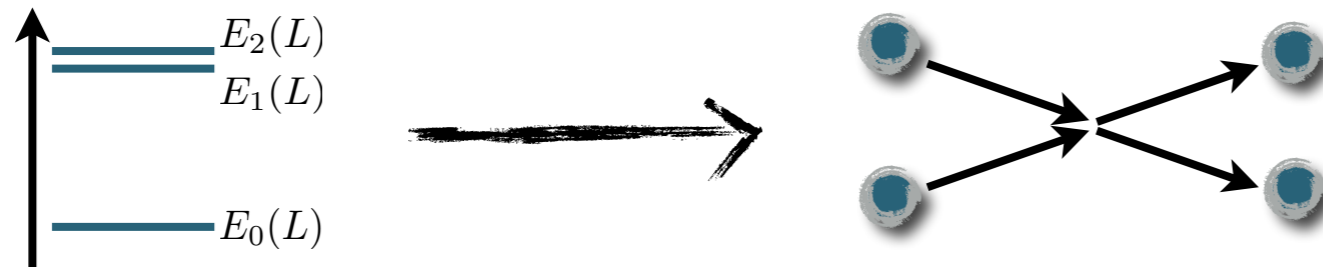


Today - *Steve*

Multi-hadron processes from LQCD

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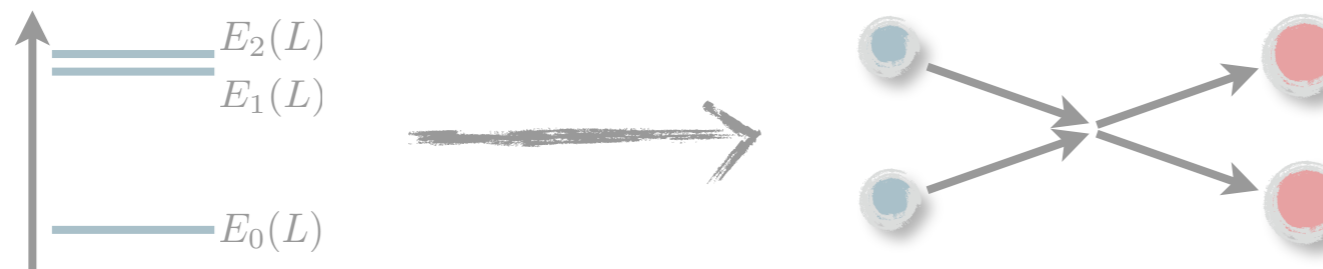
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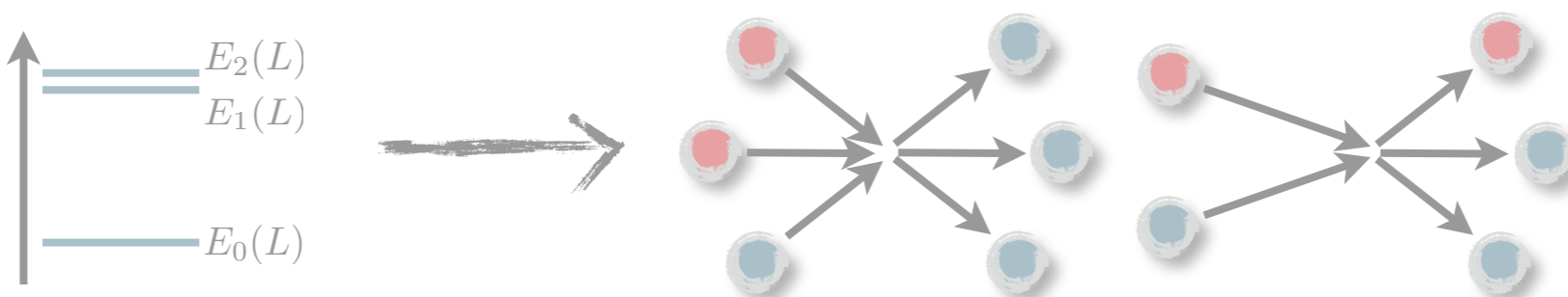
Coupled-channel two-to-two scattering + spin



Wed - *Jo*

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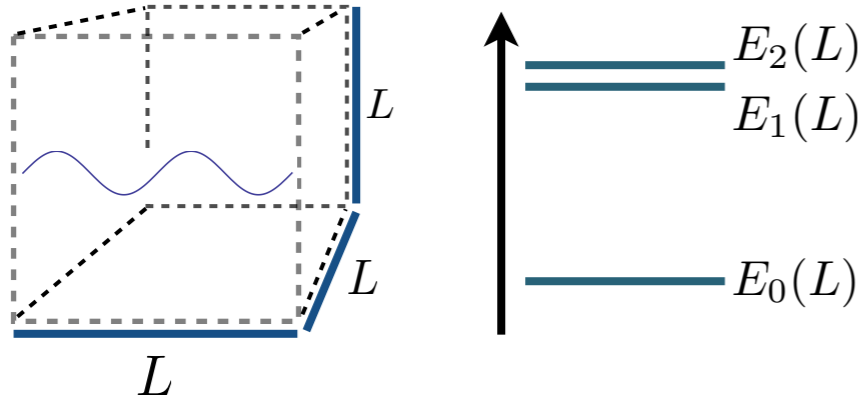
Two-to-three and three-to-three scattering



Today - *Steve*

The finite-volume as a tool

□ Finite-volume set-up



□ **cubic**, spatial volume (extent L)

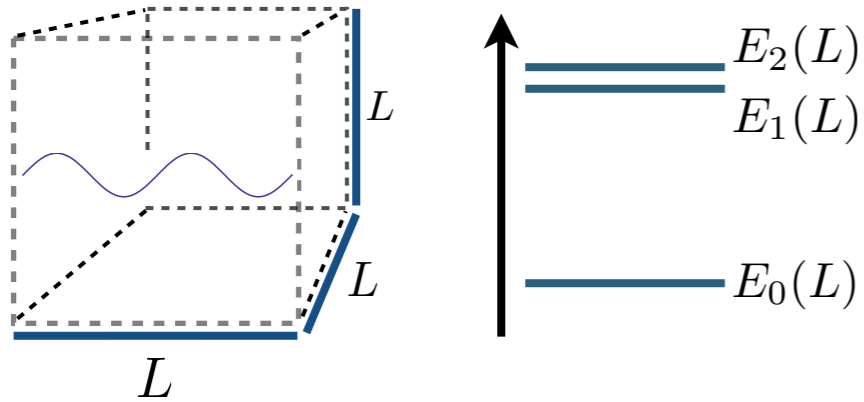
□ **periodic**

$$\vec{p} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3$$

□ L is large enough to neglect $e^{-M_\pi L}$

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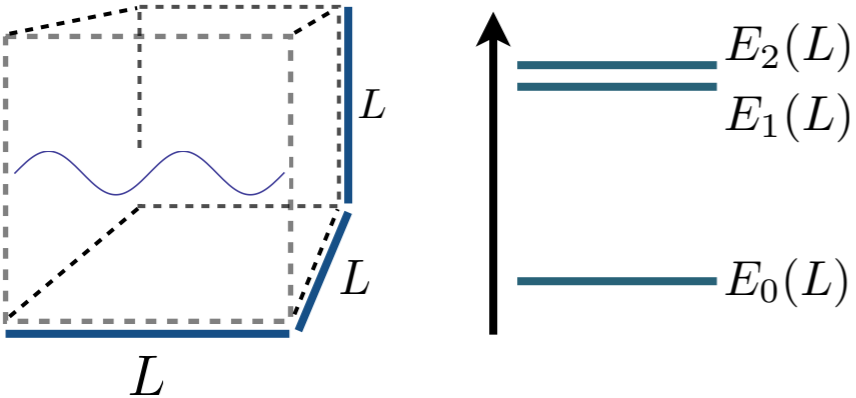
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The finite-volume as a tool

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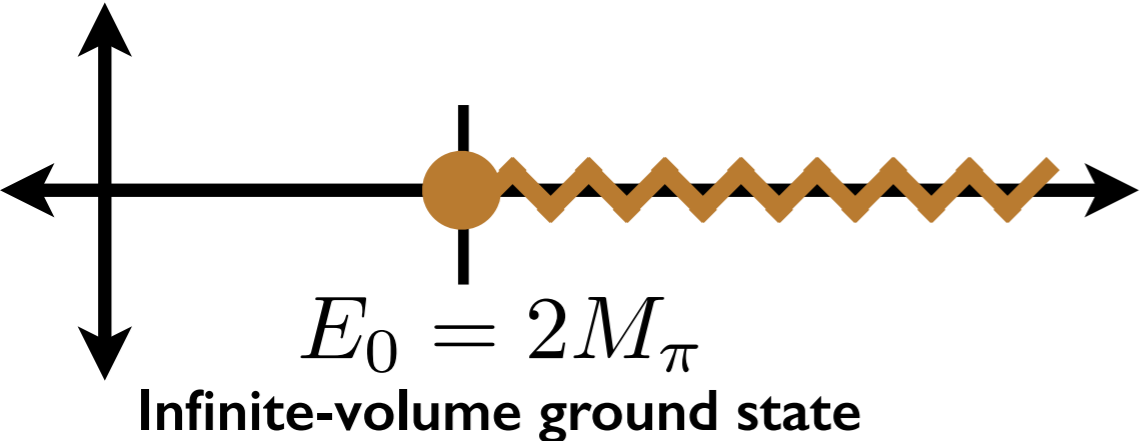
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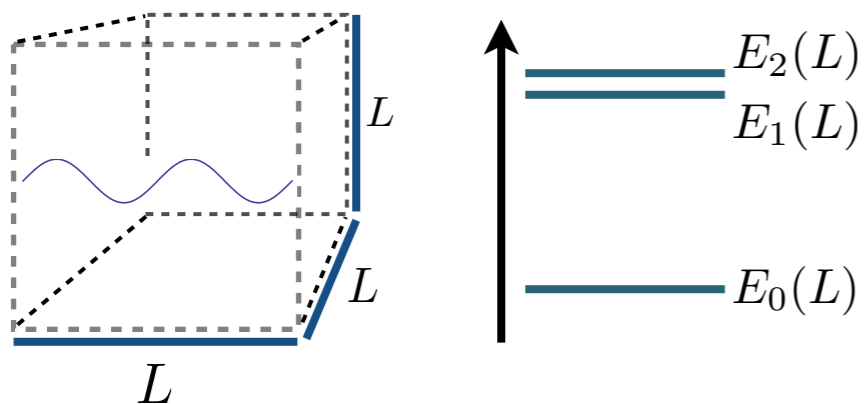
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$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

The finite-volume as a tool

Finite-volume set-up



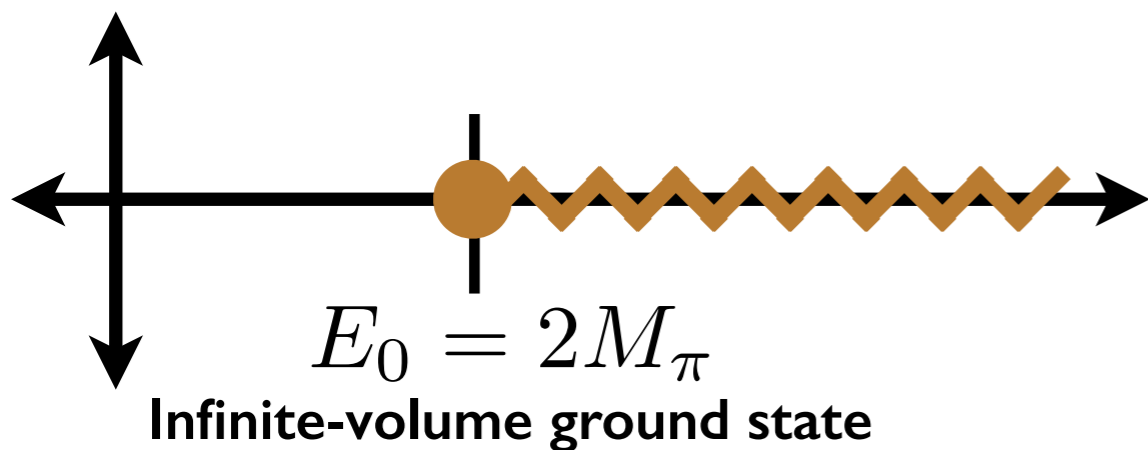
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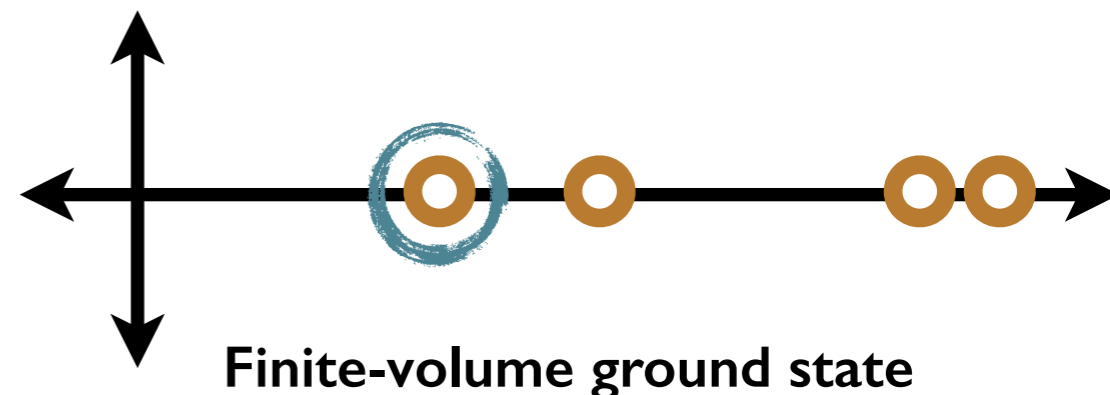
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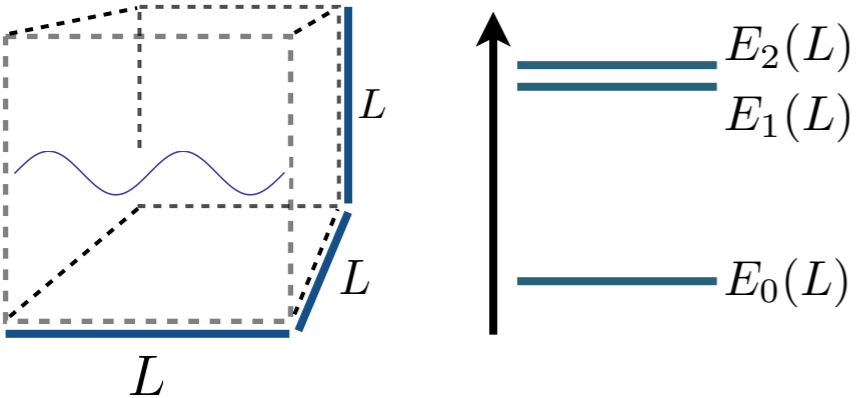


$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

Huang, Yang (1958)

The finite-volume as a tool

- Finite-volume set-up



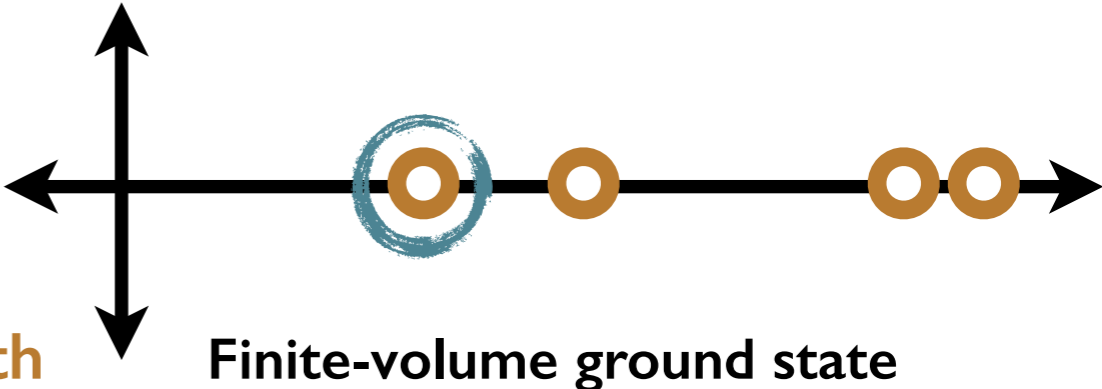
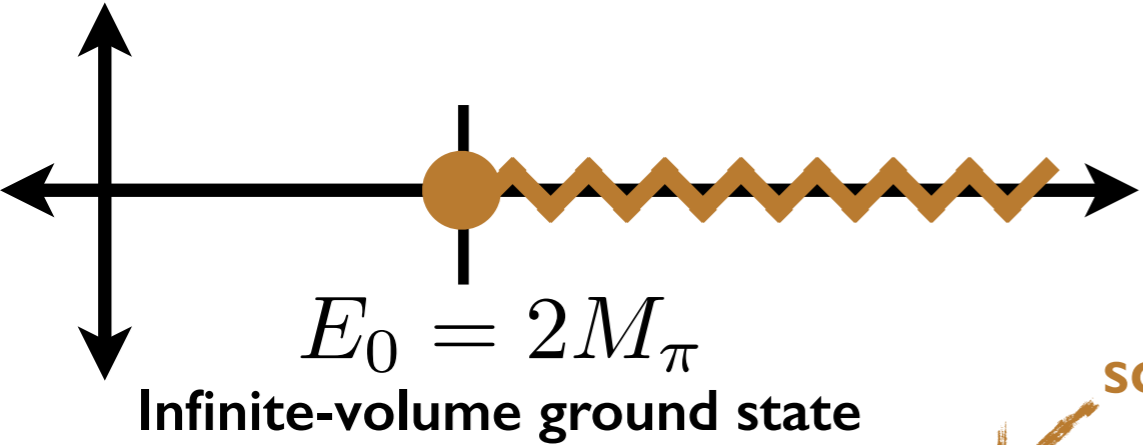
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Huang, Yang (1958)

Hint of the derivation

□ In the infinite-volume world...

$$\mathcal{M}_{\ell=0}(E_{\text{cm}}) = \text{X} + \dots = -\lambda + \mathcal{O}(\lambda^2) \quad \leftarrow \text{Low-energy degrees of freedom (e.g. pions in QCD)}$$

Hint of the derivation

□ In the infinite-volume world...

$$\mathcal{M}_{\ell=0}(E_{\text{cm}}) = \text{X} + \dots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

scattering length

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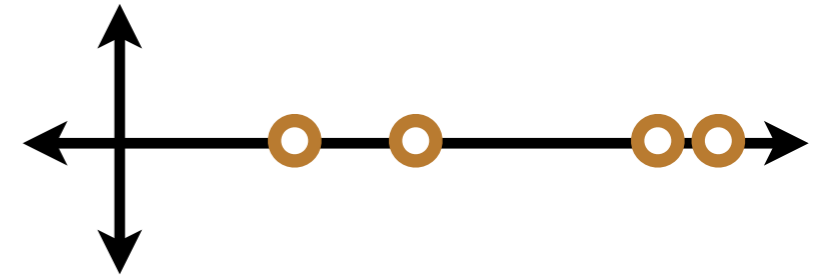
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□ In the finite-volume world...

$$\mathcal{M}_L(E_{\text{cm}}) = \text{X} + \dots$$
$$= -\lambda + \dots$$

Leading order \rightarrow no poles



Hint of the derivation

□ In the infinite-volume world...

$$\mathcal{M}_{\ell=0}(E_{\text{cm}}) = \text{diagram} + \dots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

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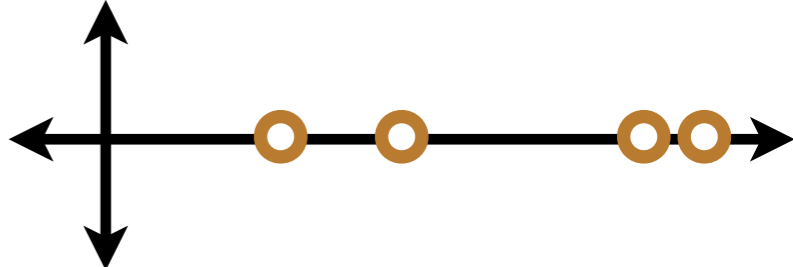
□ In the finite-volume world...

$$\begin{aligned} \mathcal{M}_L(E_{\text{cm}}) &= \text{diagram} + \text{diagram} + \dots \\ &= -\lambda - \lambda \frac{1}{2} \frac{1}{L^3} \sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2 (E_{\text{cm}} - 2\omega_{\mathbf{k}})} \lambda + \dots \end{aligned}$$

Leading order → no poles

Next-to-leading order → poles of **two non-interacting particles**

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M_\pi^2} \quad \mathbf{k} = \frac{2\pi}{L} \mathbf{n}$$



Hint of the derivation

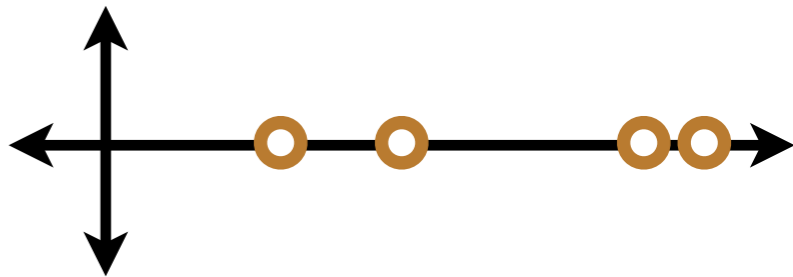
□ In the infinite-volume world...

$$\mathcal{M}_{\ell=0}(E_{\text{cm}}) = \text{[diagram: a circle with four lines crossing at the center]} + \dots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

scattering length

□ In the finite-volume world...

$$\begin{aligned} \mathcal{M}_L(E_{\text{cm}}) &= \text{[diagram: a circle with four lines crossing at the center]} + \text{[diagram: a circle with two lines crossing at the left and right, and two arcs connecting the top and bottom]} + \dots \\ &= -\lambda - \lambda \frac{1}{2} \frac{1}{L^3} \sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2 (E_{\text{cm}} - 2\omega_{\mathbf{k}})} \lambda + \dots \end{aligned}$$



Leading order → no poles

Next-to-leading order → poles of **two non-interacting particles** $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M_\pi^2}$ $\mathbf{k} = \frac{2\pi}{L} \mathbf{n}$

□ Expansion fails for: $E_{\text{cm}} - 2M_\pi = \mathcal{O}(\lambda)$

Hint of the derivation

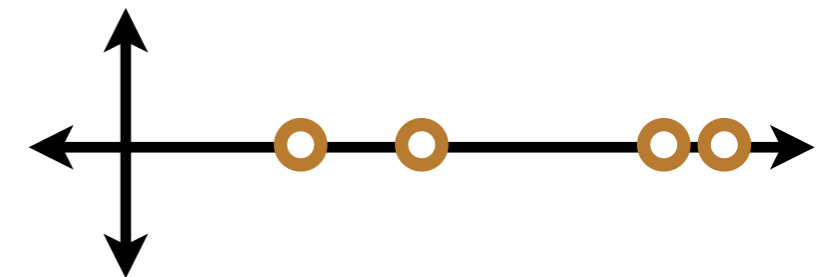
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scattering length

- In the finite-volume world...

$$\begin{aligned} \mathcal{M}_L(E_{\text{cm}}) &= \text{diagram} + \text{diagram} + \dots \\ &= -\lambda - \lambda \frac{1}{2} \frac{1}{L^3} \sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2 (E_{\text{cm}} - 2\omega_{\mathbf{k}})} \lambda + \dots \end{aligned}$$



Leading order → no poles

Next-to-leading order → poles of **two non-interacting particles** $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M_\pi^2}$ $\mathbf{k} = \frac{2\pi}{L} \mathbf{n}$

- Expansion fails for: $E_{\text{cm}} - 2M_\pi = \mathcal{O}(\lambda)$

$$= -\lambda \sum_{n=0}^{\infty} [f(E_{\text{cm}}, L) \lambda]^n = \frac{1}{-1/\lambda + f(E_{\text{cm}}, L)}$$

- Physical pole recovered via re-summation

$$-1/\lambda + f(E_{\text{cm}}, L) = 0 \implies E_{\text{cm}} = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

Skeleton expansion derivation

$$C_L(P) = \begin{array}{c} \text{Diagram 1} + \text{Diagram 2} \\ + \text{Diagram 3} + \dots \end{array}$$

The diagram shows the skeleton expansion of the correlation function $C_L(P)$. It consists of a sum of terms, each representing a different skeleton structure. The first term is a circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right, connected by two arcs. A dashed box encloses the two arcs. The second term is a circle labeled \mathcal{O}^\dagger on the left, a circle labeled iB in the middle, and a circle labeled \mathcal{O} on the right, all connected by two arcs. Two dashed boxes enclose the arcs between \mathcal{O}^\dagger and iB , and between iB and \mathcal{O} . The third term is a circle labeled \mathcal{O}^\dagger on the left, two circles labeled iB in the middle, and a circle labeled \mathcal{O} on the right, all connected by two arcs. Three dashed boxes enclose the arcs between \mathcal{O}^\dagger and the first iB , between the two iB circles, and between the second iB and \mathcal{O} . The expansion continues with an ellipsis.

Skeleton expansion derivation

$$C_L(P) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams are Feynman diagrams representing the skeleton expansion of a correlation function. Each diagram consists of a sequence of vertices connected by lines. The vertices are represented by circles containing labels: \mathcal{O}^\dagger , \mathcal{O} , and iB . The lines are represented by arcs connecting the vertices. In the first diagram, a dashed blue box highlights a loop structure between the \mathcal{O}^\dagger and \mathcal{O} vertices, with a blue arrow pointing to it from the text below. In the subsequent diagrams, the iB vertices are enclosed in dashed boxes, indicating they are summed over.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

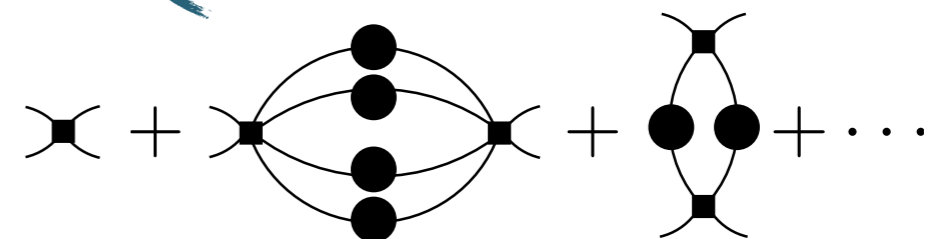
Skeleton expansion derivation

$$C_L(P) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams show a sequence of terms in the skeleton expansion of $C_L(P)$. Each term consists of a chain of circles representing operators. The first circle is \mathcal{O}^\dagger and the last is \mathcal{O} . Intermediate circles are labeled iB . The first term has two internal vertices (black dots) connected by a dashed box. The second term has two iB circles, each with two internal vertices. The third term has two iB circles, each with two internal vertices. The expansion continues with more terms.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$



Bethe Salpeter kernel

Skeleton expansion derivation

$$C_L(P) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

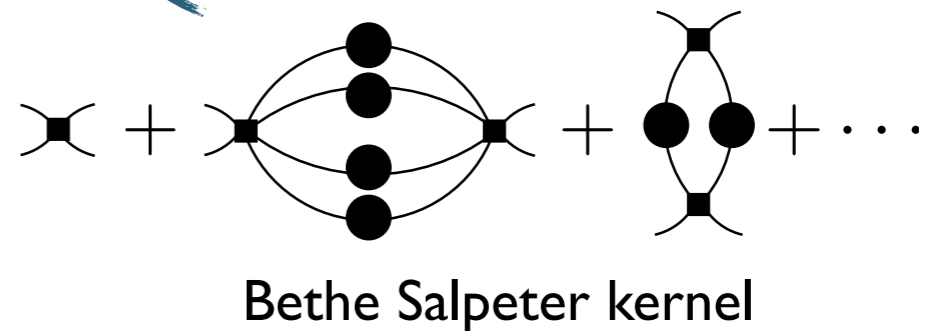
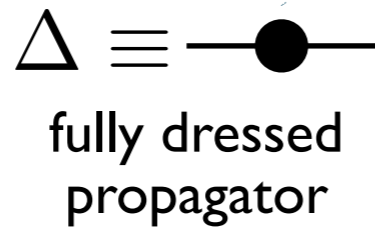
Diagram 1: A circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right. Two black dots are connected by a vertical dashed line between them.

Diagram 2: A circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right. A circle labeled iB is in the middle. Two black dots are connected by a vertical dashed line between \mathcal{O}^\dagger and iB , and another between iB and \mathcal{O} .

Diagram 3: A circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right. Two circles labeled iB are in the middle. Two black dots are connected by a vertical dashed line between \mathcal{O}^\dagger and the first iB , and another between the first iB and the second iB , and a third between the second iB and \mathcal{O} .

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$



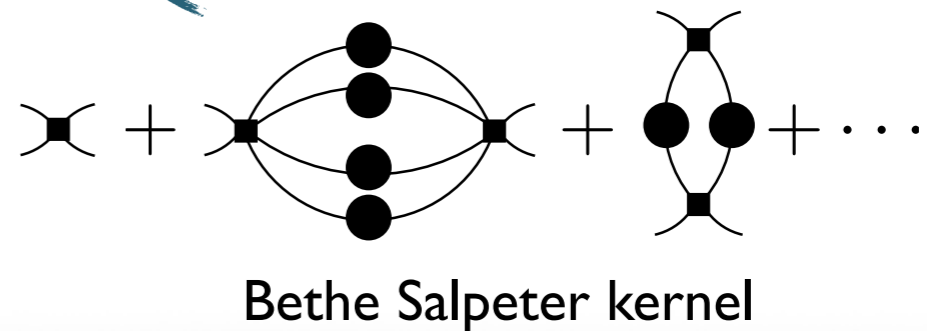
Skeleton expansion derivation

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagrams show a sequence of terms in the skeleton expansion. Each term consists of a chain of circles representing operators. The first term is \mathcal{O}^\dagger followed by a loop of two black dots, then \mathcal{O} . The second term is \mathcal{O}^\dagger , a loop of two black dots, a circle labeled iB , another loop of two black dots, and \mathcal{O} . The third term is \mathcal{O}^\dagger , a loop of two black dots, a circle labeled iB , a loop of two black dots, a circle labeled iB , another loop of two black dots, and \mathcal{O} . Blue arrows point from the first and second terms to the definition of the fully dressed propagator below.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$



If $E^* < 4m$ **then**

$$B_L = B_\infty + \mathcal{O}(e^{-mL})$$

$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

where we define $E^{*2} = E^2 - \vec{P}^2$

Skeleton expansion derivation

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of diagrams representing the skeleton expansion of the correlation function $C_L(P)$. Each diagram consists of a chain of circles. The first circle is labeled \mathcal{O}^\dagger and the last is labeled \mathcal{O} . The intermediate circles are labeled \mathcal{O} , iB , iB , etc. Each circle has two external legs, represented by black dots. Dashed boxes enclose the internal propagator regions between the circles. Arrows from the diagrams below point to the corresponding regions in this expansion.

$$\frac{1}{L^3} \sum_{\vec{k}} \text{diagram}_A = \int_{\vec{k}} \text{diagram}_B + \underbrace{\text{diagram}_C}_F$$

The diagram below shows the derivation of the skeleton expansion. It starts with a diagram representing a two-point function with a dashed box indicating a region to be summed over. This is equated to an integral over \vec{k} of a diagram with a solid line, plus a term F which is noted to contain all power-law corrections.

Skeleton expansion derivation

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of diagrams representing the skeleton expansion of the two-point correlation function $C_L(P)$. Each diagram consists of two external vertices, \mathcal{O}^\dagger and \mathcal{O} , connected by a chain of internal vertices. The first diagram has two internal vertices. The second diagram has three internal vertices, with the middle one labeled iB . The third diagram has four internal vertices, with the two middle ones labeled iB . Dashed boxes enclose the internal vertices in each diagram. Arrows point from the first two diagrams to the first two terms of the expansion below.

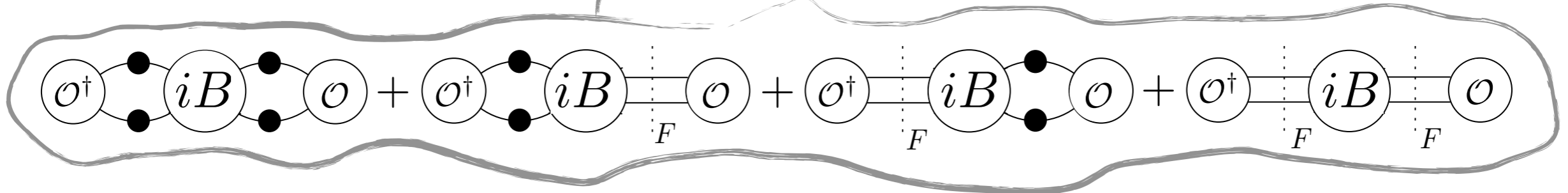
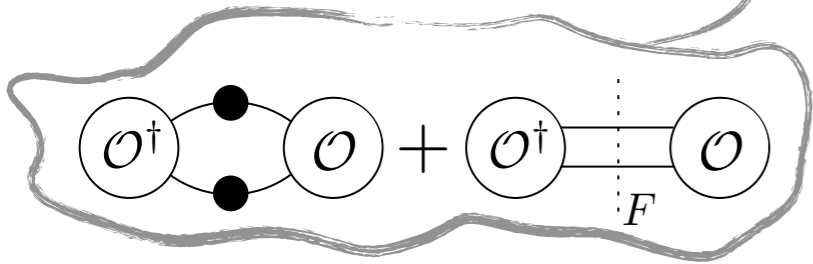
$$\frac{1}{L^3} \sum_{\vec{k}} \text{diagram}_1 = \int_{\vec{k}} \text{diagram}_2 + \underbrace{\text{diagram}_3}_F$$

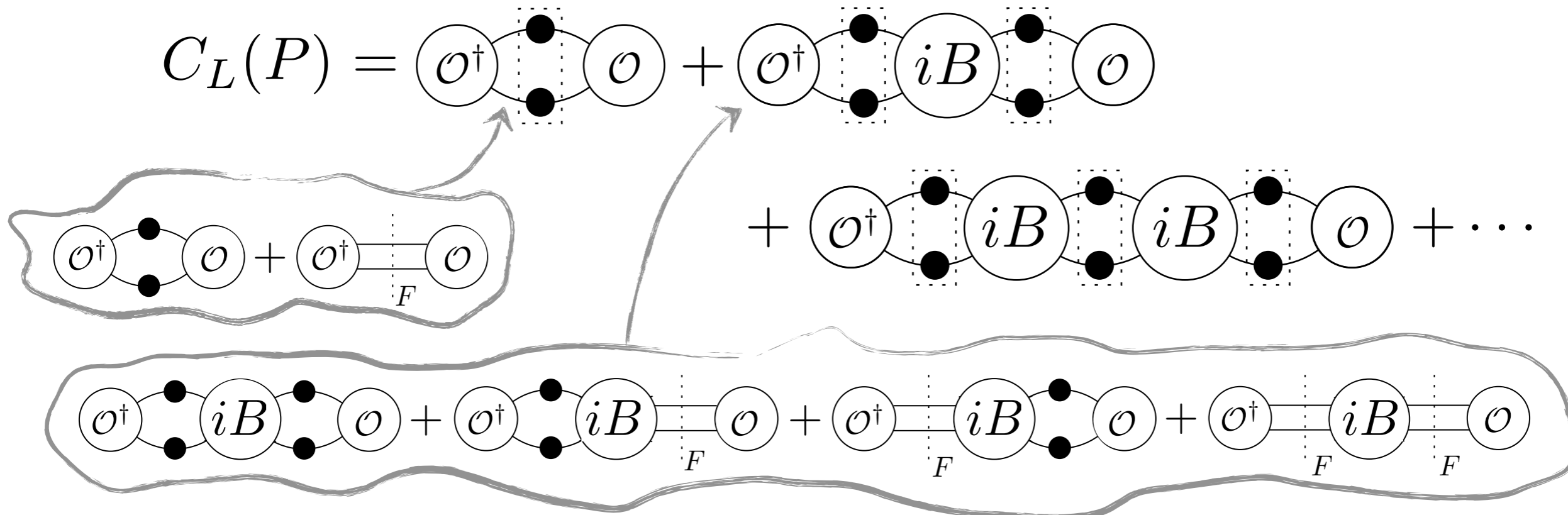
The diagram shows the derivation of the skeleton expansion. On the left, a diagram with two external vertices and a dashed box around the internal vertices is equated to the integral of a diagram with two external vertices and a solid internal line, plus a term labeled F . The term F is underlined and labeled "contains all power-law corrections".

In  all four-momenta are projected on shell.

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2$$

$$+ \text{diagram}_3 + \dots$$

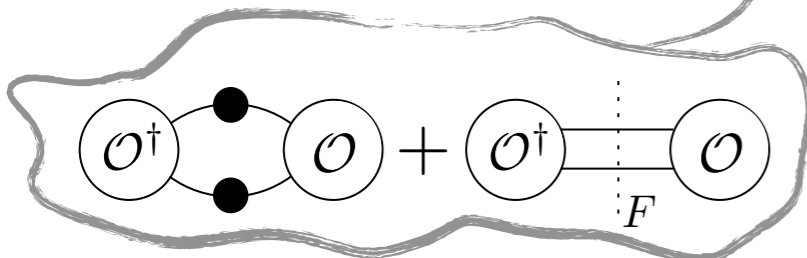




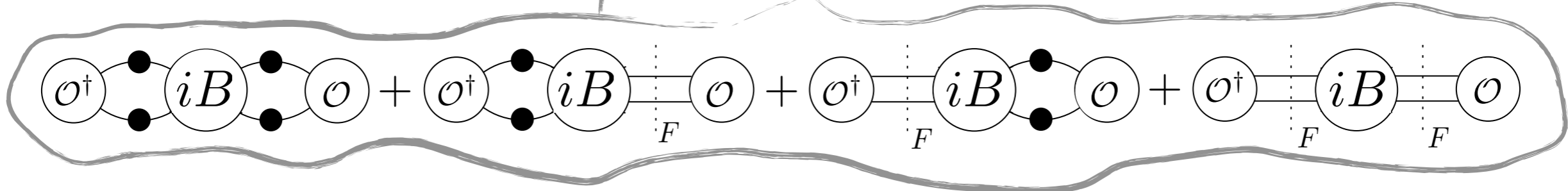
□ Regroup by number of F cuts
 zero F s

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) +$$

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2$$



$$+ \text{diagram}_3 + \text{diagram}_4 + \dots$$

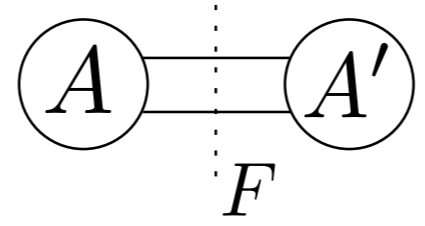


□ Regroup by number of F cuts

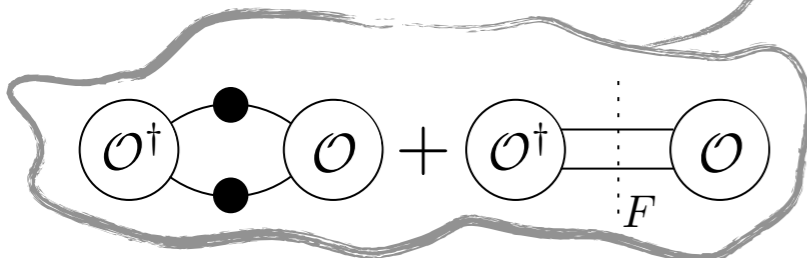
zero Fs

one F

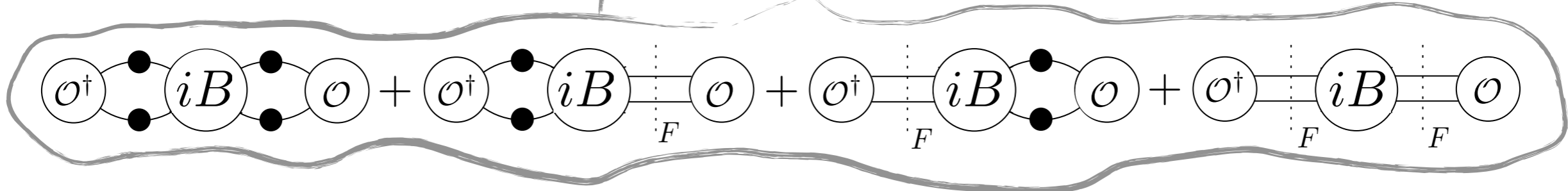
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram}_5 +$$



$$C_L(P) = \text{diagram}_1 + \text{diagram}_2$$



$$+ \text{diagram}_3 + \text{diagram}_4 + \dots$$

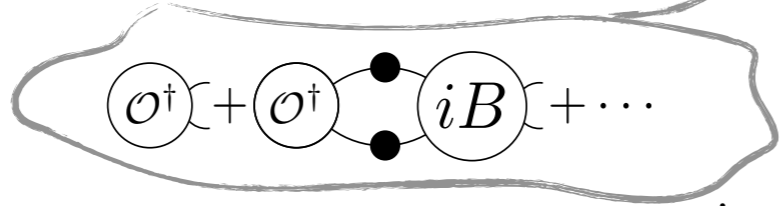


□ Regroup by number of F cuts

zero Fs

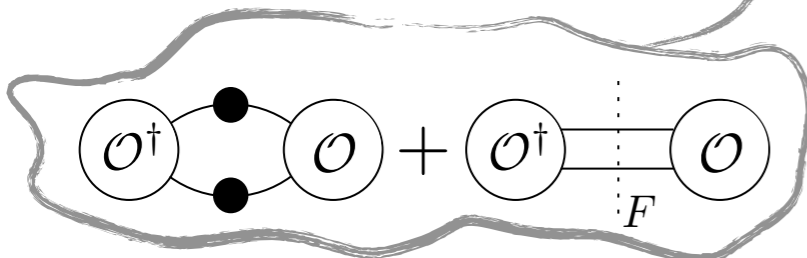
one F

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram}_5 +$$

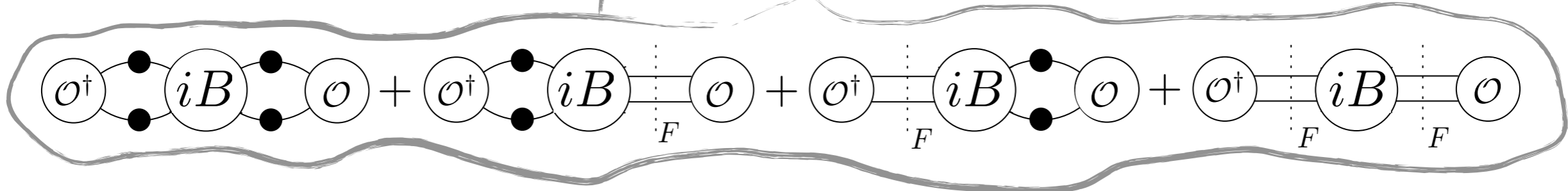


$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$

$$C_L(P) = \text{diagram} + \text{diagram}$$



$$+ \text{diagram} + \text{diagram} + \dots$$



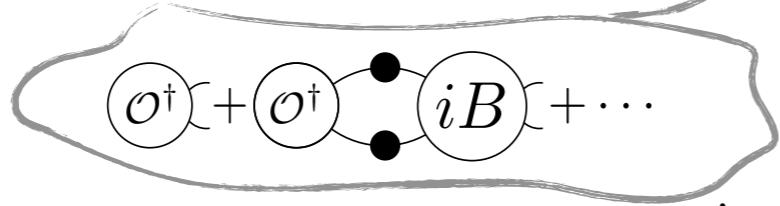
□ Regroup by number of F cuts

zero Fs

one F

two Fs

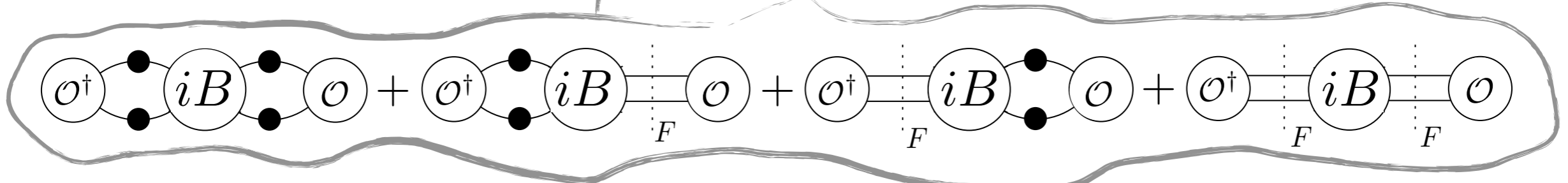
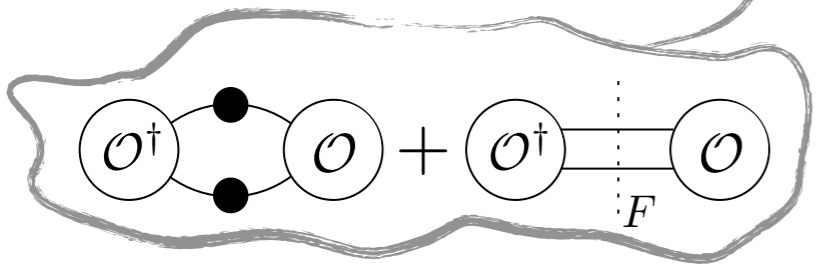
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram} + \text{diagram} + \dots$$



$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$

$$C_L(P) = \text{diagram} + \text{diagram} + \text{diagram} + \dots$$

The equation shows a series of diagrams representing terms in a sum. Each diagram consists of circles connected by lines, with some circles containing black dots. The first term is \mathcal{O}^\dagger connected to \mathcal{O} . The second term is \mathcal{O}^\dagger connected to iB connected to \mathcal{O} . The third term is \mathcal{O}^\dagger connected to iB connected to iB connected to \mathcal{O} . Dashed boxes labeled 'F' indicate cuts between the first and second terms, and between the second and third terms.



□ Regroup by number of F cuts

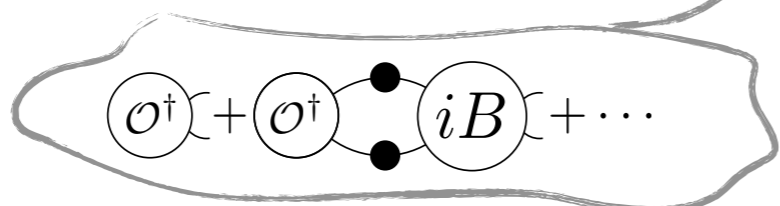
zero Fs

one F

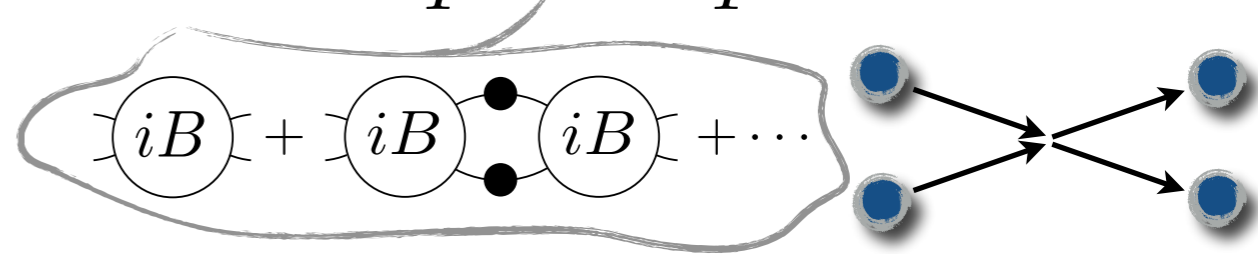
two Fs

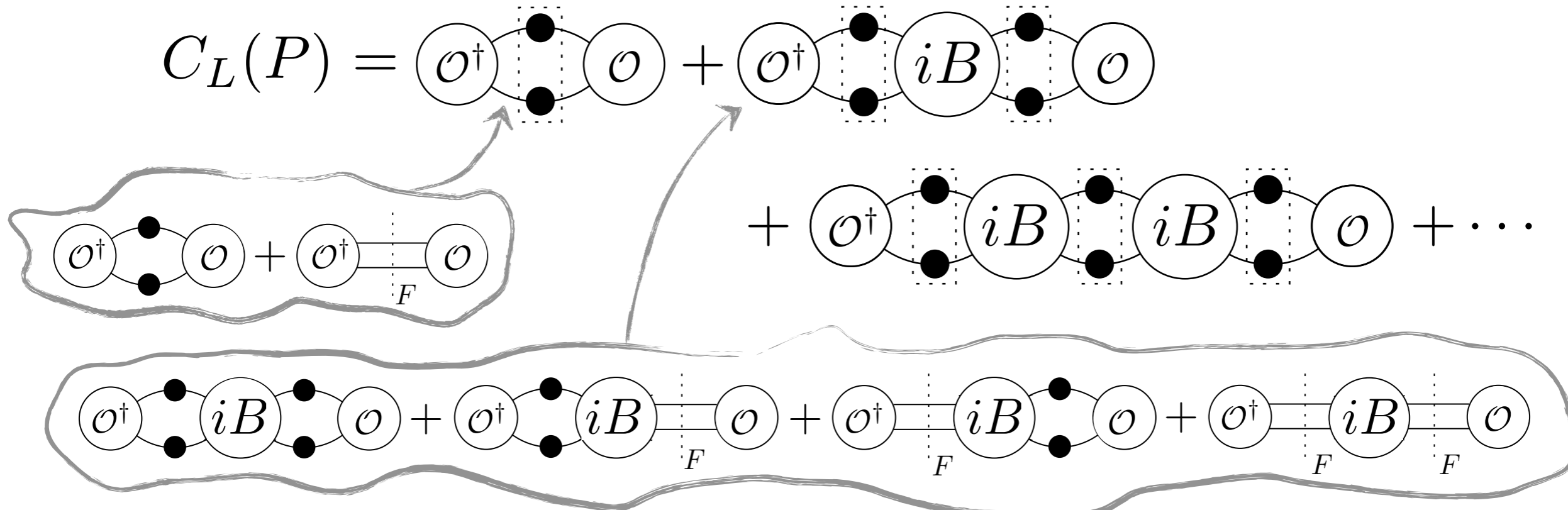
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram} + \text{diagram} + \dots$$

The equation shows a series of diagrams representing terms in a sum. The first term is $C_\infty(E, \vec{P})$. The second term is A connected to A' with a dashed line labeled 'F' below it. The third term is A connected to iM connected to A' with dashed lines labeled 'F' below the connections. The fourth term is iB connected to iB connected to iB with dashed lines labeled 'F' below the connections.



$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$





□ Regroup by number of F cuts

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{one } F + \text{two } Fs + \dots$$

The equation shows $C_L(E, \vec{P})$ as a sum of terms grouped by the number of F cuts. The first term is $C_\infty(E, \vec{P})$. The second term, labeled "one F", shows a chain of circles A and A' with a vertical dashed line F between them. The third term, labeled "two Fs", shows a chain of circles A , $i\mathcal{M}$, and A' with two vertical dashed lines F between A and $i\mathcal{M}$, and between $i\mathcal{M}$ and A' . Ellipses indicate further terms.

$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$

The equation shows the vacuum expectation value $\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$. Below it, a diagram shows a chain of circles \mathcal{O}^\dagger and iB with vertical dashed lines F between them, representing the infinite-volume part. To the right, a diagram shows four blue circles with arrows pointing from two to two, representing the physical observables.

When we factorize diagrams and group infinite-volume parts...

physical observables emerge

Review

$$C_L(P) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

1

The diagram shows the expansion of the Lüscher correction $C_L(P)$ as a sum of Feynman diagrams. The first row contains two diagrams: the first has a \mathcal{O}^\dagger circle on the left and a \mathcal{O} circle on the right, with two vertices between them enclosed in a dashed box; the second has a \mathcal{O}^\dagger circle, a iB circle, and a \mathcal{O} circle, with two vertices between \mathcal{O}^\dagger and iB , and two between iB and \mathcal{O} , all enclosed in dashed boxes. The second row shows a diagram with \mathcal{O}^\dagger , iB , iB , and \mathcal{O} circles, with dashed boxes around the vertex pairs. A blue arrow points from the second iB circle to a set of diagrams on the right representing self-energy corrections to the iB propagator, including a tadpole diagram and a sunset diagram.

Review

$$C_L(P) = \text{2} \left(\text{1} \right) + \text{2} \left(\text{1} \right) + \dots$$

The diagram illustrates the perturbative expansion of the Lüscher correction $C_L(P)$. It is presented as a sum of terms, with the first two terms explicitly shown and the rest indicated by ellipses. The terms are labeled with '1' and '2'.

Term 1: A sequence of diagrams representing the expansion of the propagator $\frac{1}{P^2}$. The first diagram shows two external legs, \mathcal{O}^\dagger and \mathcal{O} , connected by a loop with two internal vertices (black dots). The second diagram shows the same loop structure but with an insertion of the operator iB between the two vertices. This is followed by a series of diagrams showing higher-order corrections involving multiple iB insertions and self-energy corrections on the internal lines, all connected by plus signs and ellipses.

Term 2: A sequence of diagrams representing the expansion of the two-point function $\langle \mathcal{O}^\dagger \mathcal{O} \rangle$. The first diagram shows two external legs connected by a loop with two internal vertices. The second diagram shows the same loop structure but with a self-energy correction on one of the internal lines, indicated by a vertical dashed line. This is followed by ellipses.

Blue arrows indicate the mapping from the diagrams in Term 2 to the corresponding diagrams in Term 1.

Review

1

$$C_L(P) = \text{diagram 1} + \text{diagram 2} + \dots$$

2

$$\text{diagram 3} + \text{diagram 4} + \dots$$

3

$$C_L(P) = C_\infty(P)$$

3

$$+ \text{diagram 5} + \text{diagram 6} + \dots$$

3

$$+ \text{diagram 7} + \dots$$

3

$$+ \text{diagram 8} + \dots$$

Review

1

$$C_L(P) = \text{diagram 1} + \text{diagram 2} + \dots$$

2

$$\text{diagram 2} = \text{diagram 2a} + \text{diagram 2b}$$

Diagram 1 shows a series of terms in a sum for $C_L(P)$. The first term is a circle labeled \mathcal{O}^\dagger connected to a circle labeled \mathcal{O} with two internal black dots. The second term is \mathcal{O}^\dagger connected to a circle labeled iB (with two internal black dots), which is then connected to \mathcal{O} (with two internal black dots). The third term is \mathcal{O}^\dagger connected to iB (with two internal black dots), which is connected to another iB (with two internal black dots), which is then connected to \mathcal{O} (with two internal black dots). The diagram shows a sequence of such terms with ellipses indicating further terms. A blue arrow points from the first term to the second term in the second equation.

$$C_L(P) = C_\infty(P)$$

3

$$+ \text{diagram 3a} + \text{diagram 3b} + \dots$$

Diagram 3a shows a sequence of circles: A connected to A' via a horizontal line labeled F . Diagram 3b shows A connected to $i\mathcal{M}$ via F , $i\mathcal{M}$ connected to $i\mathcal{M}$ via F , and $i\mathcal{M}$ connected to A' via F . Diagram 3c shows a circle labeled iB with two internal black dots, followed by iB with two internal black dots connected to iB with two internal black dots, and so on. A blue arrow points from the iB terms in diagram 3c to the $i\mathcal{M}$ terms in diagram 3b.

Deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

poles in here

Two-particle quantization condition

□ Finite-volume energies = solutions to...

$$\det \left[\underbrace{\mathcal{M}_2^{-1}(E_n^*)}_{\text{scattering amplitude}} + \underbrace{F(E_n, \vec{P}, L)}_{\text{known geometric function}} \right] = 0$$

where $E_n^{*2} \equiv E_n^2 - \vec{P}^2$

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

Two-particle quantization condition

- Finite-volume energies = solutions to...

$$\det \left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L) \right] = 0$$

scattering amplitude known geometric function

where $E_n^{*2} \equiv E_n^2 - \vec{P}^2$

- Matrices in angular momentum space
- Holds only for $E_n^{*2} < (4m)^2$
- Ignores suppressed volume effects (e^{-mL})

Two-particle quantization condition

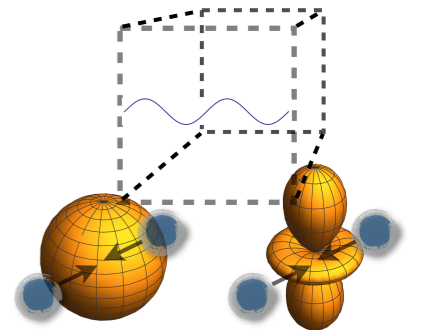
- Finite-volume energies = solutions to...

$$\det \left[\underbrace{\mathcal{M}_2^{-1}(E_n^*)}_{\text{scattering amplitude}} + \underbrace{F(E_n, \vec{P}, L)}_{\text{known geometric function}} \right] = 0$$

$$\text{where } E_n^{*2} \equiv E_n^2 - \vec{P}^2$$

- Matrices in angular momentum space
- Holds only for $E_n^{*2} < (4m)^2$
- Ignores suppressed volume effects (e^{-mL})

$$\mathcal{M}_2(E^*) = \begin{pmatrix} \bullet & & \\ & \bullet & \\ & & \bullet \end{pmatrix} \quad F(E_n, \vec{P}, L) = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$$



- Matrices block diagonalize into finite-volume irreps, e.g.

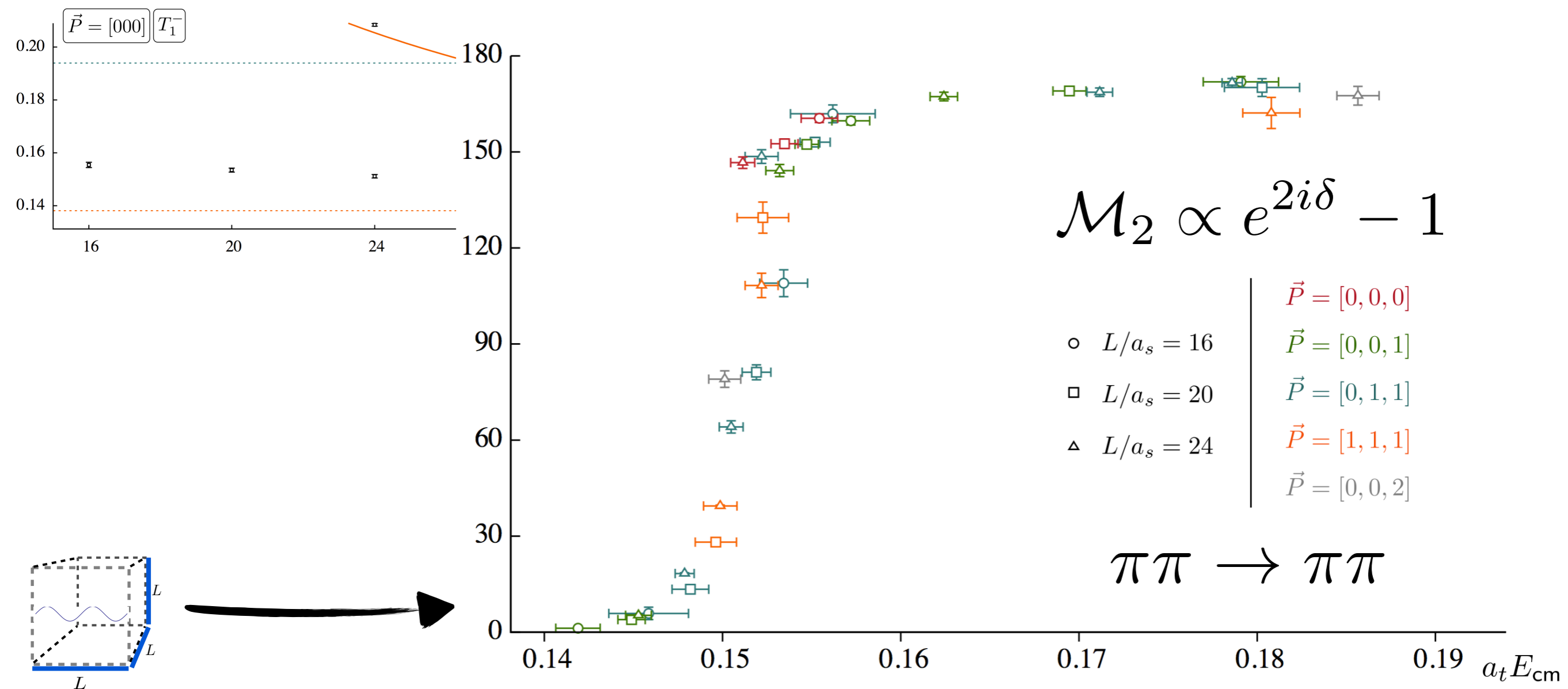
$$\det \left[\mathcal{M}_2^{-1}(E_n) + F(E_n, \vec{0}, L) \right]_{\mathbb{A}_1^+} = 0$$

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

Using the result

- Simplest case is a single angular momentum (e.g. 2 pions in a p-wave)

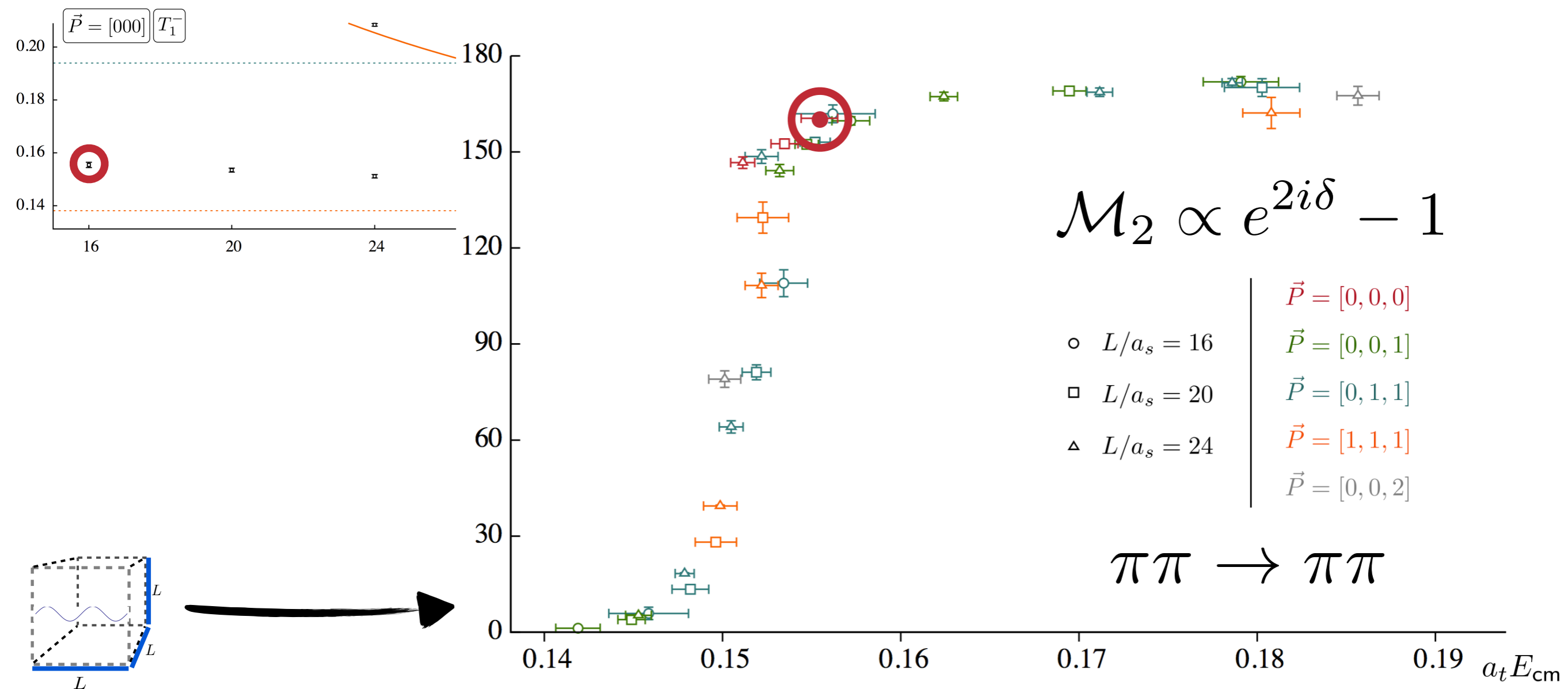
$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$



Using the result

□ Simplest case is a single angular momentum (e.g. 2 pions in a p-wave)

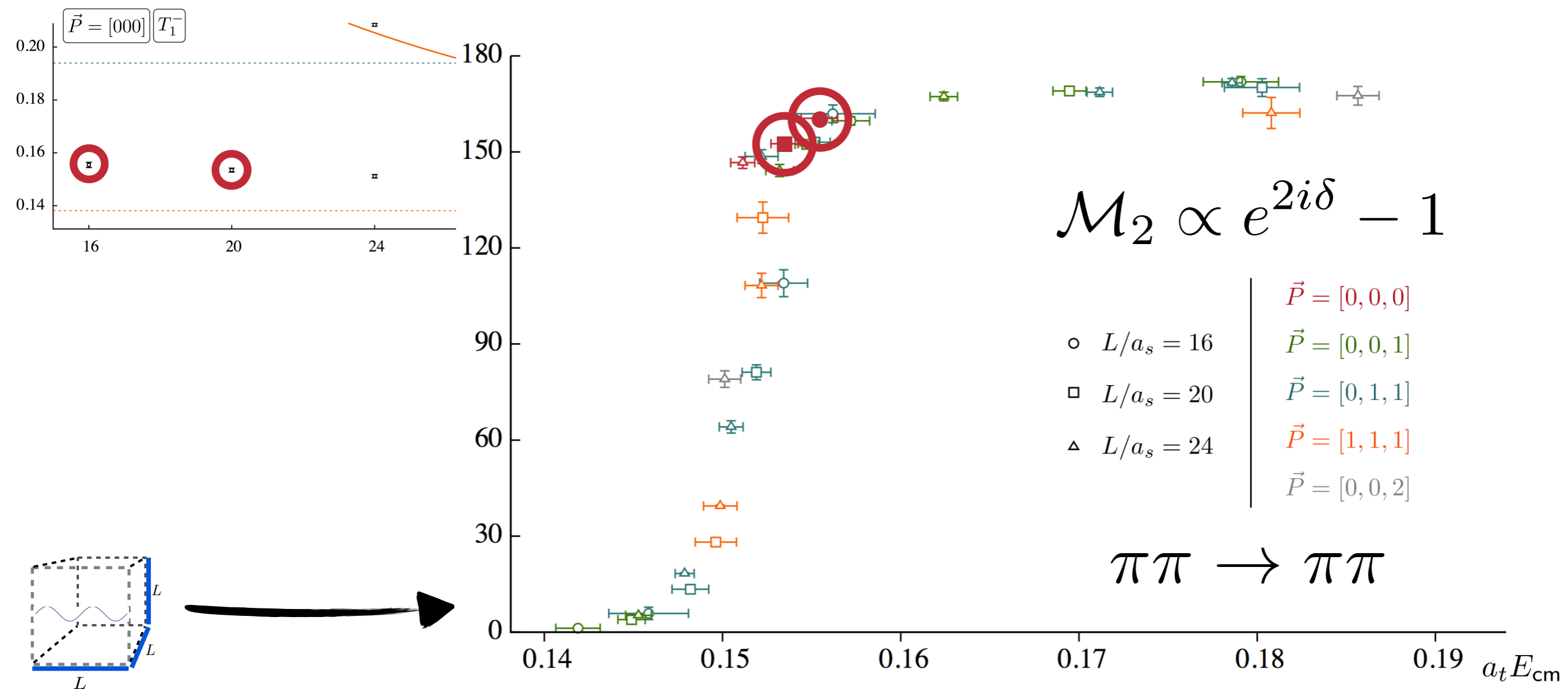
$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$



Using the result

□ Simplest case is a single angular momentum (e.g. 2 pions in a p-wave)

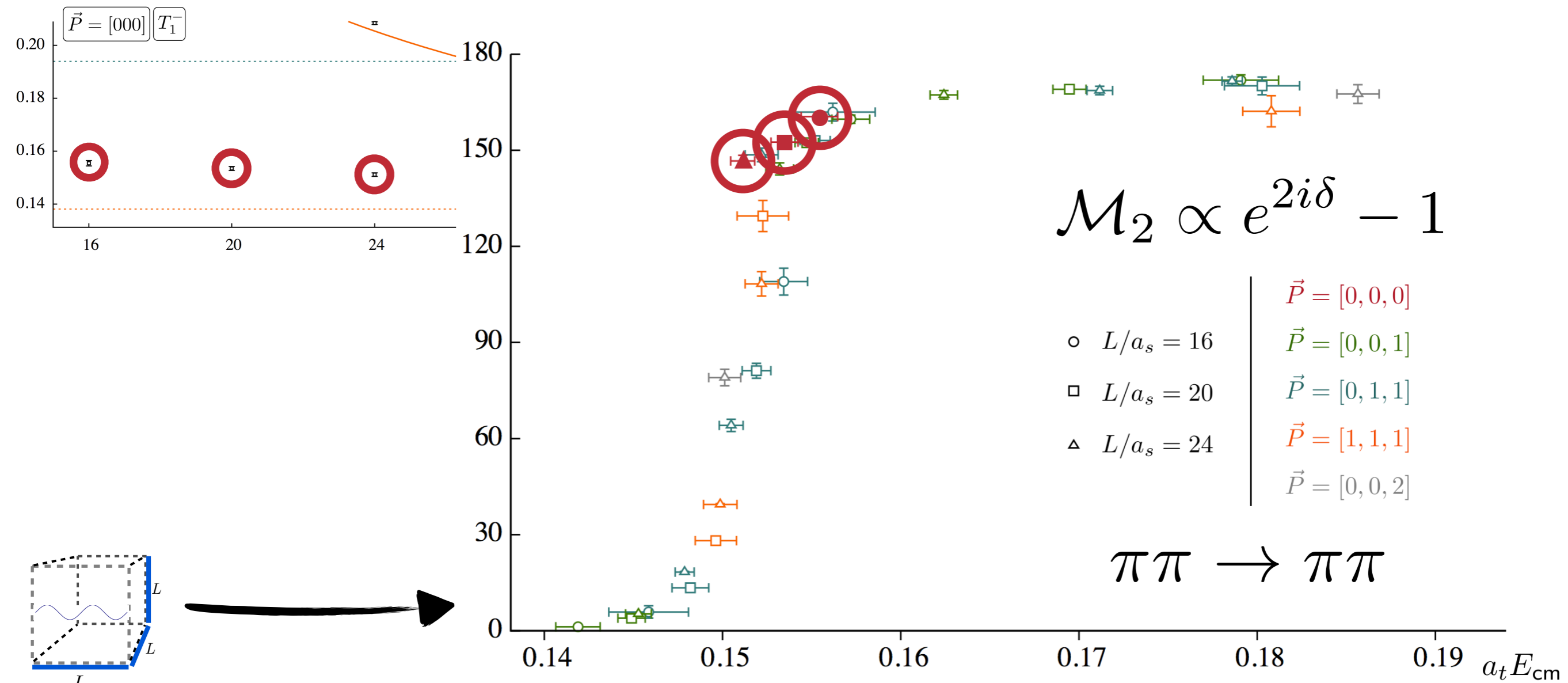
$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$



Using the result

□ Simplest case is a single angular momentum (e.g. 2 pions in a p-wave)

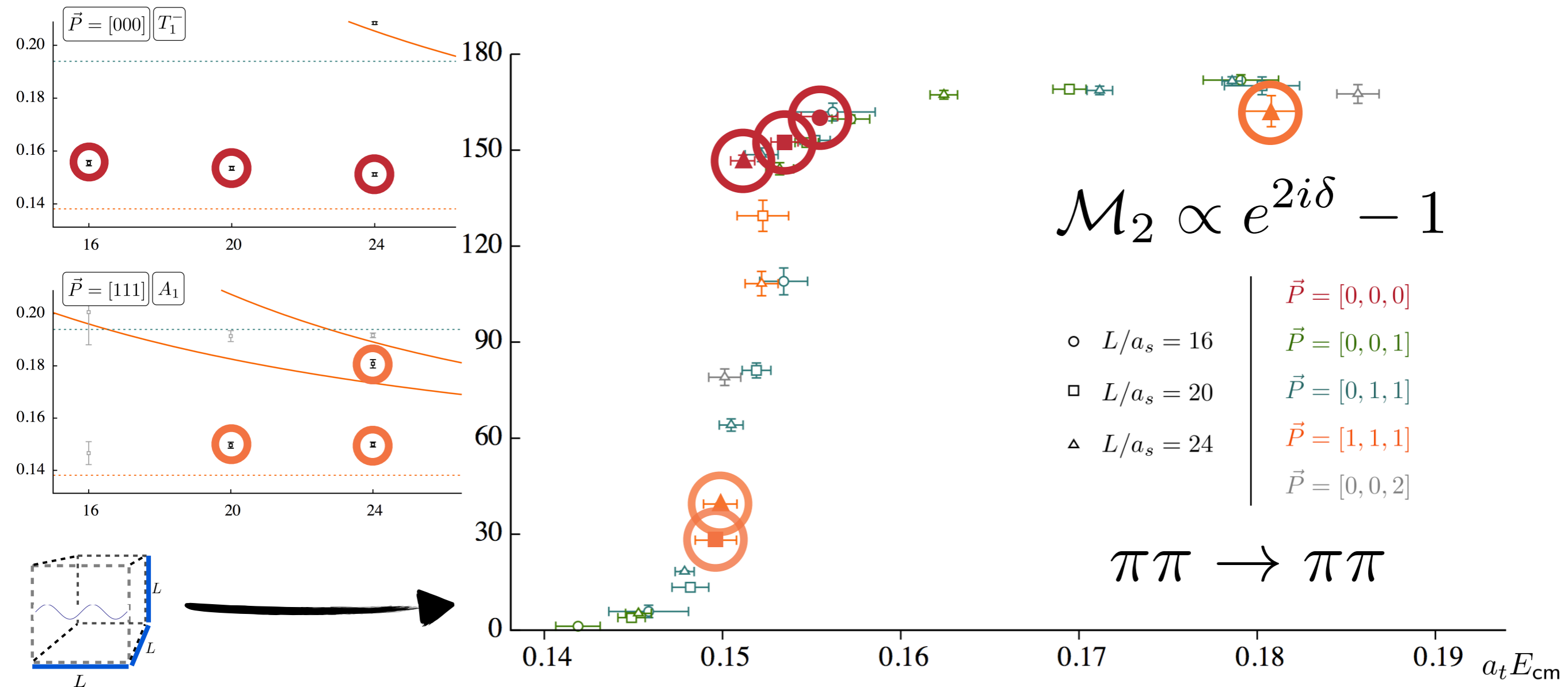
$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$

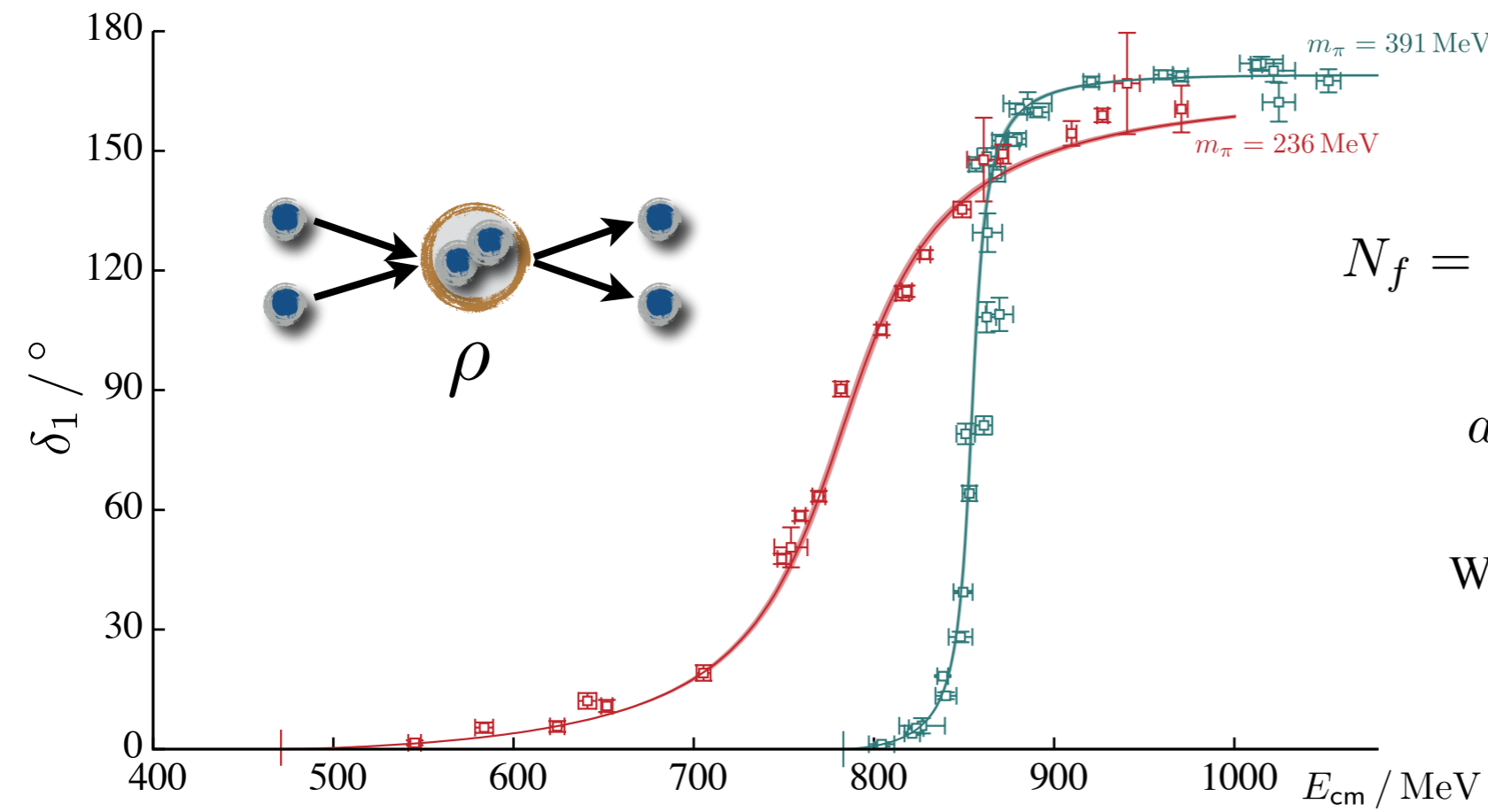


Using the result

□ Simplest case is a single angular momentum (e.g. 2 pions in a p-wave)

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$



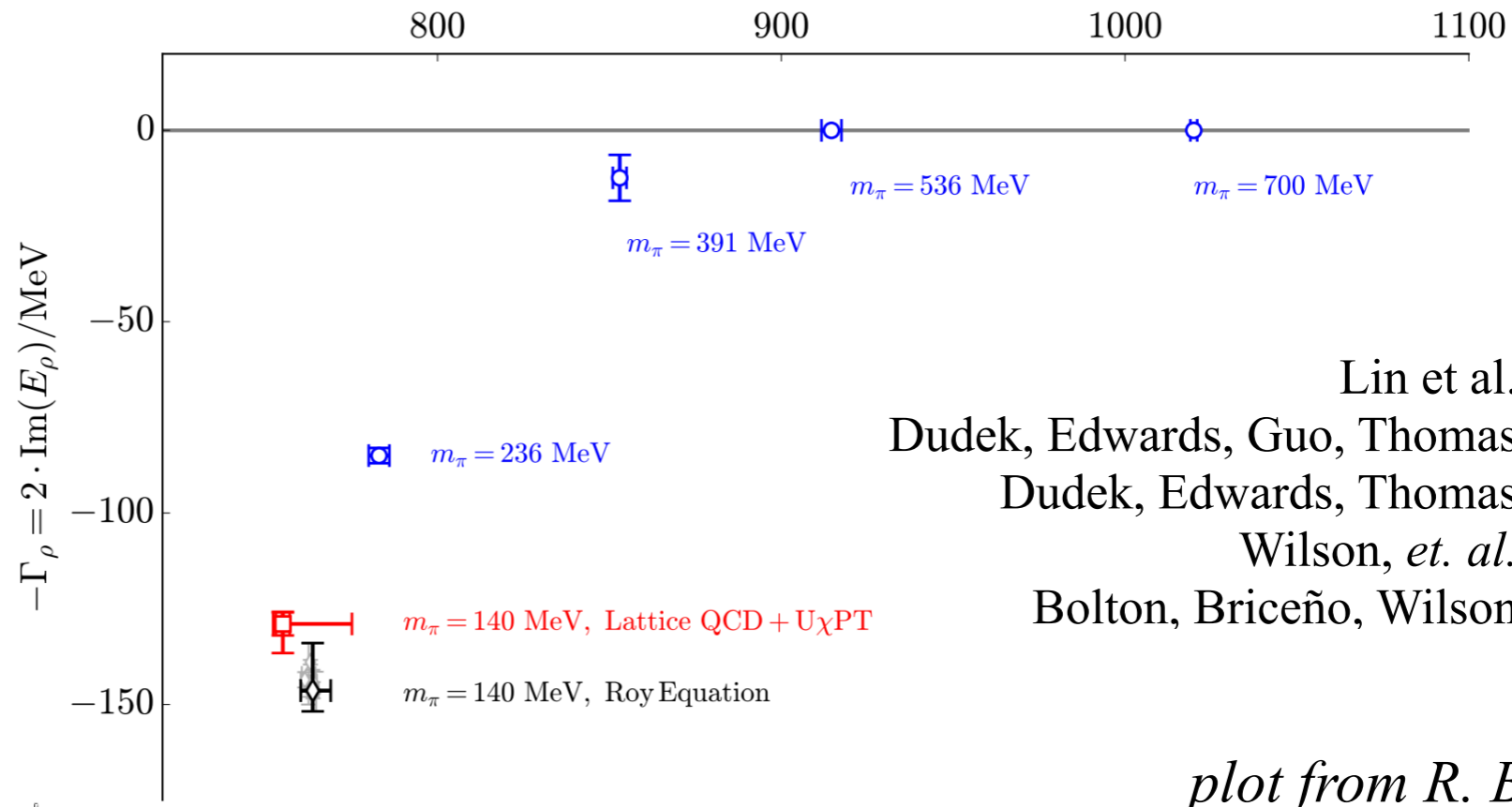
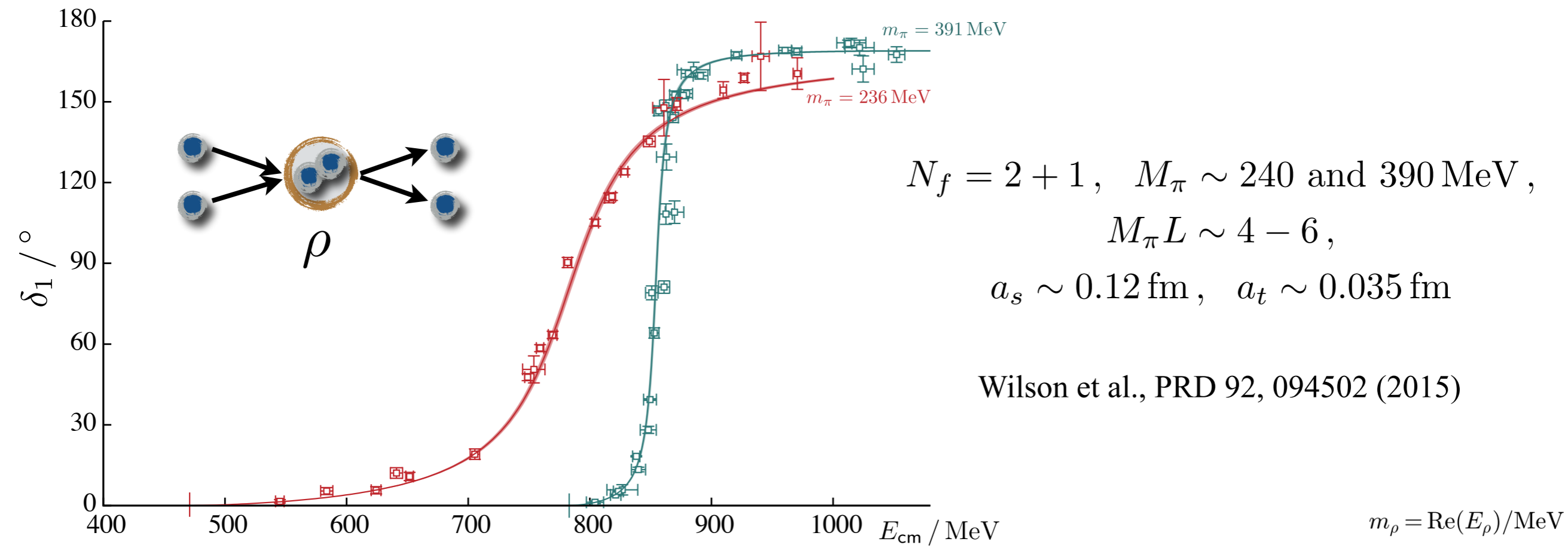


$N_f = 2 + 1, \quad M_\pi \sim 240 \text{ and } 390 \text{ MeV},$

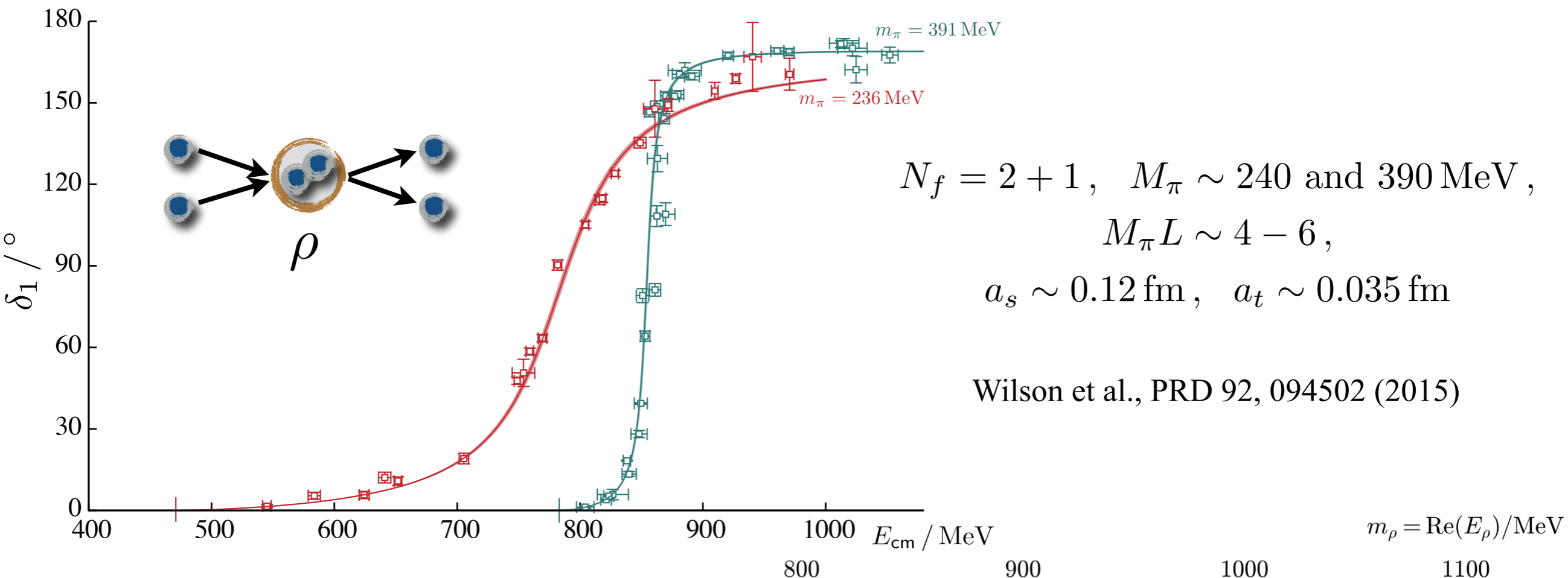
$M_\pi L \sim 4 - 6,$

$a_s \sim 0.12 \text{ fm}, \quad a_t \sim 0.035 \text{ fm}$

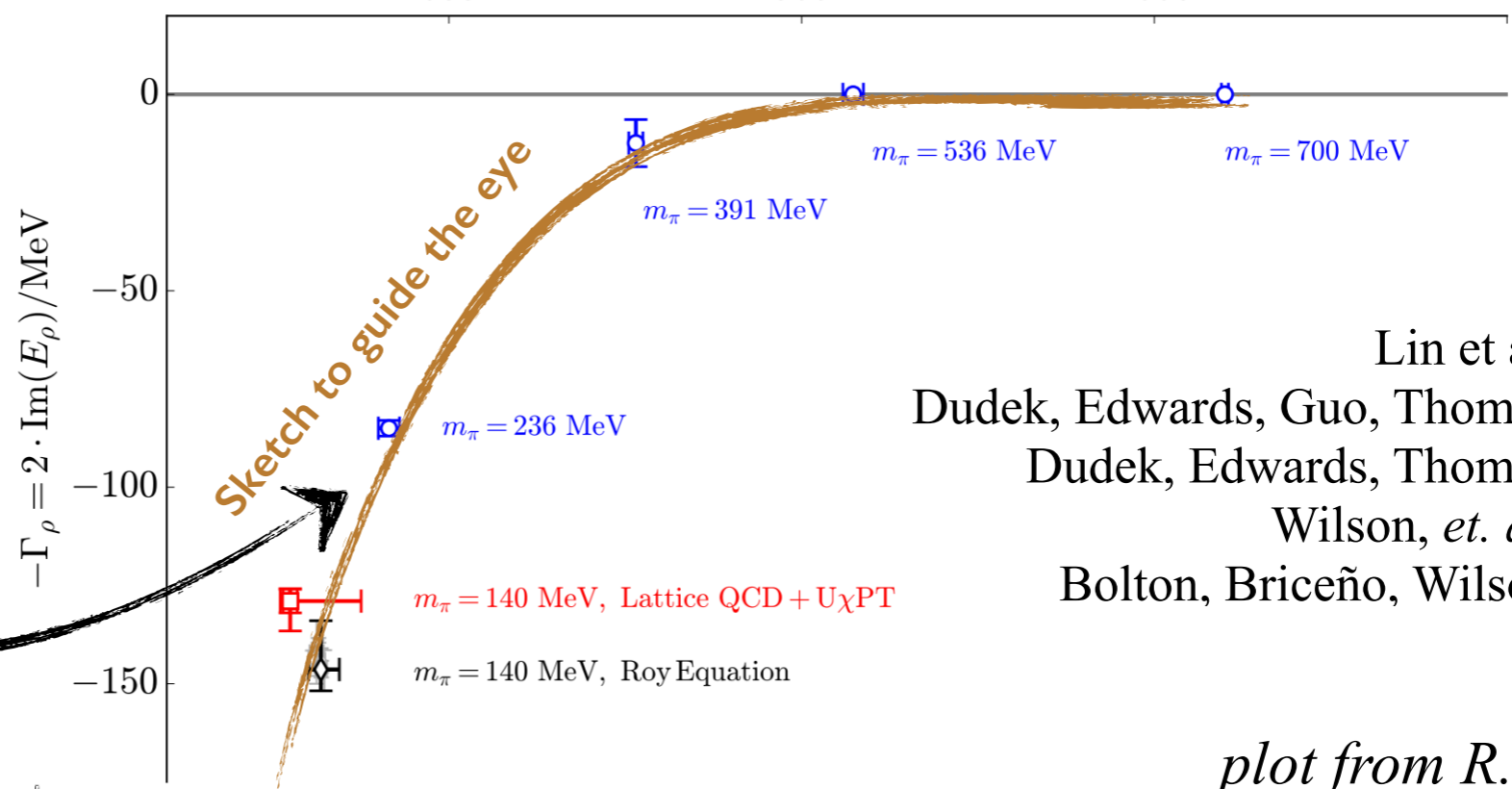
Wilson et al., PRD 92, 094502 (2015)



plot from R. Briceño



Quantitatively **track**
the pole position in the
 complex plane



Lin et al. (2009)
 Dudek, Edwards, Guo, Thomas (2013)
 Dudek, Edwards, Thomas (2012)
 Wilson, *et. al.* (2015)
 Bolton, Briceño, Wilson (2015)

plot from R. Briceño

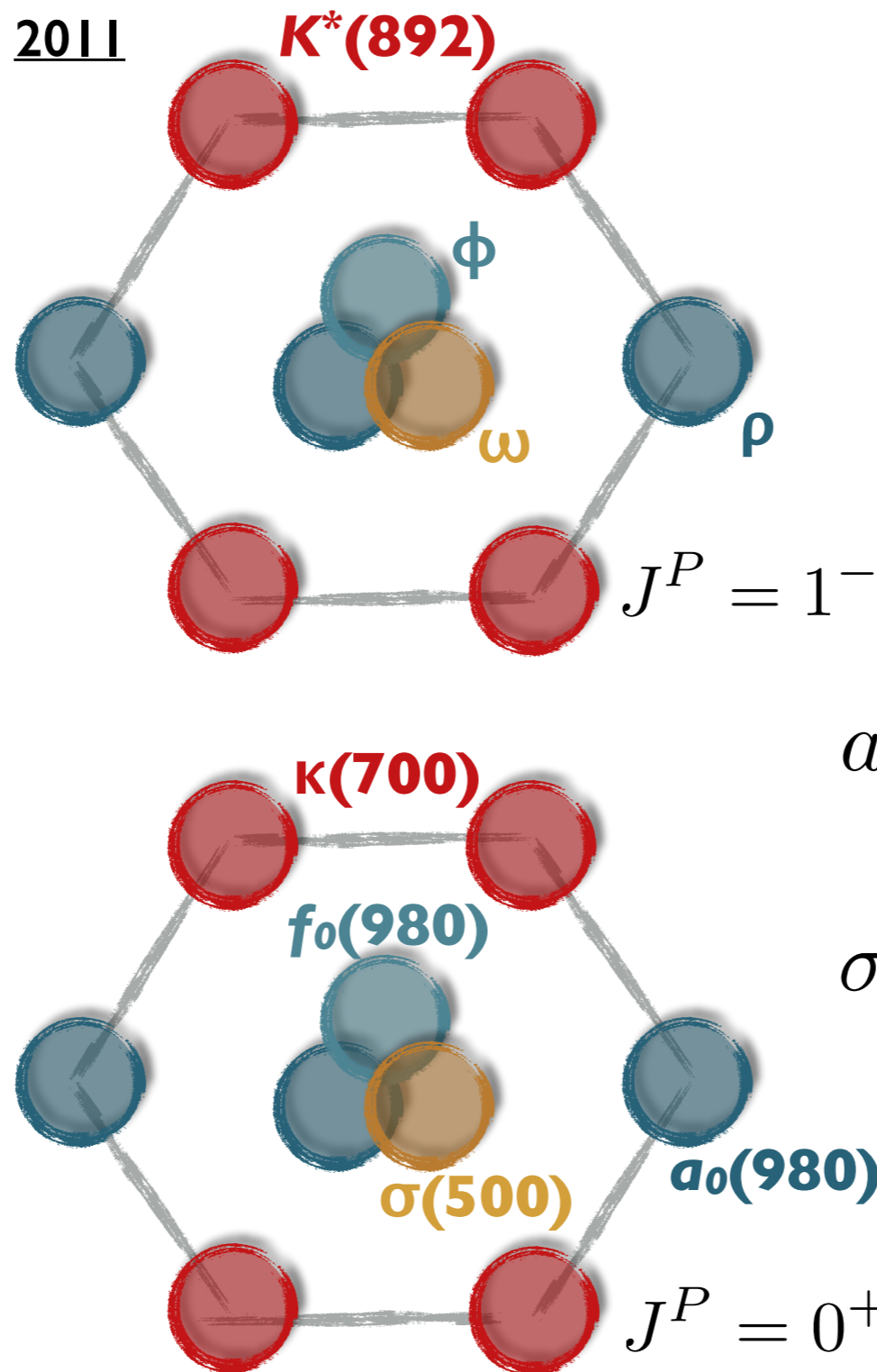
Lots of activity!

$$\rho \rightarrow \pi\pi$$

- [CP-PACS/PACS-CS 2007, 2011](#)
- [Feng et al. 2010](#)
- [Lang et al. 2011](#)
- [HadSpec 2012, 2015](#)
- [Pelissier, Alexandru 2012](#)
- [RQCD 2015](#)
- [Guo et al. 2016](#)
- [Fu, Wang 2016](#)
- [Bulava et al. 2016](#)
- [Alexandrou et al. 2017](#)
- [Andersen et al. 2018](#)

$$\sigma \rightarrow \pi\pi$$

- [Prelovsek et al. 2010](#)
- [Fu 2013](#)
- [Wakayama 2015](#)
- [Howarth, Giedt 2017](#)
- [Briceño et al. 2017](#)
- [Molina et al. 2018](#)



$$K^* \rightarrow K\pi$$

- [Lang et al. 2012](#)
- [Prelovsek et al. 2013](#)
- [Wilson et al. 2015](#)
- [RQCD 2015](#)
- [Brett et al. 2018](#)
- [Wilson et al. 2019](#)

$$a_0(980) \rightarrow \pi\eta, K\bar{K}$$

- [Dudek et al. 2016](#)

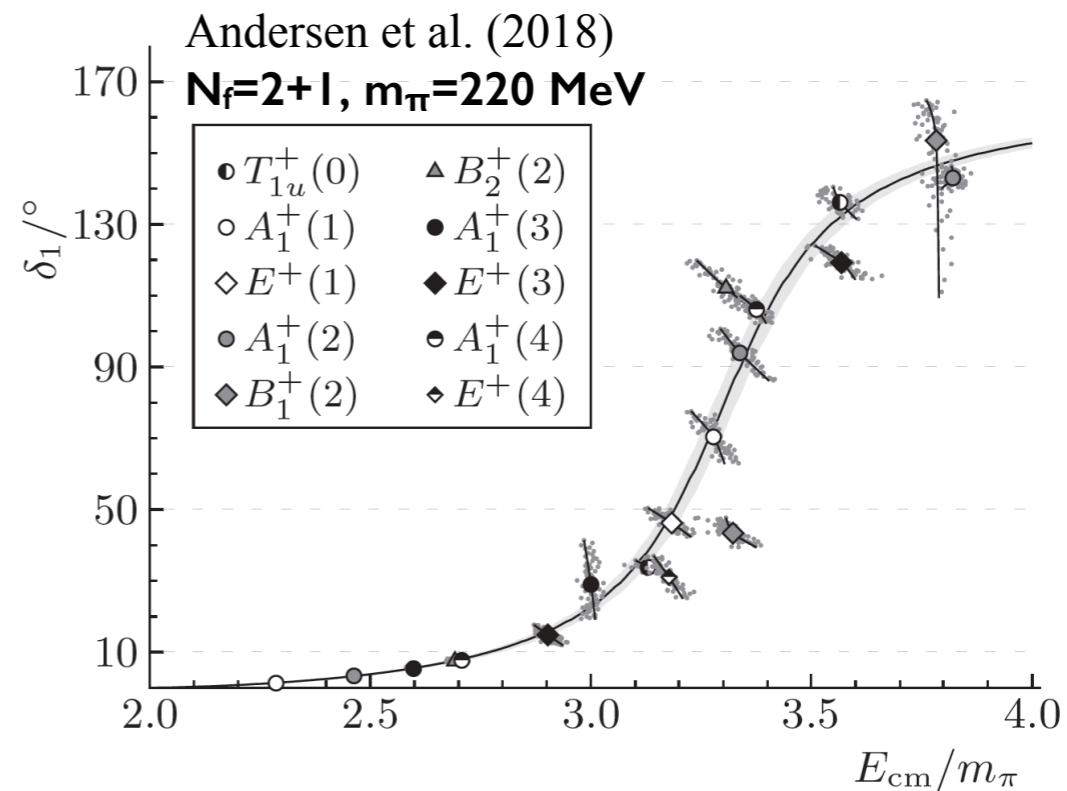
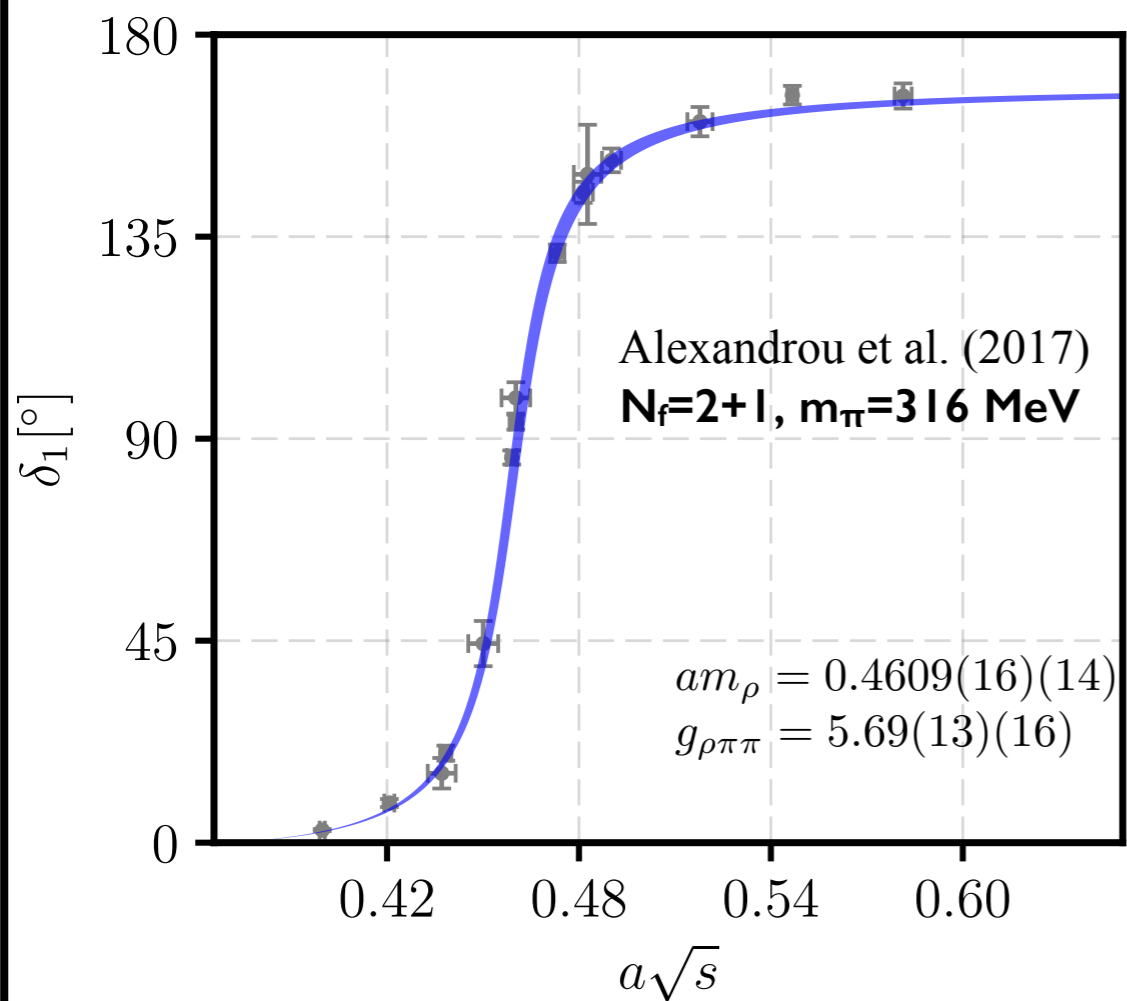
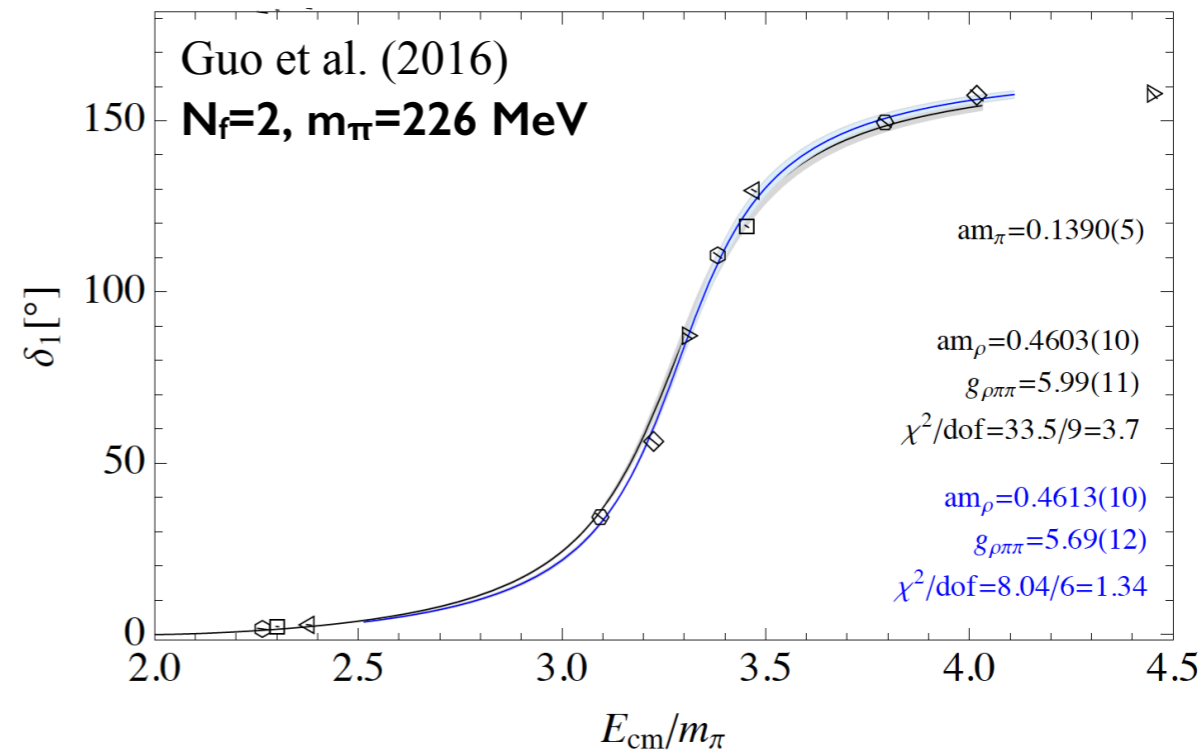
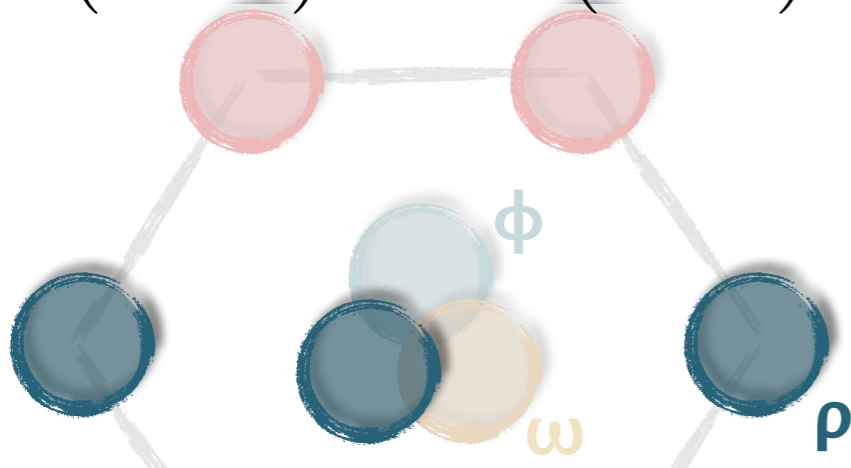
$$\sigma, f_0, f_2 \rightarrow \pi\pi, K\bar{K}, \eta\eta$$

- [Briceño et al. 2017](#)

[See the recent review by Briceño, Dudek and Young](#)

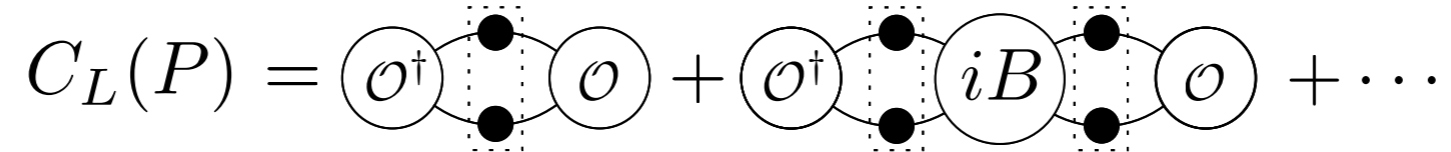
$$\rho \rightarrow \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$



Coupled channels

□ Recall the single-channel skeleton expansion

$$C_L(P) = \text{Diagram 1} + \text{Diagram 2} + \dots$$


Coupled channels

- Recall the single-channel skeleton expansion

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \dots$$

The diagram shows the single-channel skeleton expansion. It consists of three terms: 1) a circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right, connected by two grey lines that meet at two grey dots in the middle, which are enclosed in a dashed box; 2) a similar structure but with a circle labeled iB in the middle; 3) followed by an ellipsis.

- For coupled channels this becomes

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \text{diagram}_4 + \text{diagram}_5 + \text{diagram}_6 + \dots$$

The diagram shows the expansion for coupled channels. It consists of six terms: 1) \mathcal{O}^\dagger and \mathcal{O} connected by two blue lines; 2) \mathcal{O}^\dagger and \mathcal{O} connected by two red lines; 3) \mathcal{O}^\dagger and \mathcal{O} connected by two blue lines with a circle labeled iB in the middle; 4) \mathcal{O}^\dagger and \mathcal{O} connected by two red lines with a circle labeled iB in the middle; 5) \mathcal{O}^\dagger and \mathcal{O} connected by two red lines with a circle labeled iB in the middle; 6) followed by an ellipsis.

Coupled channels

□ Recall the single-channel skeleton expansion

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \dots$$

The diagram shows the skeleton expansion for a single channel. It consists of a sequence of diagrams: a self-energy loop on a propagator, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, and so on. The diagrams are connected by plus signs and an ellipsis.

□ For coupled channels this becomes

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagram shows the skeleton expansion for coupled channels. It consists of a sequence of diagrams: a self-energy loop on a propagator, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, and so on. The diagrams are connected by plus signs and an ellipsis.

$$= \begin{pmatrix} \text{diagram}_1 & \text{diagram}_2 \end{pmatrix} \left[\begin{pmatrix} \text{diagram}_3 & \text{diagram}_4 \end{pmatrix} + \begin{pmatrix} \text{diagram}_5 & \text{diagram}_6 \end{pmatrix} \begin{pmatrix} iB & iB \\ iB & iB \end{pmatrix} \begin{pmatrix} \text{diagram}_7 & \text{diagram}_8 \end{pmatrix} \right] \begin{pmatrix} \text{diagram}_9 \\ \text{diagram}_{10} \end{pmatrix} + \dots$$

The diagram shows the skeleton expansion for coupled channels in matrix form. It consists of a sequence of diagrams: a self-energy loop on a propagator, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, followed by a self-energy loop on a vertex, and so on. The diagrams are connected by plus signs and an ellipsis.

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$$= \left(\text{diagram} \quad \text{diagram} \right) \left[\left(\text{diagram} \quad \text{diagram} \right) + \left(\text{diagram} \quad \text{diagram} \right) \left(\text{diagram} \quad \text{diagram} \right) \left(\text{diagram} \quad \text{diagram} \right) \right] \left(\text{diagram} \quad \text{diagram} \right) + \dots$$

Full derivation goes through with this **new matrix structure**

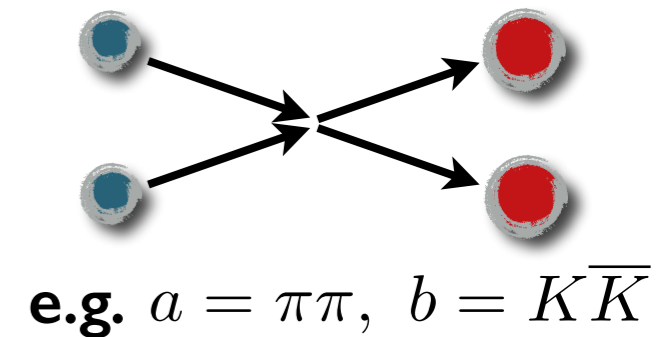


Coupled quantization condition

□ Finite-volume energies = solutions to...

$$\det \left[\begin{array}{cc} \mathcal{M}_{a \rightarrow a} & \mathcal{M}_{a \rightarrow b} \\ \mathcal{M}_{b \rightarrow a} & \mathcal{M}_{b \rightarrow b} \end{array} \right]^{-1} + \begin{array}{cc} F_a & 0 \\ 0 & F_b \end{array} = 0$$

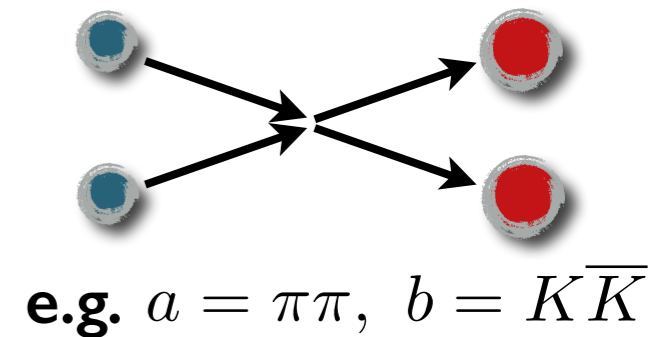
scattering amplitude known functions



Coupled quantization condition

- Finite-volume energies = solutions to...

$$\det \left[\underbrace{\begin{pmatrix} \mathcal{M}_{a \rightarrow a} & \mathcal{M}_{a \rightarrow b} \\ \mathcal{M}_{b \rightarrow a} & \mathcal{M}_{b \rightarrow b} \end{pmatrix}^{-1}}_{\text{scattering amplitude}} + \underbrace{\begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix}}_{\text{known functions}} \right] = 0$$



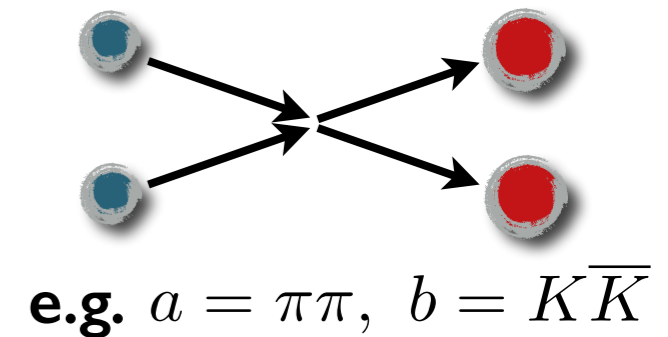
- Matrices in angular momentum space
- Holds only for $E_n^{*2} < (4m)^2$
- Ignores suppressed volume effects (e^{-mL})

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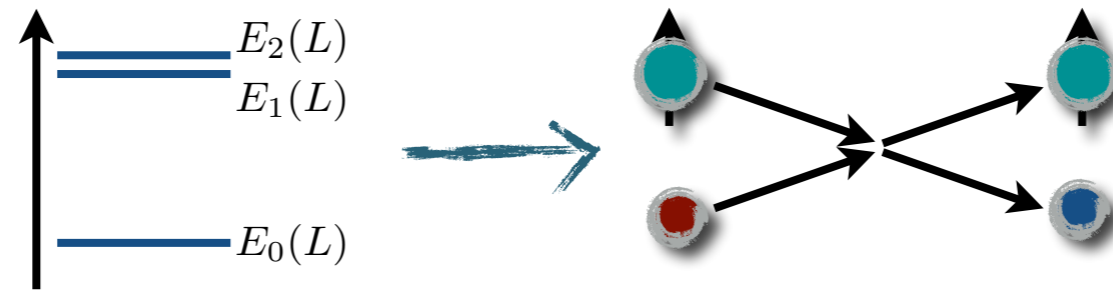


- Matrices in angular momentum space
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- Ignores suppressed volume effects (e^{-mL})

$$\mathcal{M}_2(E^*) = \begin{pmatrix} \bullet & & \bullet & & \bullet & & \bullet \\ & \bullet & & & & & & \\ & & \bullet & & & & & \\ & & & \bullet & & & & \\ \bullet & & & & \bullet & & & \\ & \bullet & & & & & \bullet & \\ & & \bullet & & & & & \bullet \\ & & & \bullet & & & & \bullet \end{pmatrix} \quad F(E_n, \vec{P}, L) = \begin{pmatrix} \bullet & & & & & & & \\ & \bullet & & & & & & \\ & & \bullet & & & & & \\ & & & \bullet & & & & \\ & & & & \bullet & & & \\ \bullet & & & & & \bullet & & \\ & \bullet & & & & & \bullet & \\ & & \bullet & & & & & \bullet \\ & & & \bullet & & & & \bullet \end{pmatrix}$$

General two-to-two scattering

- Lüscher's formalism + extensions give a general mapping



- All results are contained in a **generalized quantization condition**

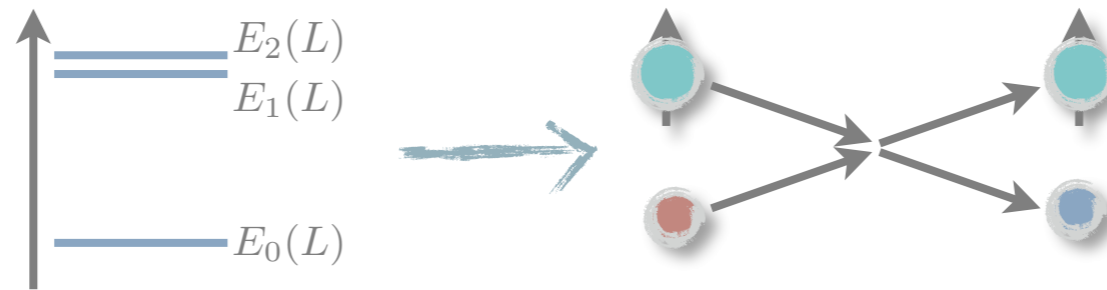
$$\det \left[\mathcal{M}_2^{-1} (E_n^*) + F(E_n, \vec{P}, L) \right] = 0$$

scattering amplitude known geometric function

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
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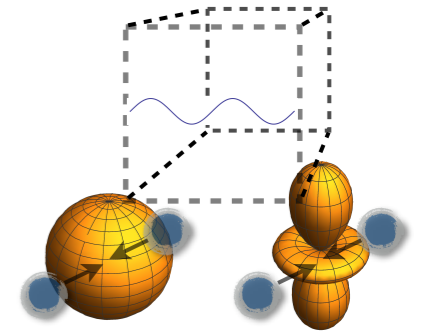
- Matrices in **angular momentum, spin** and **channel** space
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Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
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Not a one-to-one mapping

□ The cubic volume mixes different partial waves...

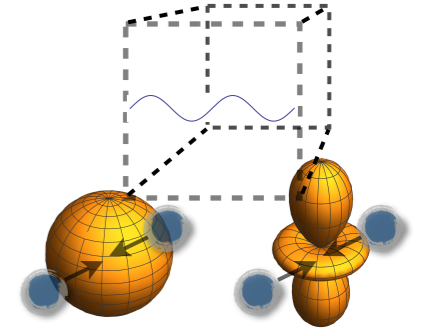
$$\text{e.g. } \begin{matrix} K\pi \rightarrow K\pi \\ \vec{P} \neq 0 \end{matrix} \longrightarrow \det \left[\begin{pmatrix} \mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0$$



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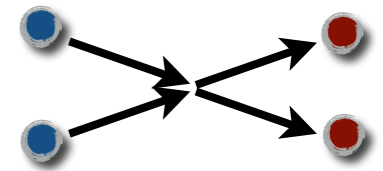
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...as well as different flavor channels...

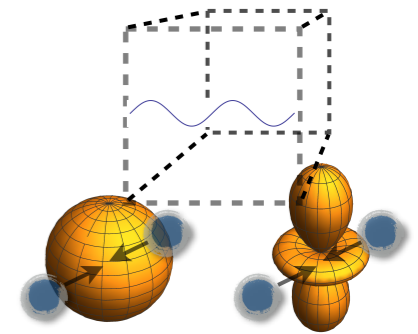
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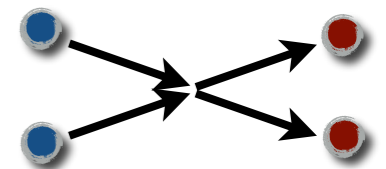
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□ Strategy:

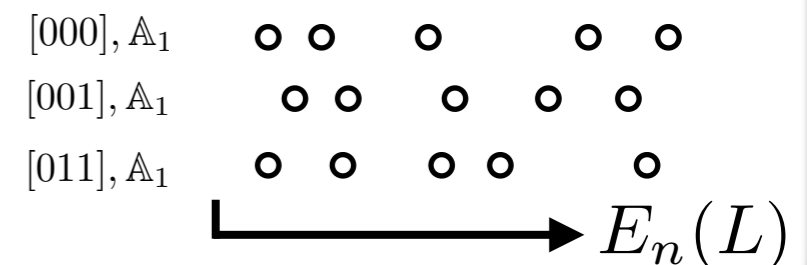
Matrix of correlators
with varied operators

$$\langle \mathcal{O}_a(\tau) \mathcal{O}_b^\dagger(0) \rangle$$

Diagonalize (GEVP) to
extract energies

$$\langle \Omega_m(\tau) \Omega_m^\dagger(0) \rangle \sim e^{-E_m(L)\tau}$$

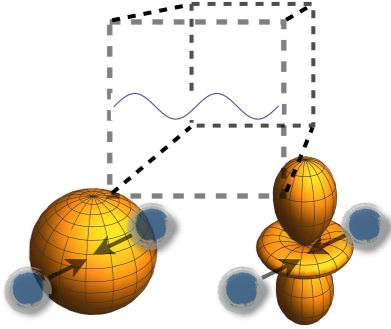
Vary L and P to recover a
dense set of energies



Not a one-to-one mapping

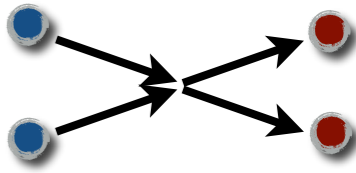
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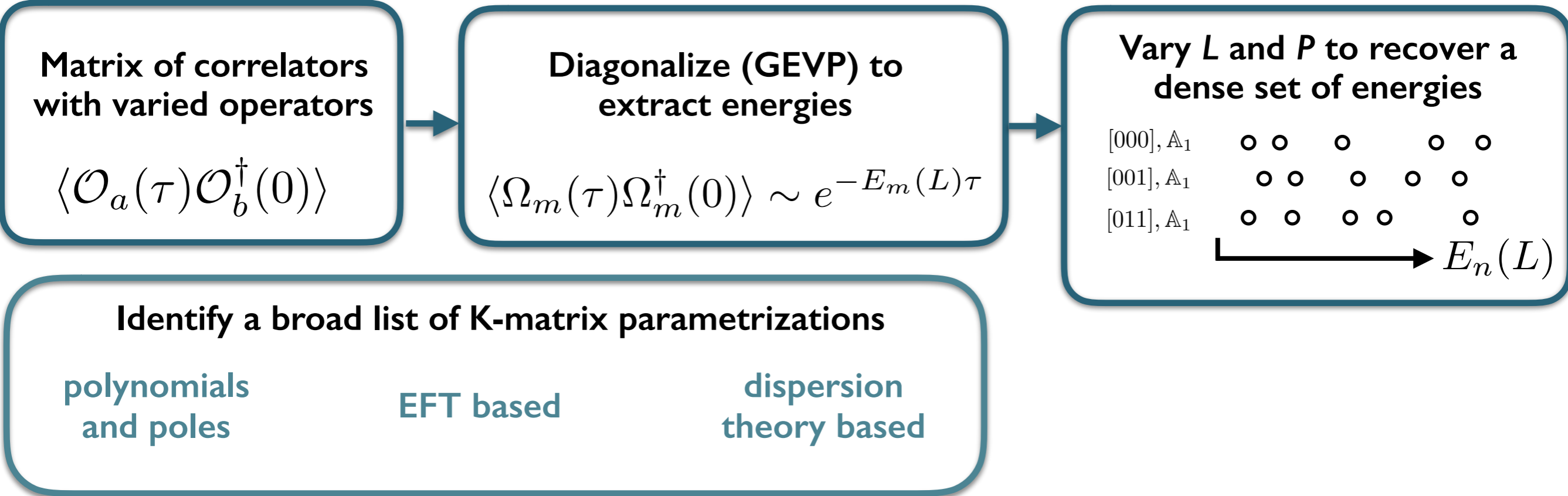


...as well as different flavor channels...

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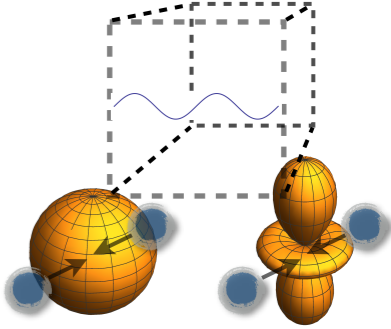
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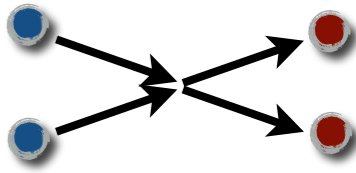
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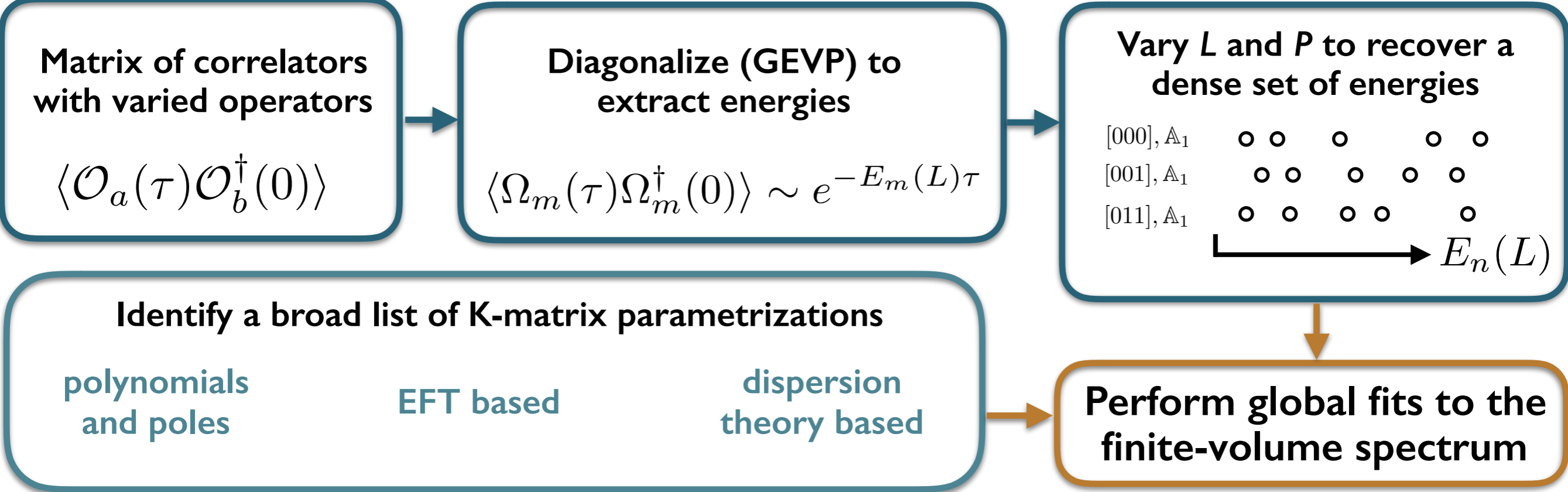


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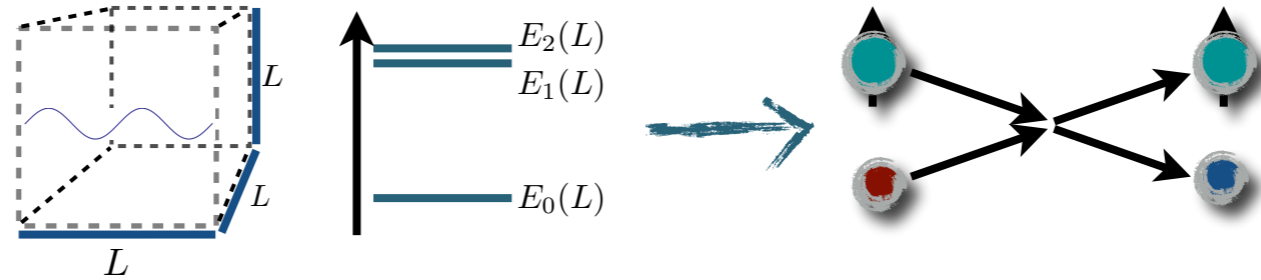


□ Strategy:



Conclusions and outlook

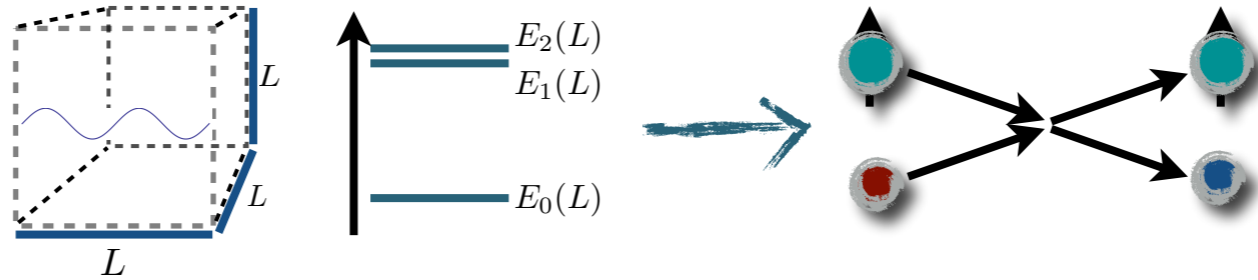
- ❑ Multi-hadron observables accesible in LQCD
- ❑ Use the ***finite-volume as a tool***



- ❑ Numerical ***implementation well underway***... still lots to come
- ❑ Room for thought:
 - ❑ Incorporating lattice/residual volume effects
 - ❑ Confronting the non-one-to-one mapping
 - ❑ Extending beyond two-to-two! (*Steve's talk*)

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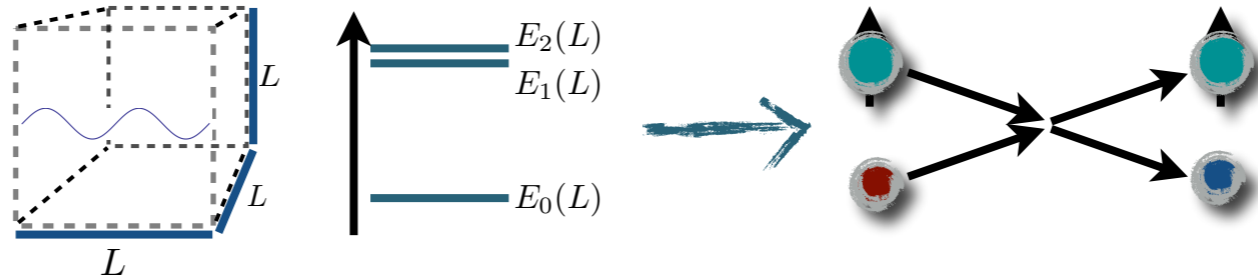
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Approaches not covered here

- ❑ Finite-volume hamiltonian method \longrightarrow W2 - Tues - *Derek, Finn*
- ❑ HALQCD potential method
- ❑ Reconstruction of the spectral function \longrightarrow Fri - *John, Antonin*
- ❑ Lüscher + KSS methods for long-range matrix elements Tues - *Zohreh*

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Thanks for listening - Enjoy the workshop!