Introduction to scattering on the lattice

Maxwell T. Hansen

July 22nd, 2019

QCD Fock space

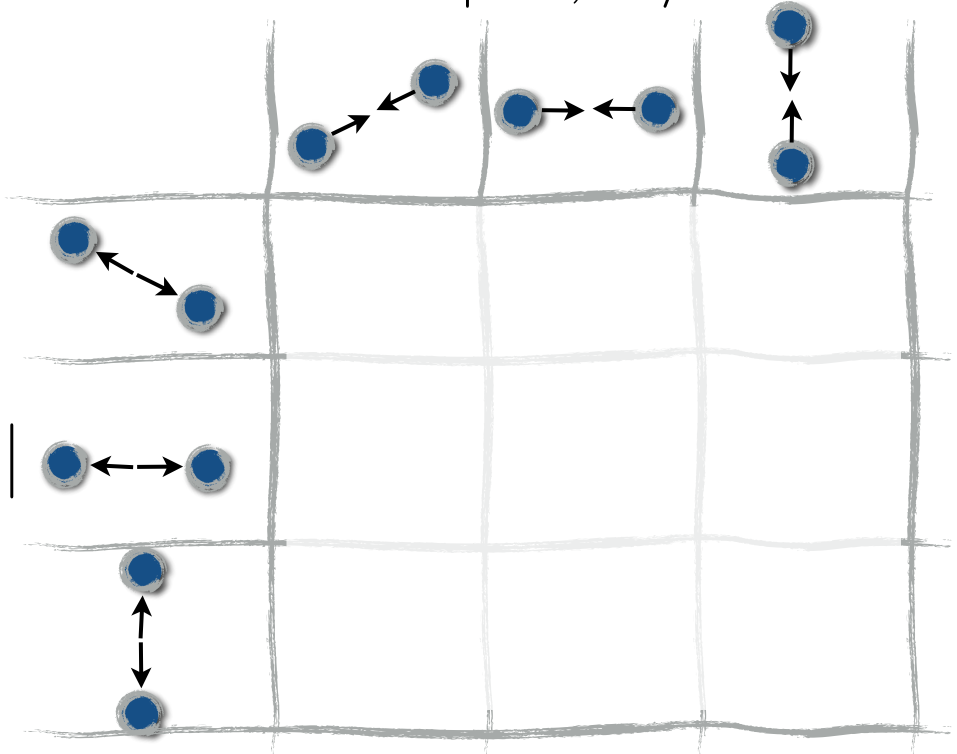
- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$

$$|\pi\pi, \mathrm{in}\rangle$$

$$S(E) \equiv \langle \pi\pi, \mathrm{out}|$$



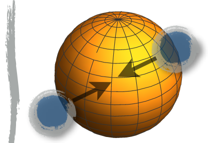
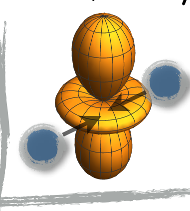
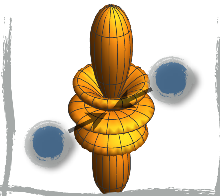
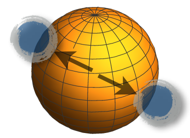
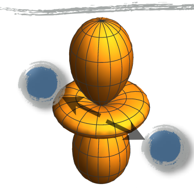
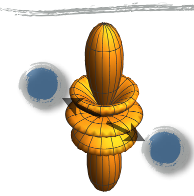
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$

$|\pi\pi, \mathrm{in}\rangle$

$S(E) \equiv \langle \pi\pi, \mathrm{out}|$

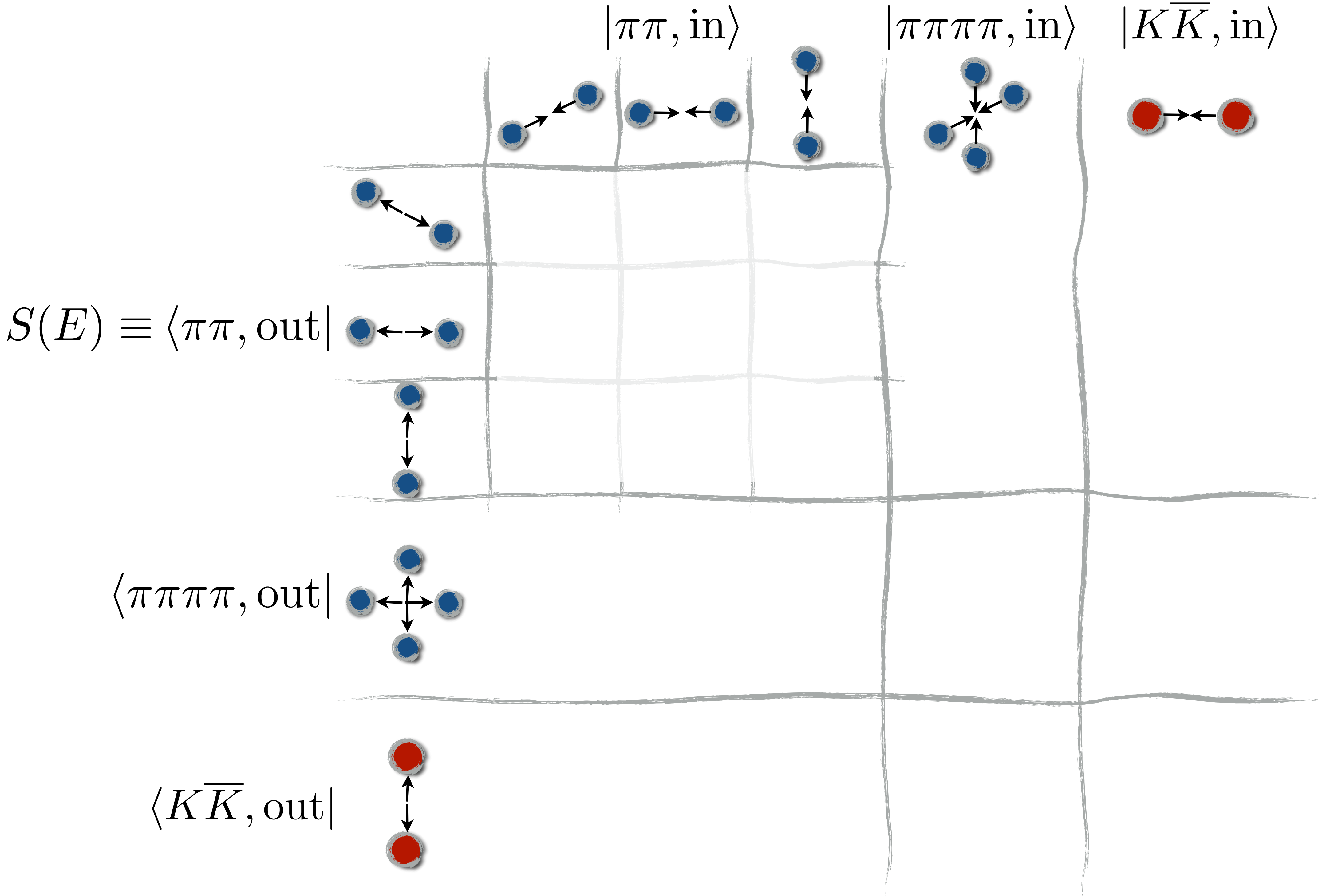
			
	$S_0(E)$	0	0
	0	$S_1(E)$	0
	0	0	$S_2(E)$

- Overlaps of multi-hadron ***asymptotic states*** → S matrix
- Diagonalized in angular-momentum basis

$$S_0(E) = e^{2i\delta_0(E)} \qquad \mathcal{M}_0(E) \propto e^{2i\delta_0(E)} - 1$$

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$

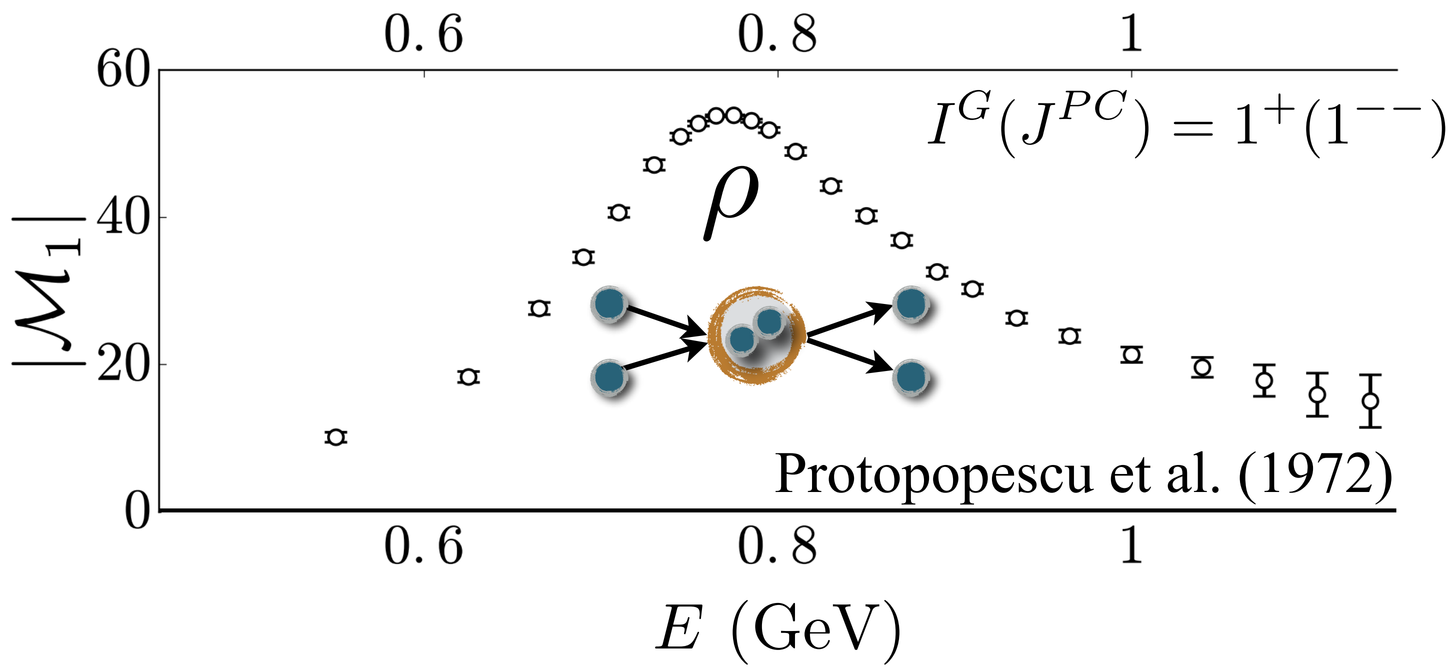


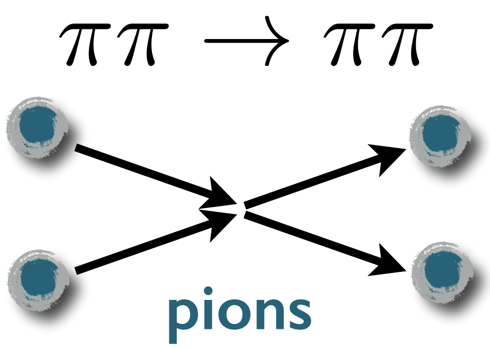
Infinite-dimensional matrix for each set of quantum numbers

Extracting resonance properties

- Roughly speaking, a bump in: $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

scattering rate **unitarity relation**

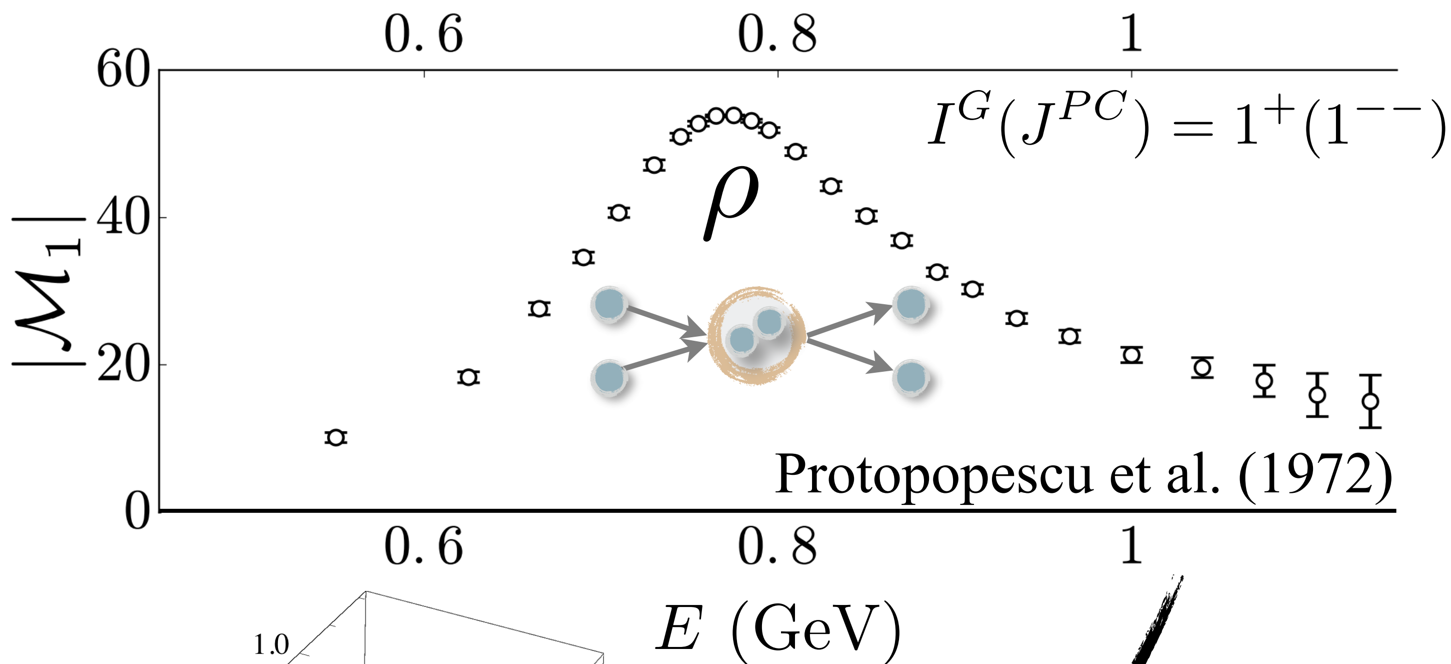


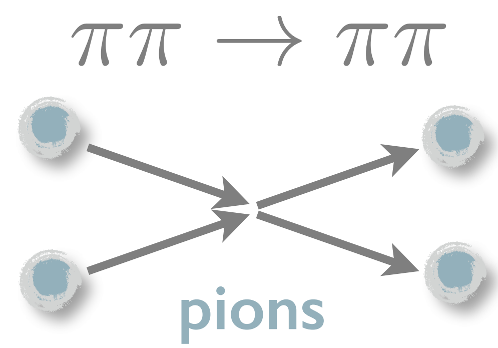


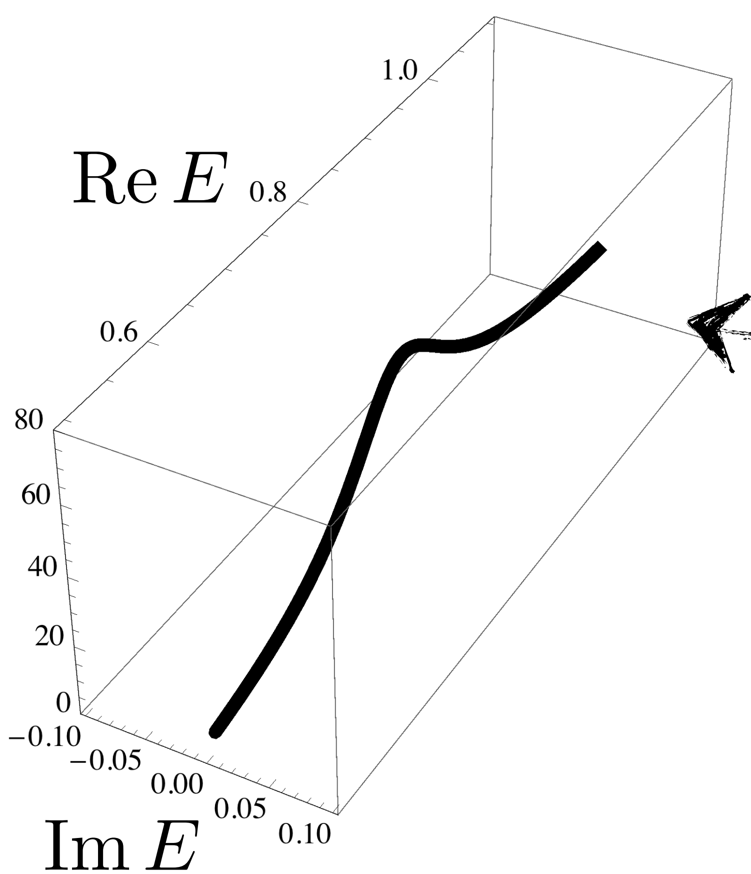
Extracting resonance properties

- ☐ Roughly speaking, a bump in: $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

scattering rate — unitarity relation





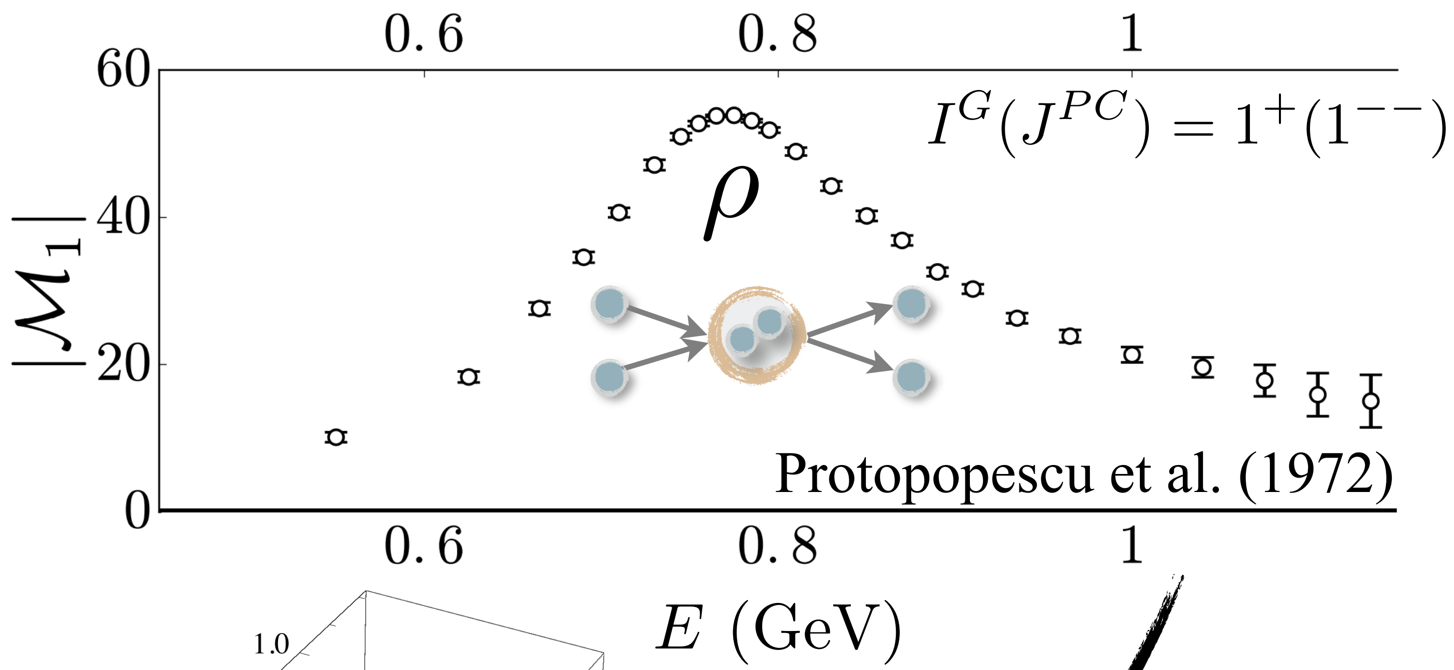


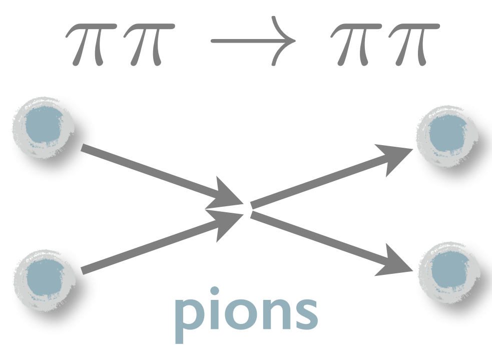
Extend to the **complex plane**

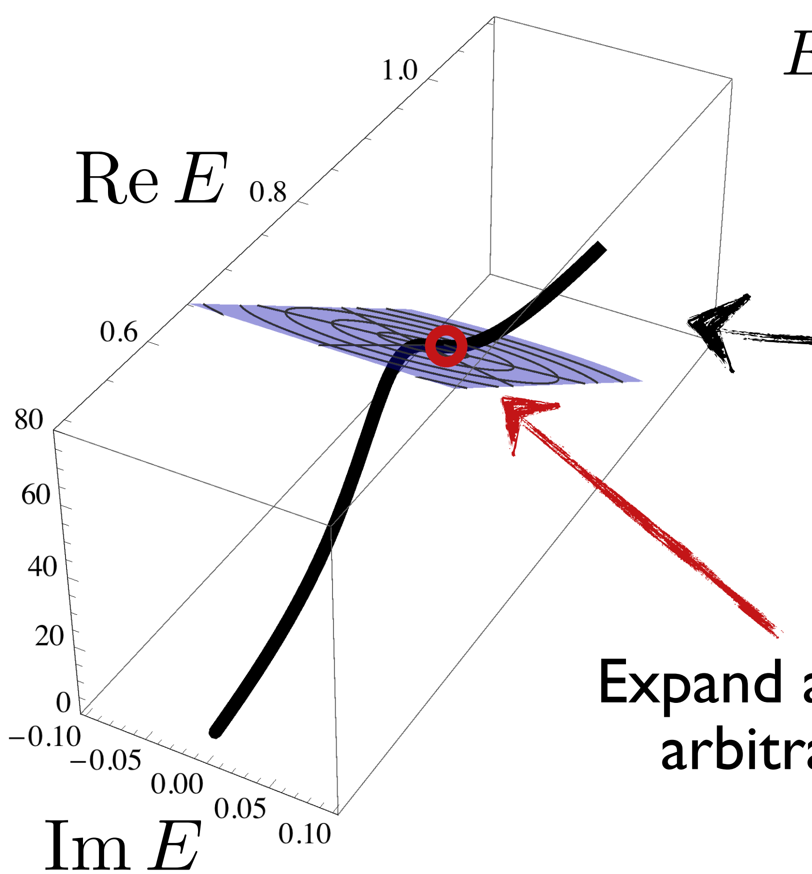
Extracting resonance properties

- ☐ Roughly speaking, a bump in: $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

scattering rate — unitarity relation







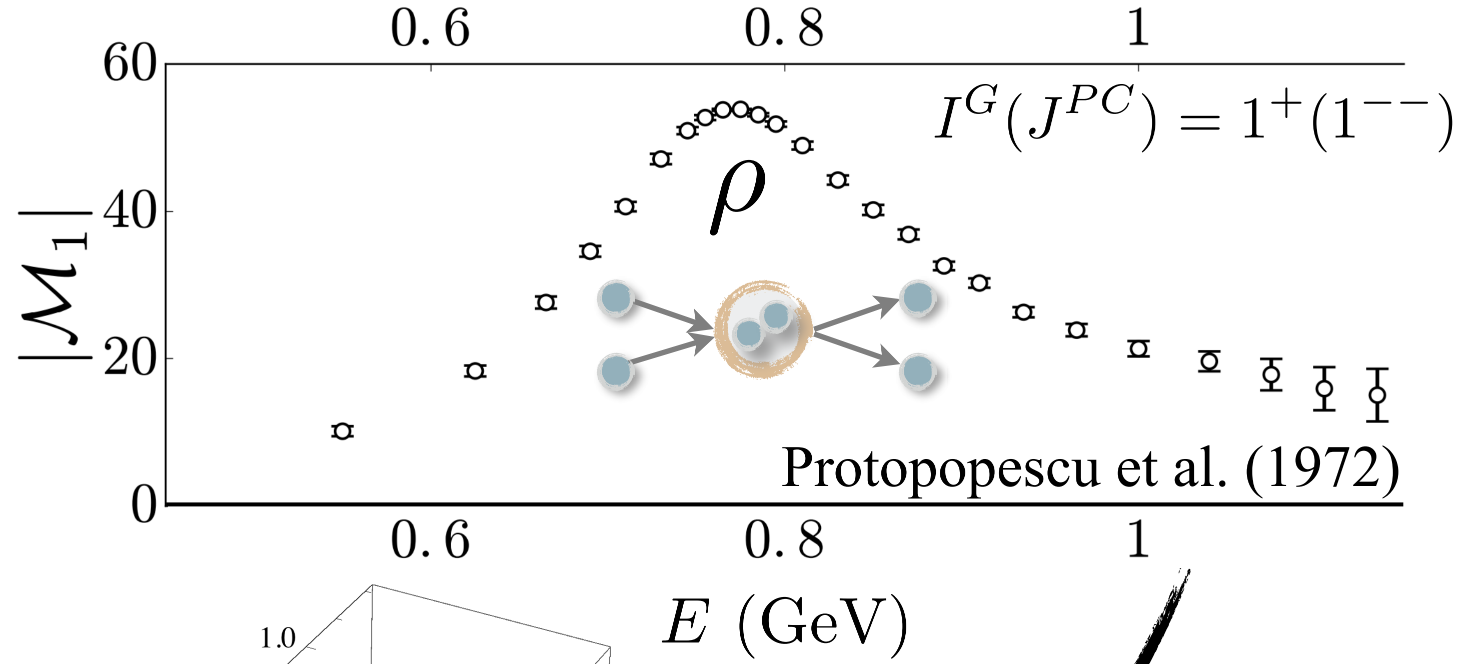
Extend to the **complex plane**

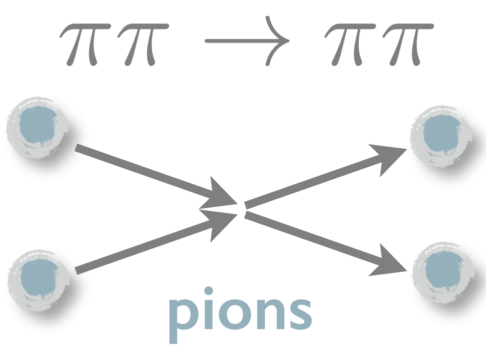
Expand about some arbitrary point

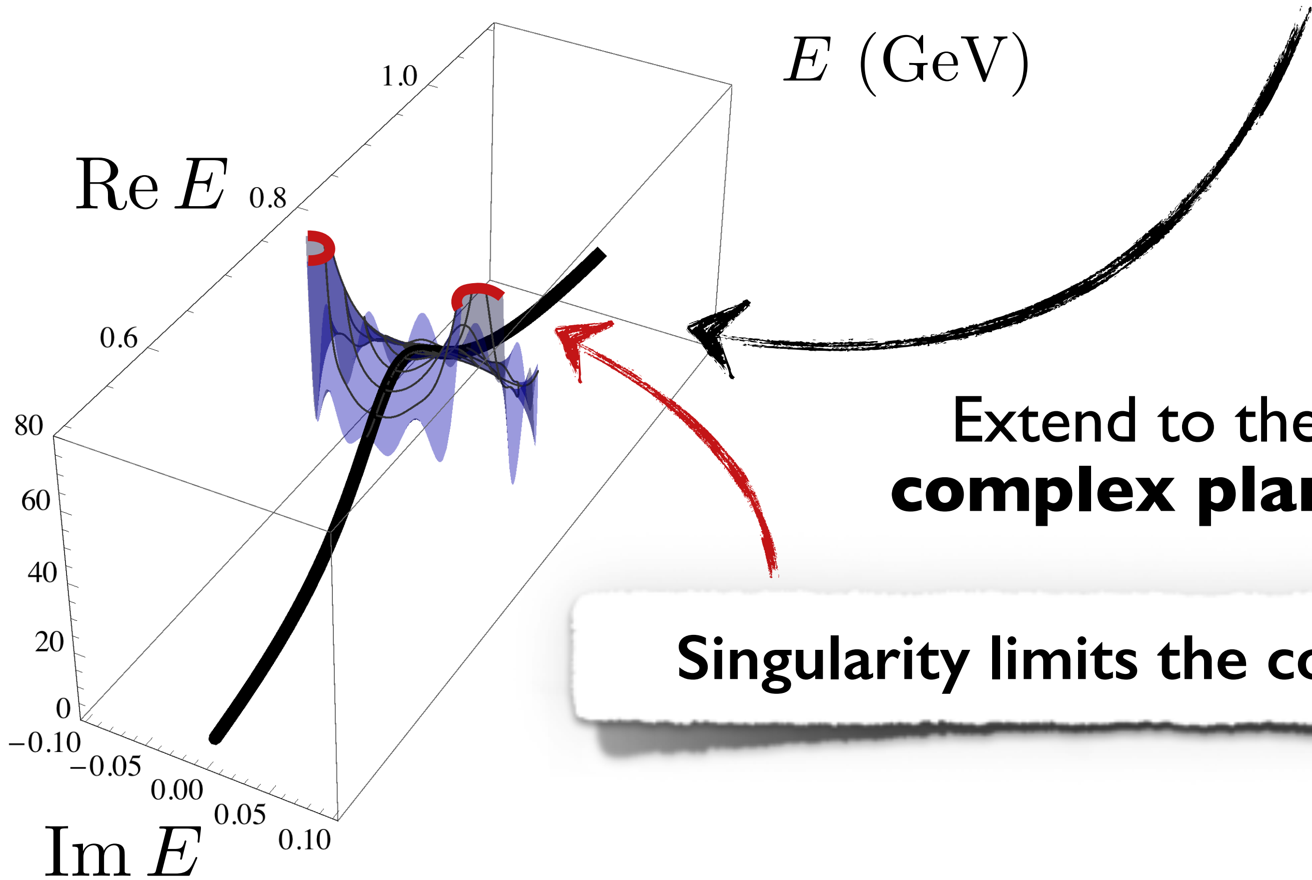
Extracting resonance properties

- Roughly speaking, a bump in: $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

scattering rate — unitarity relation







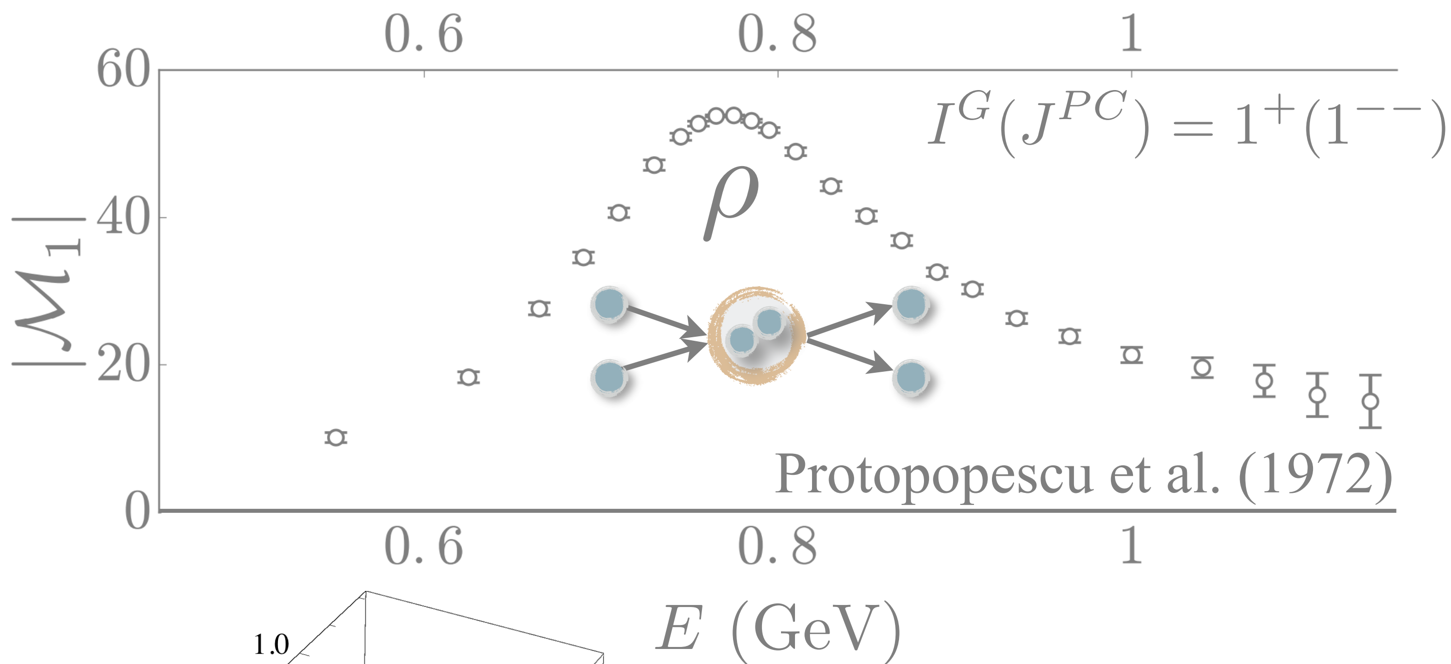
Extend to the **complex plane**

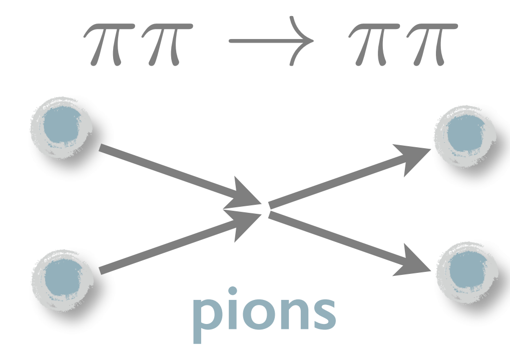
Singularity limits the convergence

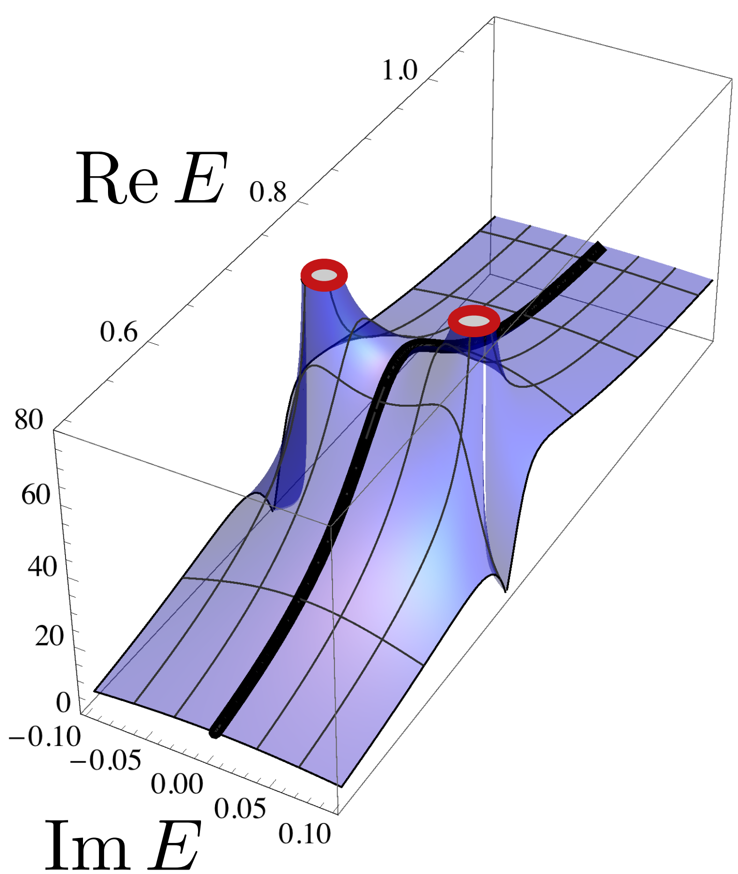
Extracting resonance properties

- Roughly speaking, a bump in: $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2 \delta(E)$

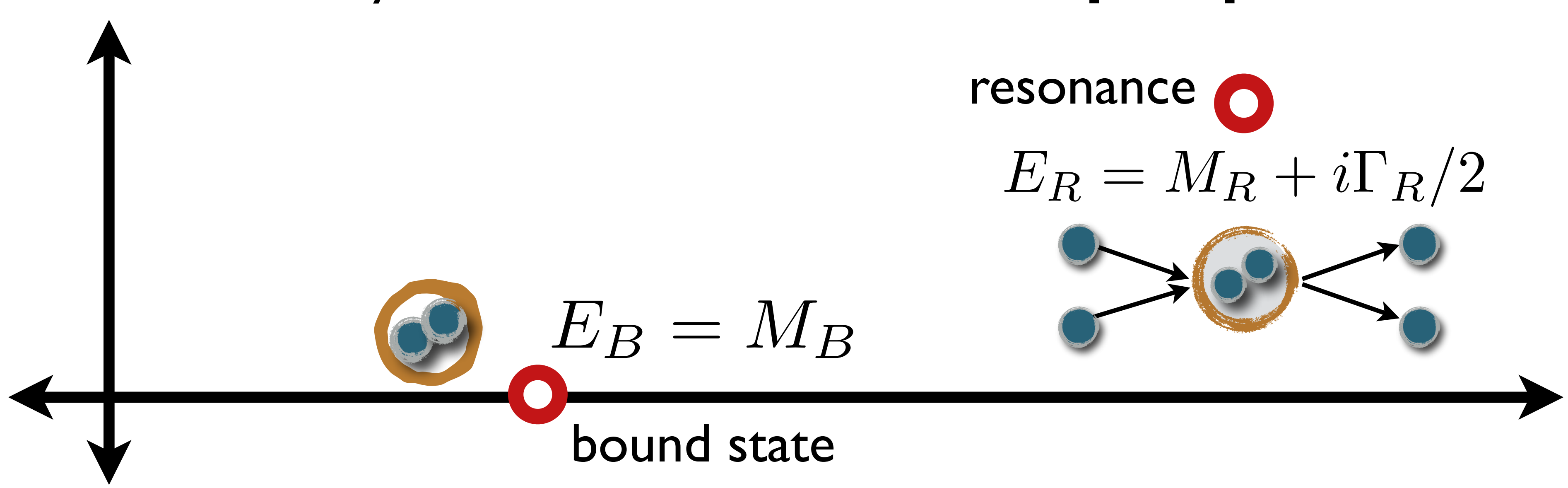
scattering rate | unitarity relation







Analytic continuation reveals a **complex pole**



Riemann sheets

- Most useful to analytically continue the scattering ***amplitude***

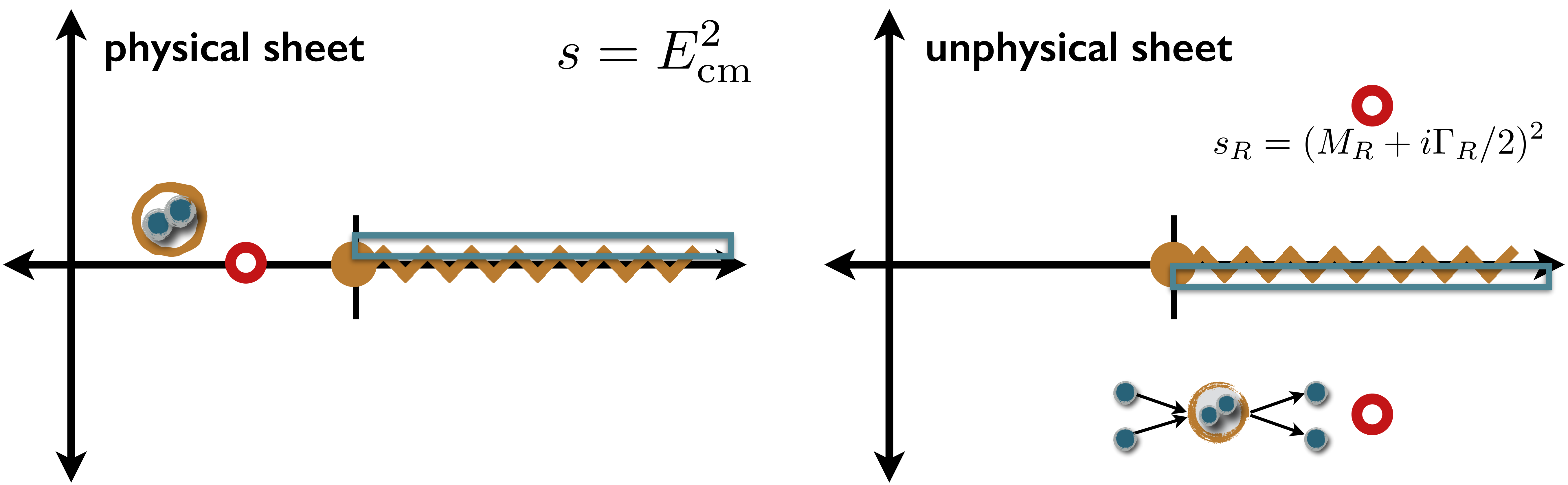
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} + \rho(s)} \qquad \rho(s) \propto -i\sqrt{s-(2M_\pi)^2}$$

Riemann sheets

- Most useful to analytically continue the scattering ***amplitude***

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} + \rho(s)} \qquad \rho(s) \propto -i\sqrt{s-(2M_\pi)^2}$$

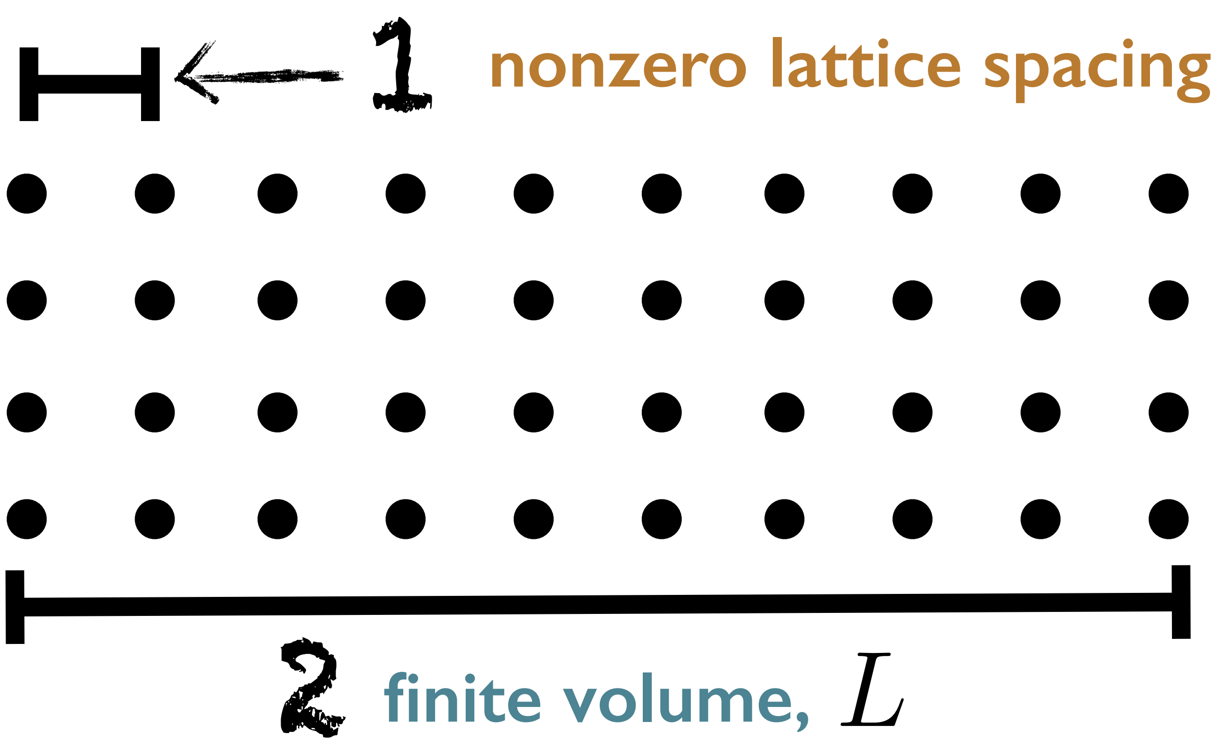
- Each channel generates a *square-root cut* → ***doubles*** the number of sheets

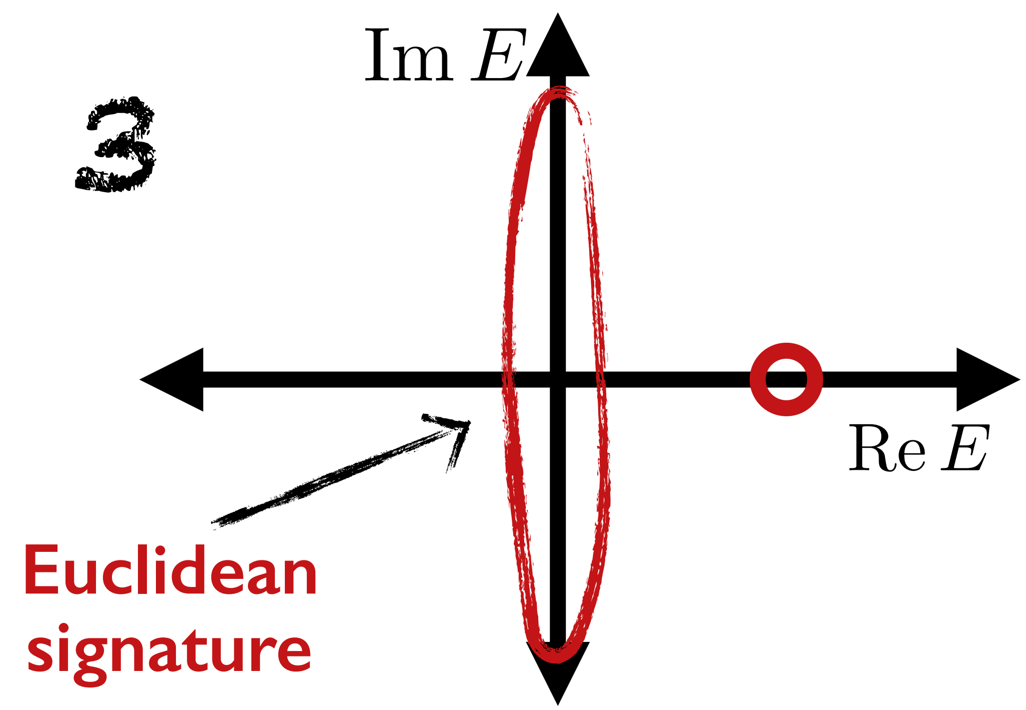


Lattice QCD

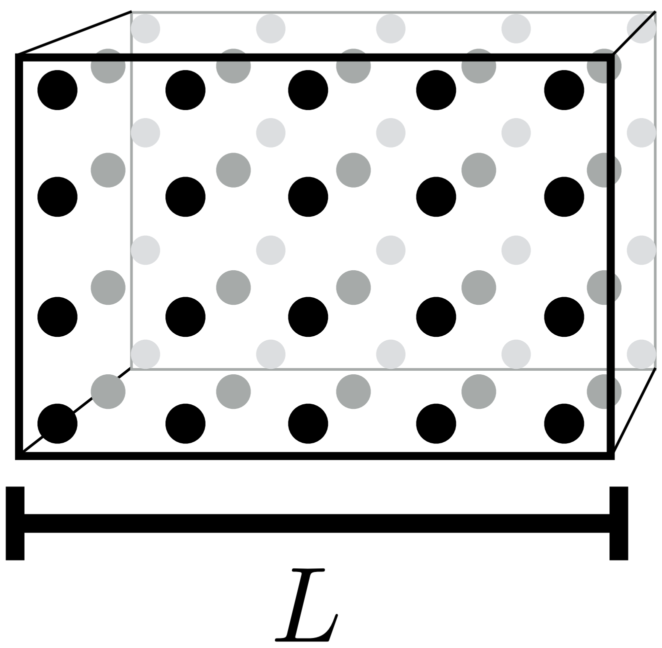
$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make **three modifications**



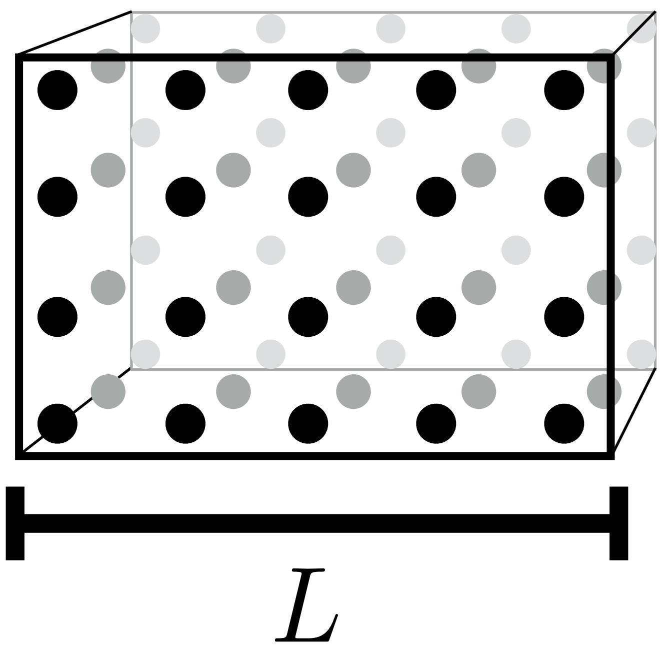


Difficulties for multi-hadron observables



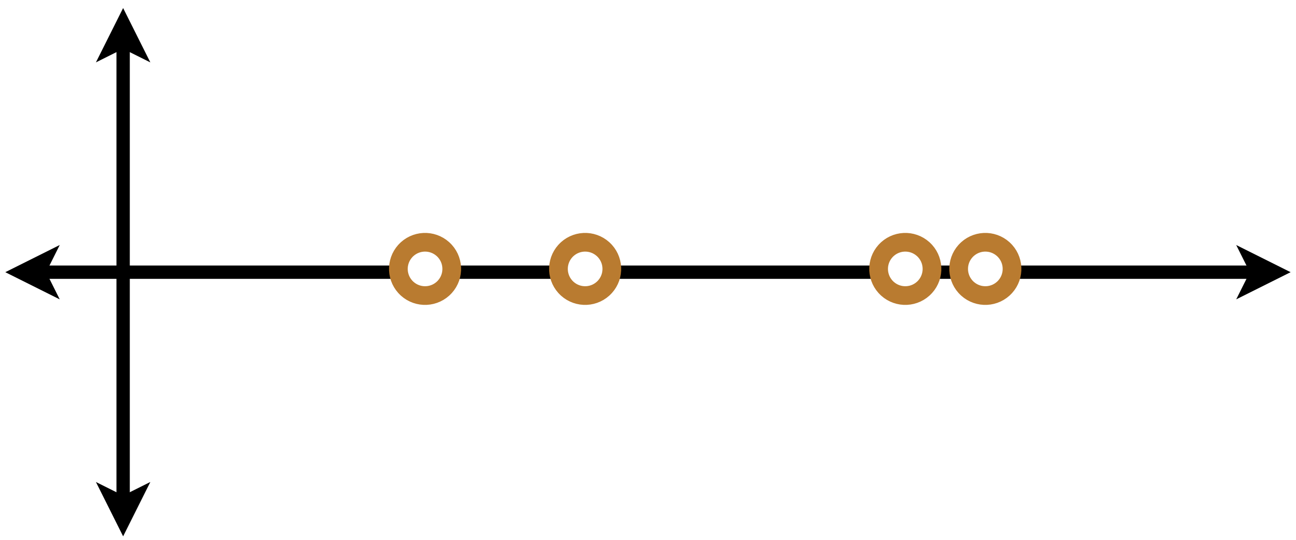
- The **finite volume**…
  - **Discretizes** the spectrum
  - **Eliminates** the branch cuts and extra sheets
  - **Hides** the resonance poles

Difficulties for multi-hadron observables

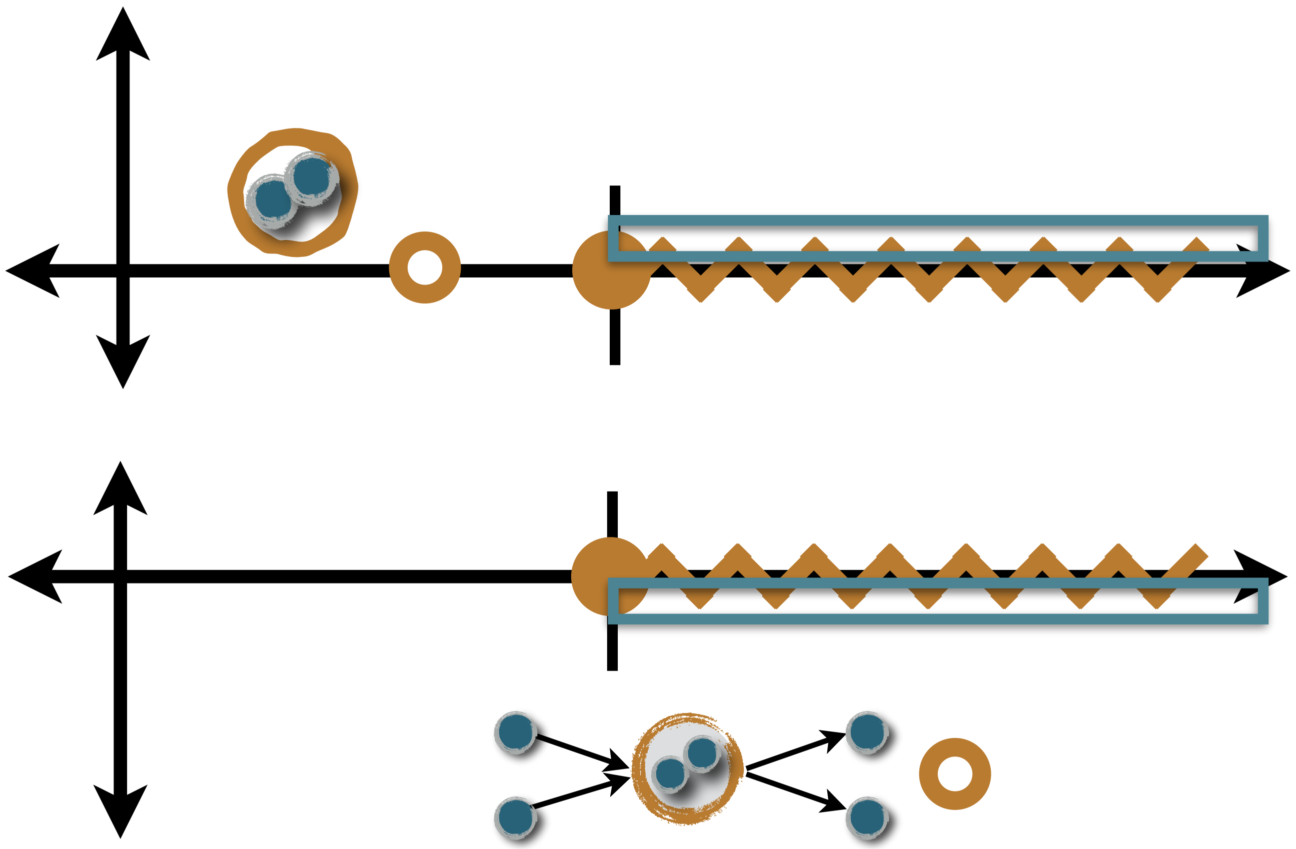


- The **finite volume…**
  - **Discretizes** the spectrum
  - **Eliminates** the branch cuts and extra sheets
  - **Hides** the resonance poles

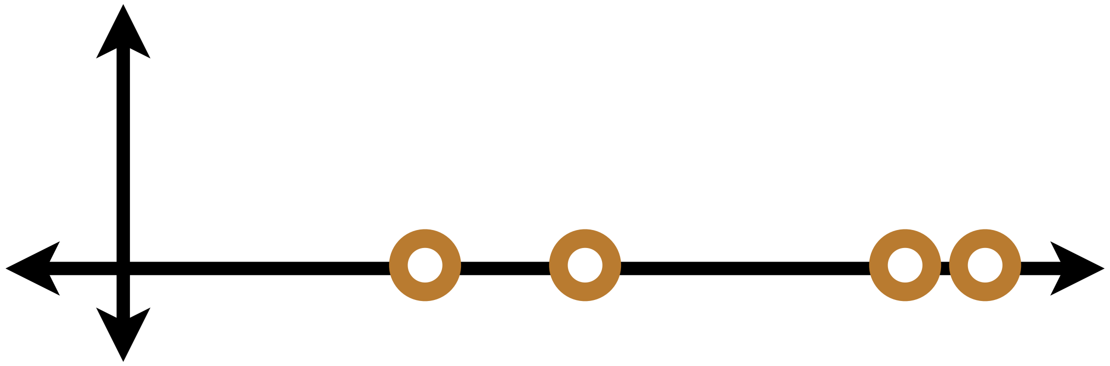
Finite-volume analytic structure



Infinite-volume analytic structure



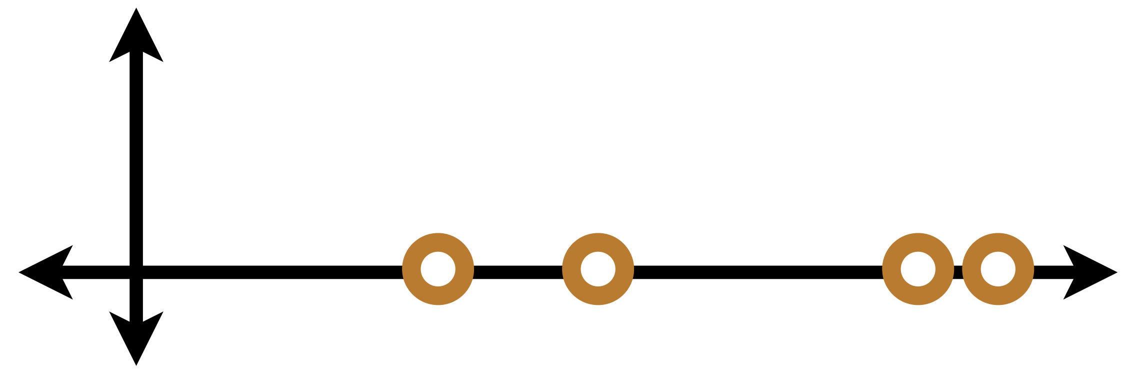
Observables available in LQCD



- LQCD → **Energies** and **matrix elements**

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z^*_{n,i}$$

Observables available in LQCD



- LQCD → **Energies** and **matrix elements**

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

- Determine optimized operators by *diagonalizing* (GEVP) - requires ***distillation***

$$\big\langle \Omega_m(\tau)\Omega_m^\dagger(0)\big\rangle \sim e^{-E_m(L)\tau} + \cdots$$

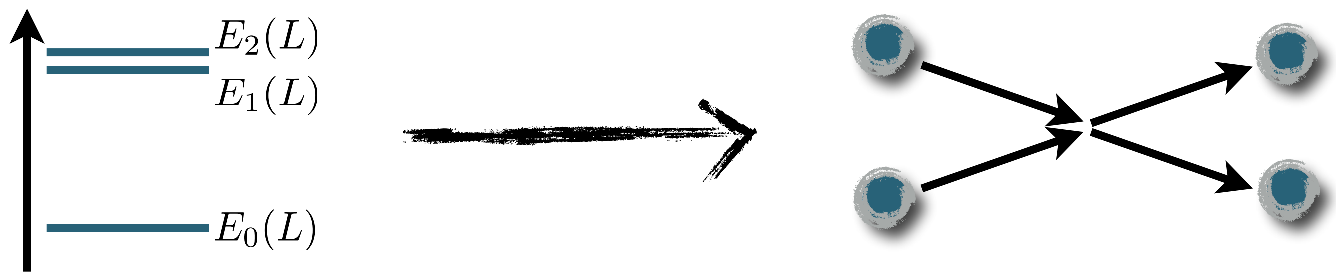
$$\big\langle \Omega_{m'}(\tau)\ \mathcal{J}(0)\ \Omega_m^\dagger(-\tau)\big\rangle \sim e^{-E_{m'}\tau}\ e^{-E_m\tau}\ \langle E_{m'}|\mathcal{J}(0)|E_m\rangle + \cdots$$

- Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to **experimental observables**

Multi-hadron processes from LQCD

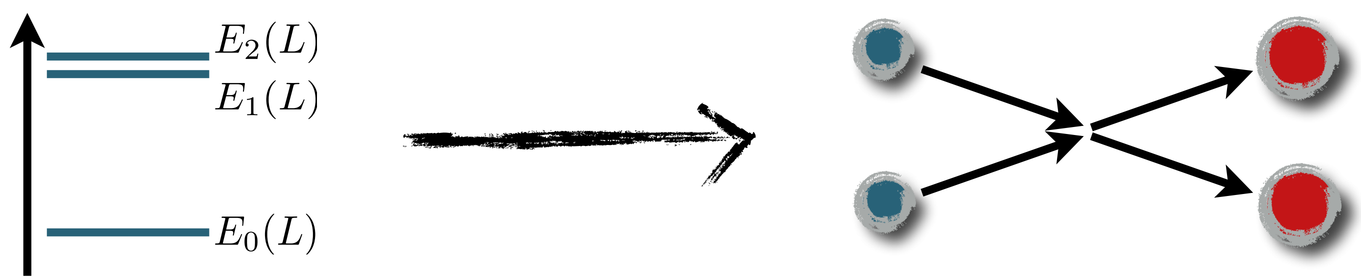
KEY IDEA: **We can use the finite volume as a tool to extract multi-hadron observables**

- **Single-channel two-to-two scattering**



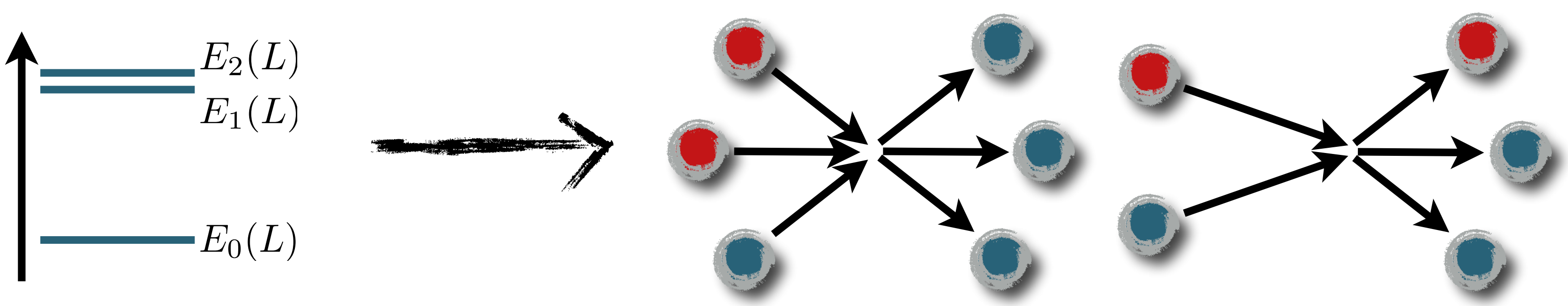
W2 - Fri - *Aaron*
W3 - Thurs - *Antonio, Dave*

- **Coupled-channel two-to-two scattering + spin**



Wed - *Jo*
Thurs - *Christopher, Antoni*

- **Two-to-three and three-to-three scattering**

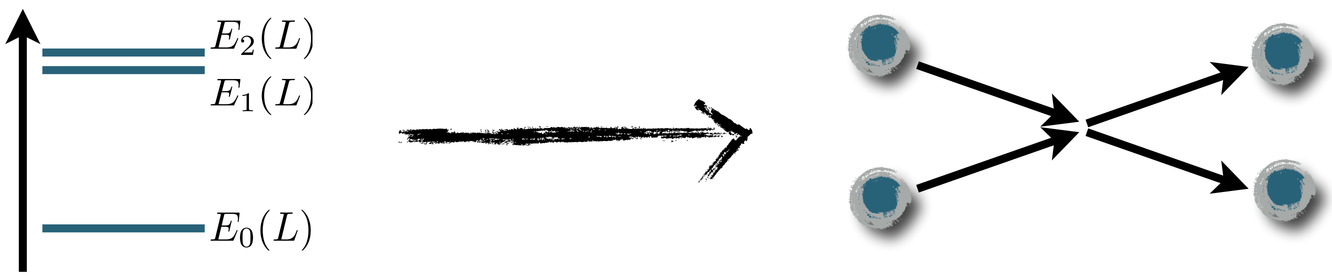


Today - *Steve*

Multi-hadron processes from LQCD

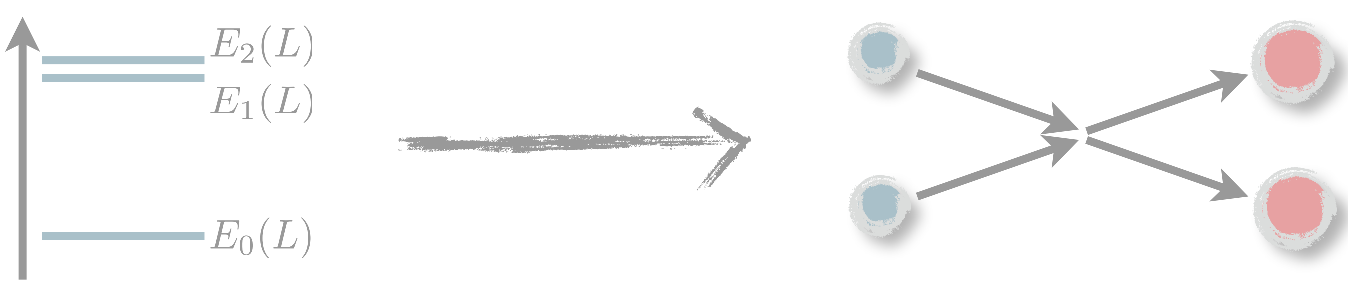
KEY IDEA: We can use the finite volume as a **tool** to extract multi-hadron observables

- ☐ **Single-channel two-to-two scattering**



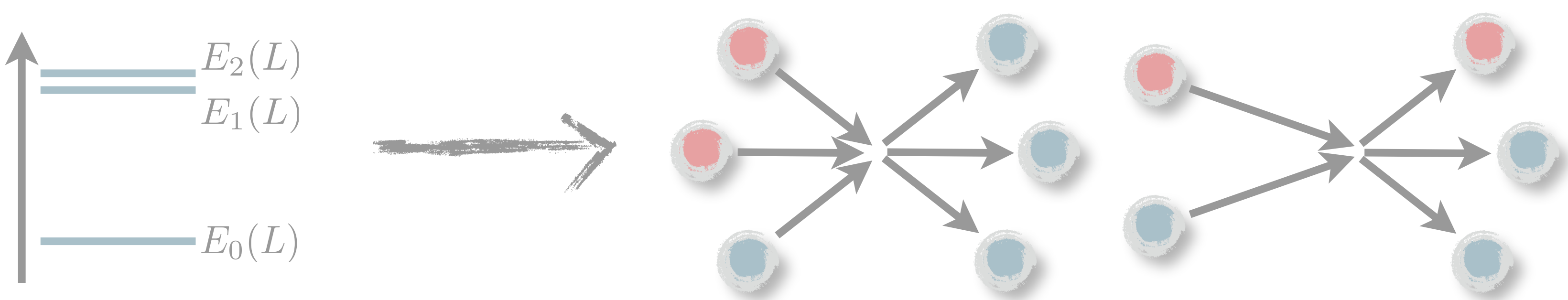
W2 - Fri - *Aaron*
W3 - Thurs - *Antonio, Dave*

- ☐ **Coupled-channel two-to-two scattering + spin**



Wed - *Jo*
Thurs - *Christopher, Antoni*

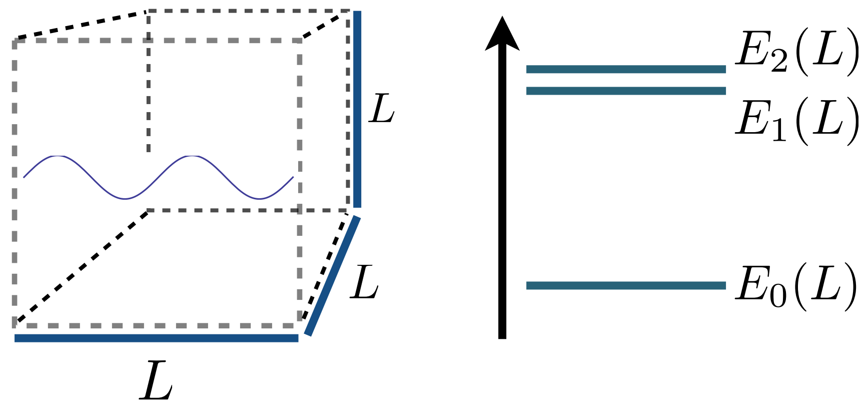
- ☐ **Two-to-three and three-to-three scattering**



Today - *Steve*

The finite-volume as a tool

- Finite-volume set-up



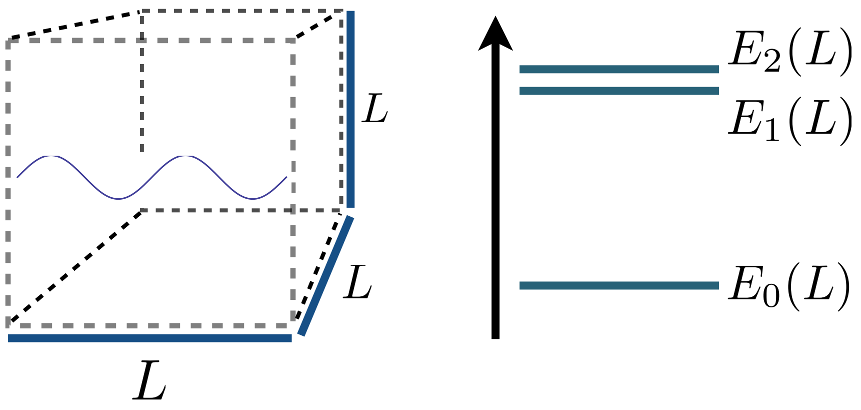
- **cubic**, spatial volume (extent *L*)
- **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$

The finite-volume as a tool

- Finite-volume set-up



- **cubic**, spatial volume (extent *L*)
- **periodic**

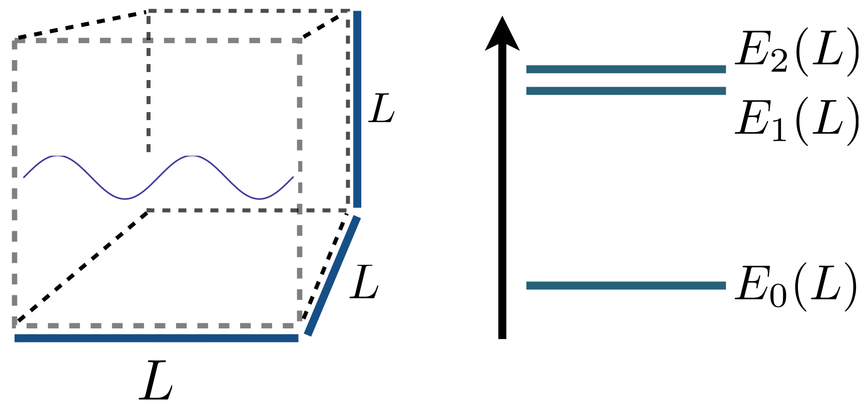
$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$

- Scattering leaves an ***imprint*** on finite-volume quantities

The finite-volume as a tool

- Finite-volume set-up

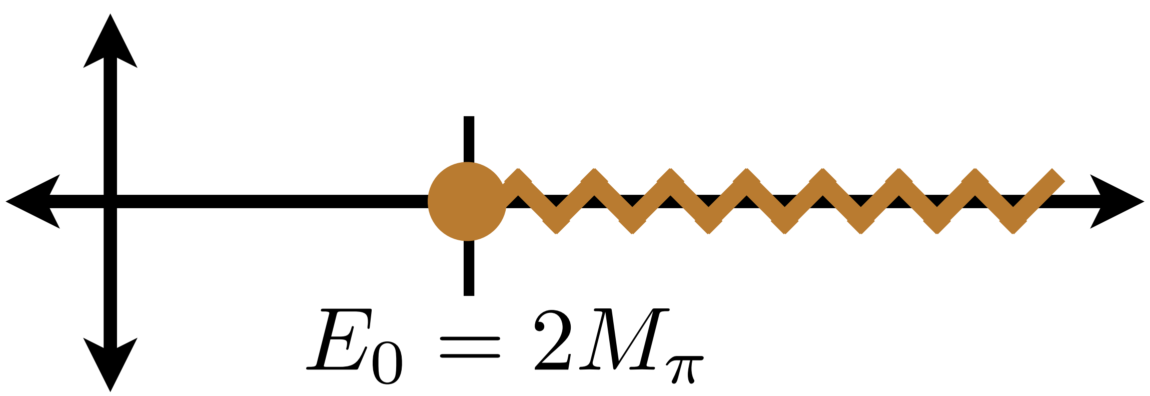


- **cubic**, spatial volume (extent *L*)
- **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$

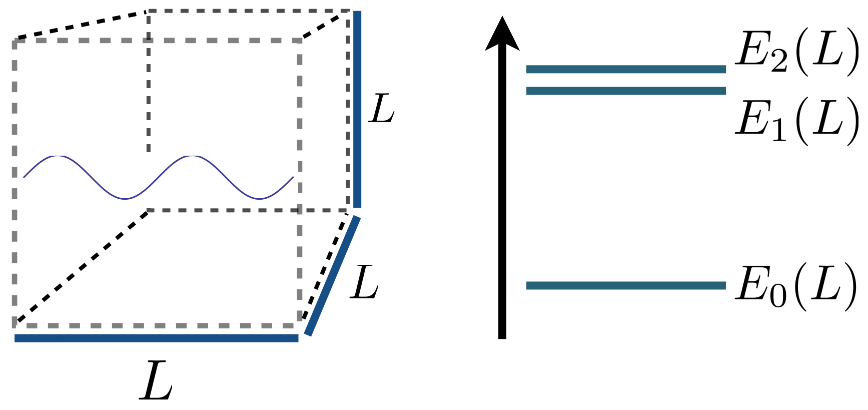
- Scattering leaves an ***imprint*** on finite-volume quantities



$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

The finite-volume as a tool

- Finite-volume set-up

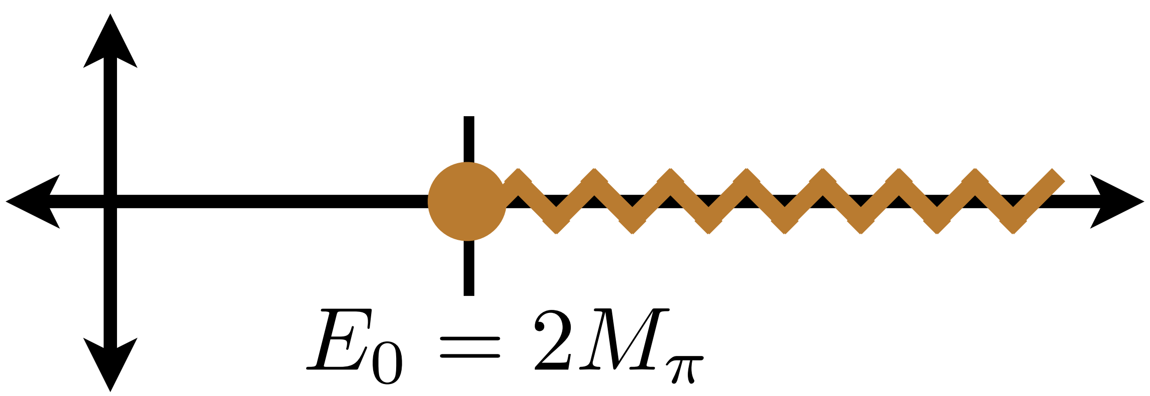


- cubic, spatial volume (extent *L*)
- periodic

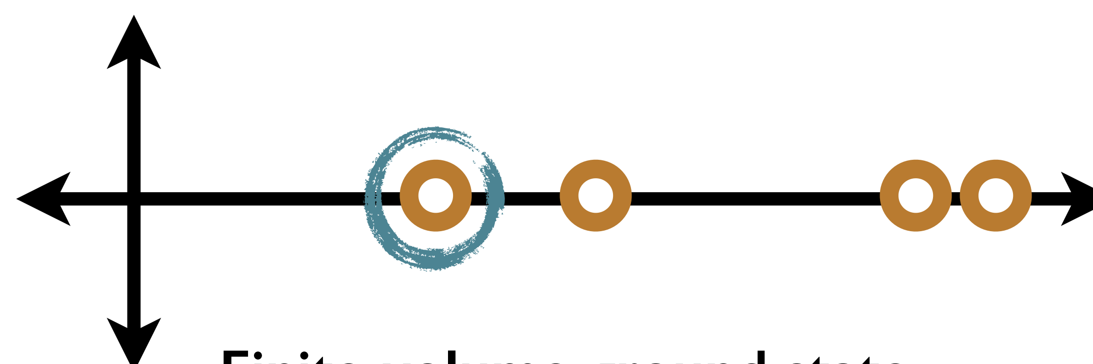
$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$

- Scattering leaves an ***imprint*** on finite-volume quantities



Infinite-volume ground state



Finite-volume ground state

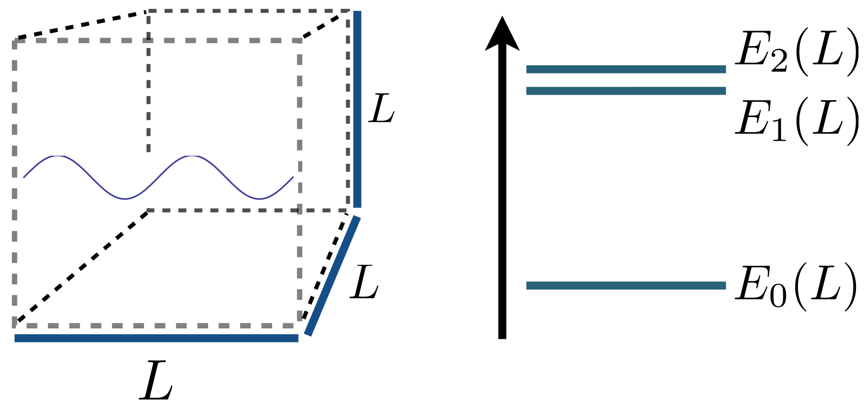
$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

Huang, Yang (1958)

The finite-volume as a tool

- Finite-volume set-up

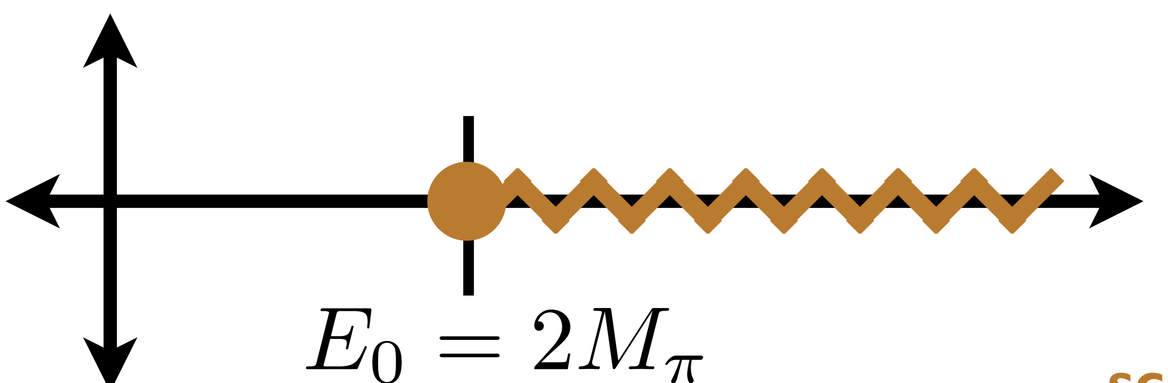


- cubic, spatial volume (extent *L*)
- periodic

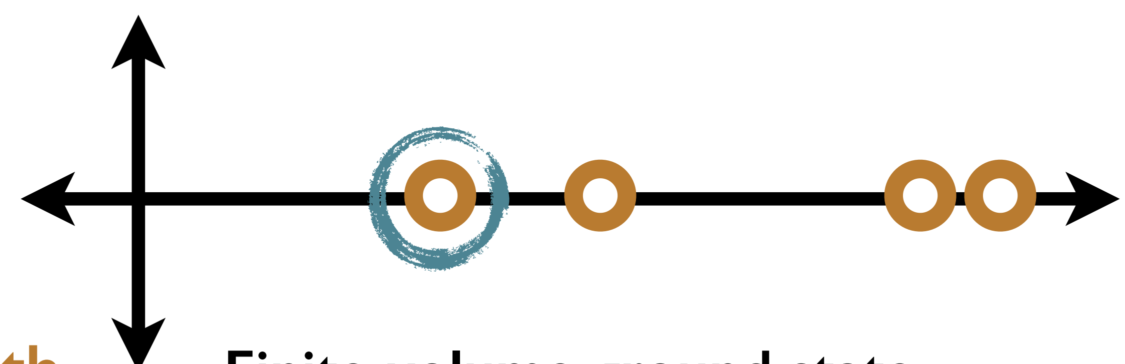
$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$

- Scattering leaves an ***imprint*** on finite-volume quantities



Infinite-volume ground state



Finite-volume ground state

scattering length

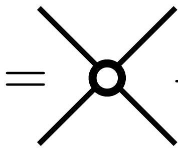
$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

Huang, Yang (1958)

Hint of the derivation

- In the infinite-volume world…

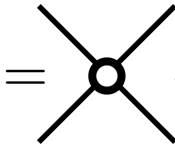
$$\mathcal{M}_{\ell=0}(E_{\text{cm}}) =$$  $$+ \cdots = -\lambda + \mathcal{O}(\lambda^2)$$

← **Low-energy degrees of freedom**
(e.g. pions in QCD)

Hint of the derivation

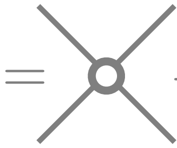
- In the infinite-volume world…

scattering length

$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$  $+ \cdots = -\lambda + \mathcal{O}(\lambda^2)$ $\longrightarrow$ $a = \dfrac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$

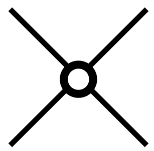
Hint of the derivation

- In the infinite-volume world…

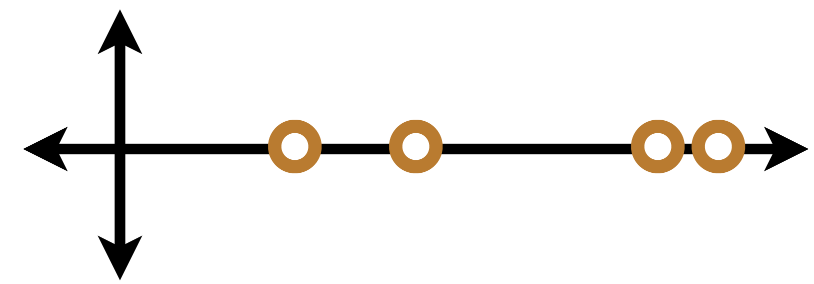
$\mathcal{M}_{\ell=0}(E_{\text{cm}}) =$  $+ \cdots = -\lambda + \mathcal{O}(\lambda^2)$ $\longrightarrow$ $a = \dfrac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$

scattering length

- In the finite-volume world…

$\mathcal{M}_L(E_{\text{cm}}) =$  $+ \cdots$

$= -\lambda + \cdots$

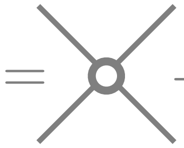


Leading order → no poles

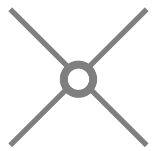
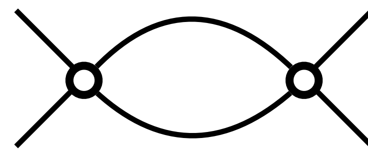
Hint of the derivation

- In the infinite-volume world…

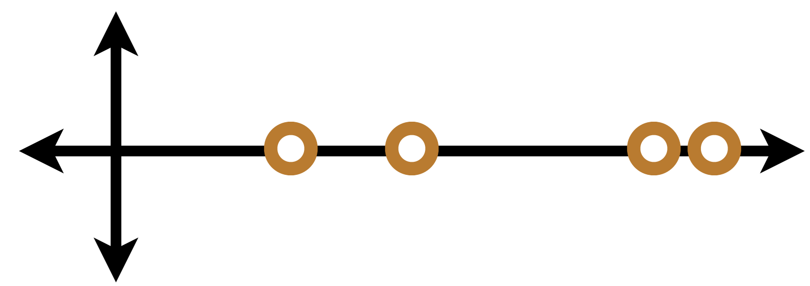
scattering length

$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$  $+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \dfrac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$

- In the finite-volume world…

$\mathcal{M}_L(E_{\rm cm}) =$  $+$  $+ \cdots$

$$= -\lambda - \lambda \frac{1}{2}\frac{1}{L^3}\sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2(E_{\rm cm} - 2\omega_{\mathbf{k}})}\lambda + \cdots$$



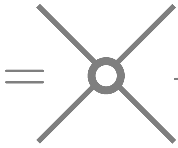
Leading order → no poles

Next-to-leading order → poles of two non-interacting particles $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M_\pi^2}$ $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$

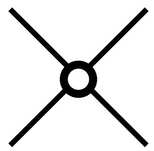
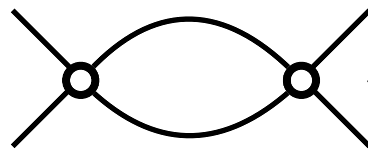
Hint of the derivation

- In the infinite-volume world…

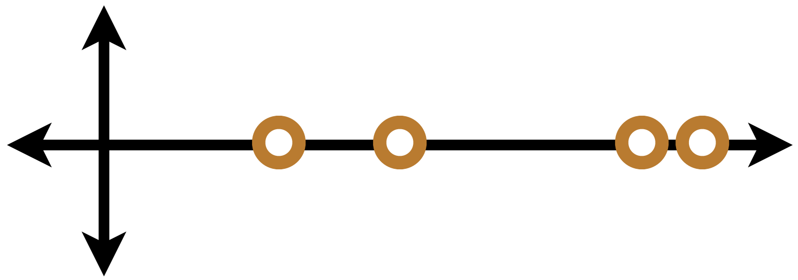
scattering length

$$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$$  $$+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

- In the finite-volume world…

$$\mathcal{M}_L(E_{\rm cm}) =$$  $$+$$  $$+ \cdots$$

$$= -\lambda - \lambda \frac{1}{2}\frac{1}{L^3}\sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2(E_{\rm cm} - 2\omega_{\mathbf{k}})}\lambda + \cdots$$



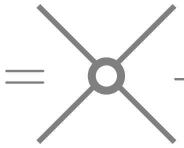
Leading order → no poles

Next-to-leading order → poles of two non-interacting particles $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M_\pi^2}$ $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$

- Expansion fails for: $E_{\rm cm} - 2M_\pi = \mathcal{O}(\lambda)$

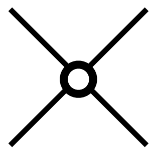
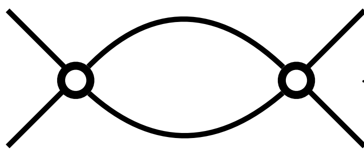
Hint of the derivation

- In the infinite-volume world…

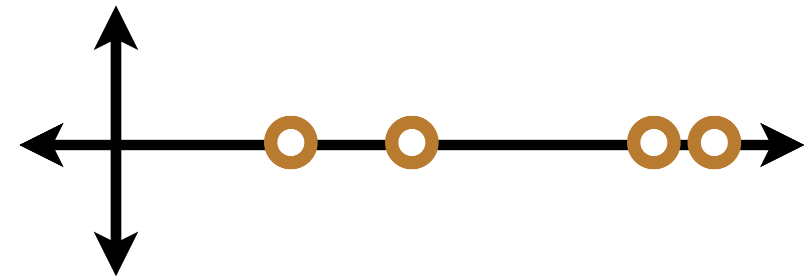
$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$  $+ \cdots = -\lambda + \mathcal{O}(\lambda^2)$ ⟶ $a = \dfrac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$

scattering length

- In the finite-volume world…

$\mathcal{M}_L(E_{\rm cm}) =$  $+$  $+ \cdots$

$$= -\lambda - \lambda \frac{1}{2}\frac{1}{L^3}\sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2 (E_{\rm cm} - 2\omega_{\mathbf{k}})} \lambda + \cdots$$



Leading order → no poles

Next-to-leading order → poles of **two non-interacting particles** $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M_\pi^2}$ $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$

- Expansion fails for: $E_{\rm cm} - 2M_\pi = \mathcal{O}(\lambda)$

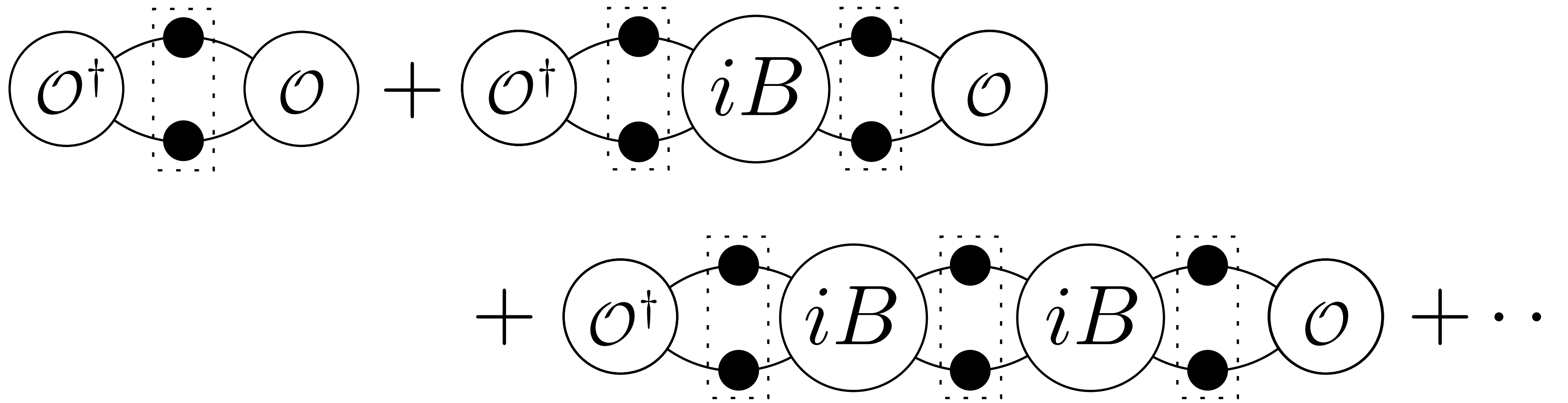
$$= -\lambda \sum_{n=0}^{\infty} [f(E_{\rm cm}, L)\ \lambda]^n = \frac{1}{-1/\lambda + f(E_{\rm cm}, L)}$$

- Physical pole recovered via re-summation

$$-1/\lambda + f(E_{\rm cm}, L) = 0 \implies E_{\rm cm} = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

Skeleton expansion derivation

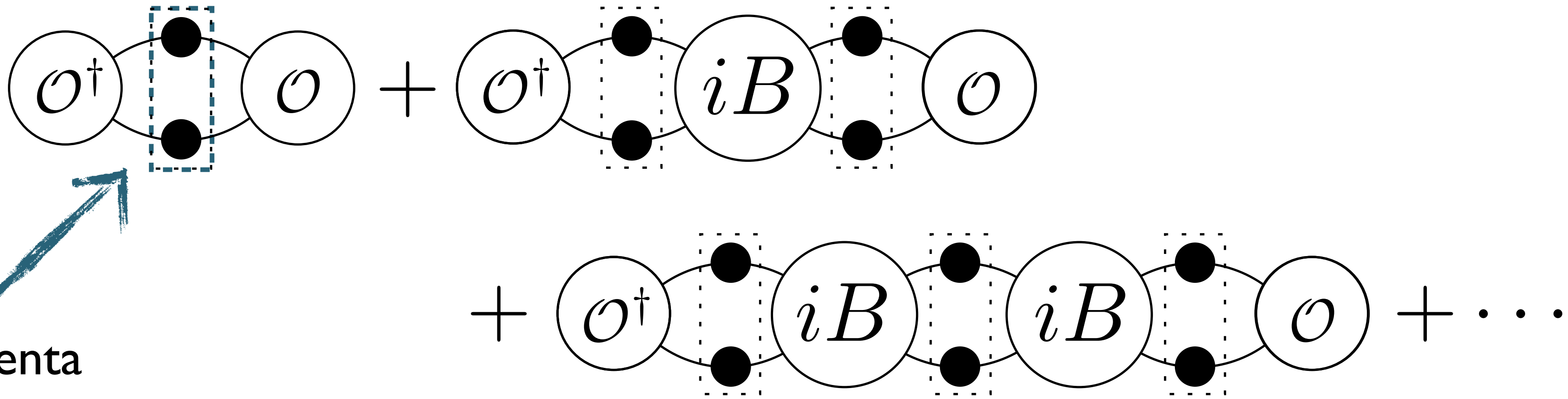
$$C_L(P) =$$



Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

Skeleton expansion derivation

$C_L(P) =$



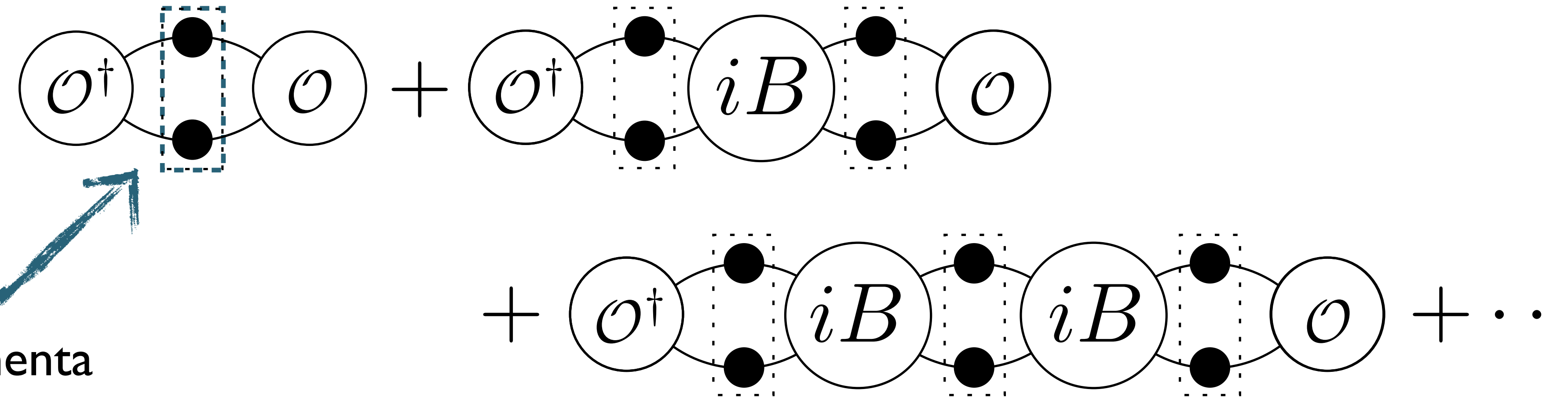
spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

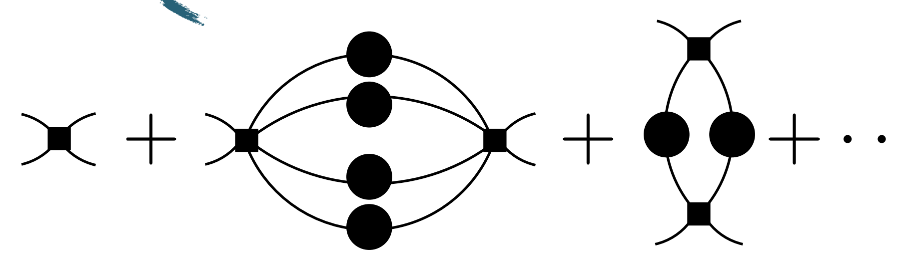
Skeleton expansion derivation

$C_L(P) =$



spatial loop momenta
are summed

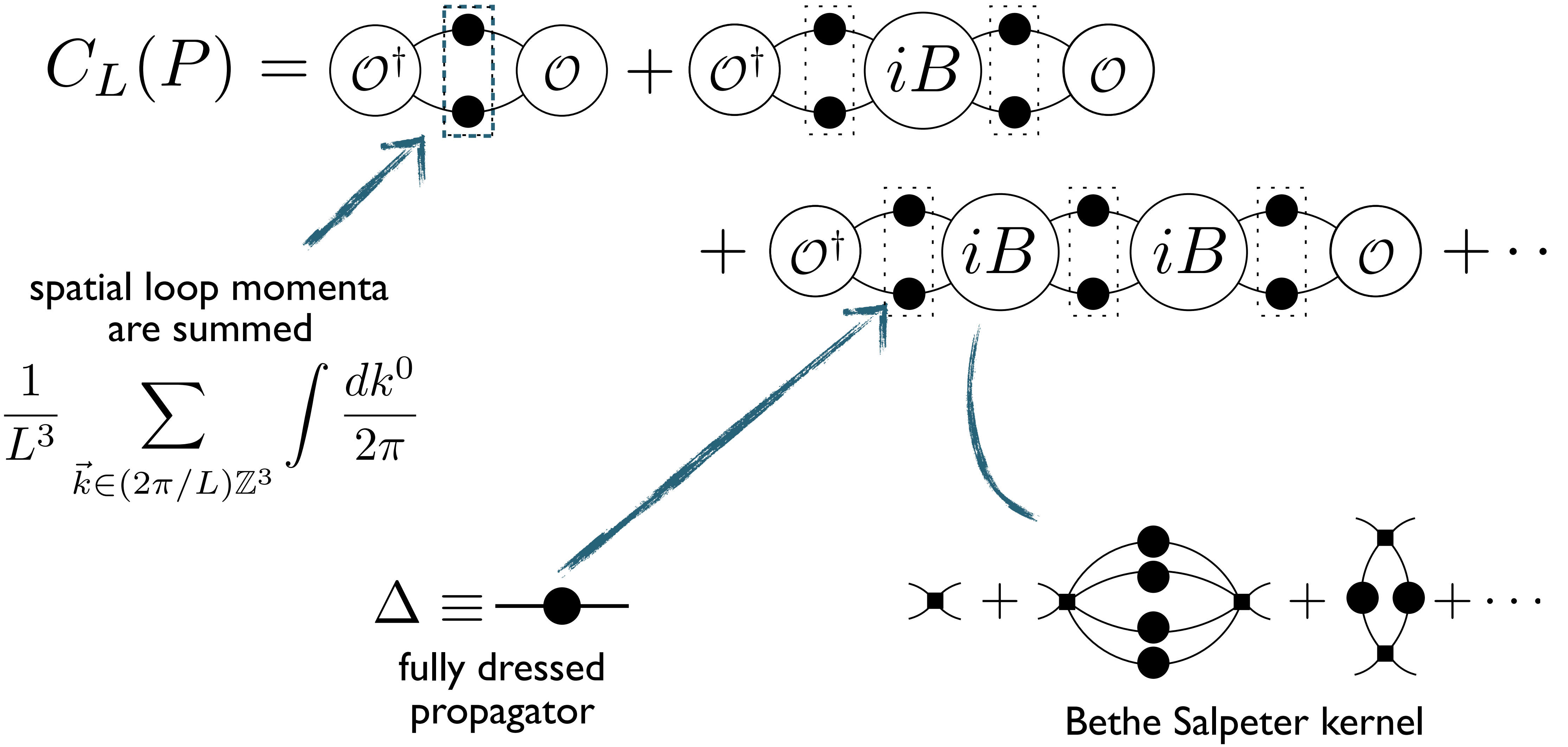
$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$



Bethe Salpeter kernel

Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

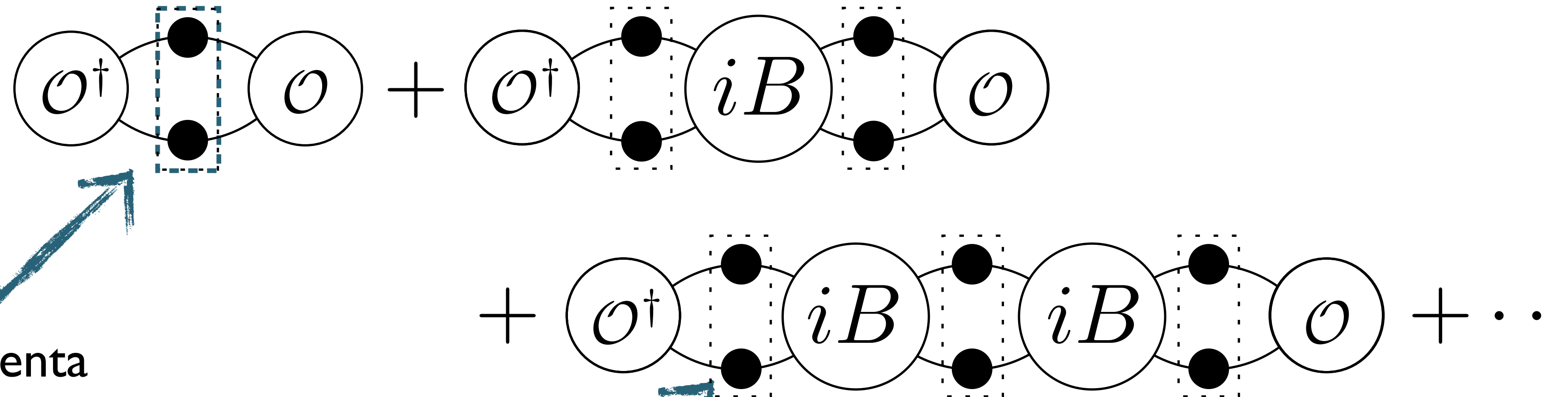
Skeleton expansion derivation



Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

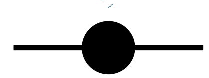
Skeleton expansion derivation

$C_L(P) =$

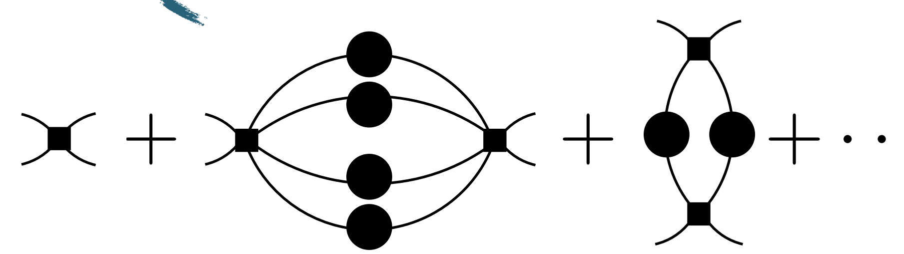


spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

$\Delta \equiv$ 

fully dressed
propagator



Bethe Salpeter kernel

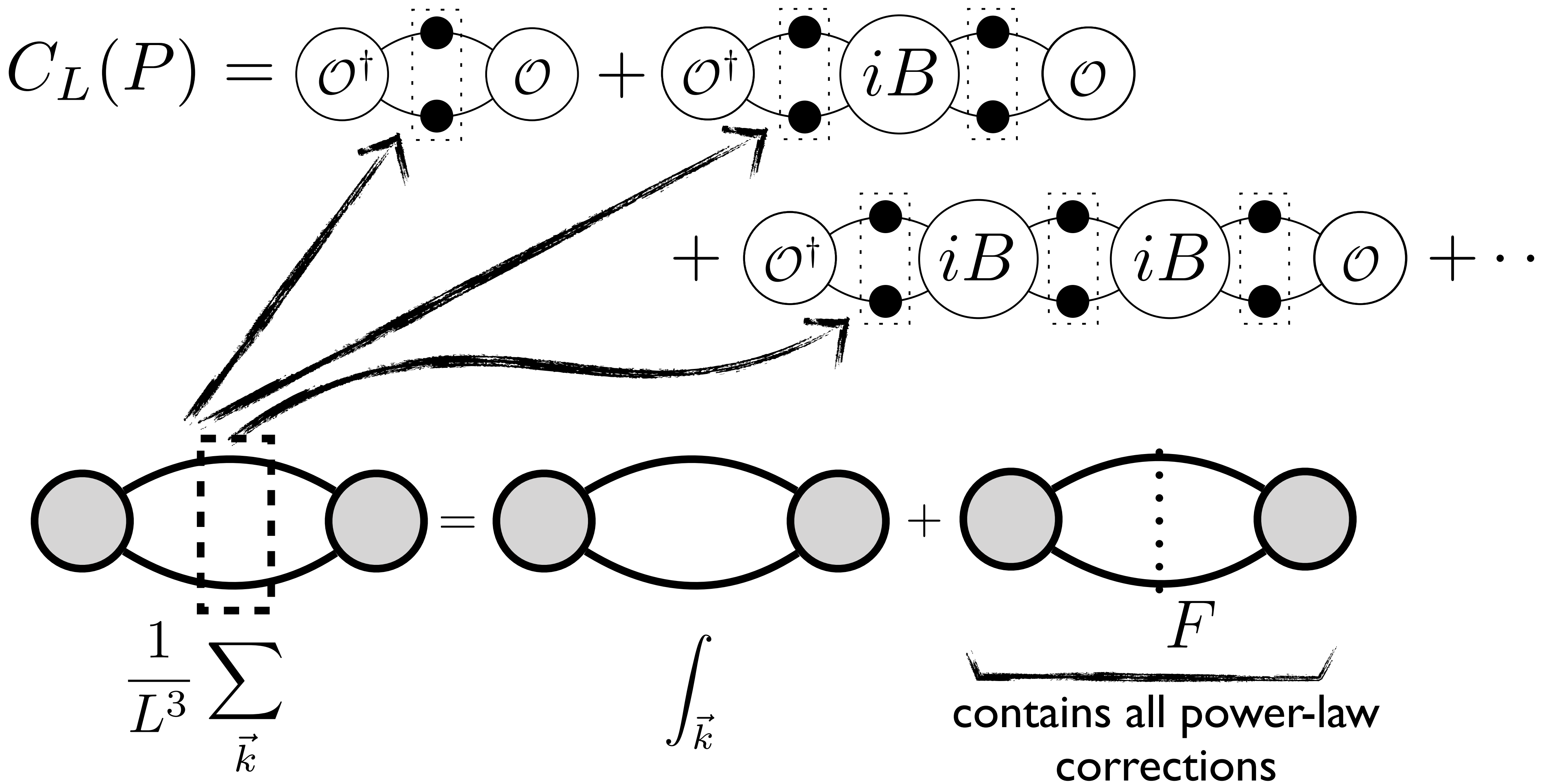
If $E^* < 4m$ **then**

$$B_L = B_\infty + \mathcal{O}(e^{-mL})$$
$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

where we define $E^{*2} = E^2 - \vec{P}^2$

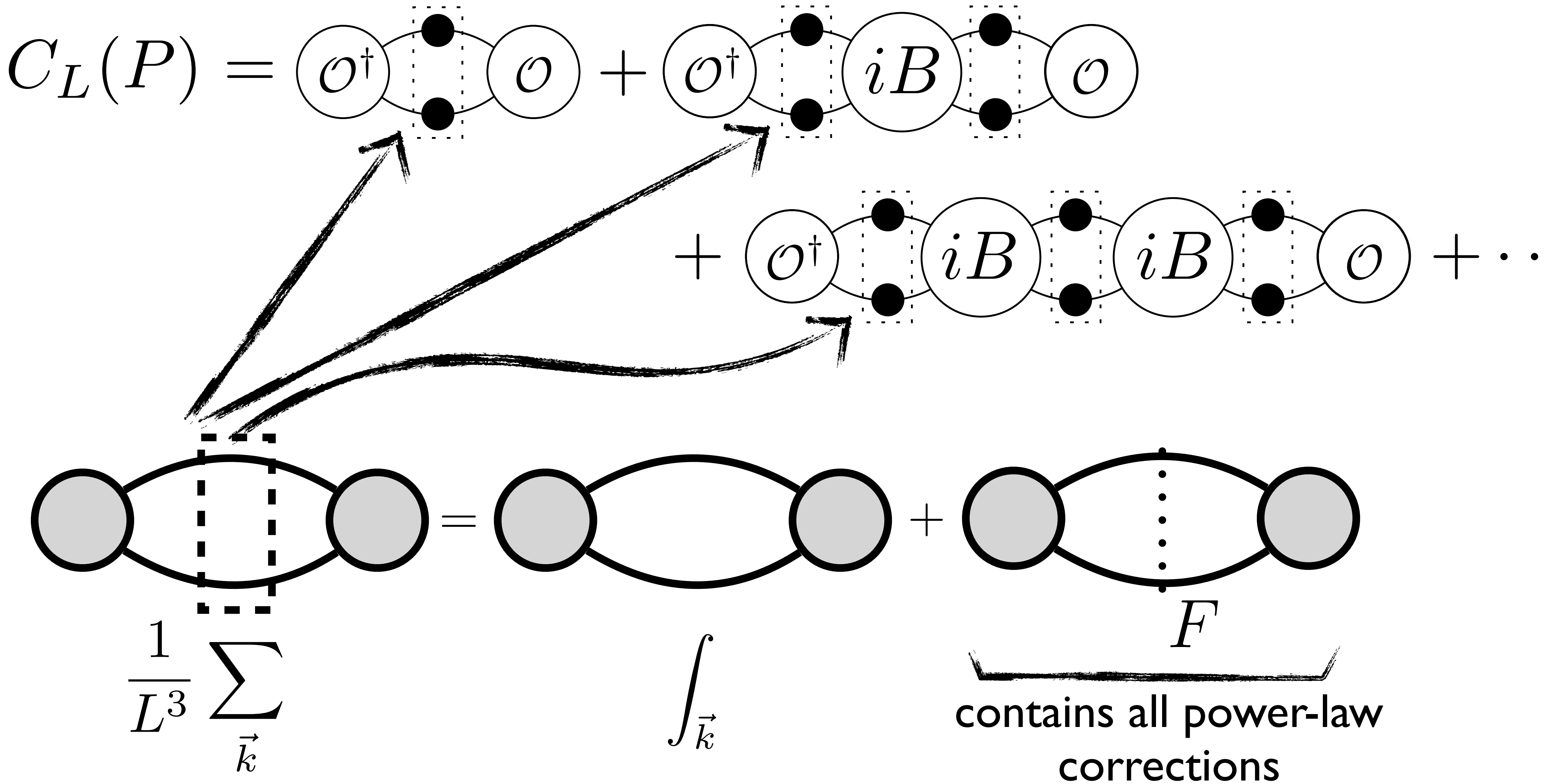
Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

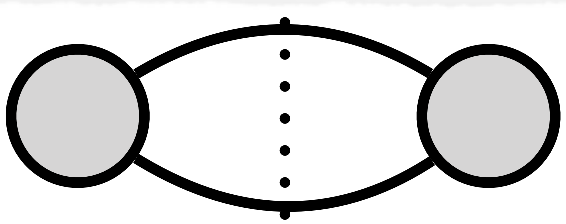
Skeleton expansion derivation



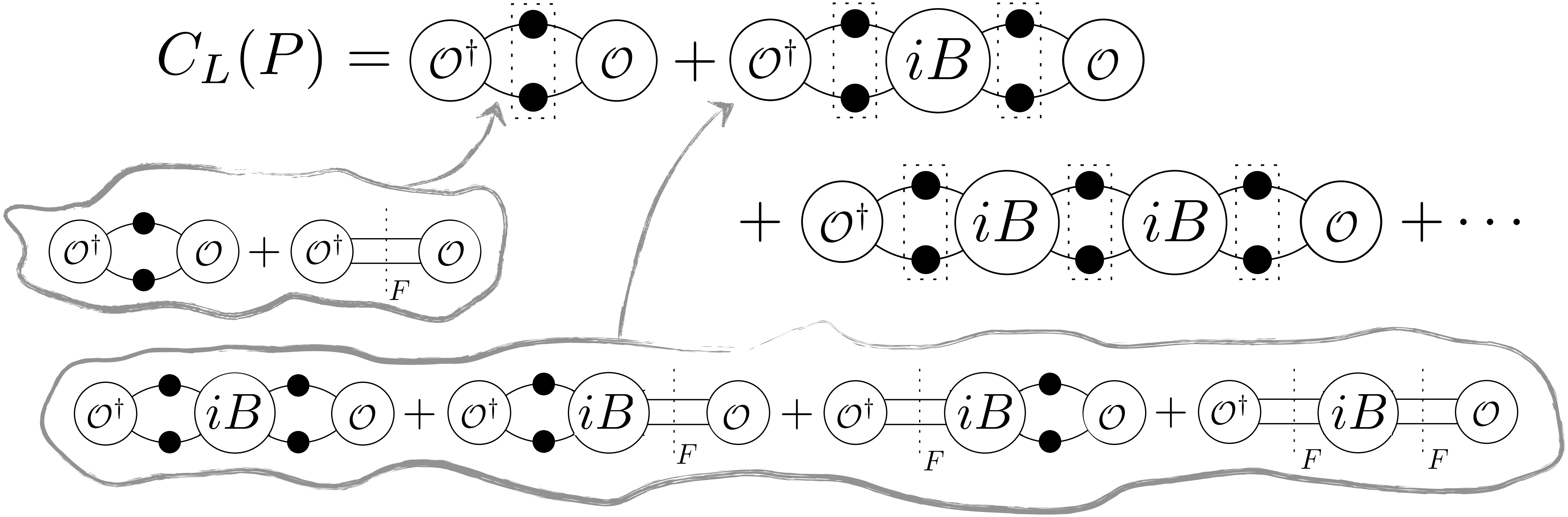
Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

Skeleton expansion derivation

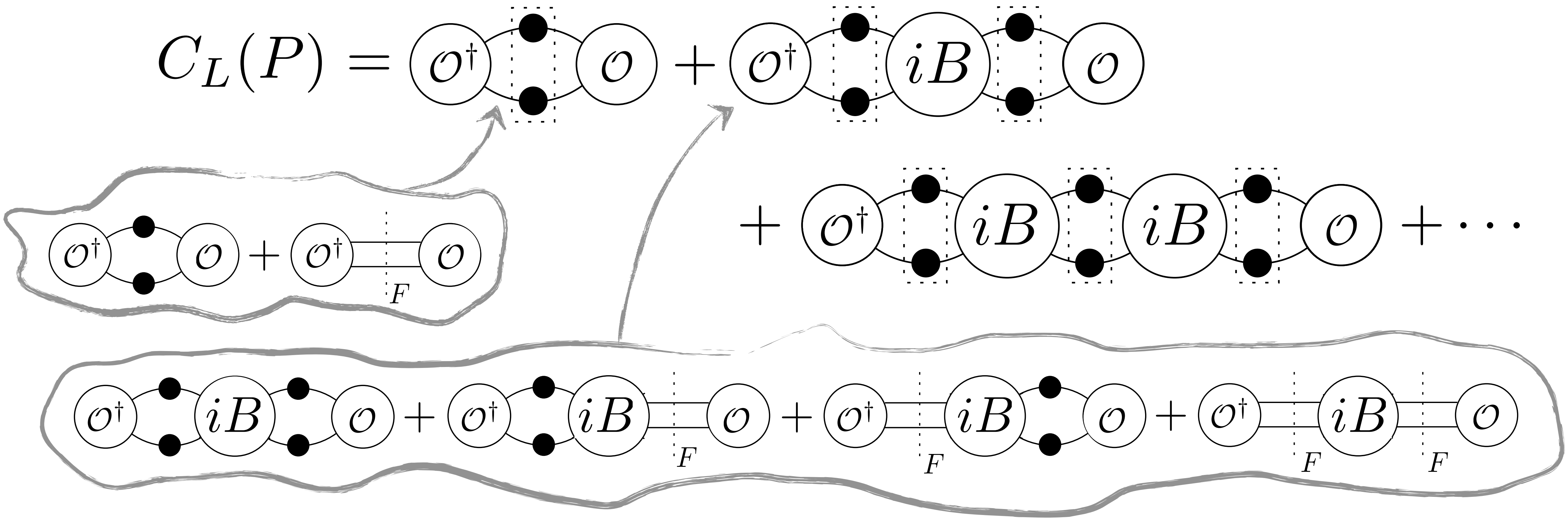


In  all four-momenta are projected on shell.

Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)



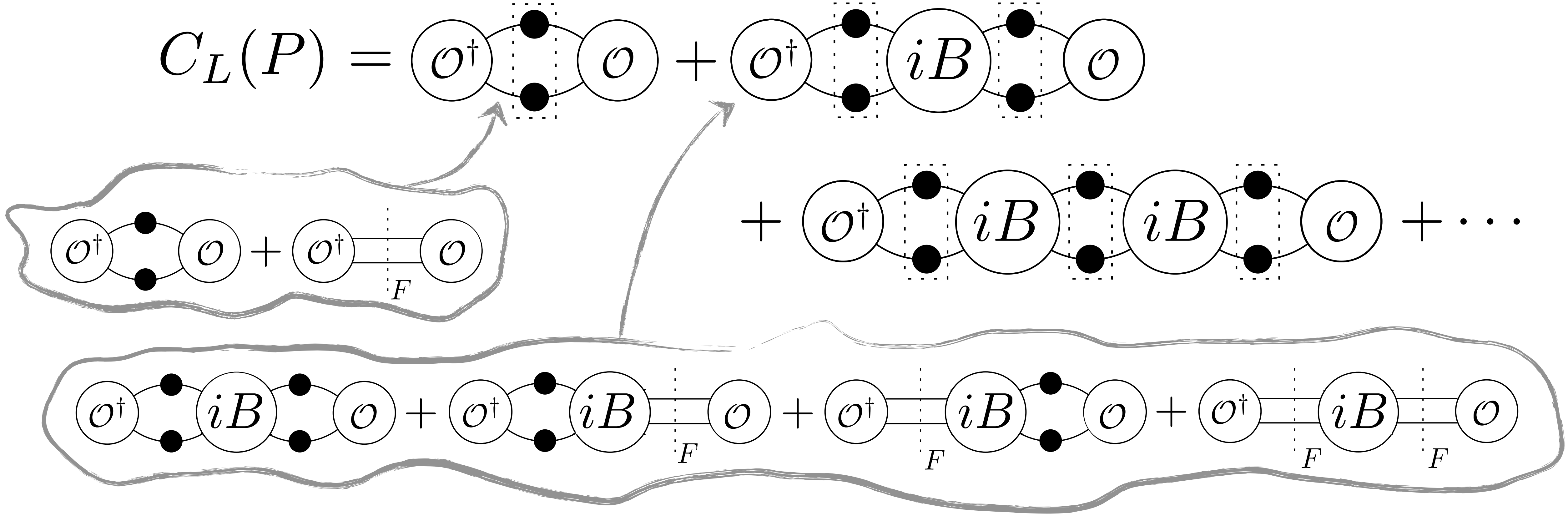
$C_L(P) =$
$\mathcal{O}^\dagger$
iB
$\mathcal{O}$
F



- Regroup by number of F cuts

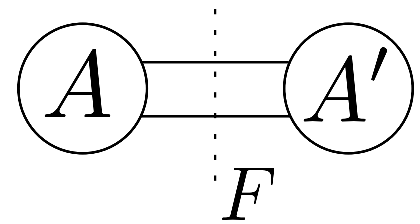
zero Fs

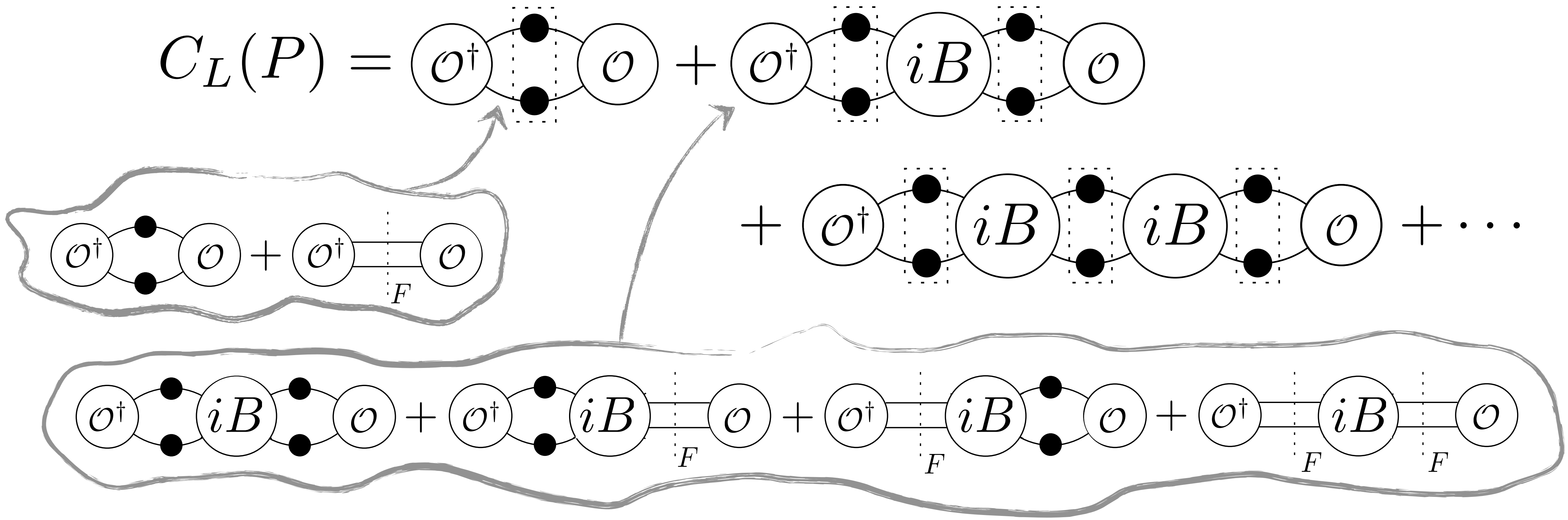
$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) +$$



- Regroup by number of F cuts

zero Fs **one F**

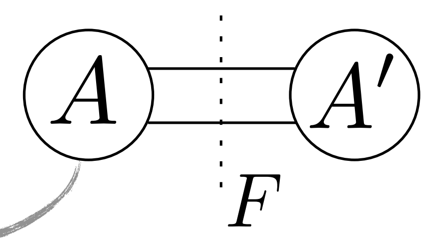
$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) +$$  $+$



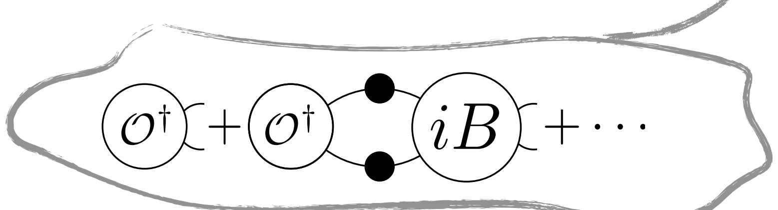
- Regroup by number of F cuts

zero Fs **one F**

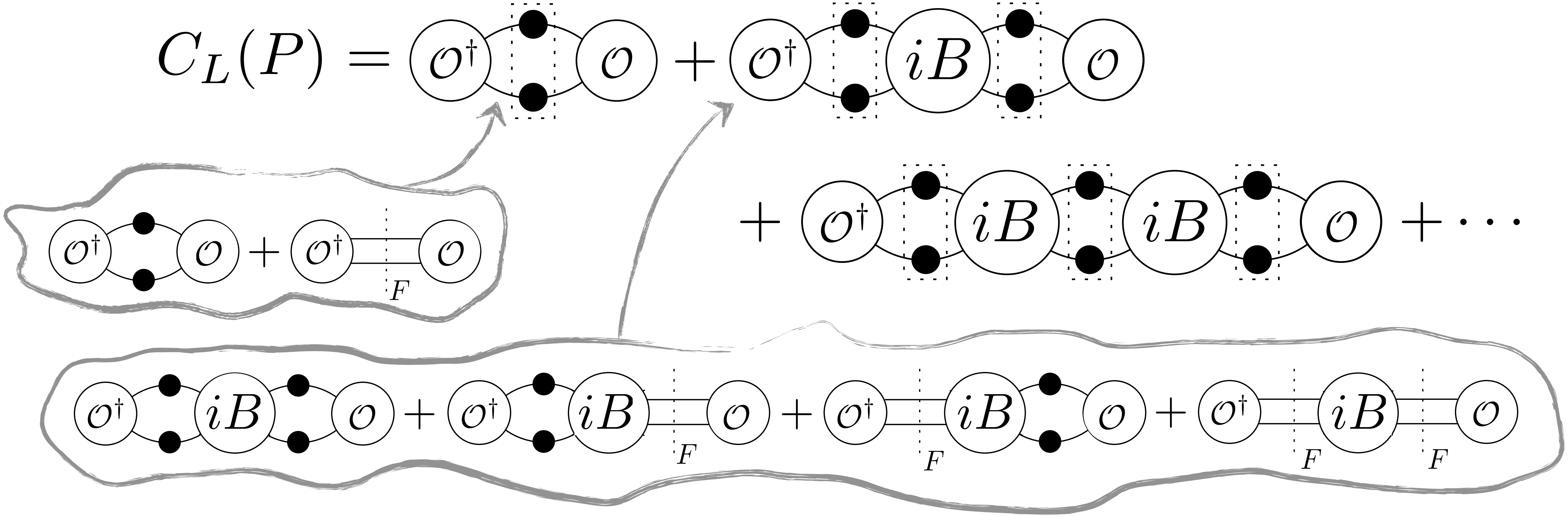
$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) +$



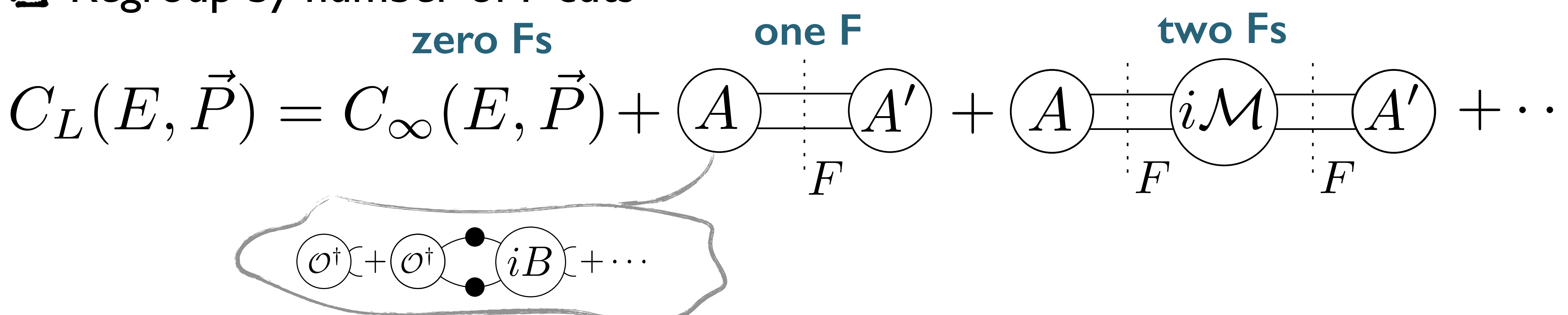
$+$



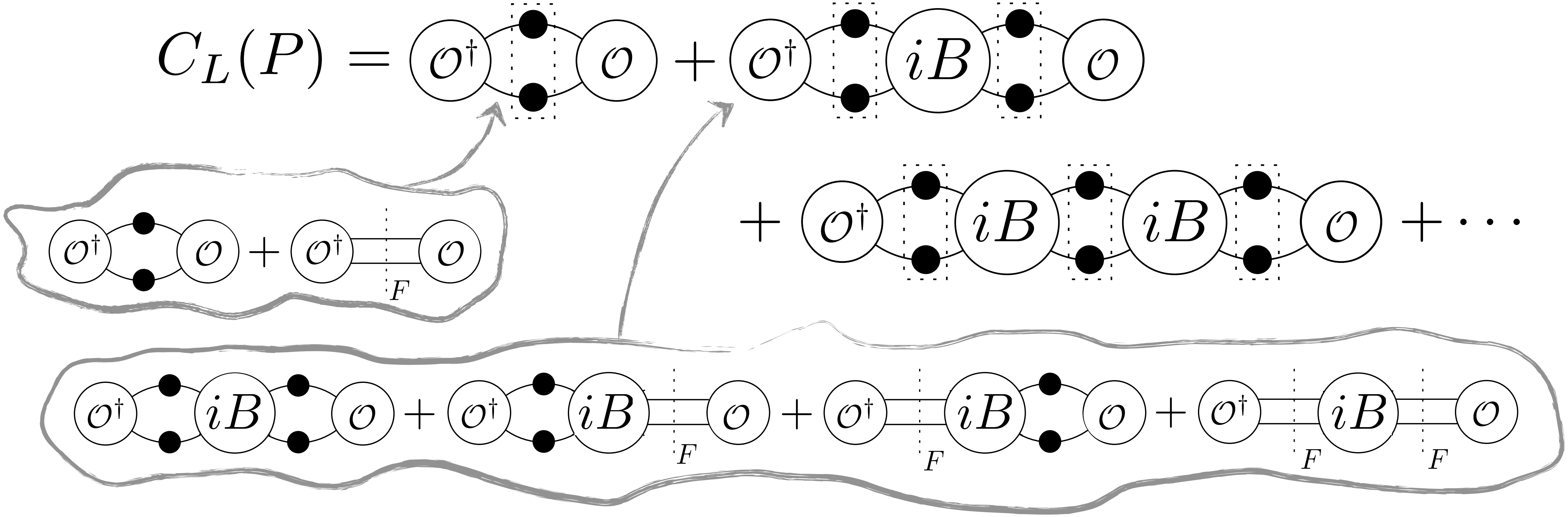
$= \langle \pi\pi, \mathrm{out} | \mathcal{O}^\dagger | 0 \rangle$



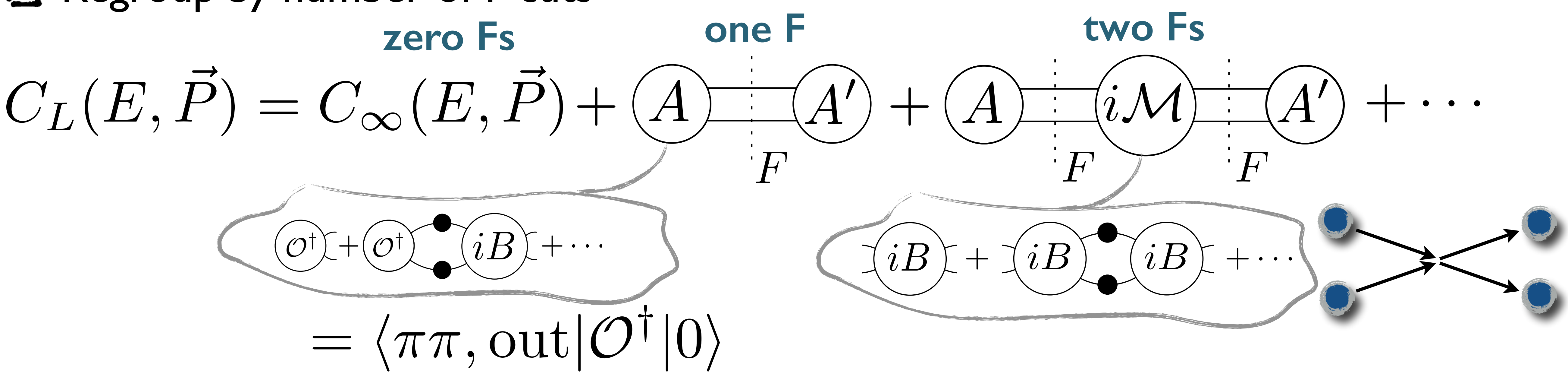
- Regroup by number of F cuts

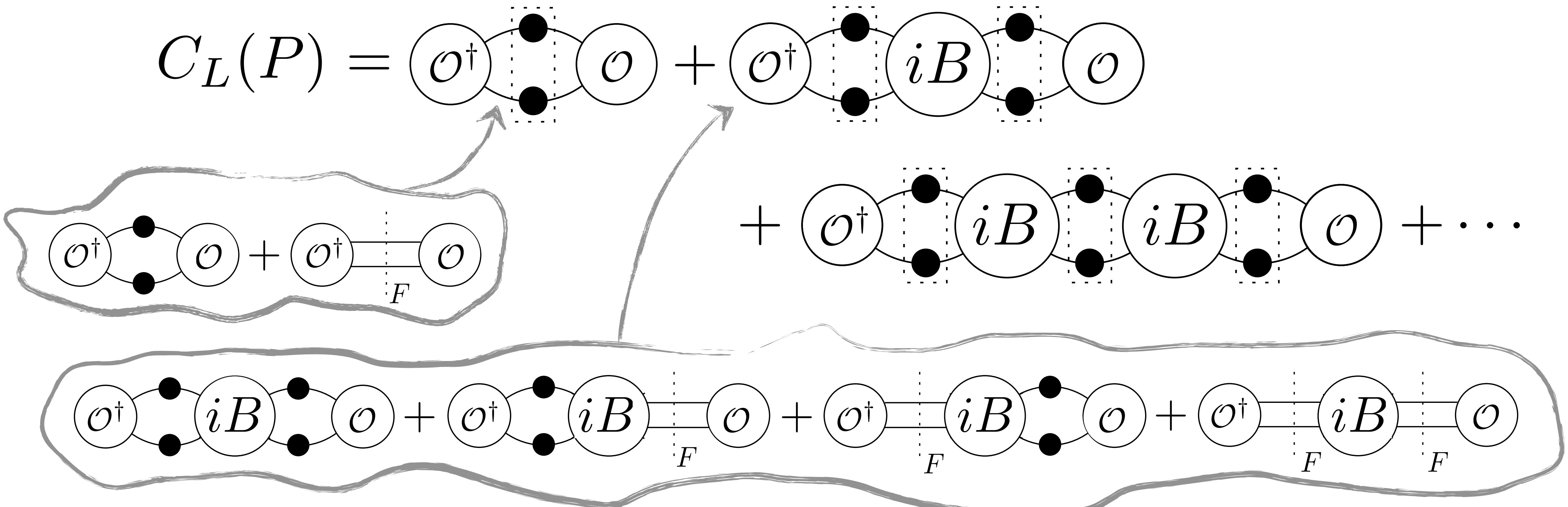


$$= \langle \pi\pi, \mathrm{out} | \mathcal{O}^\dagger | 0 \rangle$$

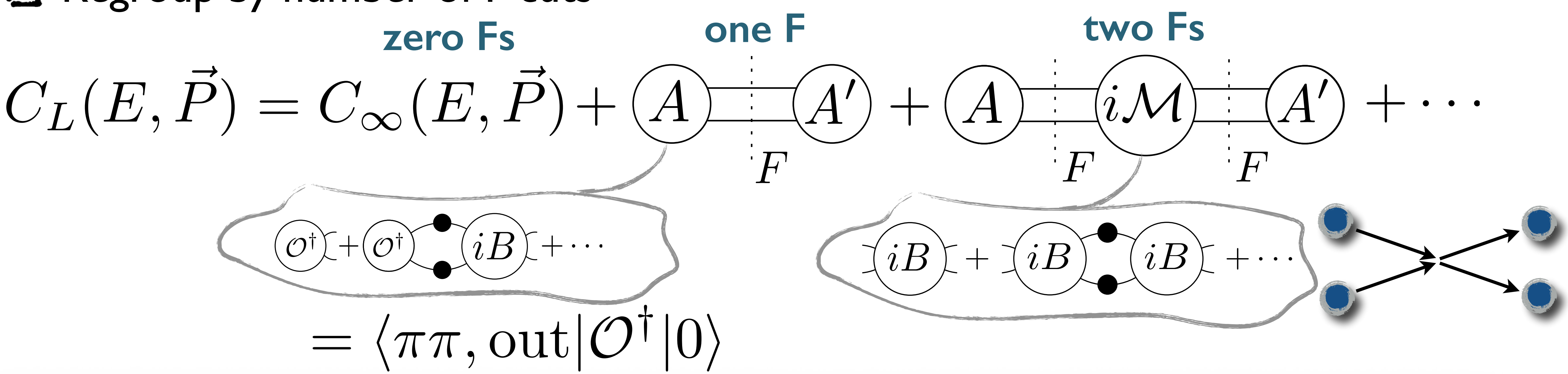


- Regroup by number of F cuts





- Regroup by number of F cuts

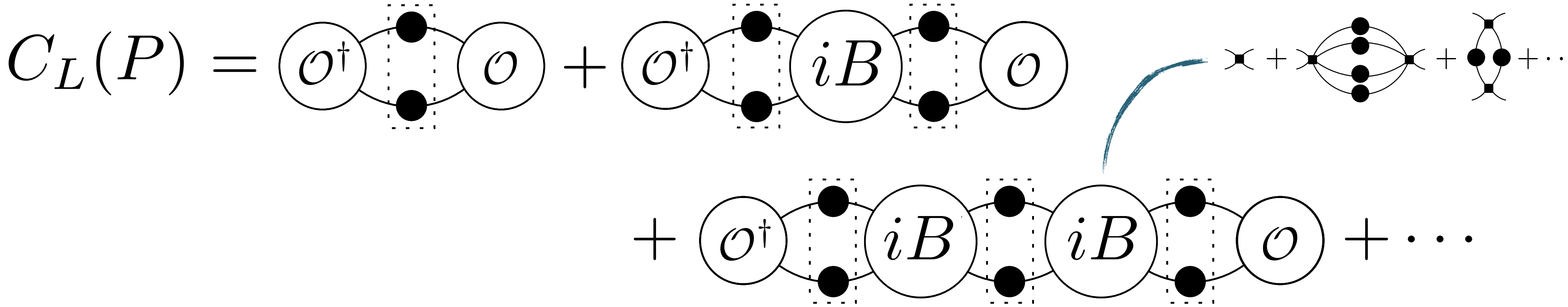


When we factorize diagrams and group infinite-volume parts…

physical observables emerge

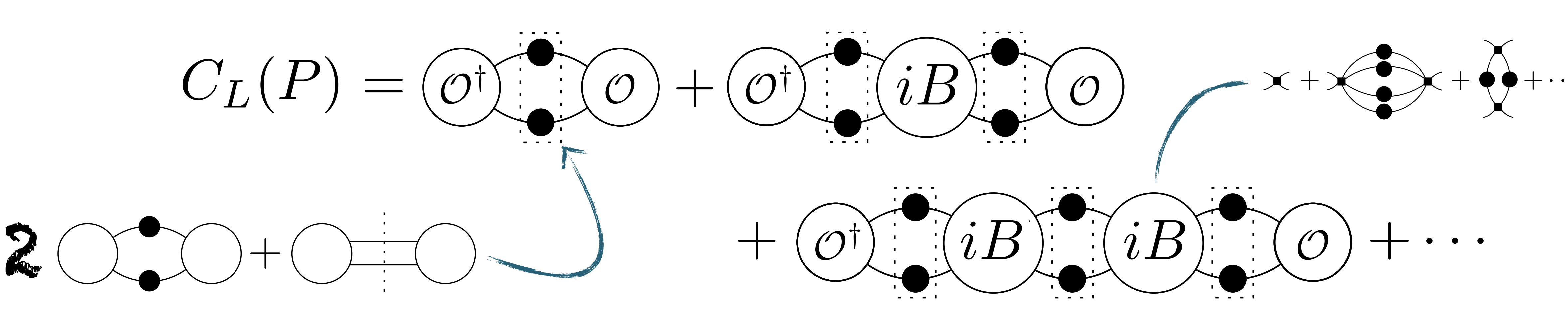
Review

1



Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

Review

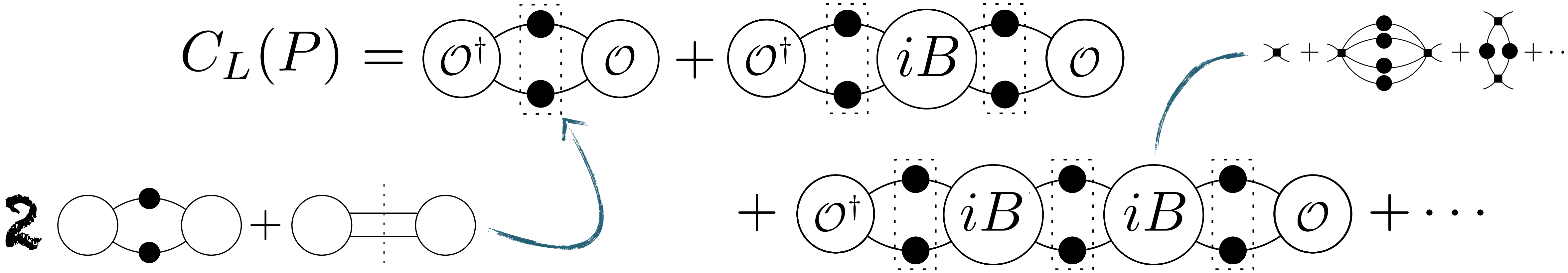


Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

Review

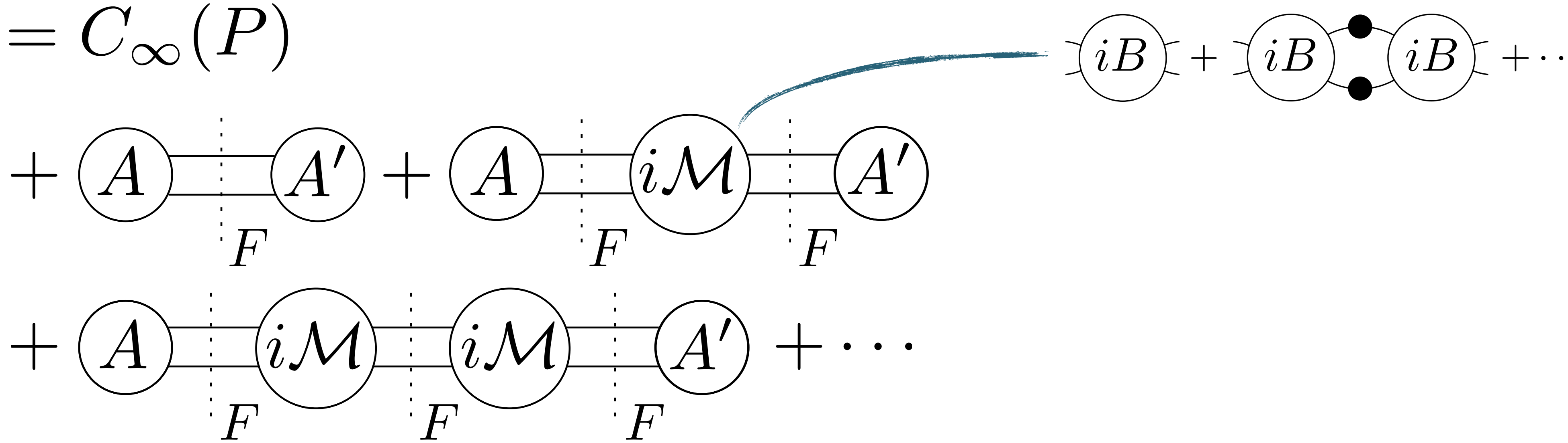
1

$$C_L(P) =$$



$$C_L(P) = C_\infty(P)$$

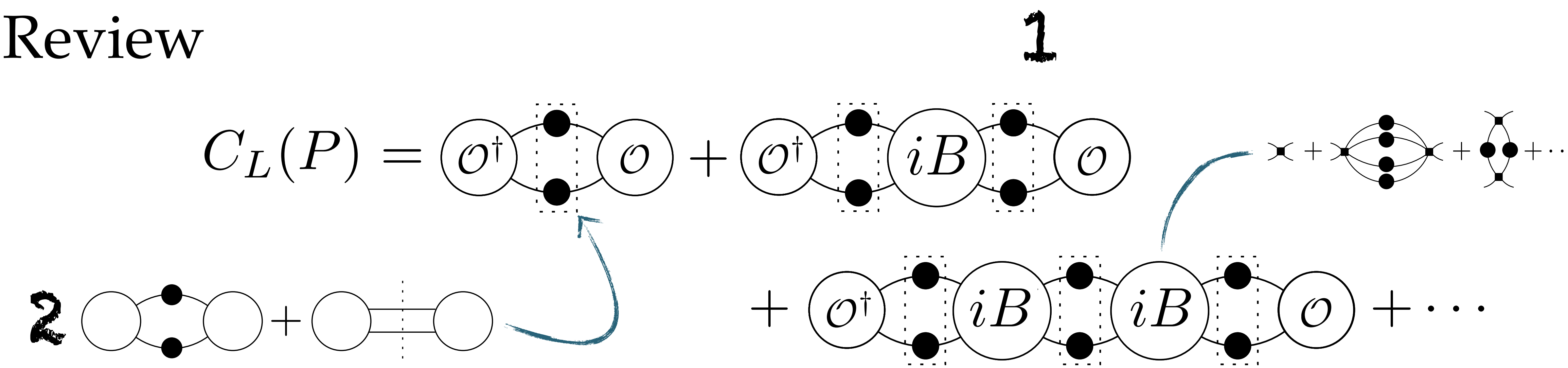
3



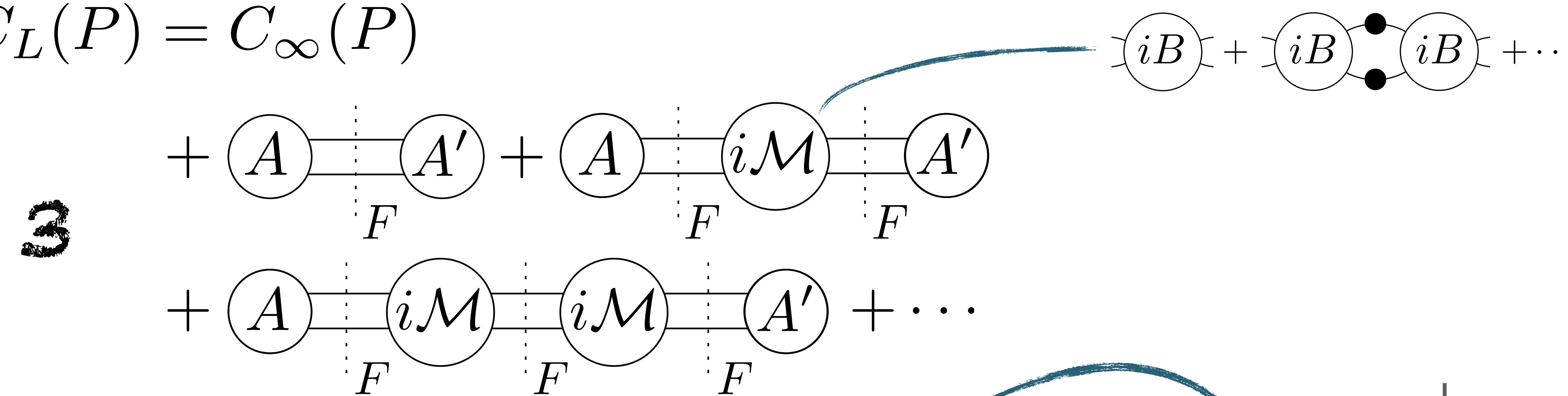
Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

Review

$C_L(P) =$



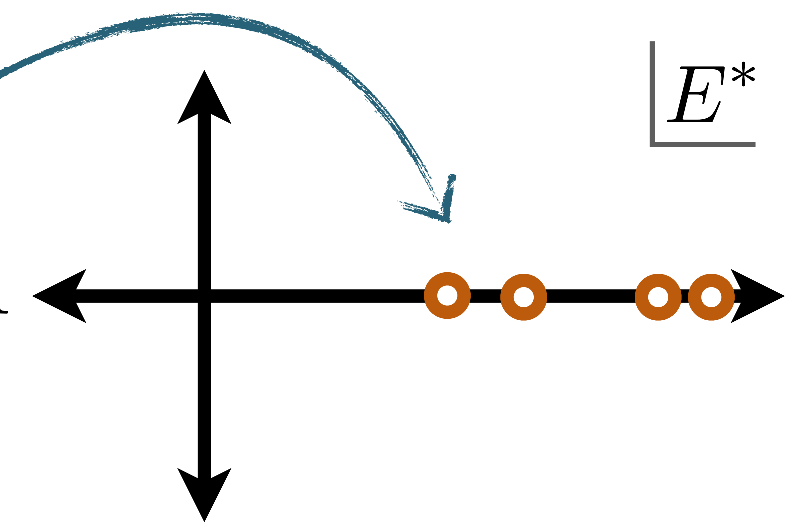
$C_L(P) = C_\infty(P)$



Deduce…

poles in here

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$



Lüscher (1986, 1991) • Kim, Sachrajda, Sharpe (2005)

Two-particle quantization condition

- Finite-volume energies = solutions to…

$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude — known geometric function

where $E_n^{*2} \equiv E_n^2 - \vec{P}^2$

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

Two-particle quantization condition

- Finite-volume energies = solutions to…

$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude known geometric function

where $E_n^{*2} \equiv E_n^2 - \vec{P}^2$

- Matrices in angular momentum space
- Holds only for $E_n^{*2} < (4m)^2$
- Ignores suppressed volume effects (e^{-mL})

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

Two-particle quantization condition

- Finite-volume energies = solutions to…

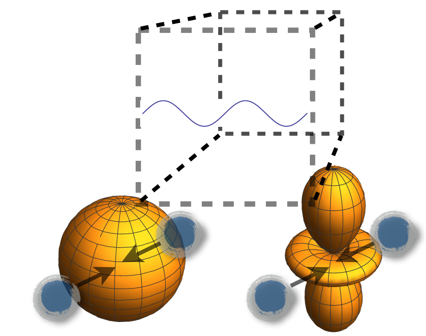
$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude — known geometric function

where $E_n^{*2} \equiv E_n^2 - \vec{P}^2$

- Matrices in angular momentum space
- Holds only for $E_n^{*2} < (4m)^2$
- Ignores suppressed volume effects (e^{-mL})

$$\mathcal{M}_2(E^*) = \begin{pmatrix} \bullet & & \\ & \bullet & \\ & & \bullet \end{pmatrix} \qquad F(E_n, \vec{P}, L) = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$$



- Matrices block diagonalize into finite-volume irreps, e.g.

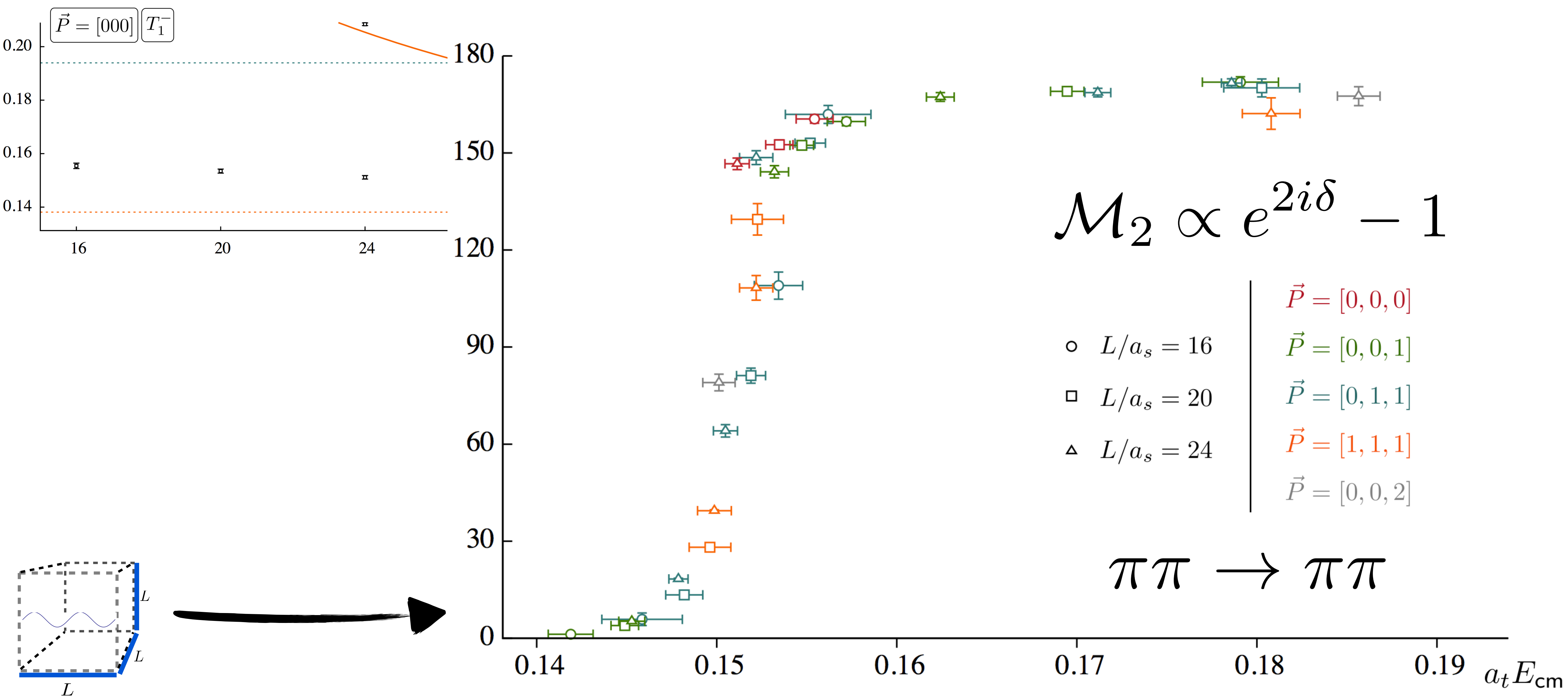
$$\det\left[\mathcal{M}_2^{-1}(E_n) + F(E_n, \vec{0}, L)\right]_{\mathbb{A}_1^+} = 0$$

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

Using the result

- Simplest case is a single angular momentum *(e.g. 2 pions in a p-wave)*

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$

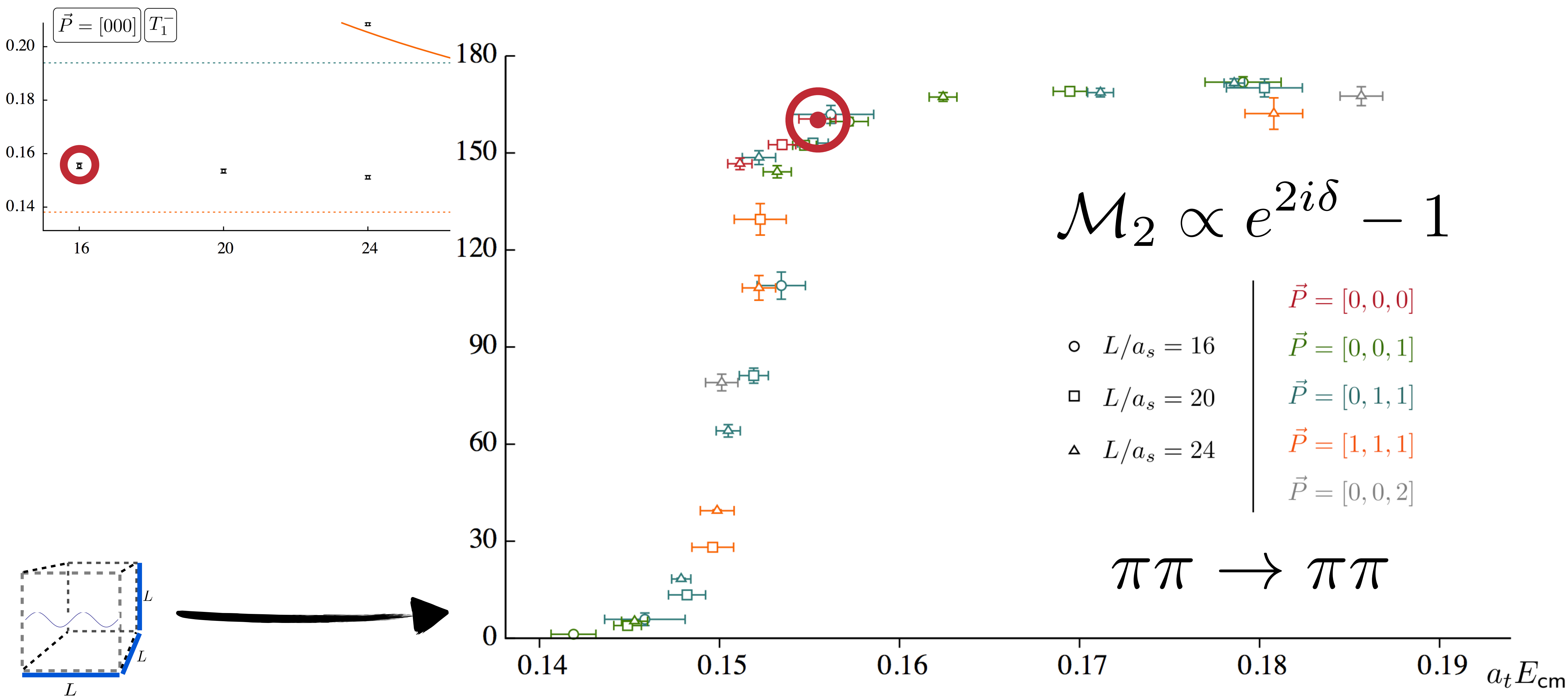


Dudek, Edwards, Thomas, PRD87 (2013) 034505

Using the result

- Simplest case is a single angular momentum *(e.g. 2 pions in a p-wave)*

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$

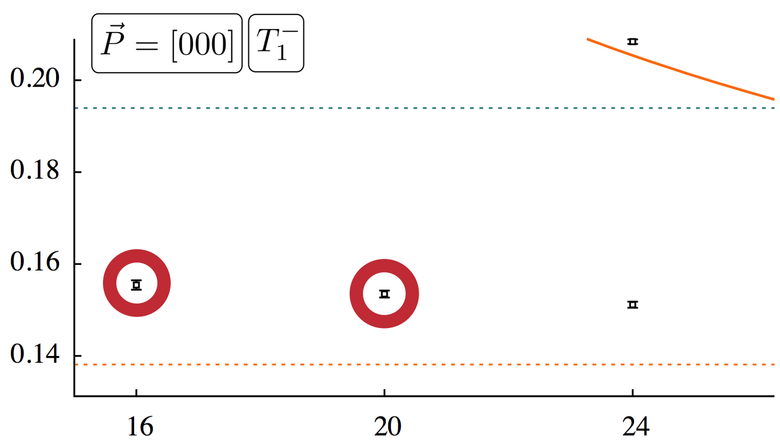


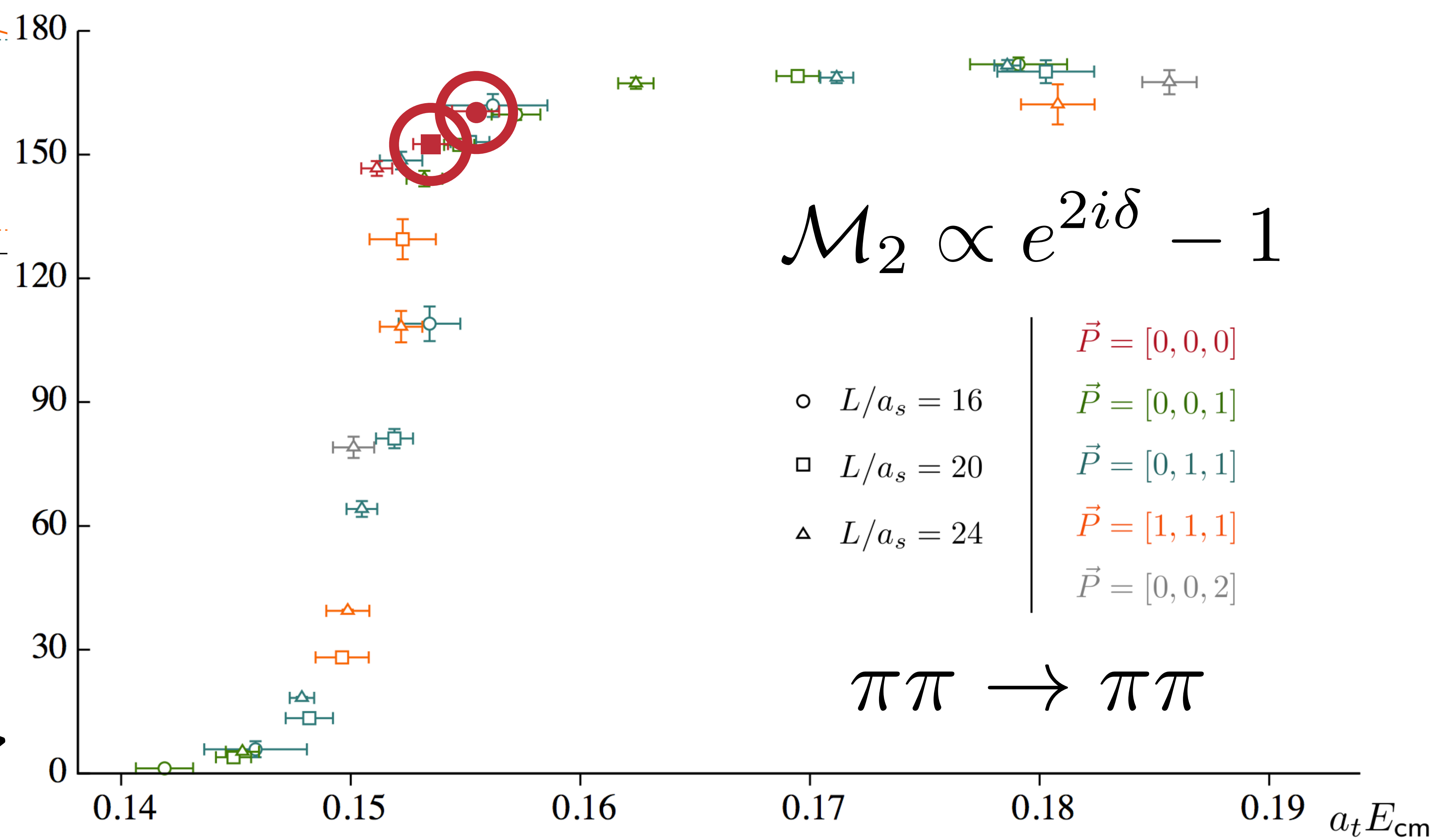
Dudek, Edwards, Thomas, PRD87 (2013) 034505

Using the result

- Simplest case is a single angular momentum *(e.g. 2 pions in a p-wave)*

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$



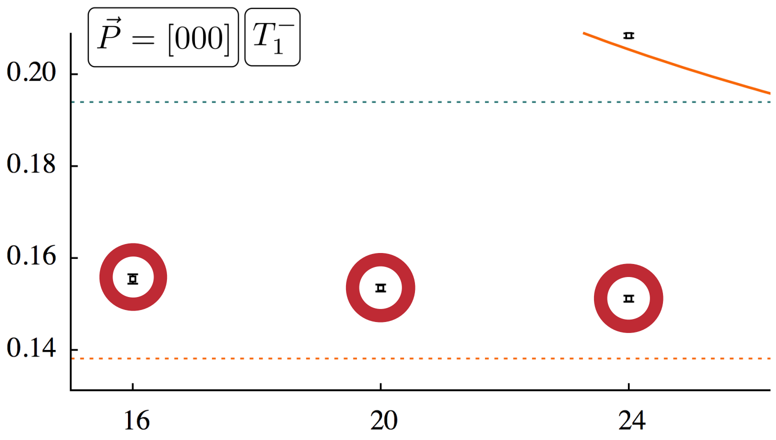


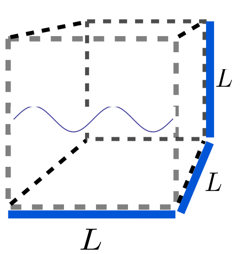
Dudek, Edwards, Thomas, PRD87 (2013) 034505

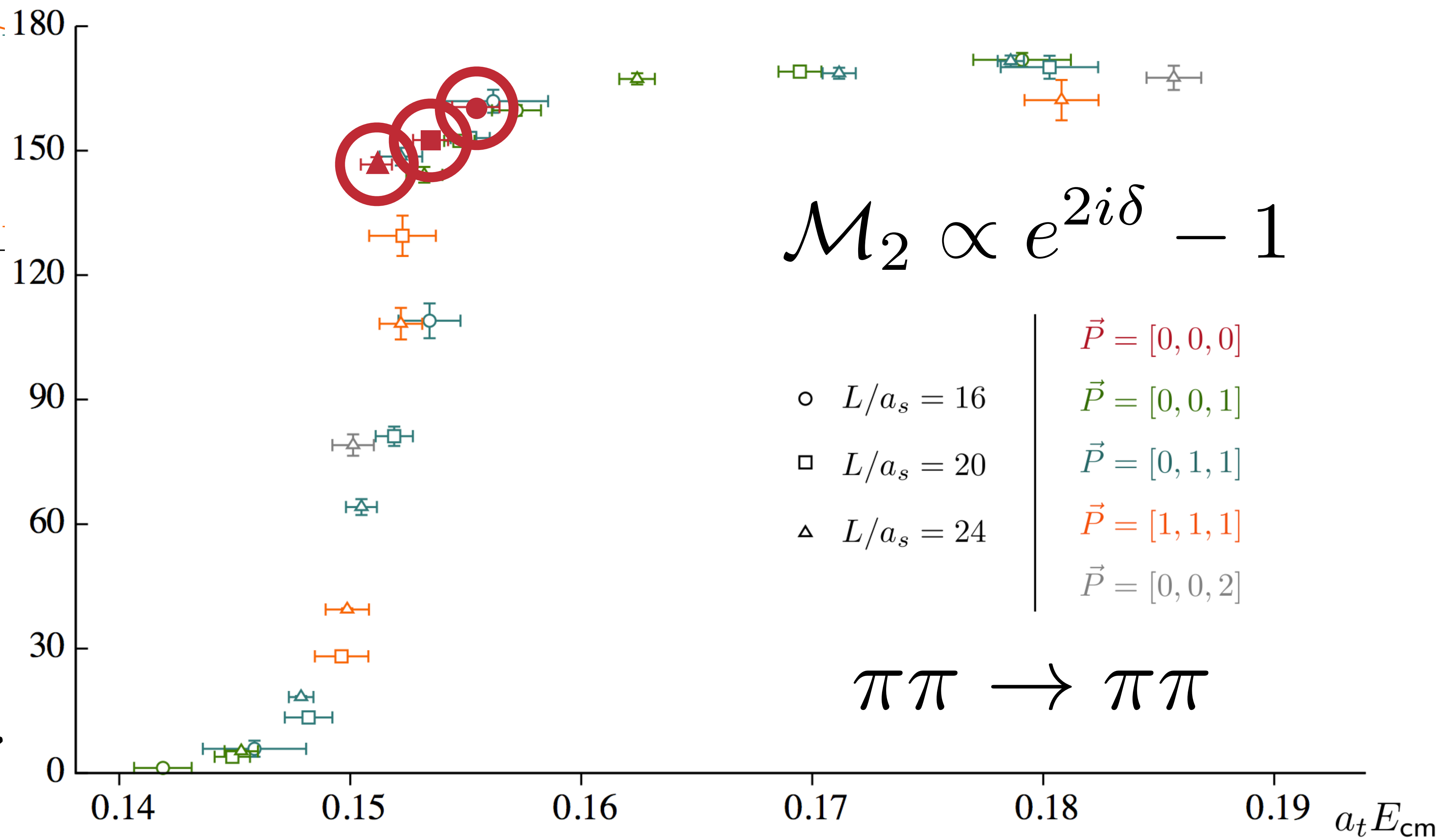
Using the result

- Simplest case is a single angular momentum *(e.g. 2 pions in a p-wave)*

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$





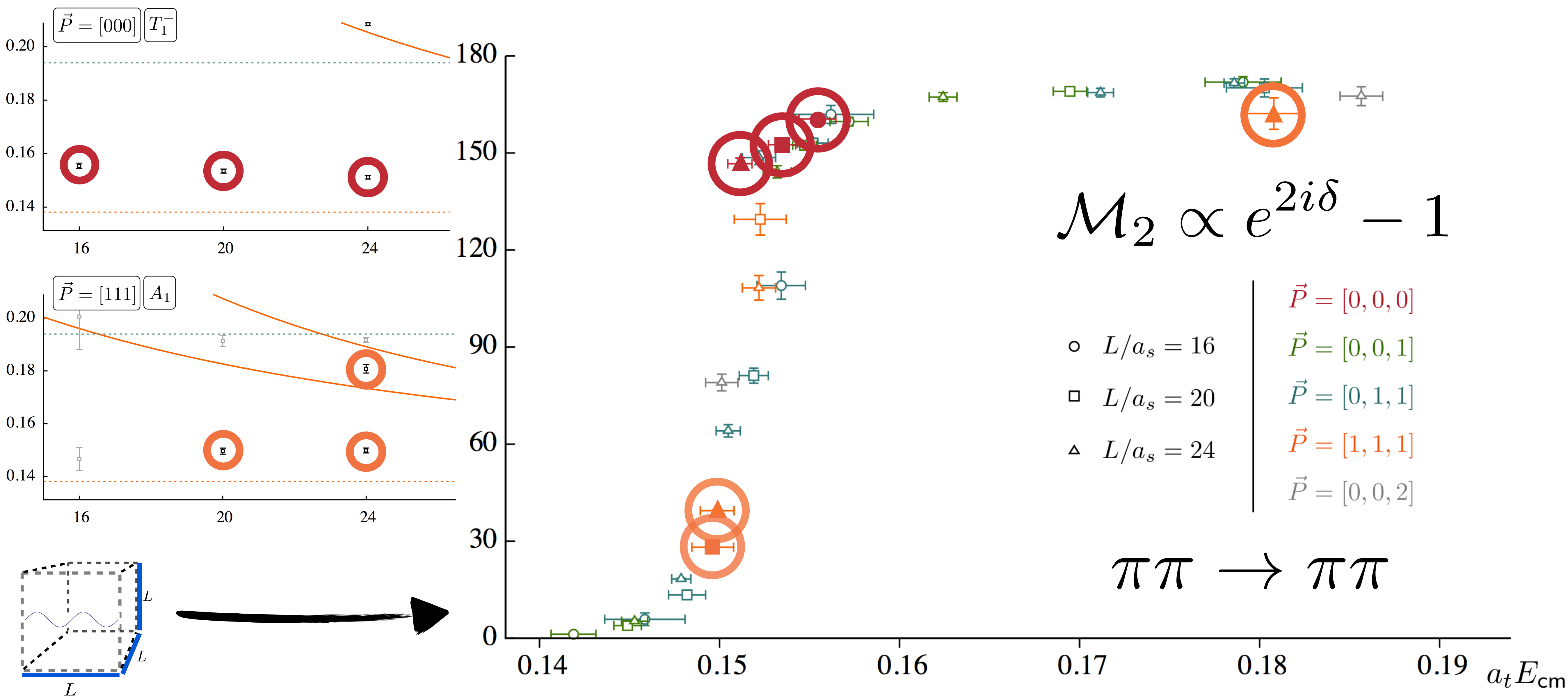


Dudek, Edwards, Thomas, PRD87 (2013) 034505

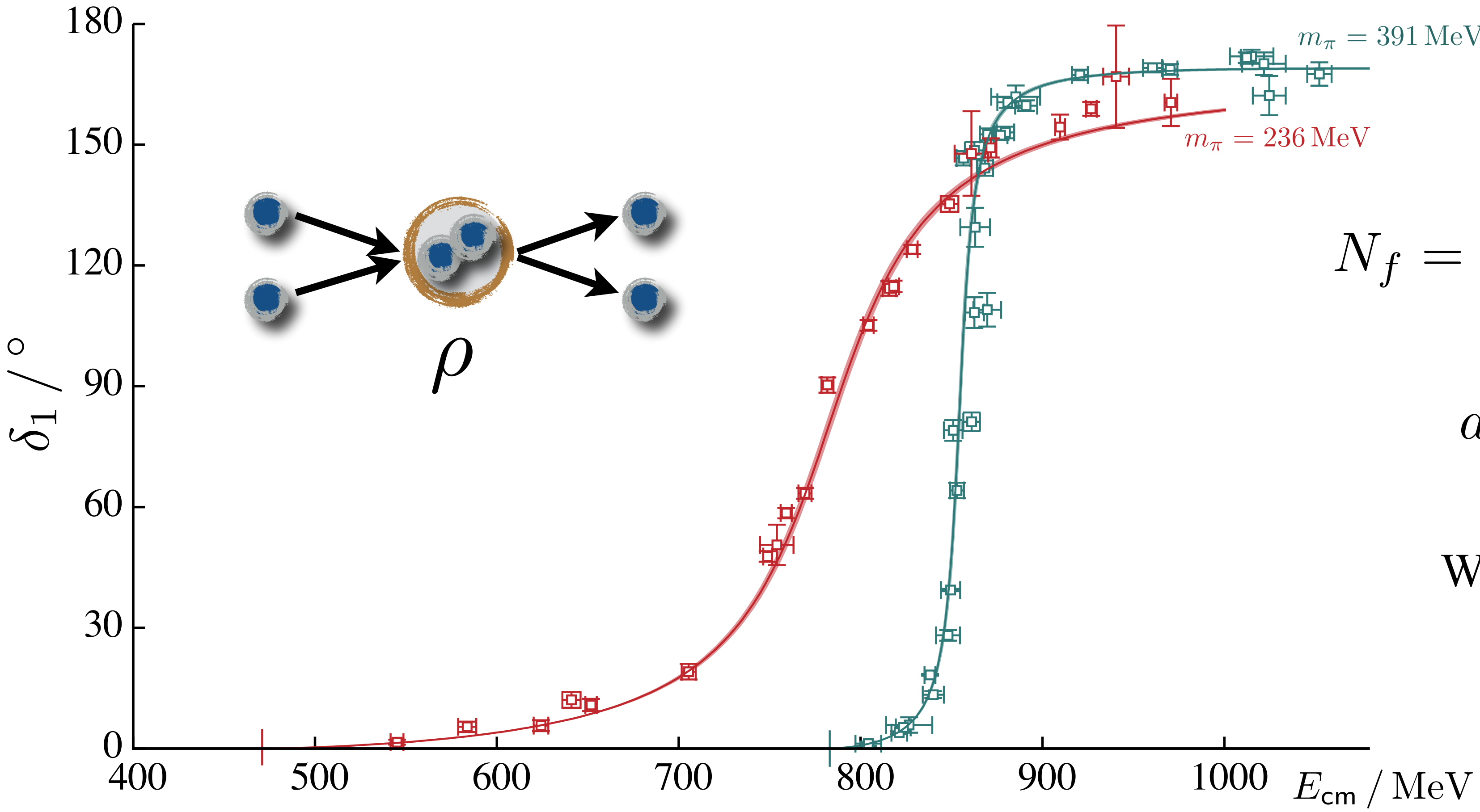
Using the result

- Simplest case is a single angular momentum *(e.g. 2 pions in a p-wave)*

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$

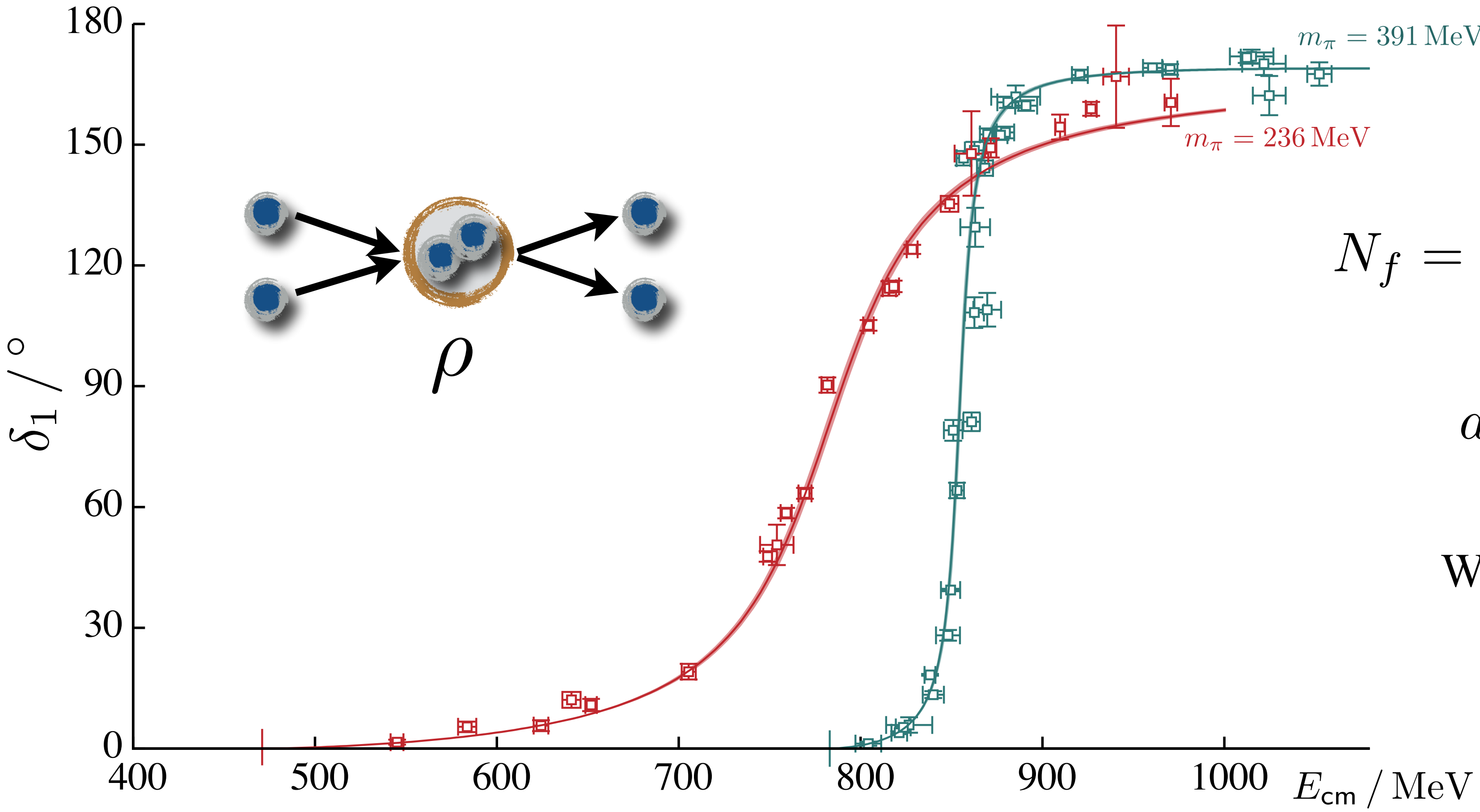


Dudek, Edwards, Thomas, PRD87 (2013) 034505



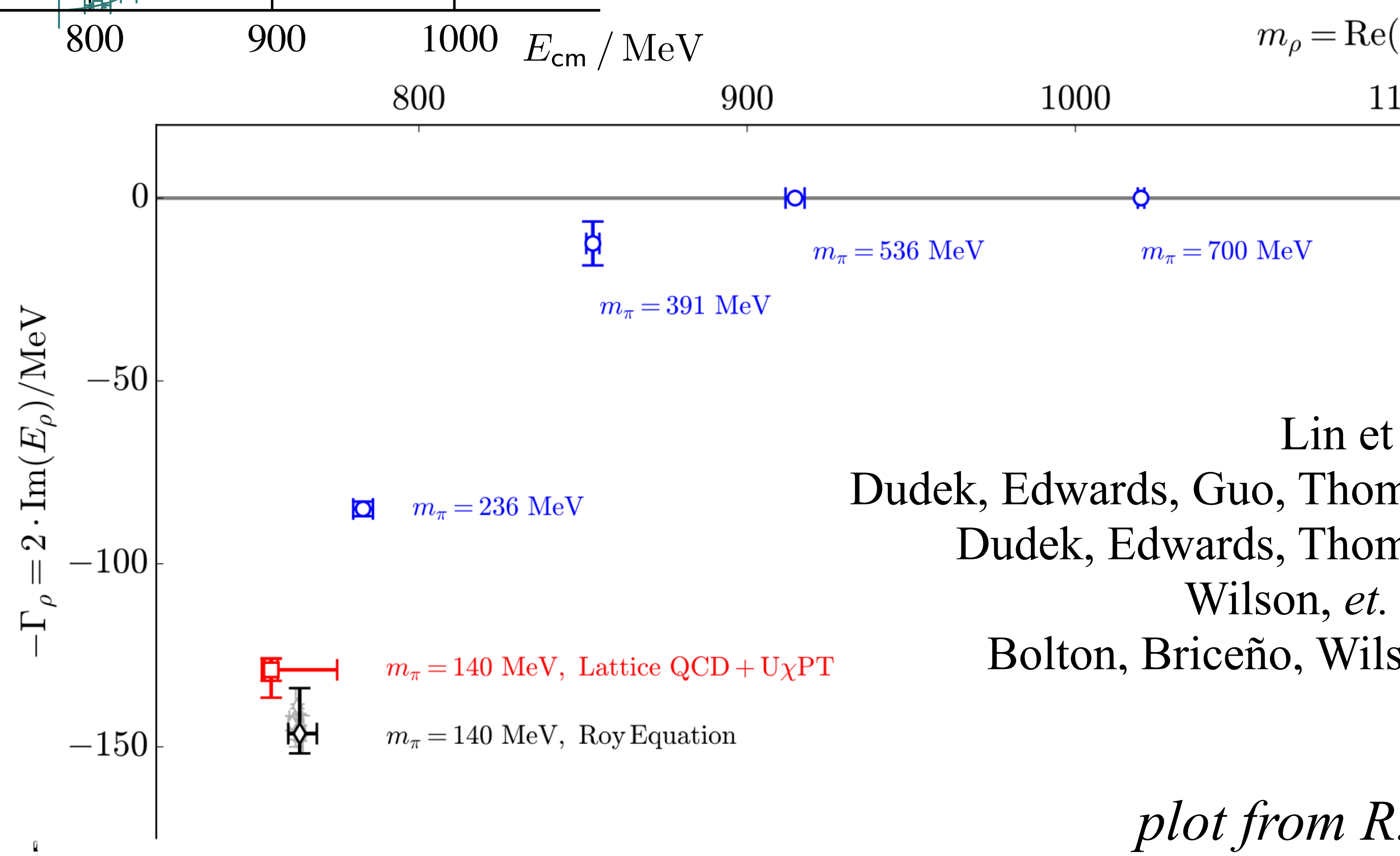
$$N_f = 2 + 1\,, \quad M_\pi \sim 240 \text{ and } 390\,\text{MeV}\,,$$
$$M_\pi L \sim 4 - 6\,,$$
$$a_s \sim 0.12\,\text{fm}\,, \quad a_t \sim 0.035\,\text{fm}$$

Wilson et al., PRD 92, 094502 (2015)



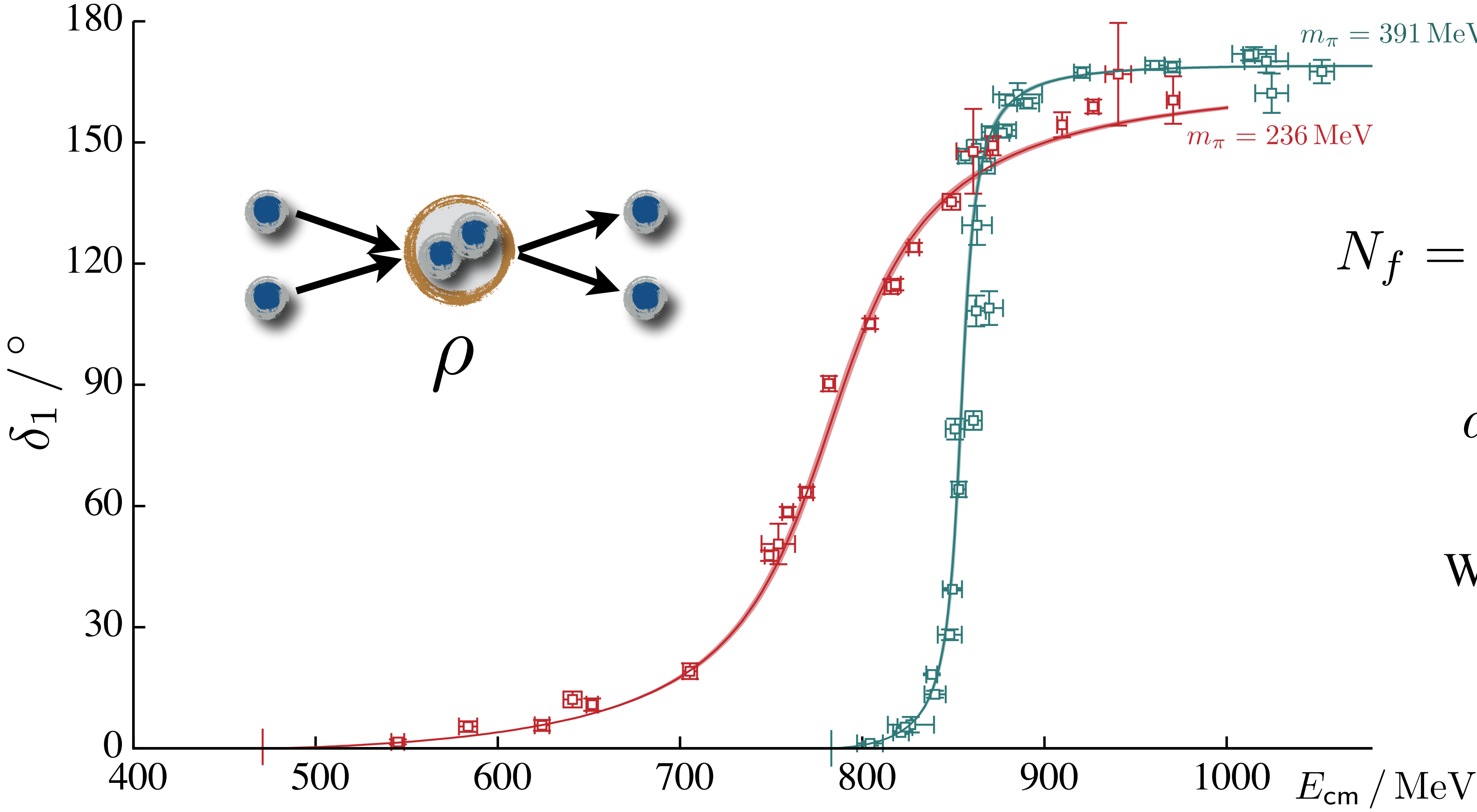
$$N_f = 2+1\,, \quad M_\pi \sim 240 \text{ and } 390\,\text{MeV}\,,$$
$$M_\pi L \sim 4-6\,,$$
$$a_s \sim 0.12\,\text{fm}\,, \quad a_t \sim 0.035\,\text{fm}$$

Wilson et al., PRD 92, 094502 (2015)



Lin et al. (2009)
Dudek, Edwards, Guo, Thomas (2013)
Dudek, Edwards, Thomas (2012)
Wilson, *et. al.* (2015)
Bolton, Briceño, Wilson (2015)

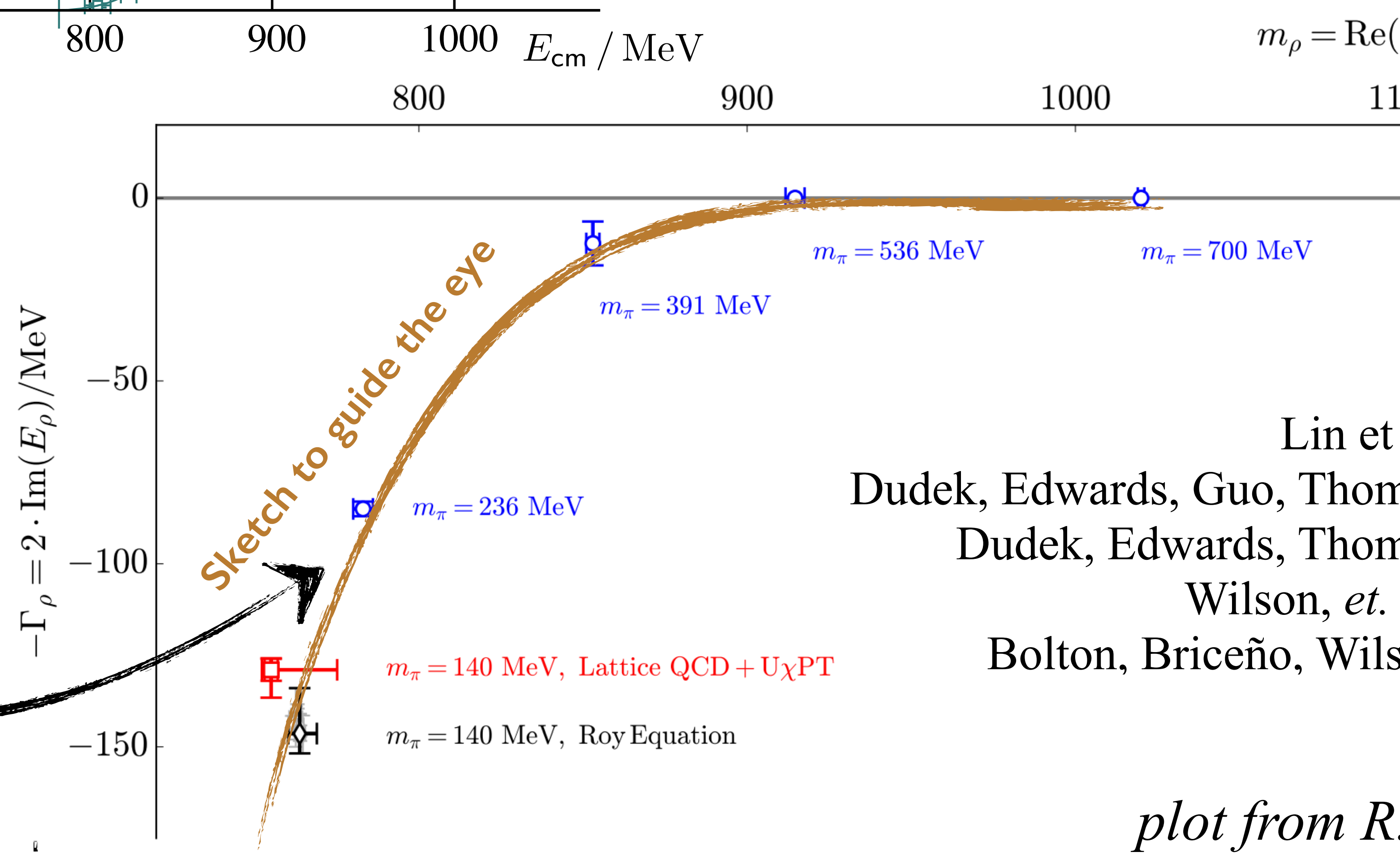
plot from R. Briceño



$$N_f = 2+1\,,\quad M_\pi \sim 240 \text{ and } 390\,\text{MeV}\,,$$
$$M_\pi L \sim 4-6\,,$$
$$a_s \sim 0.12\,\text{fm}\,,\quad a_t \sim 0.035\,\text{fm}$$

Wilson et al., PRD 92, 094502 (2015)

Quantitatively **track the pole** position in the complex plane



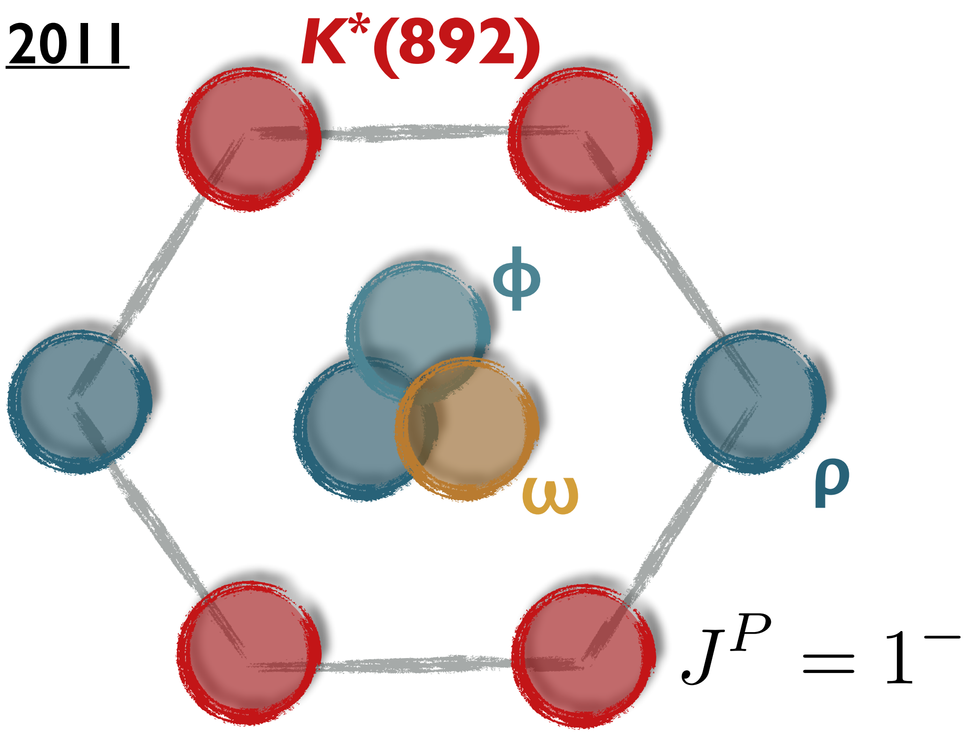
Lin et al. (2009)
Dudek, Edwards, Guo, Thomas (2013)
Dudek, Edwards, Thomas (2012)
Wilson, *et. al.* (2015)
Bolton, Briceño, Wilson (2015)

plot from R. Briceño

Lots of activity!

$\rho \to \pi\pi$

- CP-PACS/PACS-CS 2007, 2011
- Feng et al. 2010
- Lang et al. 2011
- HadSpec 2012, 2015
- Pelissier, Alexandru 2012
- RQCD 2015
- Guo et al. 2016
- Fu, Wang 2016
- Bulava et al. 2016
- Alexandrou et al. 2017
- Andersen et al. 2018

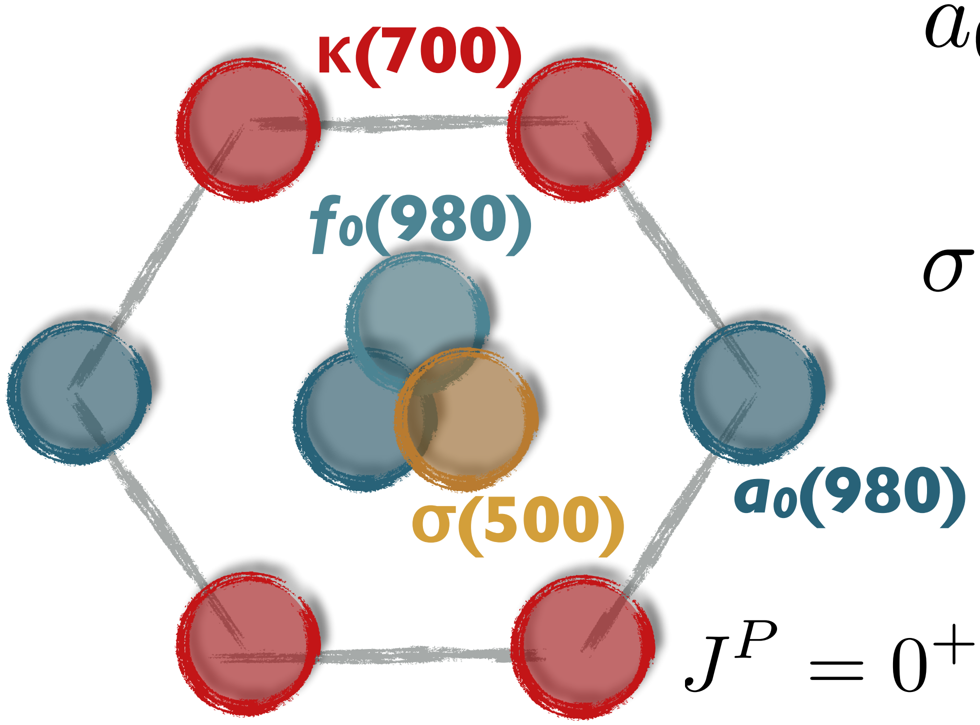


$K^* \to K\pi$

- Lang et al. 2012
- Prelovsek et al. 2013
- **Wilson et al. 2015**
- RQCD 2015
- **Brett et al. 2018**
- Wilson et. al 2019

$\sigma \to \pi\pi$

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth, Giedt 2017
- Briceño et al. 2017
- Molina et al. 2018



$a_0(980) \to \pi\eta, K\overline{K}$

- **Dudek et al. 2016**

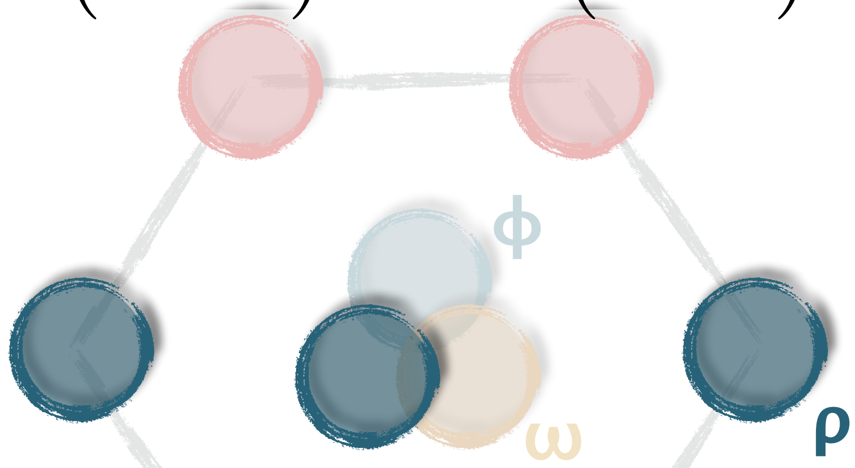
$\sigma, f_0, f_2 \to \pi\pi, K\overline{K}, \eta\eta$

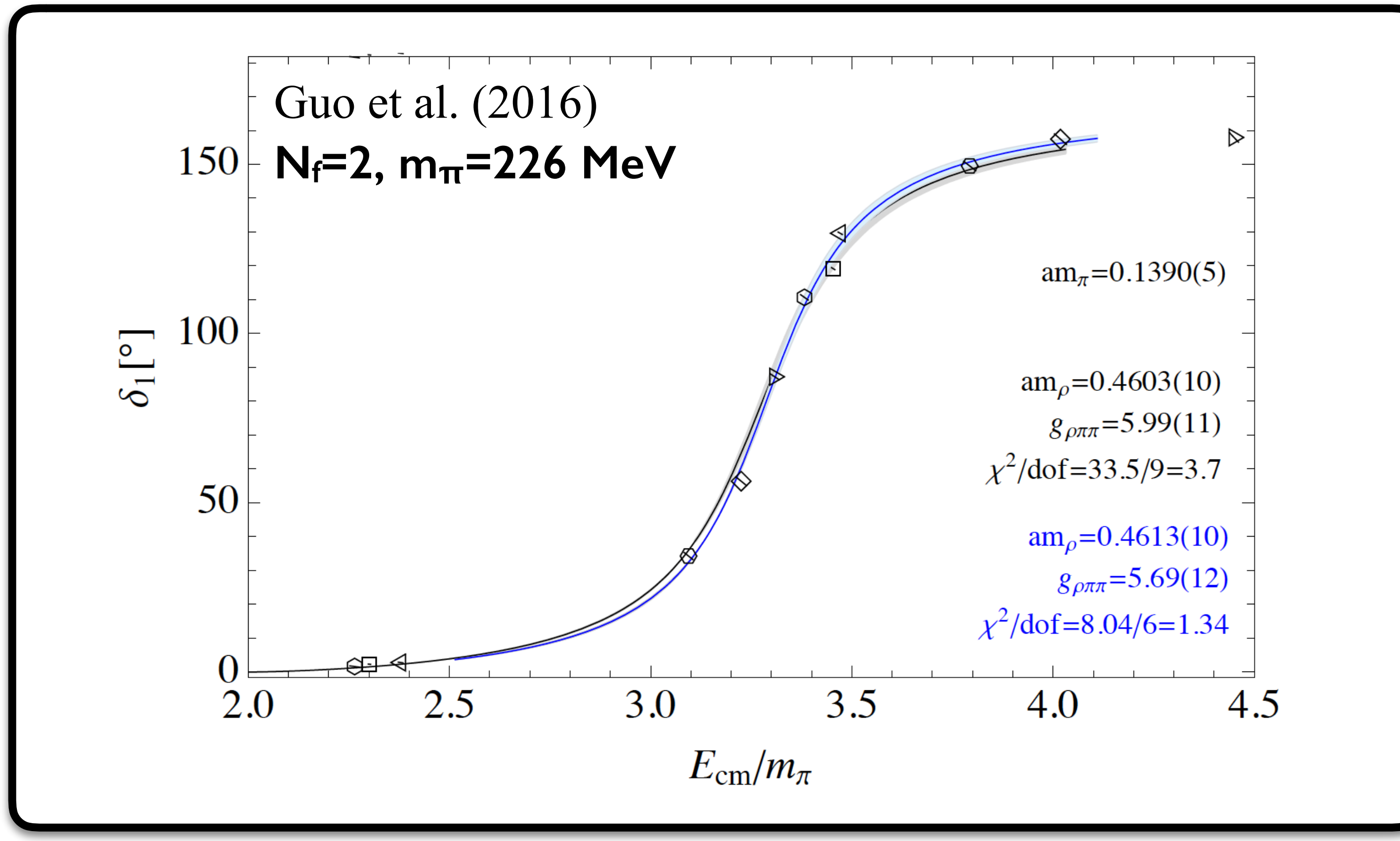
- **Briceño et al. 2017**

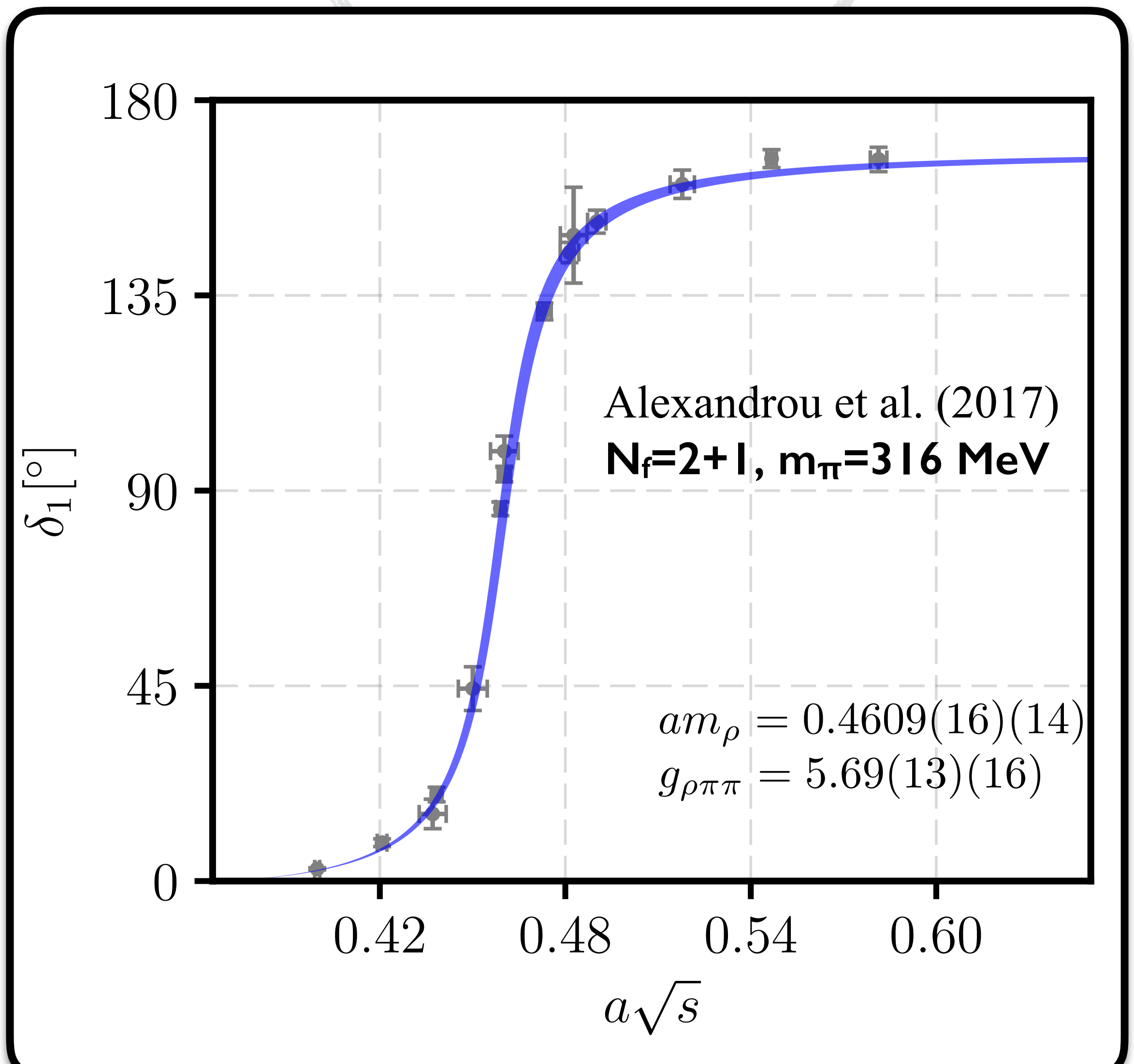
See the recent review by
Briceño, Dudek and Young

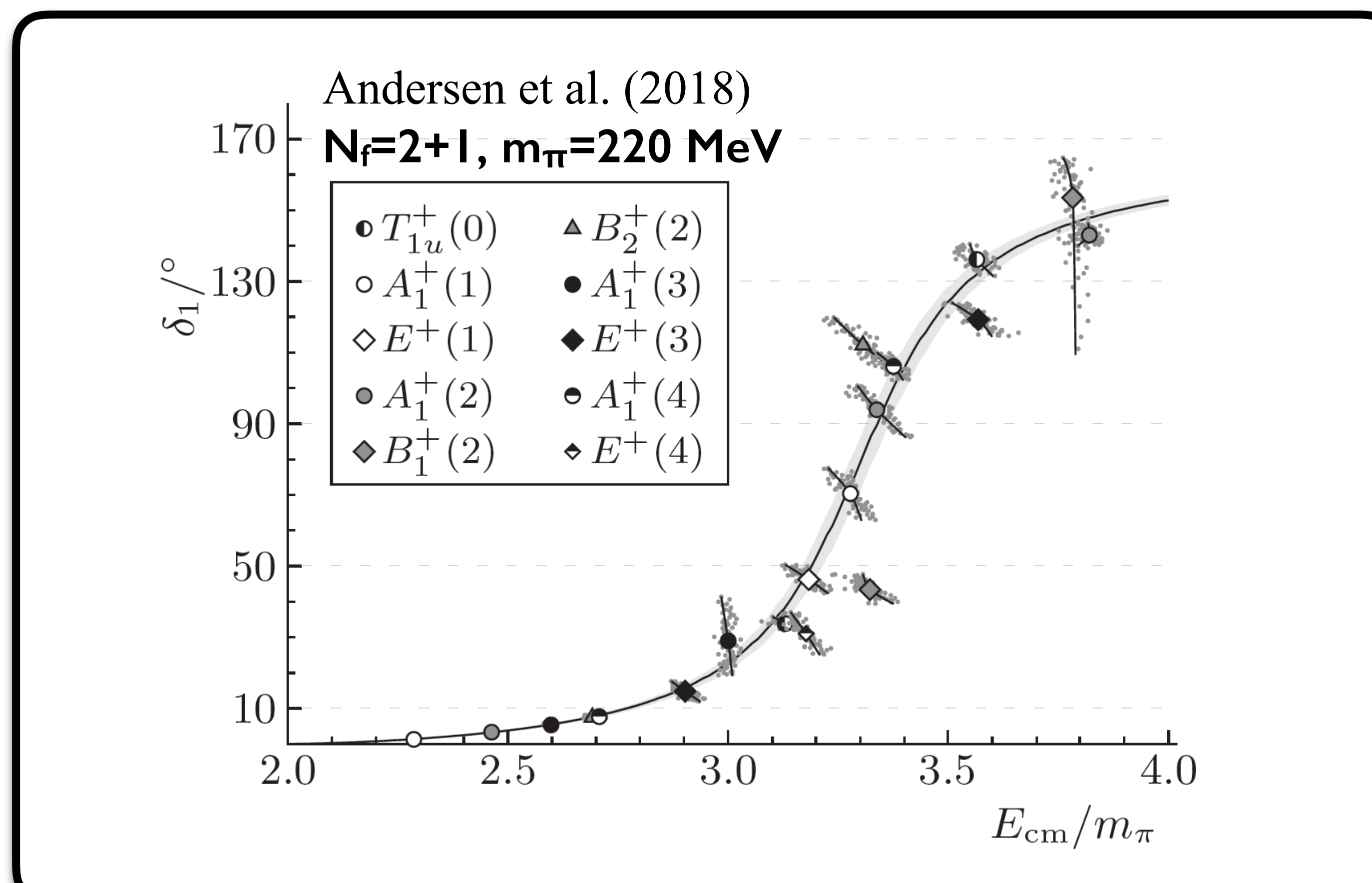
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$



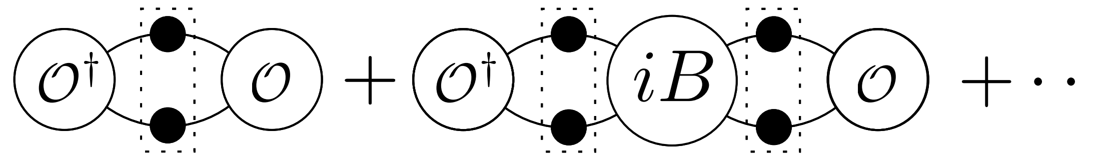






Coupled channels

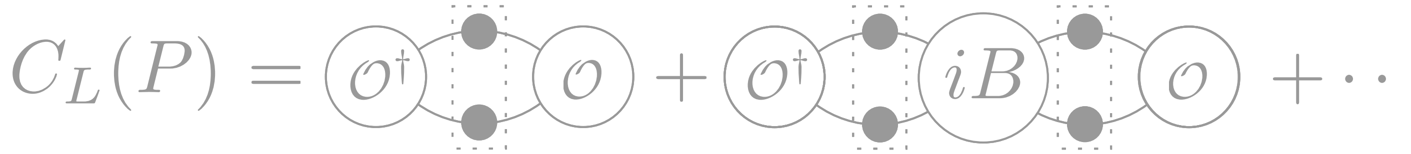
- Recall the single-channel skeleton expansion

$C_L(P) =$ 

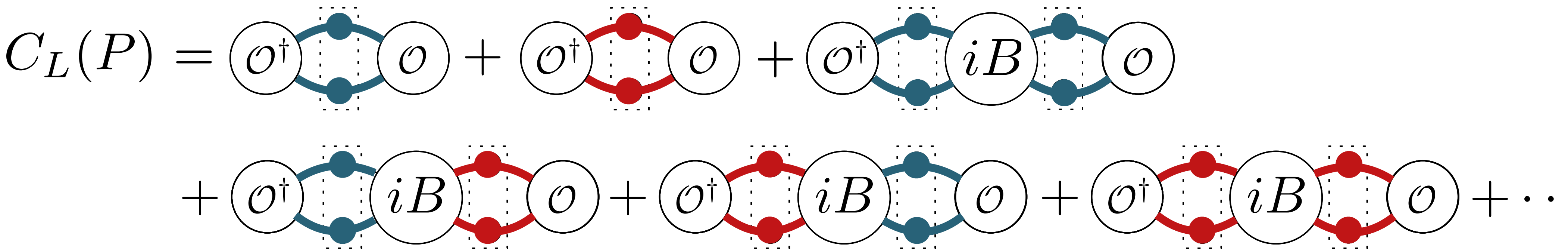
MTH, Sharpe (2012) • Briceño, Davoudi (2012)

Coupled channels

- Recall the single-channel skeleton expansion



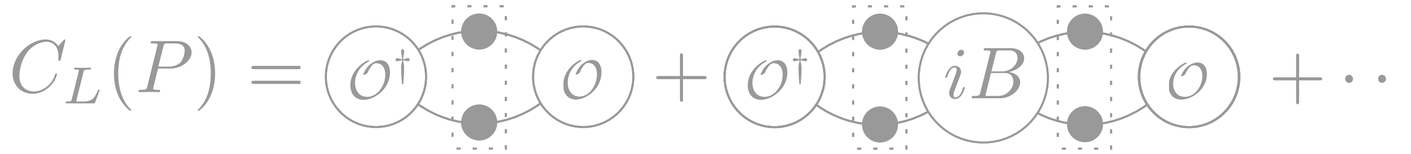
- For coupled channels this becomes



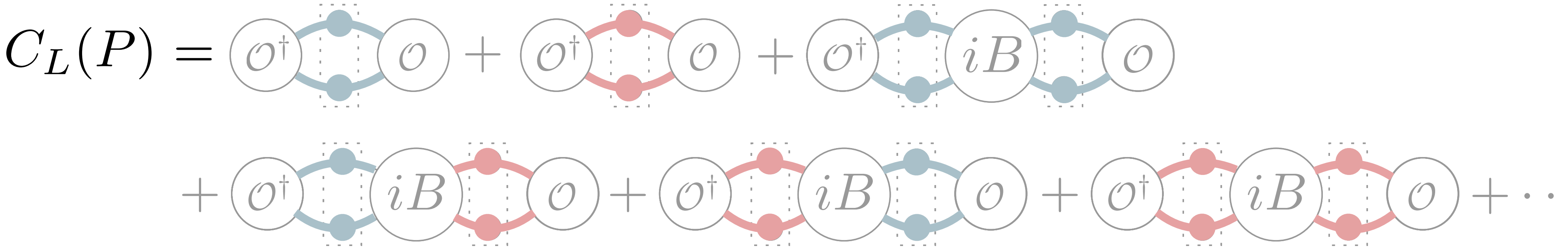
MTH, Sharpe (2012) • Briceño, Davoudi (2012)

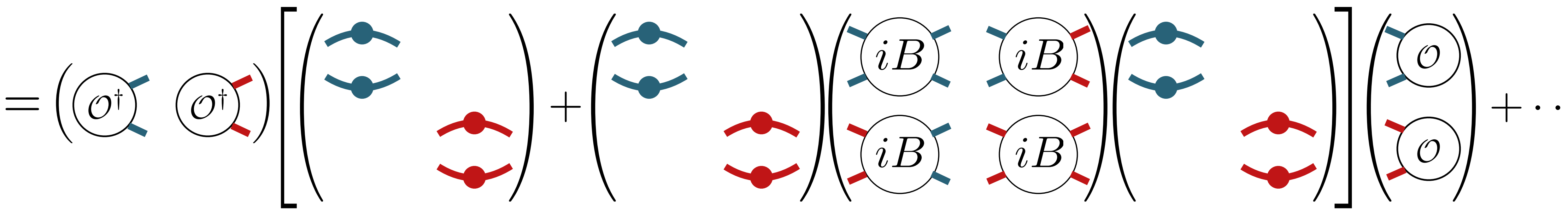
Coupled channels

- Recall the single-channel skeleton expansion



- For coupled channels this becomes

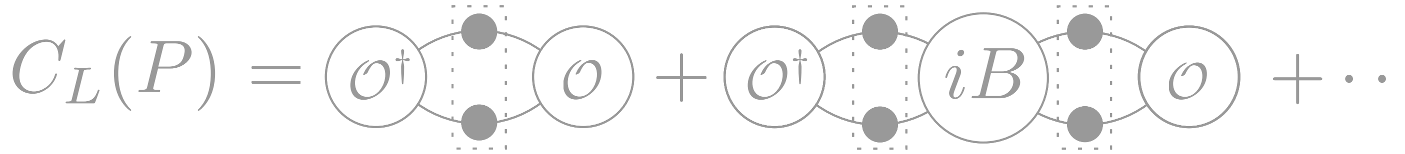




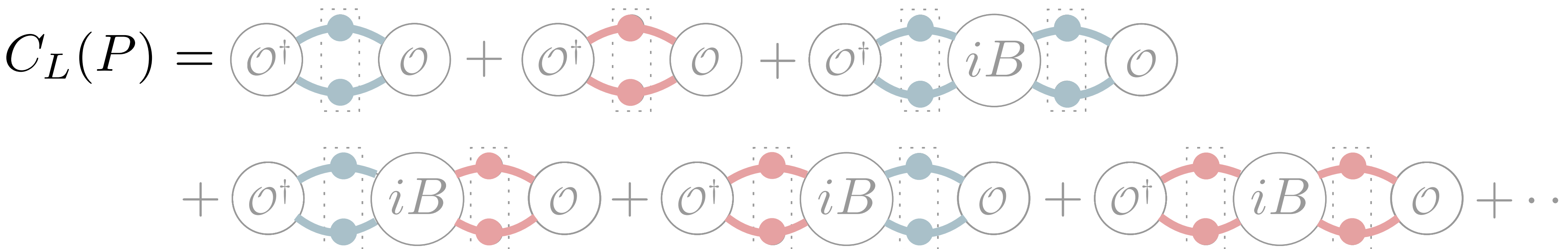
MTH, Sharpe (2012) • Briceño, Davoudi (2012)

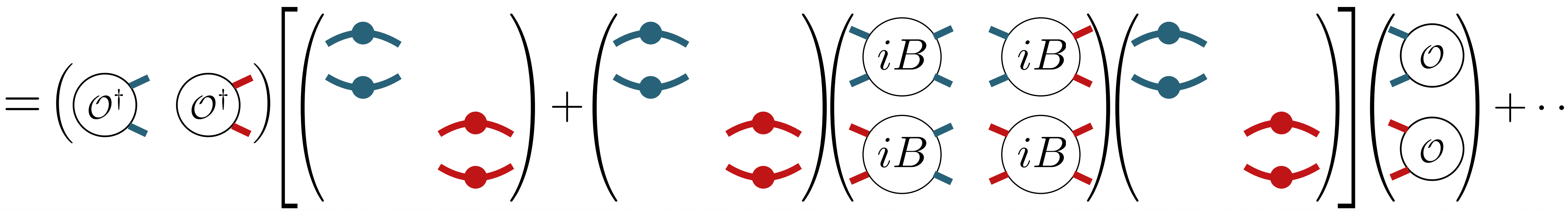
Coupled channels

- Recall the single-channel skeleton expansion



- For coupled channels this becomes





Full derivation goes through with this ***new matrix structure***

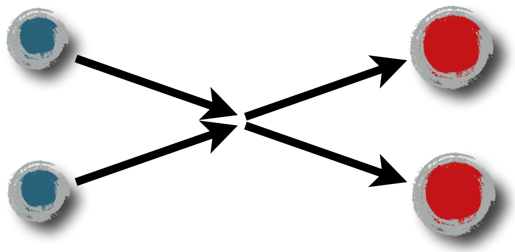
MTH, Sharpe (2012) • Briceño, Davoudi (2012)

Coupled quantization condition

- Finite-volume energies = solutions to…

$$\det\left[\begin{pmatrix}\mathcal{M}_{a\to a} & \mathcal{M}_{a\to b}\\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b}\end{pmatrix}^{-1}+\begin{pmatrix}F_a & 0\\ 0 & F_b\end{pmatrix}\right]=0$$

scattering amplitude   known functions



e.g. $a=\pi\pi,\ b=K\overline{K}$

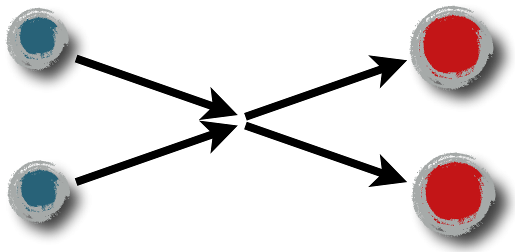
MTH, Sharpe (2012) • Briceño, Davoudi (2012)

Coupled quantization condition

- Finite-volume energies = solutions to…

$$\det\left[\begin{pmatrix} \mathcal{M}_{a\to a} & \mathcal{M}_{a\to b} \\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$$

scattering amplitude   known functions



e.g. $a = \pi\pi, \ b = K\overline{K}$

- Matrices in angular momentum space
- Holds only for $E_n^{*2} < (4m)^2$
- Ignores suppressed volume effects (e^{-mL})

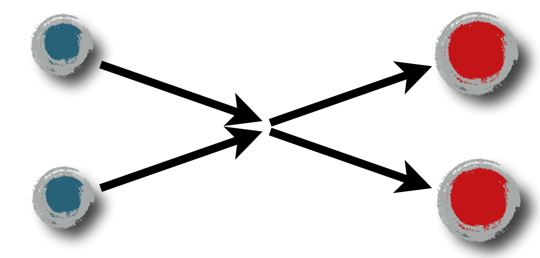
MTH, Sharpe (2012) • Briceño, Davoudi (2012)

Coupled quantization condition

- Finite-volume energies = solutions to…

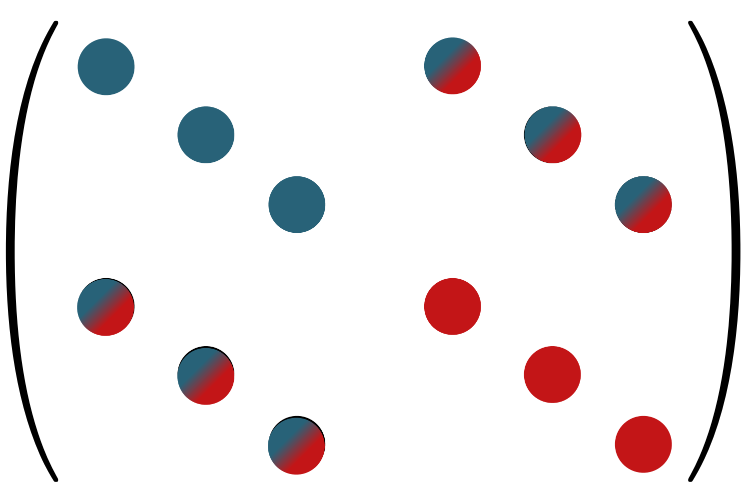
$$\det\left[\begin{pmatrix}\mathcal{M}_{a\to a} & \mathcal{M}_{a\to b}\\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b}\end{pmatrix}^{-1} + \begin{pmatrix}F_a & 0\\ 0 & F_b\end{pmatrix}\right] = 0$$

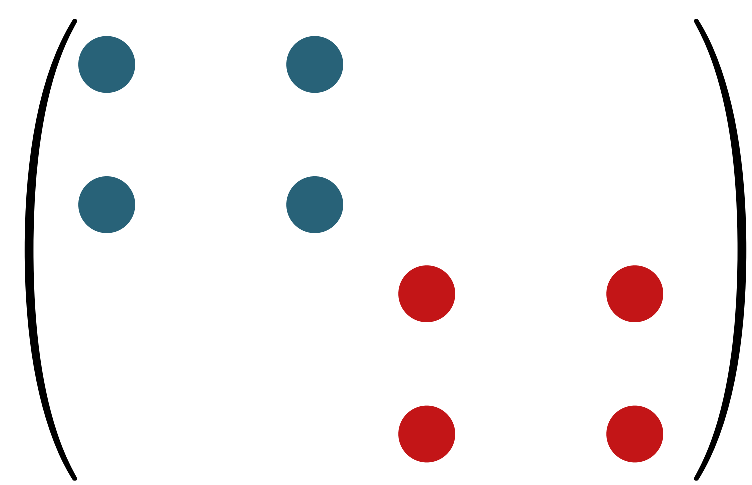
scattering amplitude — known functions



e.g. $a = \pi\pi, \ b = K\overline{K}$

- Matrices in angular momentum space
- Holds only for $E_n^{*2} < (4m)^2$
- Ignores suppressed volume effects (e^{-mL})

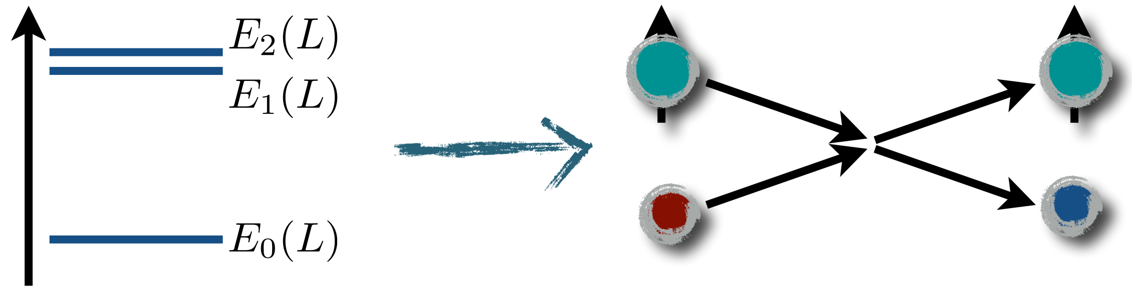
$\mathcal{M}_2(E^*) =$ 

$F(E_n, \vec{P}, L) =$ 

MTH, Sharpe (2012) • Briceño, Davoudi (2012)

General two-to-two scattering

- Lüscher's formalism + extensions give a general mapping



- All results are contained in a ***generalized quantization condition***

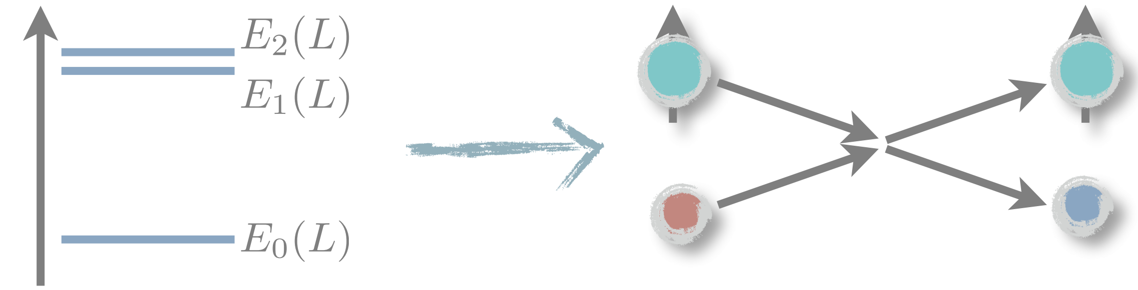
$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude — known geometric function

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Beane, Detmold, Savage (2007) • Tan (2008) • Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012)
MTH, Sharpe (2012) • Briceño, Davoudi (2012) • Li, Liu (2013) • Briceño (2014)

General two-to-two scattering

- Lüscher's formalism + extensions give a general mapping



- All results are contained in a ***generalized quantization condition***

$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude — known geometric function

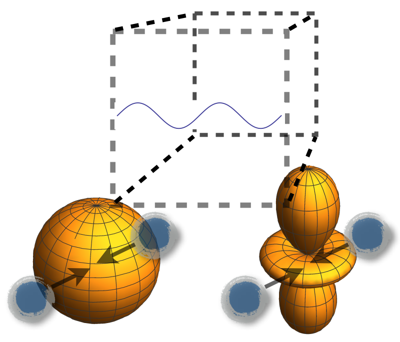
- Matrices in ***angular momentum***, ***spin*** and ***channel*** space
- Holds only for $s_{\pi\pi} < (4M_\pi)^2$
- Ignores suppressed volume effects ($e^{-M_\pi L}$)

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Beane, Detmold, Savage (2007) • Tan (2008) • Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012)
MTH, Sharpe (2012) • Briceño, Davoudi (2012) • Li, Liu (2013) • Briceño (2014)

Not a one-to-one mapping

- The cubic volume mixes different partial waves…

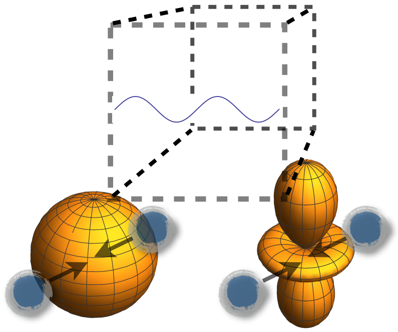
e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix}\right] = 0$



Not a one-to-one mapping

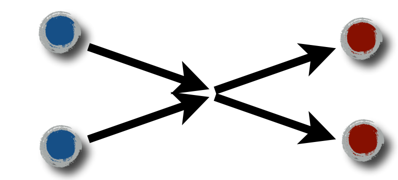
- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix}\mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1}\end{pmatrix} + \begin{pmatrix}F_{ss} & F_{sp} \\ F_{ps} & F_{pp}\end{pmatrix}\right] = 0$



…as well as different flavor channels…

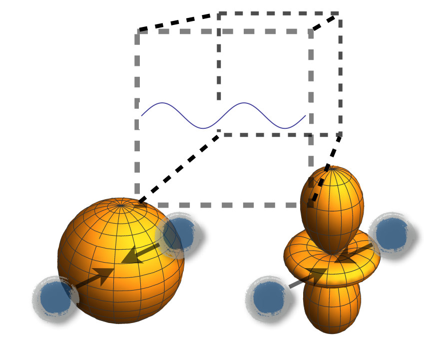
e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $\det\left[\begin{pmatrix}\mathcal{M}_{a\to a} & \mathcal{M}_{a\to b} \\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b}\end{pmatrix}^{-1} + \begin{pmatrix}F_a & 0 \\ 0 & F_b\end{pmatrix}\right] = 0$



Not a one-to-one mapping

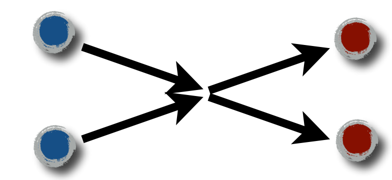
- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix}\right] = 0$



…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{M}_{a\to a} & \mathcal{M}_{a\to b} \\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix}\right] = 0$



- Strategy:

Matrix of correlators with varied operators

$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

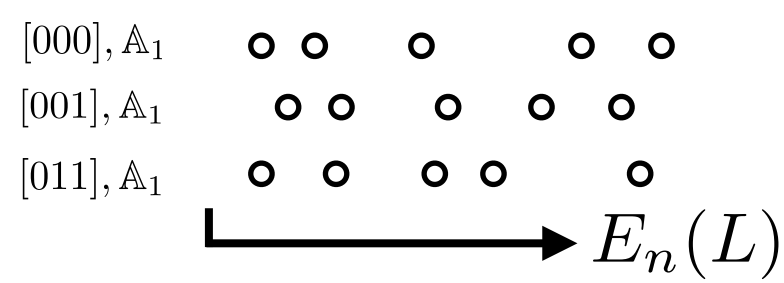
→

Diagonalize (GEVP) to extract energies

$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

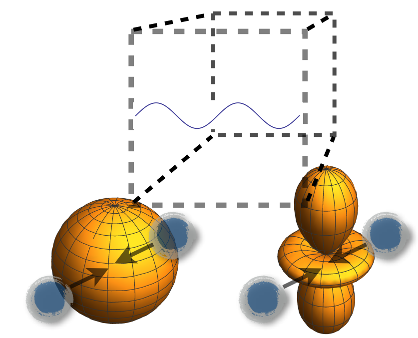
→

Vary *L* and *P* to recover a dense set of energies



Not a one-to-one mapping

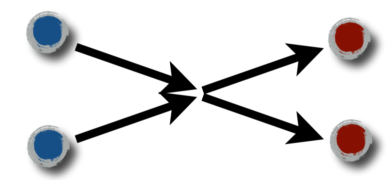
- The cubic volume mixes different partial waves…



e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix}\right] = 0$

…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{M}_{a\to a} & \mathcal{M}_{a\to b} \\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix}\right] = 0$



- Strategy:

Matrix of correlators with varied operators

$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

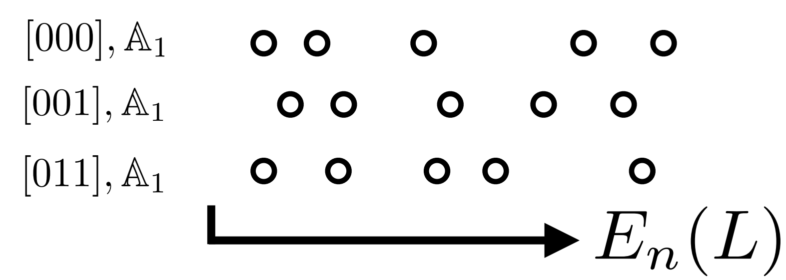
→

Diagonalize (GEVP) to extract energies

$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→

Vary *L* and *P* to recover a dense set of energies



Identify a broad list of K-matrix parametrizations

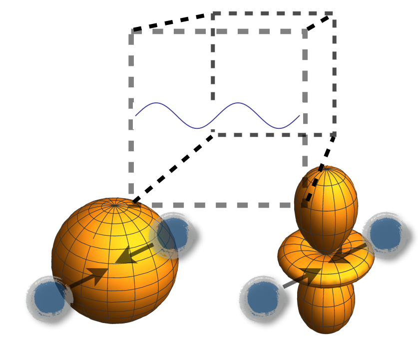
polynomials and poles

EFT based

dispersion theory based

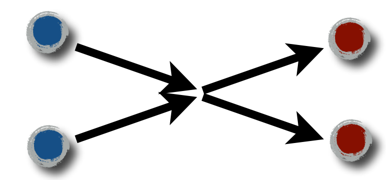
Not a one-to-one mapping

- The cubic volume mixes different partial waves…



e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix}\right] = 0$

…as well as different flavor channels…



e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{M}_{a\to a} & \mathcal{M}_{a\to b} \\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix}\right] = 0$

- Strategy:

Matrix of correlators with varied operators

$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

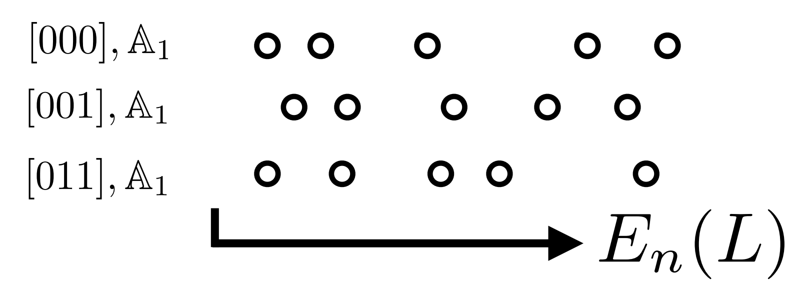
→

Diagonalize (GEVP) to extract energies

$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→

Vary *L* and *P* to recover a dense set of energies



↓

Identify a broad list of K-matrix parametrizations

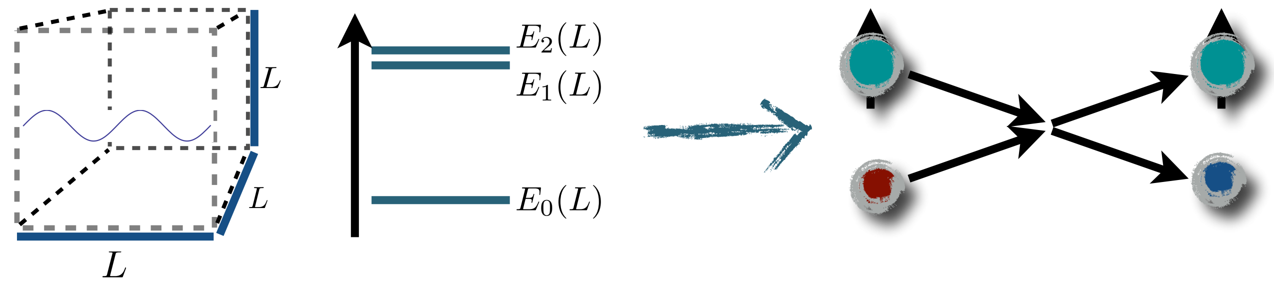
polynomials and poles | EFT based | dispersion theory based

→

Perform global fits to the finite-volume spectrum

Conclusions and outlook

- Multi-hadron observables accesible in LQCD
- Use the ***finite-volume as a tool***

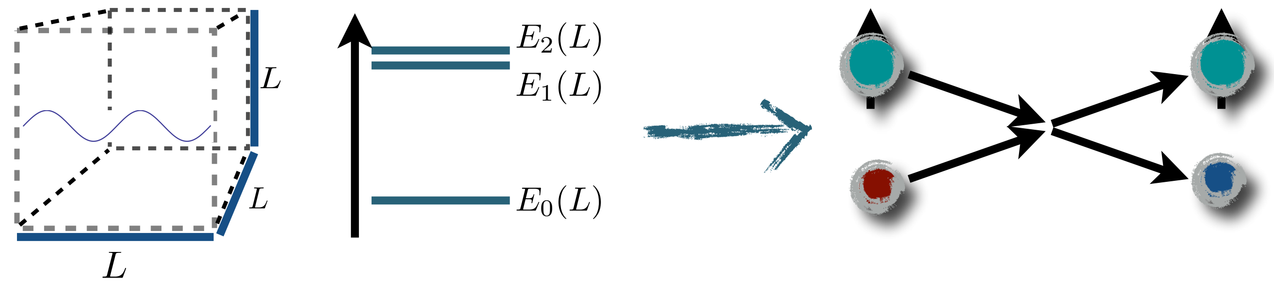


- Numerical ***implementation well underway***… still lots to come

- Room for thought:
  - Incorporating lattice/residual volume effects
  - Confronting the non-one-to-one mapping
  - Extending beyond two-to-two! *(Steve's talk)*

Conclusions and outlook

- Multi-hadron observables accesible in LQCD
- Use the ***finite-volume as a tool***



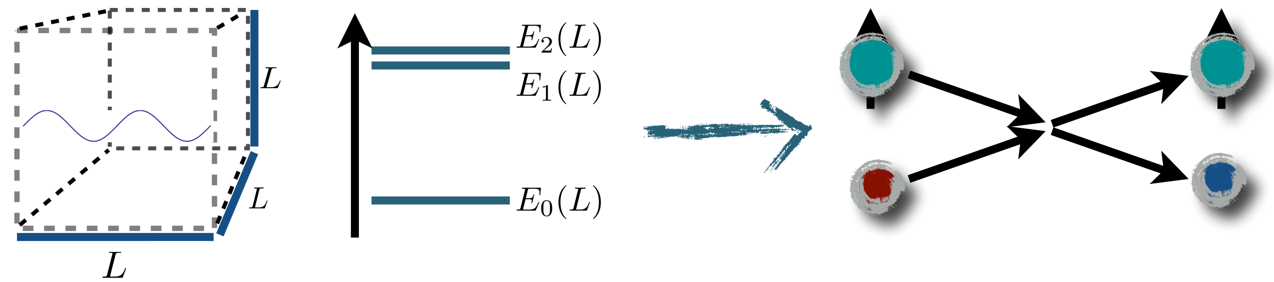
- Numerical ***implementation well underway***… still lots to come
- Room for thought:
  - Incorporating lattice/residual volume effects
  - Confronting the non-one-to-one mapping
  - Extending beyond two-to-two! *(Steve's talk)*

Approaches not covered here

- Finite-volume hamiltonian method → W2 - Tues - ***Derek, Finn***
- HALQCD potential method
- Reconstruction of the spectral function → Fri - ***John, Antonin***
- Lüscher + KSS methods for long-range matrix elements — Tues - ***Zohreh***

Conclusions and outlook

- Multi-hadron observables accesible in LQCD
- Use the ***finite-volume as a tool***



- Numerical ***implementation well underway***… still lots to come
- Room for thought:
  - Incorporating lattice/residual volume effects
  - Confronting the non-one-to-one mapping
  - Extending beyond two-to-two! *(Steve's talk)*

Approaches not covered here

- Finite-volume hamiltonian method → W2 - Tues - ***Derek, Finn***
- HALQCD potential method
- Reconstruction of the spectral function → Fri - ***John, Antonin***
- Lüscher + KSS methods for long-range matrix elements — Tues - ***Zohreh***

Thanks for listening - Enjoy the workshop!