CERN

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Analysis Electron Motion Within the Beam

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Outline

- Theoretical Oscillation
- Uniform Longitudinal Profile
- Gaussian Longitudinal Profile
- Conclusions

Outline

- **Theoretical Oscillation**
  - Frequency
  - Amplitude and Phase
  - Linear Region
  - Invariant
- Uniform Longitudinal Profile
- Gaussian Longitudinal Profile
- Conclusions

Theoretical Oscillation

From the Newton law:

$$m_e \frac{d^2x}{dt^2} = -q_e E_x$$

$$\frac{d^2x}{dt^2} + \frac{q_e}{m_e} E_x = 0 \qquad \textbf{(1)}$$

Where:

- x is the electron position in the horizontal plane;
- m_e is the electron mass;
- q_e is the electron charge;
- E_x is the horizontal electric field due to the bunch passage which acts on the electron:

Theoretical Oscillation

Electric Field of a Two-Dimensional Gaussian Charge

Bassetti-Erskine:

$$E_x(x,y) = \frac{\lambda_z}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x{}^2 - \sigma_y{}^2\right)}}\,\mathrm{Im}\left[W\left(\frac{x+iy}{\sqrt{2\left(\sigma_x{}^2-\sigma_y{}^2\right)}}\right) - e^{-\left[\frac{x^2}{2\sigma_x{}^2}+\frac{y^2}{2\sigma_y{}^2}\right]}W\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2\left(\sigma_x{}^2-\sigma_y{}^2\right)}}\right)\right]$$

Where:

- y is the electron position in the vertical plane;
- ε_0 is the vacuum permittivity;
- σ_x and σ_y are the transverse dimension of the bunch (RMS), horizontal and vertical, respectively ($\sigma_x > \sigma_y$);
- λ_z is the linear longitudinal charge density of the proton bunch;
- $W(\zeta)$ is the complex error function:

$$W(\zeta) = e^{-\zeta^2}\left[1 + \frac{2i}{\sqrt{\pi}}\int_0^{\zeta} e^{-\zeta'^2}\,d\zeta'\right]$$

Theoretical Oscillation

Electric Field of a Two-Dimensional Gaussian Charge

Complex Error Function:

Expanding in series:

$$W(\zeta) = \sum_{n=0}^{\infty} \frac{(j\zeta)^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

Where the Gamma function is defined:

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t}\, dt$$

Stopping at the first order:

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\pi^{\frac{1}{2}}$$

$$W(\zeta) \approx 1 + j\frac{2}{\sqrt{\pi}}\zeta$$

Theoretical Oscillation

Electric Field of a Two-Dimensional Gaussian Charge

$$E_x(x)\Big|_{y=0} = \frac{\lambda_z}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x{}^2 - \sigma_y{}^2\right)}} \mathrm{Im}\left[W\left(\frac{x}{\sqrt{2\left(\sigma_x{}^2 - \sigma_y{}^2\right)}}\right) - e^{-\frac{x^2}{2\sigma_x{}^2}} W\left(\frac{x\frac{\sigma_y}{\sigma_x}}{\sqrt{2\left(\sigma_x{}^2 - \sigma_y{}^2\right)}}\right)\right]$$

Expanding in series and stopping at the first order:

$$E_x(x)\Big|_{y=0} \approx \frac{\lambda_z}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x{}^2 - \sigma_y{}^2\right)}} \mathrm{Im}\left[1 + \frac{2j}{\sqrt{\pi}}\frac{x}{\sqrt{2\left(\sigma_x{}^2 - \sigma_y{}^2\right)}} - 1 - \frac{2j}{\sqrt{\pi}}\frac{x\frac{\sigma_y}{\sigma_x}}{\sqrt{2\left(\sigma_x{}^2 - \sigma_y{}^2\right)}}\right]$$

$$E_x(x)\Big|_{y=0} \approx \frac{\lambda_z}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x{}^2 - \sigma_y{}^2\right)}} \frac{2}{\sqrt{\pi}} \frac{1 - {}^{\sigma_y}/_{\sigma_x}}{\sqrt{2\left(\sigma_x{}^2 - \sigma_y{}^2\right)}} x$$

$$E_x(x) = \frac{\lambda_z}{2\pi\varepsilon_0\sigma_x\left(\sigma_x + \sigma_y\right)} x \qquad \textbf{(2)}$$

The electric field is linear in the area near the centre of the bunch.

Theoretical Oscillation

Substituting (2) in (1):

$$\frac{d^2x}{dt^2} + \frac{q_e}{m_e}\frac{\lambda_z}{2\pi\varepsilon_0\sigma_x(\sigma_x+\sigma_y)}x = 0 \qquad \ddot{x} + {\omega_x}^2 x = 0$$

$$\omega_x = \sqrt{\frac{q_e\lambda_z}{2\pi\varepsilon_0 m_e\sigma_x(\sigma_x+\sigma_y)}} \qquad \textbf{(3)}$$

The linear longitudinal charge density of the proton bunch:

- Uniform distribution:

$$\lambda_z = \frac{q_e N_b}{L} \qquad \textbf{(4)}$$

N_b is the number of proton in the bunch
L is the length of the bunch ($4\sigma_z$)

- Gaussian distribution:

$$\lambda_z(z) = \frac{q_e N_b}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2{\sigma_z}^2}} \qquad \textbf{(5)}$$

Theoretical Oscillation

Equation of the Electron Transverse Motion: Harmonic Oscillator

Symmetry between Horizontal and Vertical plane

$$\ddot{x} + {\omega_x}^2 x = 0 \qquad \textbf{(6)}$$

$$\begin{cases} x(0) = x_0 & \textbf{(7.1)} \\ \dot{x}(0) = v_{x0} & \textbf{(7.2)} \end{cases}$$

Where:

- ω_x is the angular frequency of electron oscillation in the horizontal plane;
- x_0 is the electron position at bunch head;
- v_{x0} is the electron velocity at bunch head.

Theoretical Oscillation

Solution of the harmonic oscillator:

$$x(t) = A_x \cos(\omega_x t + \varphi_x) \qquad \textbf{(8)}$$

Where:

- A_x is the amplitude of electron oscillation in the horizontal plane;
- t is the arrival time of the proton slice;
- φ_x is the phase of electron oscillation in the horizontal plane.

Theoretical Oscillation

Solution of the harmonic oscillator:

$$x(t) = A_x \cos(\omega_x t + \varphi_x) \qquad \textbf{(8)}$$

Oscillation Amplitude

Oscillation Phase

Oscillation Angular Frequency

Theoretical Oscillation: Frequency

Solution of the harmonic oscillator:

$$x(t) = A_x \cos(\omega_x t + \varphi_x) \qquad \textbf{(8)}$$

Oscillation Angular Frequency

$$\omega_x = \sqrt{\frac{q_e \lambda_z}{2\pi\varepsilon_0 m_e \sigma_x(\sigma_x + \sigma_y)}} \qquad \textbf{(3)}$$

Theoretical Oscillation

Solution of the harmonic oscillator:

$$x(t) = A_x \cos(\omega_x t + \varphi_x) \qquad \textbf{(8)}$$

Oscillation Amplitude

Oscillation Phase

Theoretical Oscillation: Amplitude - Phase

Solution of the harmonic oscillator (ultra-relativistic regime $t = z/c$):

$$x(z) = A_x \cos\left[\frac{\omega_x}{c}(z - z_0) + \varphi_x\right] \quad \textbf{(9)}$$

Oscillation Amplitude (A_x)

Traslation in *z* axis: $z_0 = 2\sigma_z$

Oscillation Phase (φ_x)

$$\begin{cases} x(z_0) = x_0 & \textbf{(9.1)} \\ \dot{x}(z_0) = v_{x0} & \textbf{(9.2)} \end{cases}$$

Where:

- x_0 is the electron position at bunch head;
- v_{x0} is the electron velocity at bunch head.

Theoretical Oscillation: Amplitude - Phase

Solution of the harmonic oscillator (ultra-relativistic regime $t = z/c$):

$$x(z) = A_x \cos\left[\frac{\omega_x}{c}(z - z_0) + \varphi_x\right] \quad \textbf{(9)}$$

Oscillation Amplitude

Traslation in *z* axis: $z_0 = 2\sigma_z$

Oscillation Phase

$$\begin{cases} x(z_0) = A_x \cos(\varphi_x) = x_0 & \textbf{(10)} \\ \dot{x}(z_0) = \left[\frac{dx}{dt}\right]_{z_0} = \left[\frac{dx}{dz}\frac{dz}{dt}\right]_{z_0} = -A_x \frac{\omega_x}{c} \sin(\varphi_x)\, c = v_{x0} & \textbf{(11)} \end{cases}$$

Theoretical Oscillation: Amplitude - Phase

$$\begin{cases} A_x \cos(\varphi_x) = x_0 & \textbf{(10)} \\ A_x \sin(\varphi_x) = -\dfrac{v_{x0}}{\omega_x} & \textbf{(11)} \end{cases}$$

↓

(10)² + (11)²

$$A_x{}^2 = x_0{}^2 + \left(\frac{v_{x0}}{\omega_x}\right)^2 \qquad \textbf{(12)}$$

(11) / (10)

$$\varphi_x = -arctan_{IV}\left(\frac{v_{x0}/\omega_x}{x_0}\right) \qquad \textbf{(13)}$$

Theoretical Oscillation: Amplitude - Phase

Solution of the harmonic oscillator (ultra-relativistic regime $t = z/c$):

$$x(z) = A_x \cos\left[\frac{\omega_x}{c}(z - z_0) + \varphi_x\right] \quad \textbf{(9)}$$

Oscillation Amplitude:

$$A_x{}^2 = x_0{}^2 + \left(\frac{v_{x0}}{\omega_x}\right)^2 \quad \textbf{(12)}$$

Traslation in *z* axis:

$$z_0 = 2\sigma_z$$

Oscillation Phase:

$$\varphi_x = -arctan_{IV}\left(\frac{v_{x0}/\omega_x}{x_0}\right) \quad \textbf{(13)}$$

Oscillation Angular Frequency:

$$\omega_x = \sqrt{\frac{q_e \lambda_z}{2\pi\varepsilon_0 m_e \sigma_x(\sigma_x + \sigma_y)}} \quad \textbf{(3)}$$

Theoretical Oscillation:
Linear Region

Normalized radius

$$r_N = \frac{r}{\sigma_x}$$

$$L = 4\sigma_z$$

(14) $$E_r = \frac{N_b q_e}{2\pi\varepsilon_0 L}\frac{1}{r_N \sigma_x}\left(1 - e^{-\frac{r_N{}^2}{2}}\right)$$

(15) $$\frac{dE_r}{dr} = \frac{N_b q_e}{2\pi\varepsilon_0 L r_N{}^2 \sigma_x{}^2}\left(r_N{}^2 e^{-\frac{r_N{}^2}{2}} + e^{-\frac{r_N{}^2}{2}} - 1\right)$$

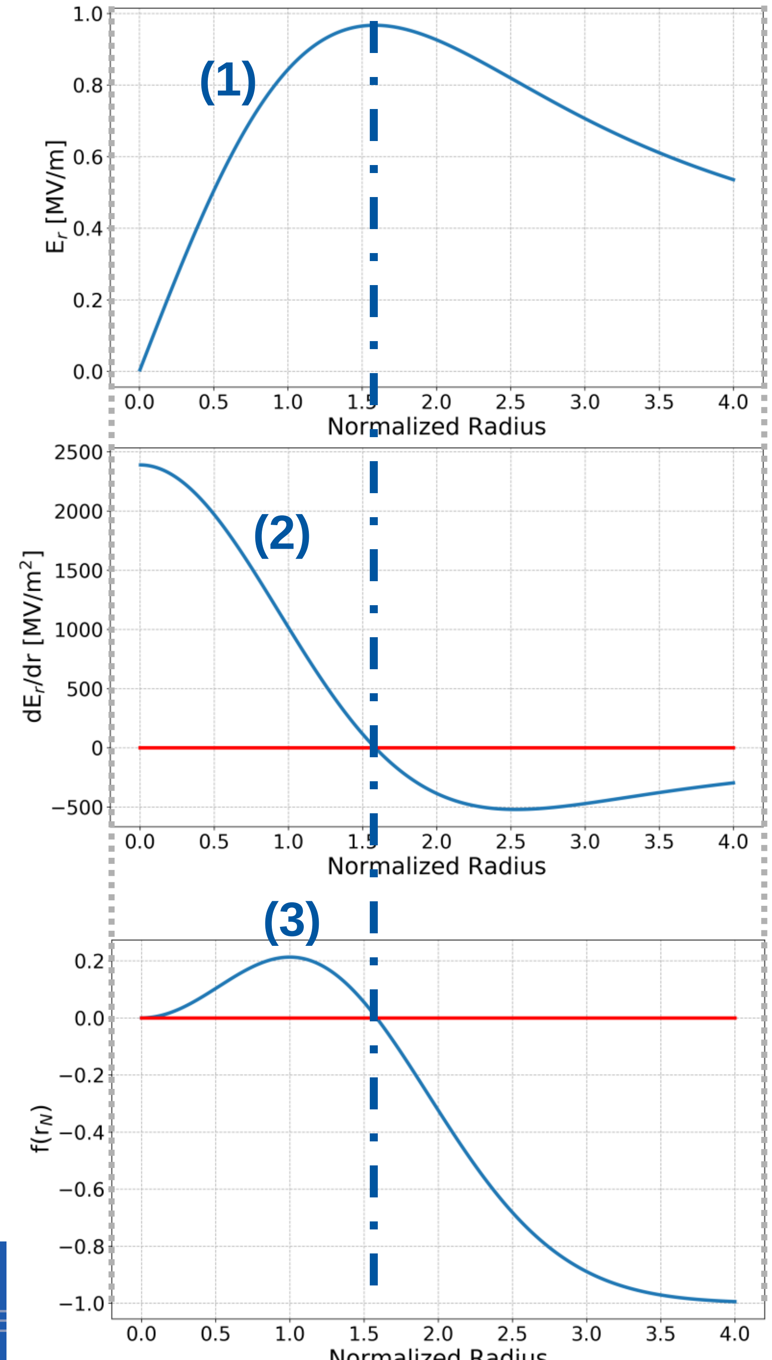
$$\frac{dE_r}{dr} = 0$$

$\sigma_x = \sigma_y$ = 448 µm
σ_z = 89.9 mm
N_b = 1.2e11

(16) $$f(r_N) = r_N{}^2 e^{-\frac{r_N{}^2}{2}} + e^{-\frac{r_N{}^2}{2}} - 1 = 0$$

from (16) $|r_N|$ << 1.59

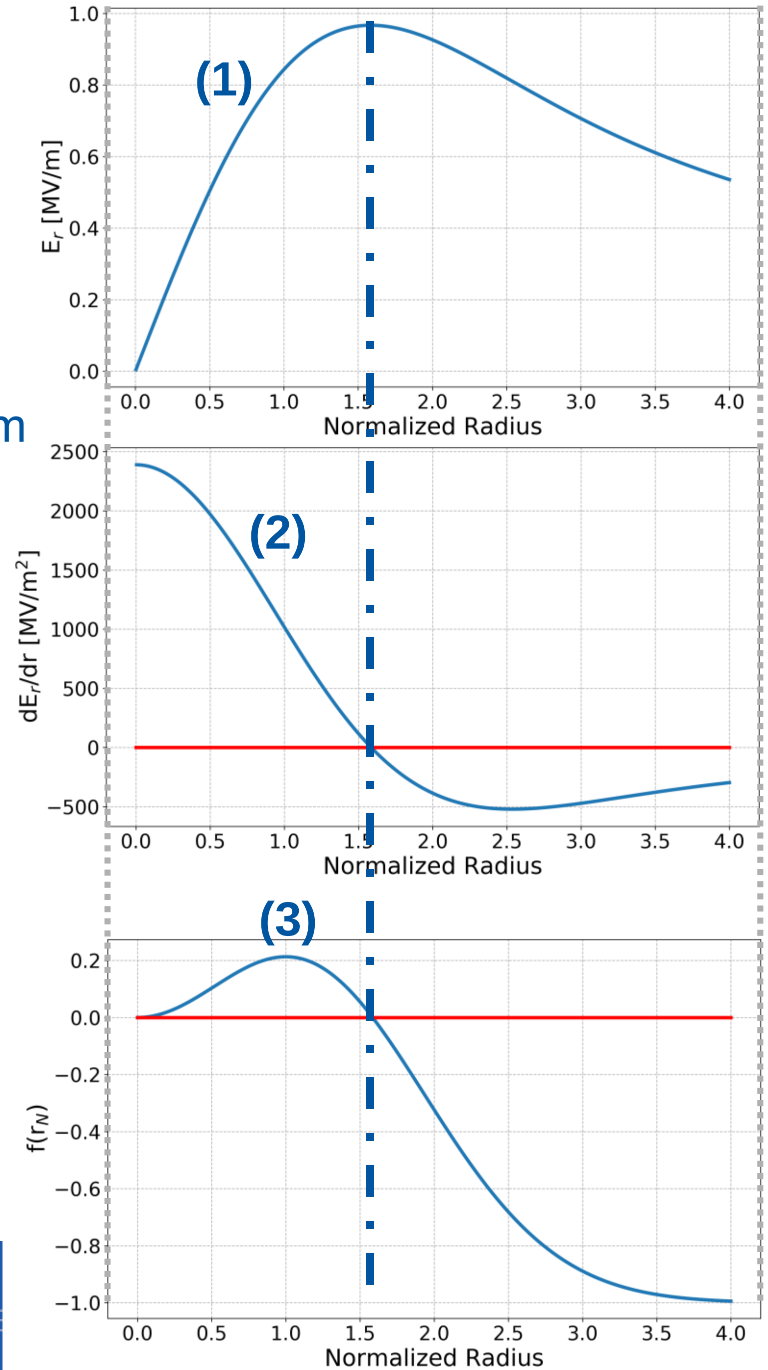
linear region of the electric field



Theoretical Oscillation: Linear Region

In our case:

- from (16) $|r_N| << 1.59$ $\sigma_x = \sigma_y = 448\ \mu m$
- $|r| << r_N * \sigma = 0.712\ \mathrm{mm}$



Theoretical Oscillation: Linear Region

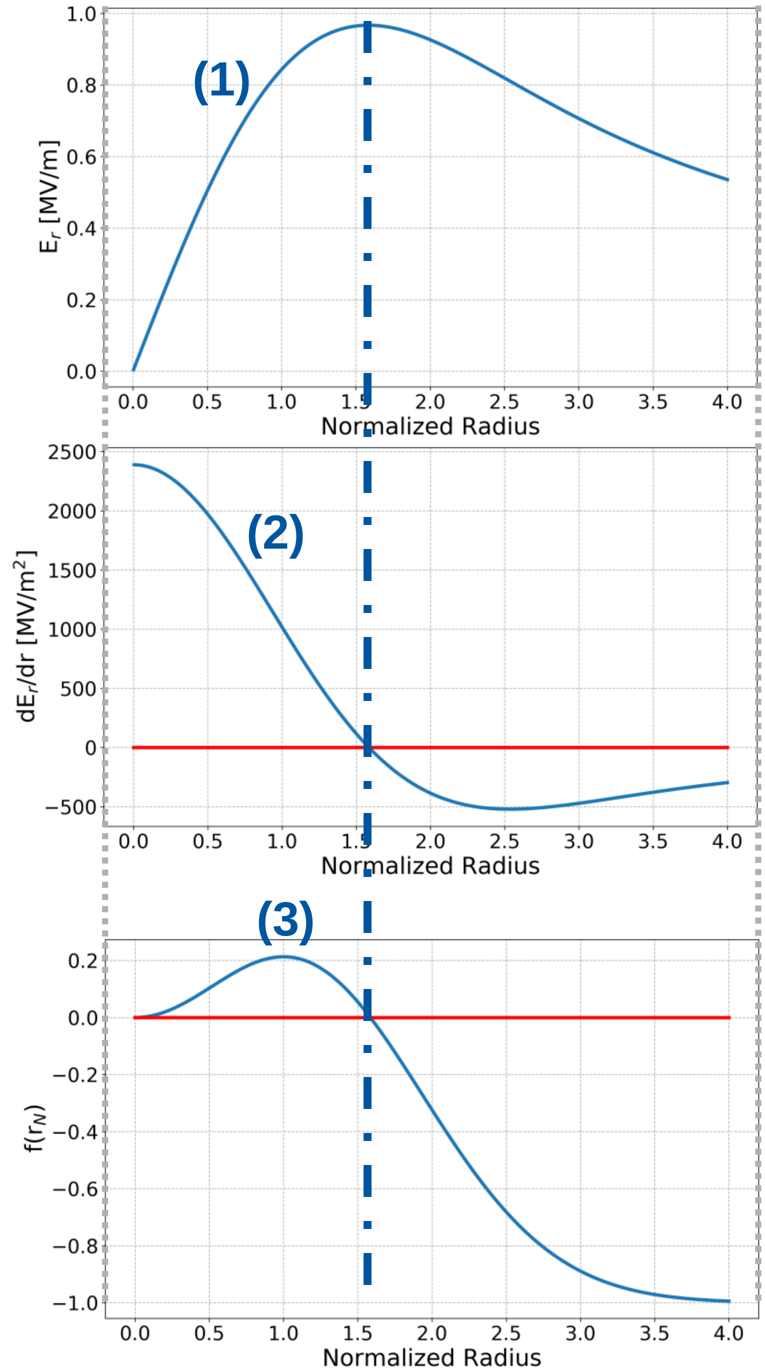
Firstly, we generate the electrons or we use a build-up simulation (no control on the velocity)
Secondly, we can choose the electron:

$$\begin{cases} A_x = \sqrt{{x_0}^2 + \left(\frac{v_{x0}}{\omega_x}\right)^2} \approx A_{xgoal} \\ A_y = \sqrt{{y_0}^2 + \left(\frac{v_{y0}}{\omega_x}\right)^2} \approx A_{ygoal} \end{cases}$$

$$r_{max} = \sqrt{{A_{xgoal}}^2 + {A_{ygoal}}^2}$$

(r_{max}: when the electron is oscillating in phase in the planes)
r_{max} has to be inside the linear region

- Uniform longitudinal profile:
  research at bunch start ($2\sigma_z$)
- Gaussian longitudinal profile:
  research at bucket start (z_{cut})

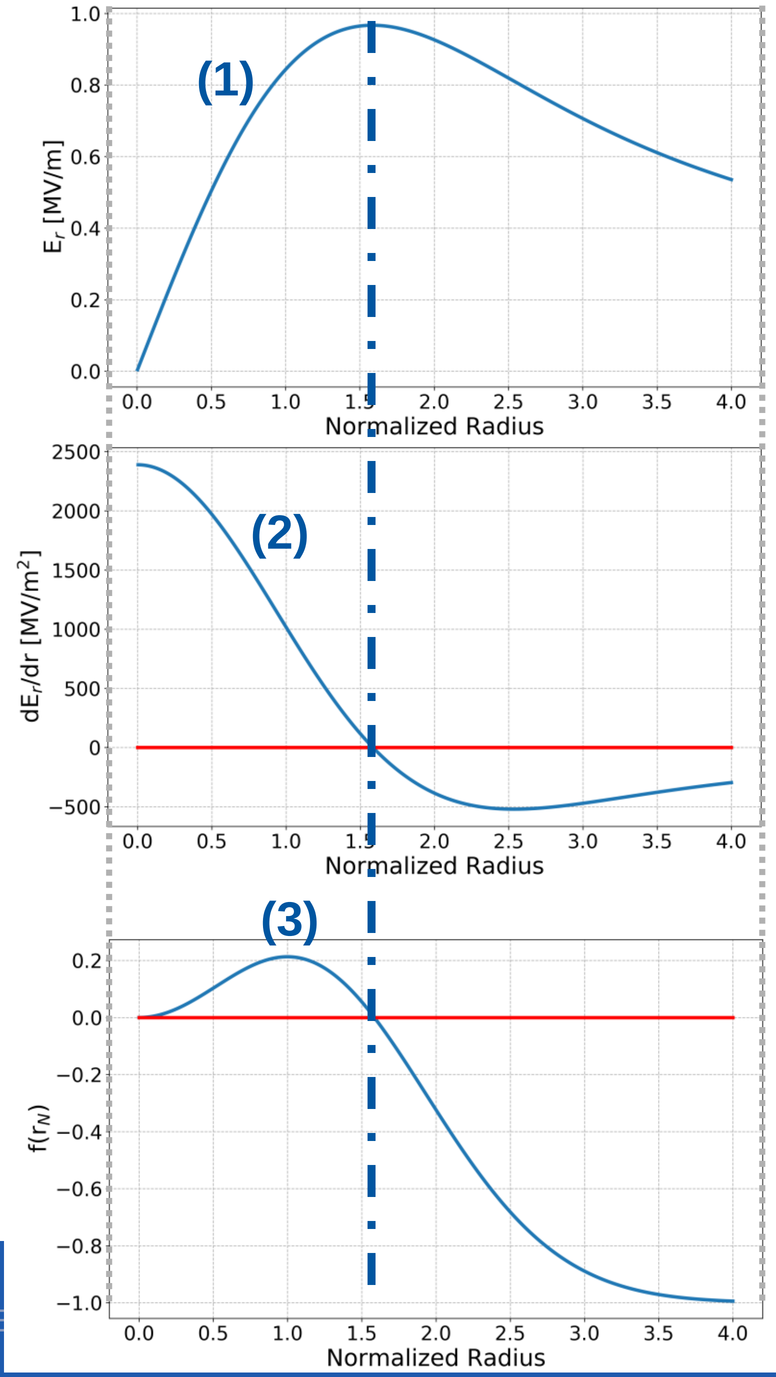


Theoretical Oscillation: Linear Region

The minimization of the mean squared error is the criteria in order to choose the electron:

$$e^{-} = \mathrm{argmin}_i(d_i)$$

$$d_i = \sqrt{\left(A_{xi} - A_{x\mathrm{goal}}\right)^2 + \left(A_{yi} - A_{y\mathrm{goal}}\right)^2}$$



Theoretical Oscillation: Invariant

$$x(t) = A_x \cos(\omega_x t + \varphi_x) \quad \textbf{(8)} \Rightarrow v_x(t) = -A_x \omega_x \sin(\omega_x t + \varphi_x) \quad \textbf{(17)}$$

The kinetic energy of the system is:

$$K(t) = \frac{1}{2} m_e {v_x}^2(t) = \frac{1}{2} m_e \, {A_x}^2 {\omega_x}^2 \sin^2(\omega_x t + \varphi_x) \quad \textbf{(18)}$$

The potential energy of the system is:

$$\mathrm{U}(t) = \frac{1}{2} m_e {\omega_x}^2 x^2(t) = \frac{1}{2} m_e \, {A_x}^2 {\omega_x}^2 \cos^2(\omega_x t + \varphi_x) \quad \textbf{(19)}$$

The total energy of the system is:

$$E = K(t) + \mathrm{U}(t) = \frac{1}{2} m_e {A_x}^2 {\omega_x}^2 = \frac{1}{2} m_e {\omega_x}^2 \left({x_0}^2 + \frac{{v_{x0}}^2}{{\omega_x}^2} \right) \quad \textbf{(20)}$$

(it does not depend on time: invariant)

Simulation Parameters

- Bunch Intensity: 1.2e11 protons per bunch
- Bunch length: 1.20 ns
- $\varepsilon_{nx} = \varepsilon_{ny} = 2.5\ \mu m$
- Energy: 7 TeV
- Electron density: 1e12 e^-/m^3 (drift, dipole), build-up (quad)
- SEY: 1.30
- $\beta_x = \beta_y = 600$ m

Numerical Parameters

- Slices = 500
- MPs/slice = 5,000
- Segments = 16

- Max Electron MPs = 900,000

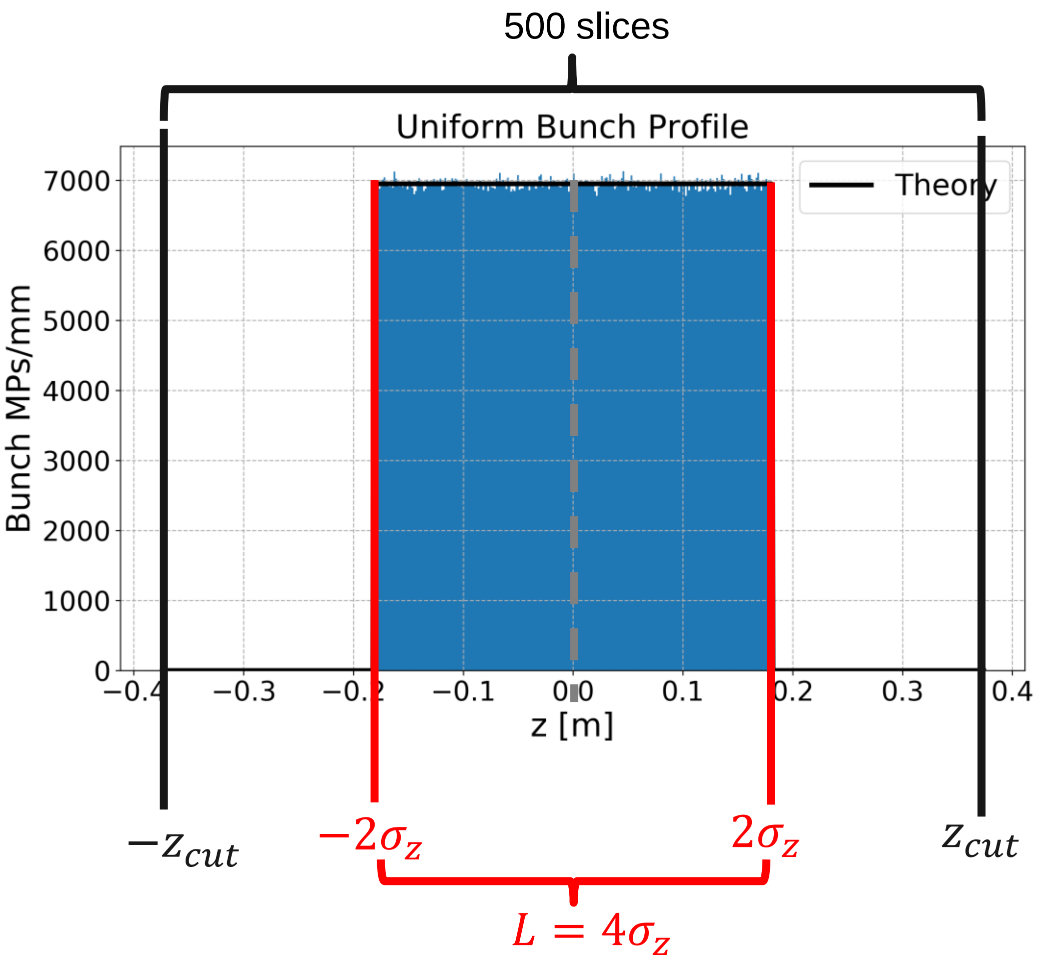
Outline

- Theoretical Oscillation
- **Uniform Longitudinal Profile**
  - Drift Space
  - Dipole
  - Arc Quadrupole
- Gaussian Longitudinal Profile
- Conclusions

Uniform Longitudinal Profile

$\sigma_z = 89.9$ mm

$z_{cut} = 375$ mm



$$\begin{cases} \frac{\text{MPs}}{L} & |z| < \frac{L}{2} \\ 0 & |z| > \frac{L}{2} \end{cases}$$

Uniform Longitudinal Profile

Theoretical Oscillation: Frequency

In the case study: uniform distribution and round bunch

$$\omega_x = \sqrt{\frac{q_e \lambda_z}{2\pi\varepsilon_0 m_e \sigma_x(\sigma_x + \sigma_y)}} \quad \textbf{(3)}$$

$$\lambda_z = \frac{q_e N_b}{L} \quad \textbf{(4)}$$

$$f_x = f_y = \frac{1}{2\pi}\omega_x = \mathbf{3.26\ GHz}$$

Where:

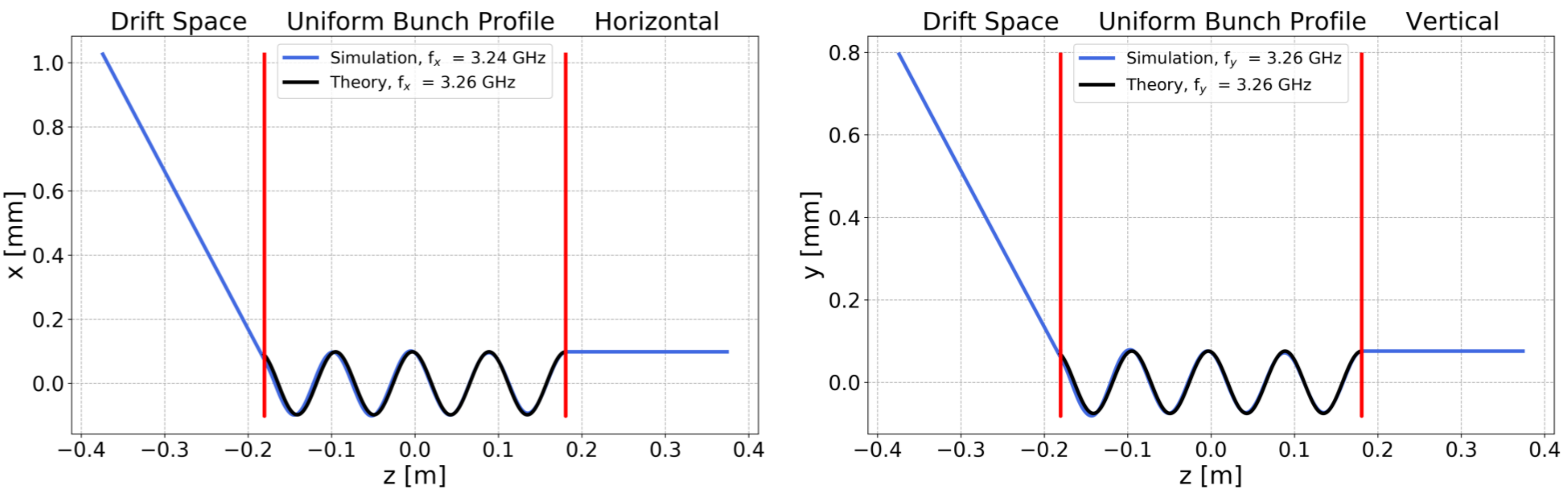
- N_b = 1.2e11;
- σ_z = 89.9 mm;
- L = 360 mm;
- $\sigma_x = \sigma_y$ = 448 μm.

Outline

- Theoretical Oscillation
- **Uniform Longitudinal Profile**
  - **Drift Space**
  - Dipole
  - Arc Quadrupole
- Gaussian Longitudinal Profile
- Comparisons

Uniform Longitudinal Profile: Drift Space

$A_{x\text{goal}} = 0.1$ mm $\quad A_{y\text{goal}} = 0.1$ mm $\quad$ at bunch start



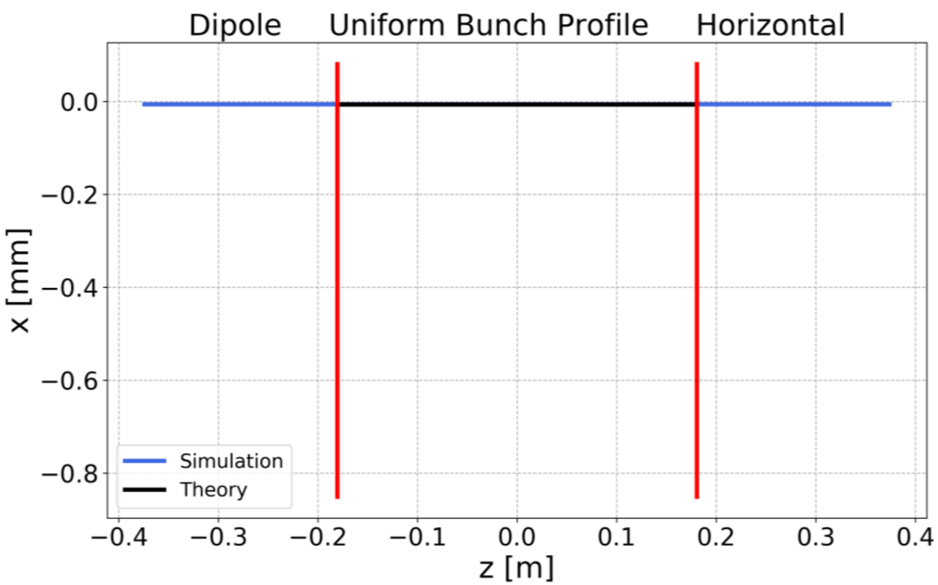
- Good agreement between simulation and theoretical prediction

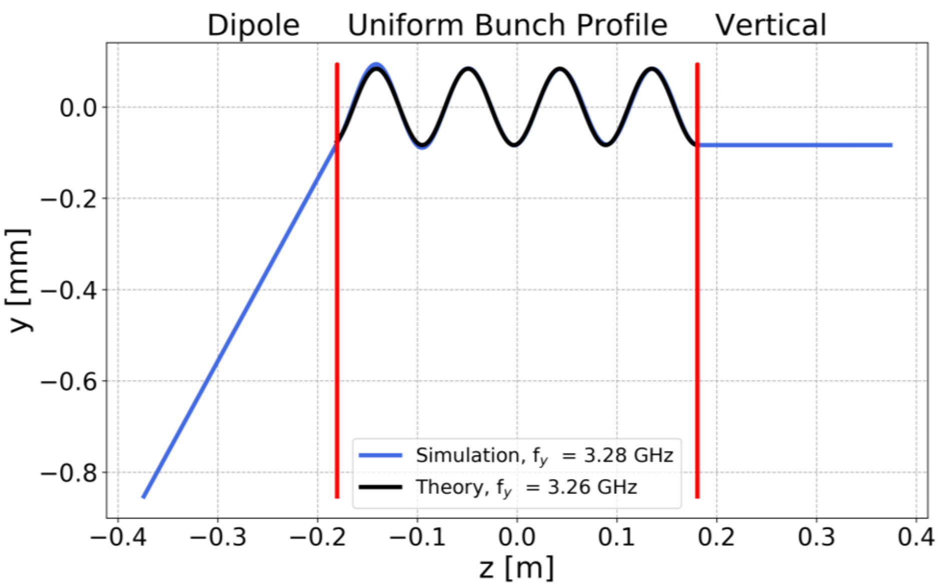
Outline

- Theoretical Oscillation
- **Uniform Longitudinal Profile**
  - Drift Space
  - **Dipole**
  - Arc Quadrupole
- Gaussian Longitudinal Profile
- Conclusions

Uniform Longitudinal Profile: Dipole

$A_{x\text{goal}} = 0.0$ mm $\quad A_{y\text{goal}} = 0.1$ mm $\quad$ at bunch start





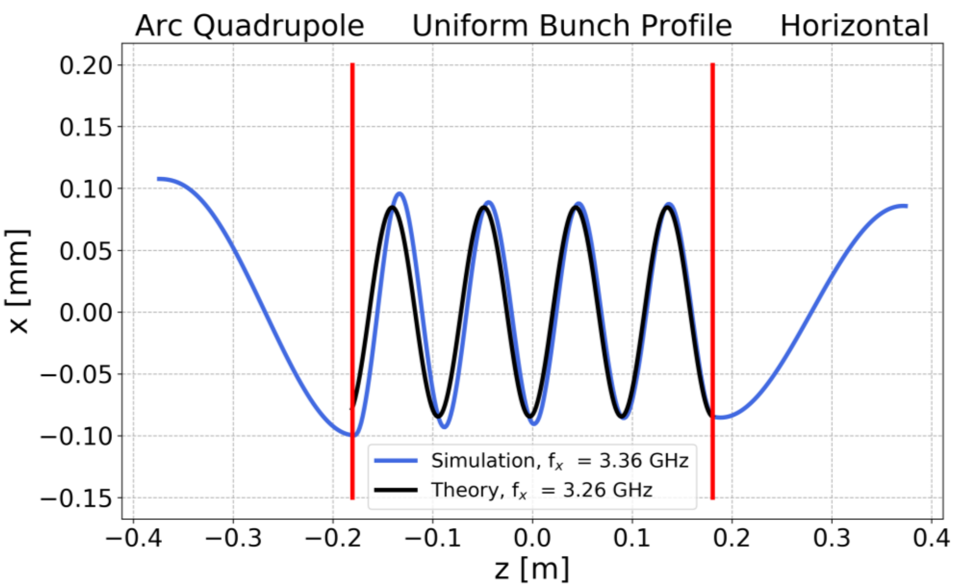
- Good agreement between simulation and theoretical prediction
- In horizontal plane, the electrons cannot move due to the presence of the dipolar magnetic field

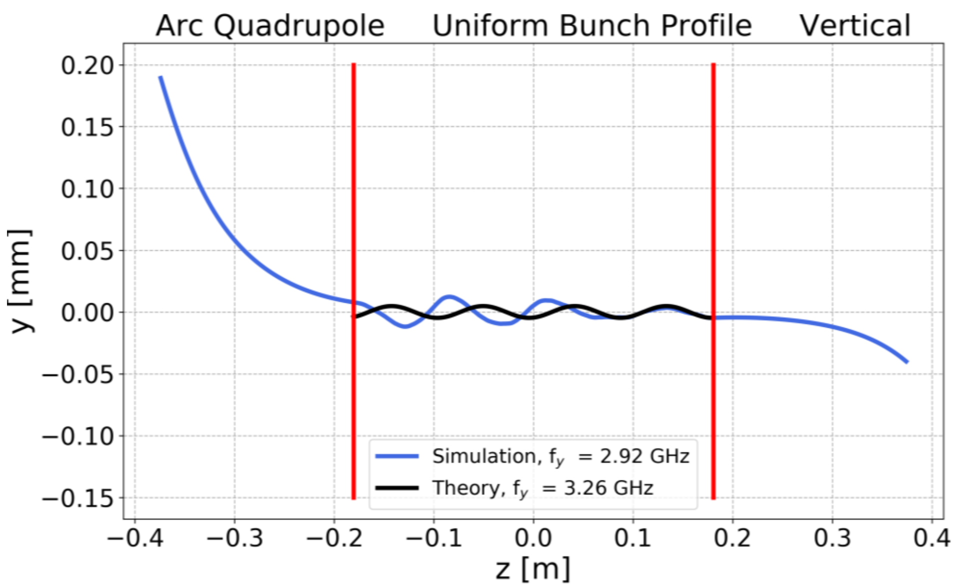
Outline

- Theoretical Oscillation
- **Uniform Longitudinal Profile**
  - Drift Space
  - Dipole
  - **Arc Quadrupole**
- Gaussian Longitudinal Profile
- Conclusions

Uniform Longitudinal Profile: Quadrupole

$A_{xgoal} = 0.1$ mm $\quad A_{ygoal} = 0.0$ mm $\quad$ at bunch start

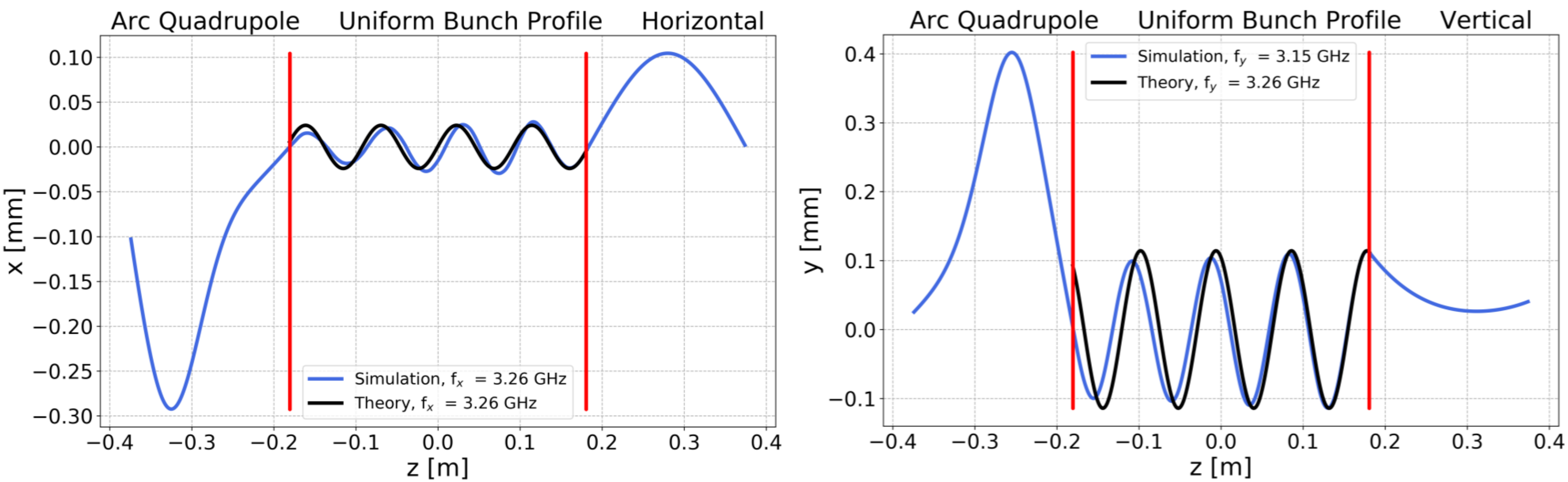




- Good agreement between simulation and theoretical prediction

Uniform Longitudinal Profile: Quadrupole

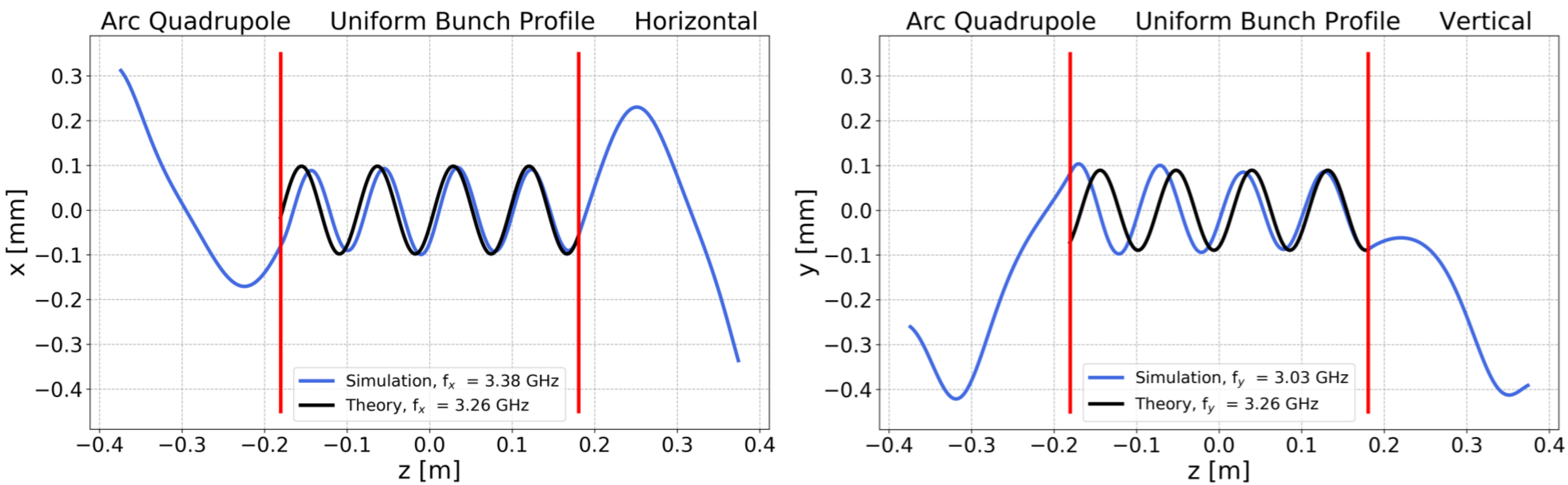
$A_{xgoal} = 0.0$ mm $\quad A_{ygoal} = 0.1$ mm $\quad$ at bunch start



- Good agreement between simulation and theoretical prediction

Uniform Longitudinal Profile: Quadrupole

$A_{x\mathrm{goal}} = 0.1$ mm $\quad A_{y\mathrm{goal}} = 0.1$ mm $\quad$ at bunch start



- The equations work well also when the electron oscillates in both the planes

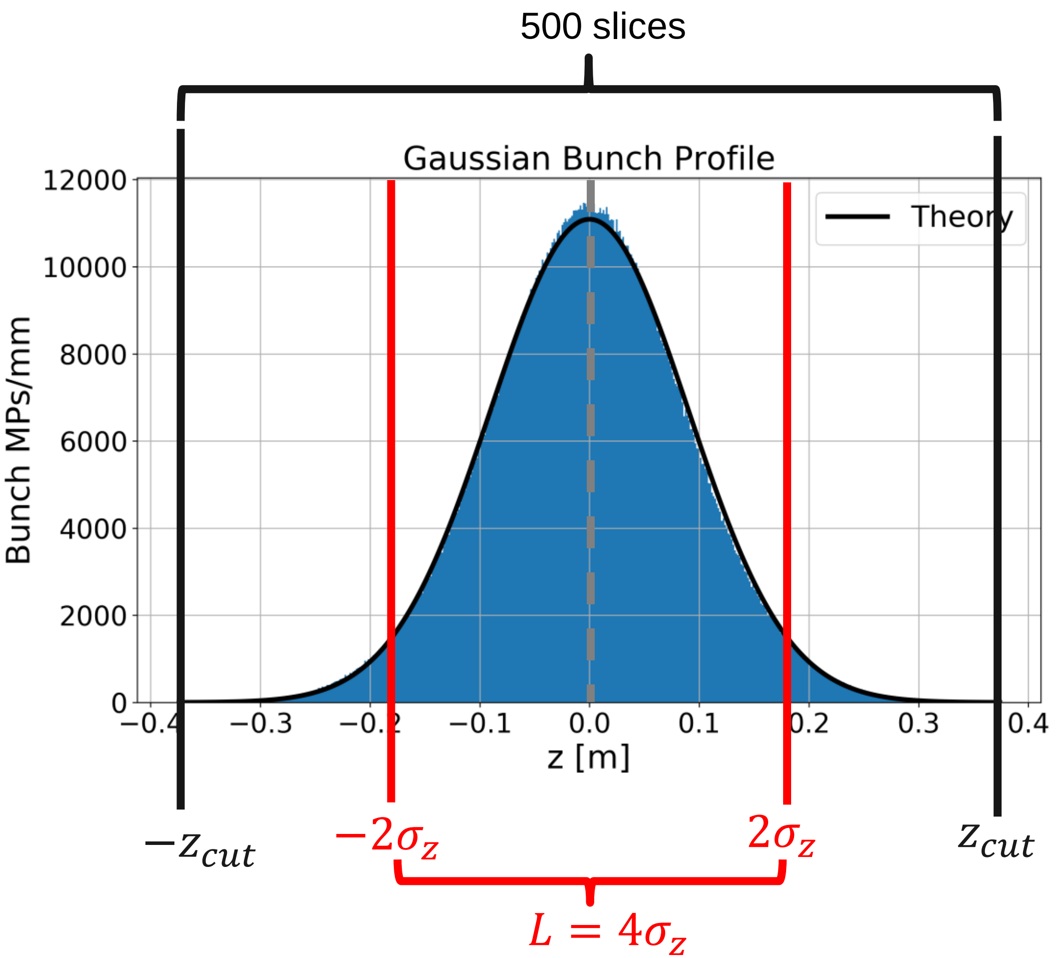
Outline

- Theoretical Oscillation
- Uniform Longitudinal Profile
- **Gaussian Longitudinal Profile**
  - Drift Space
  - Dipole
  - Arc Quadrupole
- Conclusions

Gaussian Longitudinal Profile

$\sigma_z = 89.9$ mm

$z_{cut} = 375$ mm



$$\frac{\text{MPs}}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2{\sigma_z}^2}}$$

Outline

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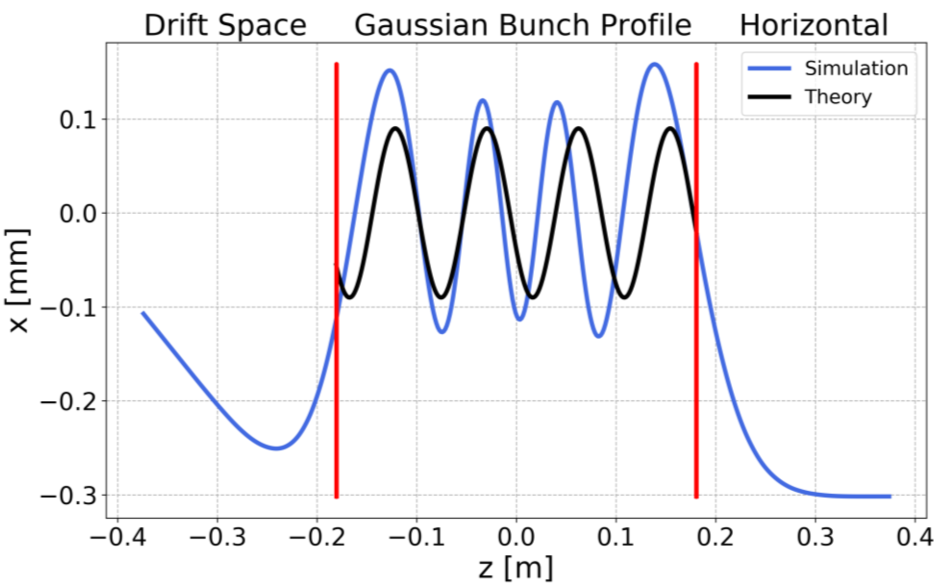
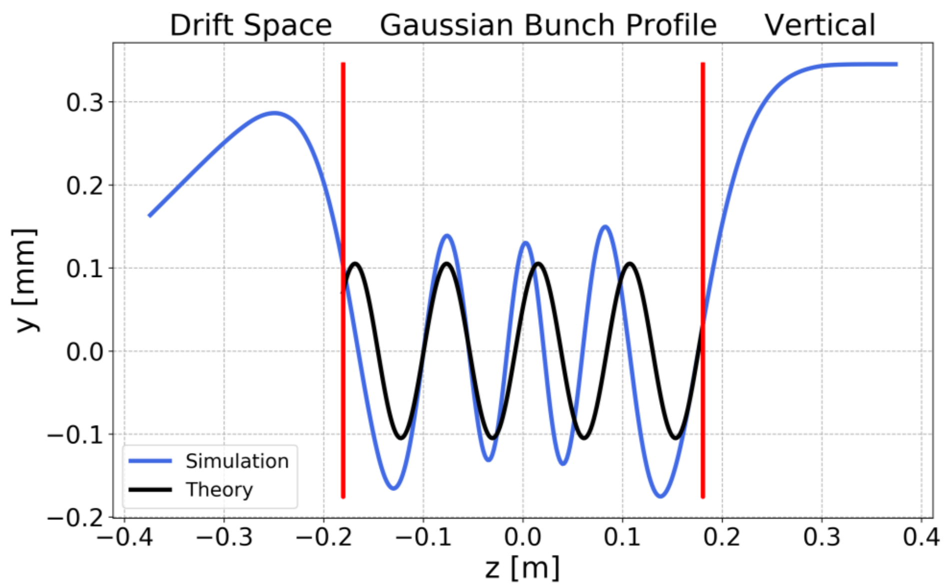
Gaussian Longitudinal Profile: Drift Space

$A_{xgoal} = 0.1$ mm $A_{ygoal} = 0.1$ mm at bunch start

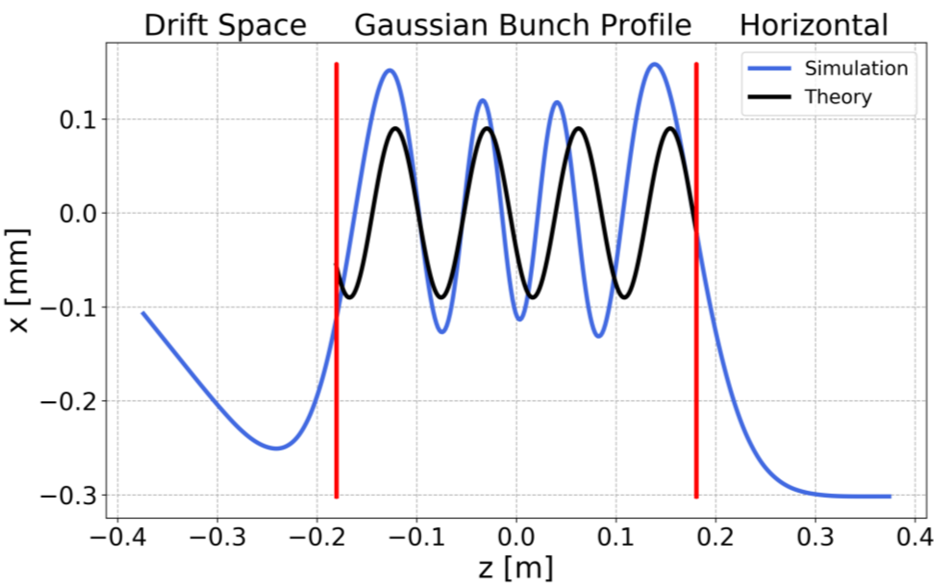


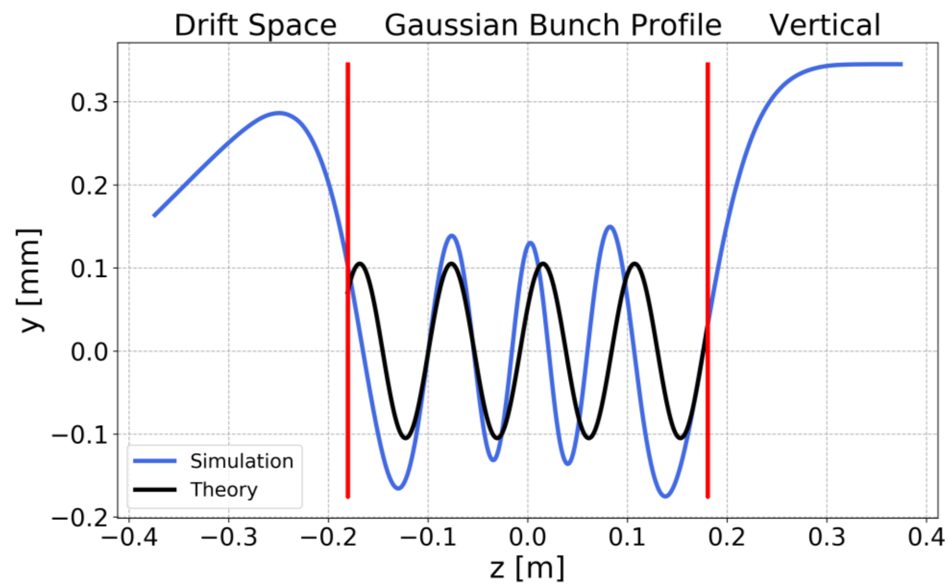


- Using the equations of the case of uniform longitudinal profile

Gaussian Longitudinal Profile: Drift Space

$A_{xgoal} = 0.1$ mm $A_{ygoal} = 0.1$ mm   at bunch start





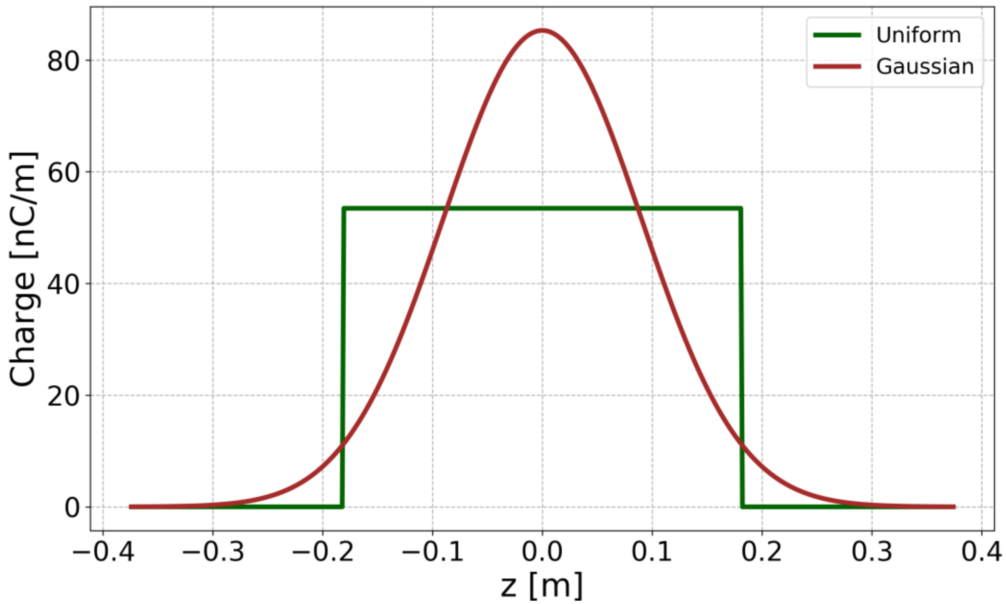
- The frequency increases in the centre of the proton bunch (more protons):

$$\omega_x = \sqrt{\frac{q_e \lambda_z}{2\pi\varepsilon_0 m_e \sigma_x(\sigma_x + \sigma_y)}} \quad \textbf{(3)} \qquad \lambda_z = \frac{q_e N_b}{L} \quad \textbf{(4)}$$

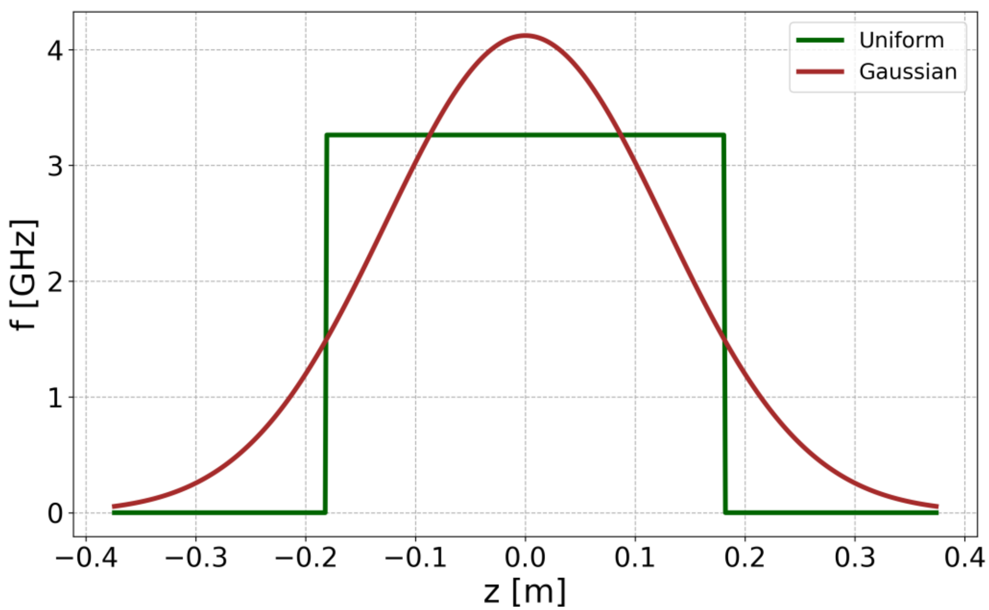
- The amplitude decreases in the centre of the proton bunch (more protons):

$$A_x{}^2 = x_0{}^2 + \left(\frac{v_{x0}}{\omega_x}\right)^2 \qquad \textbf{(12)}$$

Gaussian Longitudinal Profile



$$\lambda_z(z) = \frac{q_e N_b}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2{\sigma_z}^2}} \quad \textbf{(5)}$$



- The local frequency increases in the centre of the proton bunch

$$f_x(z) = 2\pi \sqrt{\frac{q_e \lambda_z(z)}{2\pi\varepsilon_0 m_e \sigma_x(\sigma_x + \sigma_y)}} \quad \textbf{(3)}$$

Gaussian Longitudinal Profile

In the case of Gaussian longitudinal profile the frequency depends on the time:

$$\ddot{x} + {\omega_x}^2(t)x = 0 \qquad \textbf{(6)}$$

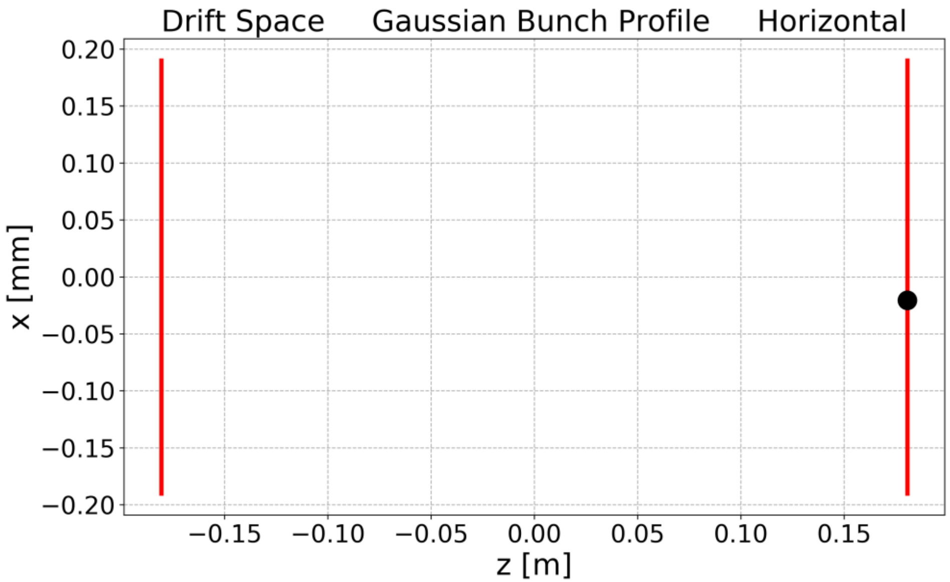
$$x(t) = A_x \cos(\omega_x(t)t + \varphi_x) \qquad \textbf{(8)}$$

Therefore, (8) is only a local approximated solution (not a global solution) of our problem.

We can use an iterative method

Gaussian Longitudinal Profile

Step 0

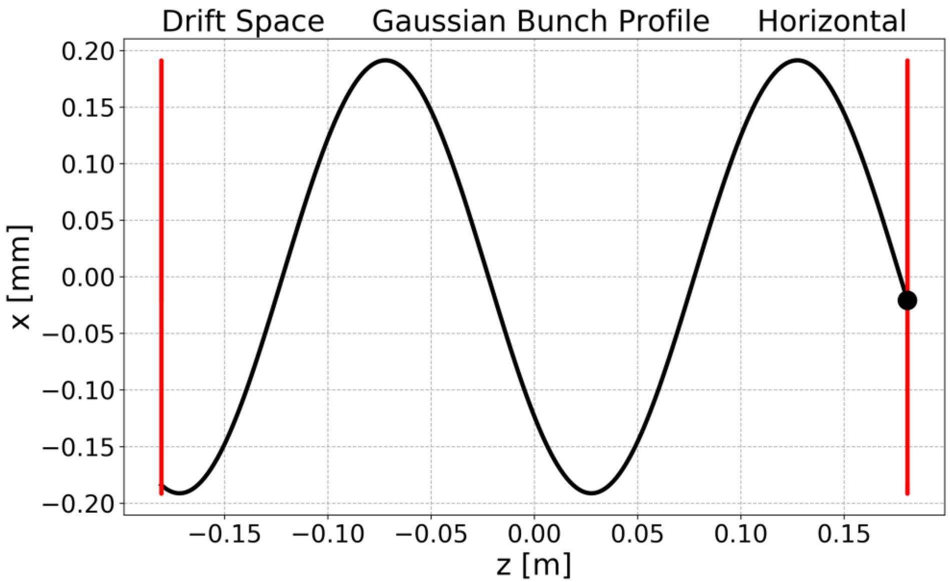


$$x(z_0) = x_0 \qquad \textbf{(9.1)}$$

The initial position of the electron is given by the initial conditions

Gaussian Longitudinal Profile

Step 1



$$x(z) = A_{x0} \cos\left[\frac{\omega_x(z_0)}{c}(z - z_0) + \varphi_{x0}\right] \quad \textbf{(21)}$$

$$A_{x0} = A_x(z_0) = \sqrt{{x_0}^2 + \left(\frac{v_{x0}}{\omega_x(z_0)}\right)^2} \quad \textbf{(12)}$$

$$\varphi_{x0} = \varphi_x(z_0) = -arctan_{IV}\left(\frac{{v_{x0}}/{\omega_x(z_0)}}{x_0}\right) \textbf{(13)}$$

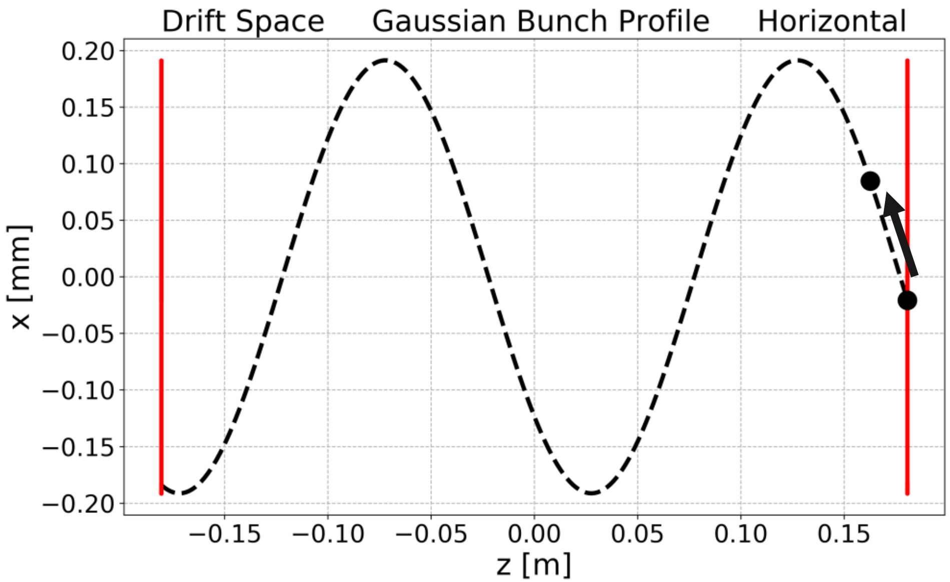
The local solution at z_0 is given by (21).

In order to simplify the mathematical notation:

$$x_{\mathrm{k}} = x(z_k)$$
$$\omega_{xk} = \omega_x(z_k)$$
$$A_{xk} = A_x(z_k)$$
$$\varphi_{xk} = \varphi_x(z_k)$$

Gaussian Longitudinal Profile

Step 1



$$x_1 = A_{x0} \cos\left[\frac{\omega_{x0}}{c}(z_1 - z_0) + \varphi_{x0}\right]$$

$$\Delta z_1 = z_1 - z_0$$

Uniform sampling:

$$\Delta z_k = \Delta z$$

$$x_1 = A_{x0} \cos\left[\frac{\omega_{x0}}{c}\Delta z + \varphi_{x0}\right] \qquad \textbf{(22)}$$

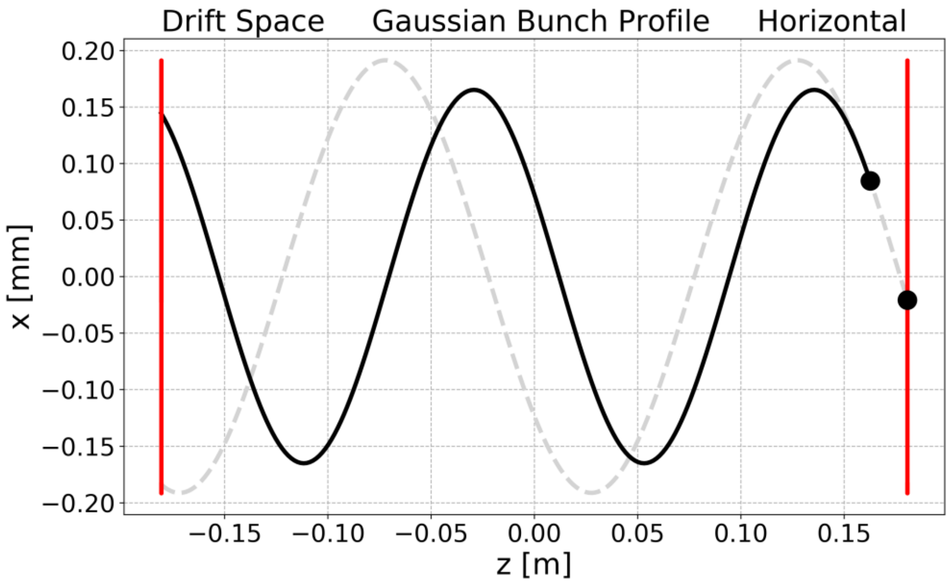
The electron position at z_1 is given by (22).

The initial conditions of the next step are:

$$\begin{cases} x_1 \\ v_{x1} = -A_{x0}\omega_{x0} \sin\left(\frac{\omega_{x0}}{c}\Delta z + \varphi_{x0}\right) \end{cases}$$

Gaussian Longitudinal Profile

Step 2



$$x(z) = A_{x1} \cos\left[\frac{\omega_{x1}}{c}(z - z_1) + \varphi_{x1}\right] \quad \textbf{(23)}$$

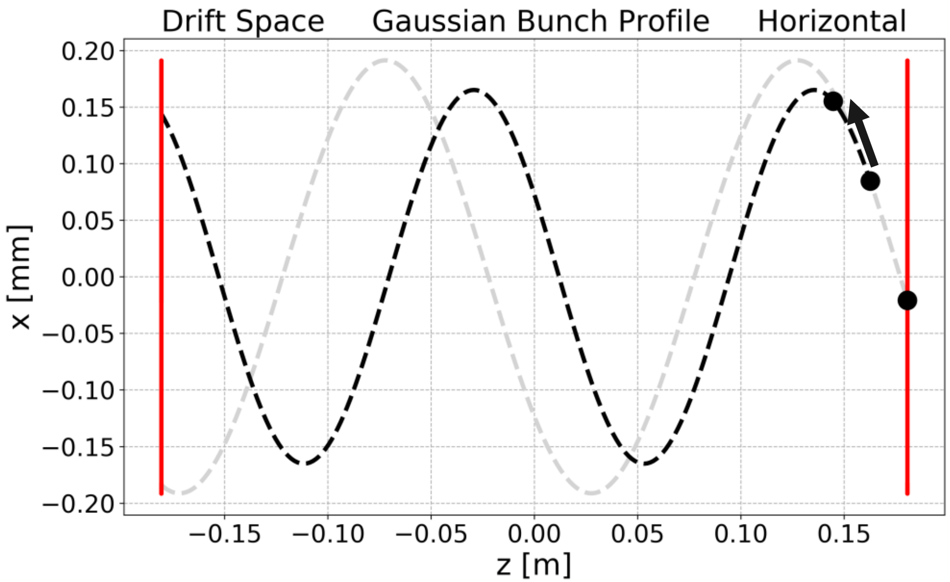
$$A_{x1} = \sqrt{{x_1}^2 + \left(\frac{v_{x1}}{\omega_{x1}}\right)^2} \quad \textbf{(12)}$$

$$\varphi_{x1} = -arctan_{IV}\left(\frac{{v_{x1}}/{\omega_{x1}}}{x_1}\right) \quad \textbf{(13)}$$

The solution at z_1 is given by (23)

Gaussian Longitudinal Profile

Step 2



$$x_2 = A_{x1} \cos\left[\frac{\omega_{x1}}{c}\Delta z + \varphi_{x1}\right] \quad \textbf{(24)}$$

The electron position at z_2 is given by (24).

The initial conditions of the next step are:

$$\begin{cases} x_2 \\ v_{x2} = -A_{x1}\omega_{x1} \sin\left(\frac{\omega_{x1}}{c}\Delta z + \varphi_{x1}\right) \end{cases}$$

And so on…

Gaussian Longitudinal Profile

Step k

Summary:

$$A_{xk} = \sqrt{{x_k}^2 + \left(\frac{v_{x\mathrm{k}}}{\omega_{xk}}\right)^2} \qquad \textbf{(25)}$$

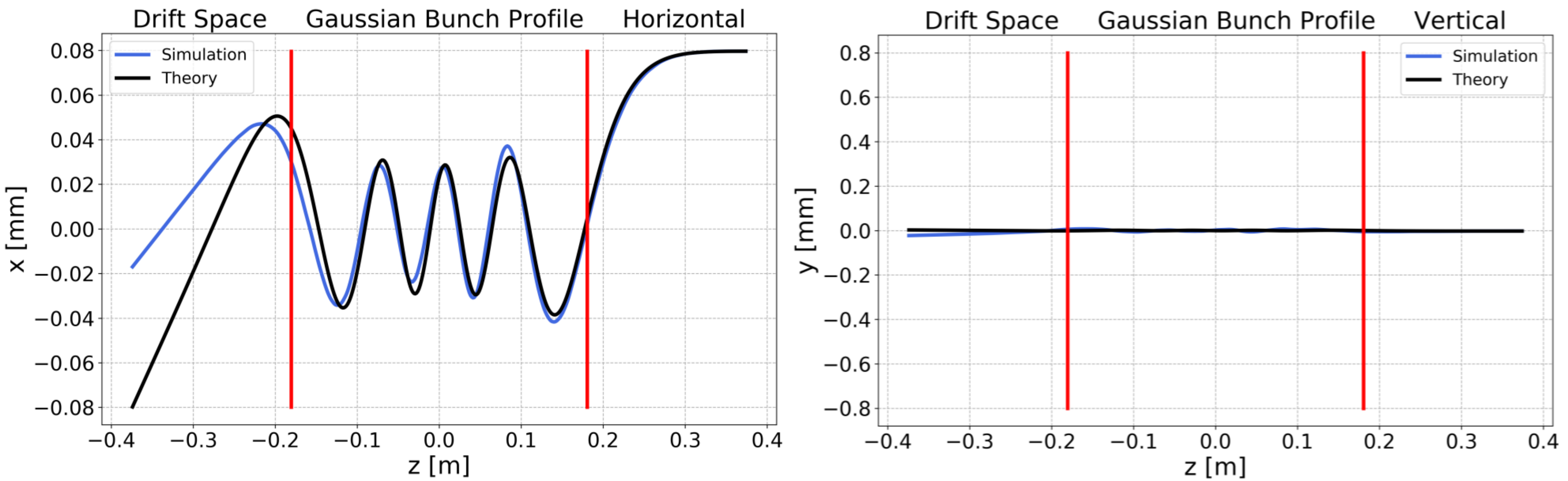
$$\varphi_{xk} = -arctan_{IV}\left(\frac{v_{x\mathrm{k}}/\omega_{xk}}{x_\mathrm{k}}\right) \qquad \textbf{(26)}$$

$$x_{\mathrm{k}+1} = A_{xk}\cos\left[\frac{\omega_{xk}}{c}\Delta z + \varphi_{xk}\right] \qquad \textbf{(27)}$$

$$v_{xk+1} = -A_{xk}\omega_{xk}\sin\left(\frac{\omega_{xk}}{c}\Delta z + \varphi_{xk}\right) \qquad \textbf{(28)}$$

Gaussian Longitudinal Profile: Drift Space

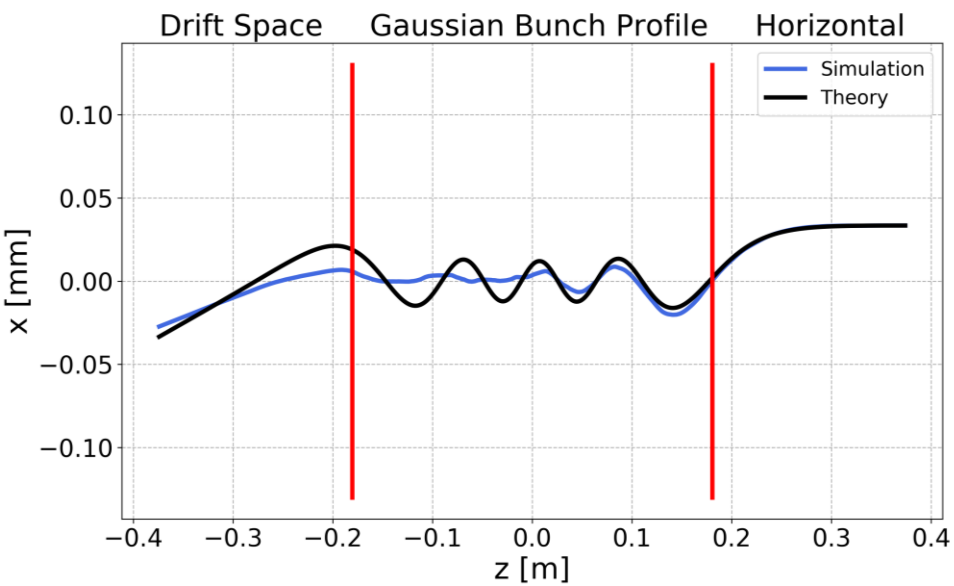
$A_{x\text{goal}} = 0.1$ mm $A_{y\text{goal}} = 0.0$ mm at bucket start

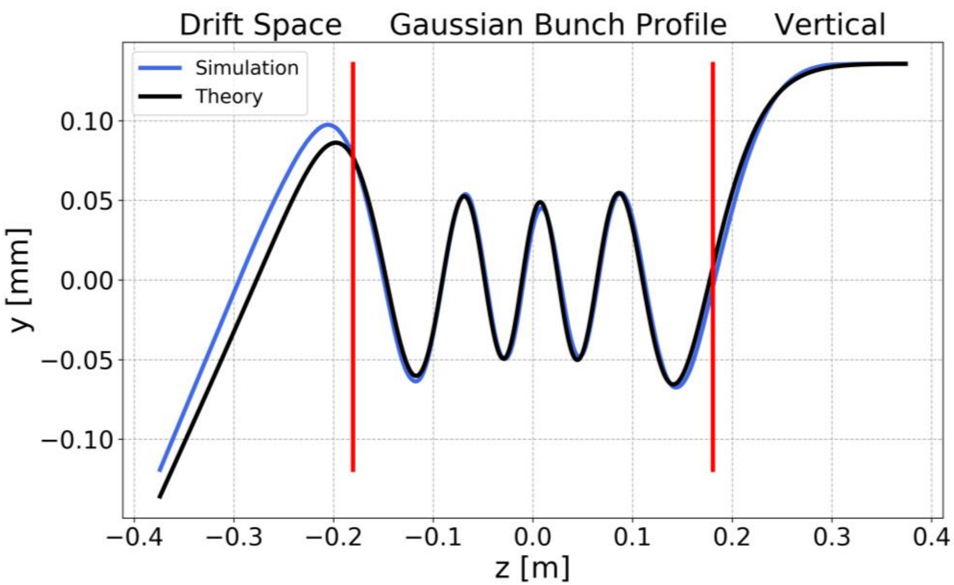


- Good agreement between simulation and theoretical prediction

Gaussian Longitudinal Profile: Drift Space

$A_{xgoal} = 0.0$ mm $A_{ygoal} = 0.1$ mm at bucket start

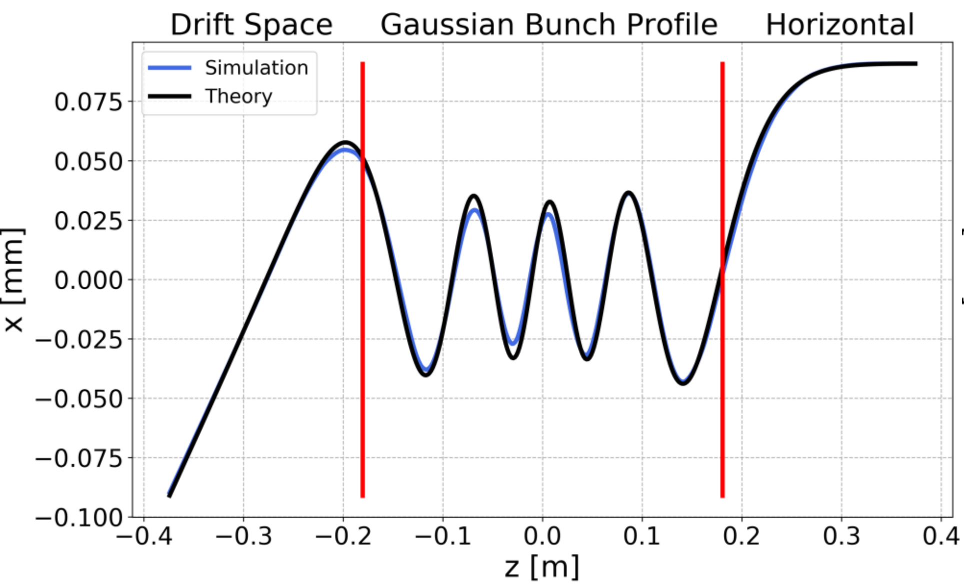


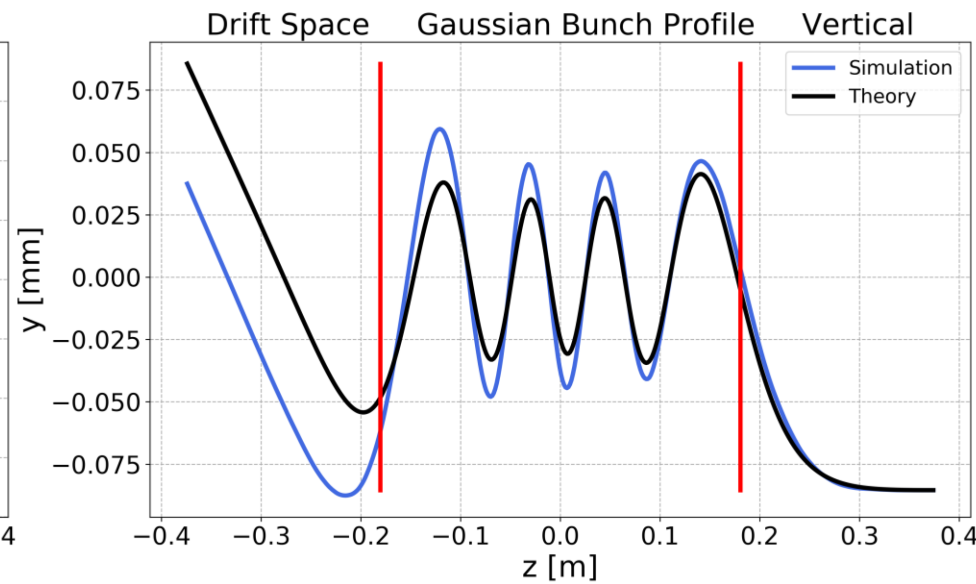


- Good agreement between simulation and theoretical prediction

Gaussian Longitudinal Profile: Drift Space

$A_{x\mathrm{goal}} = 0.1$ mm $\quad A_{y\mathrm{goal}} = 0.1$ mm $\quad$ at bucket start





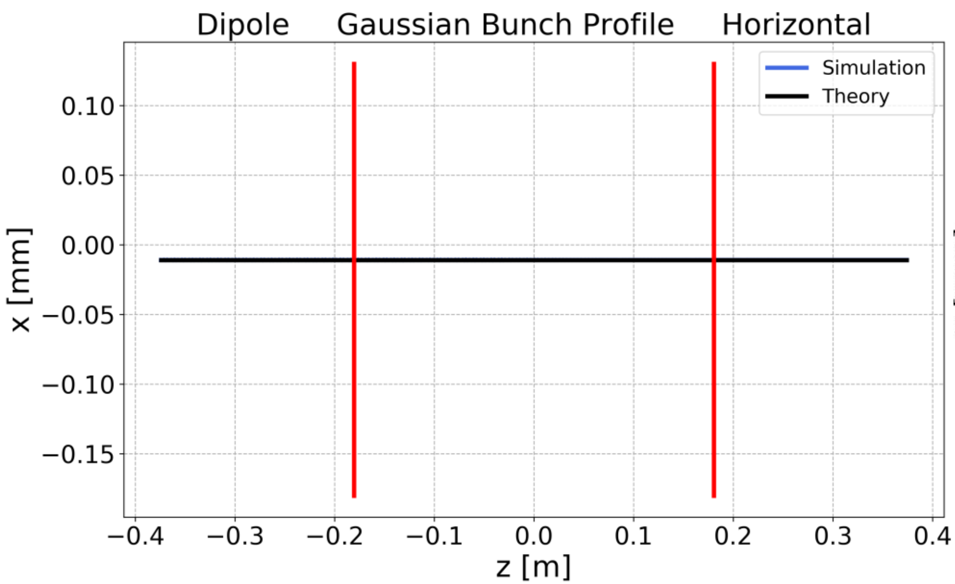
- The equations work well also when the electron oscillates in both the planes

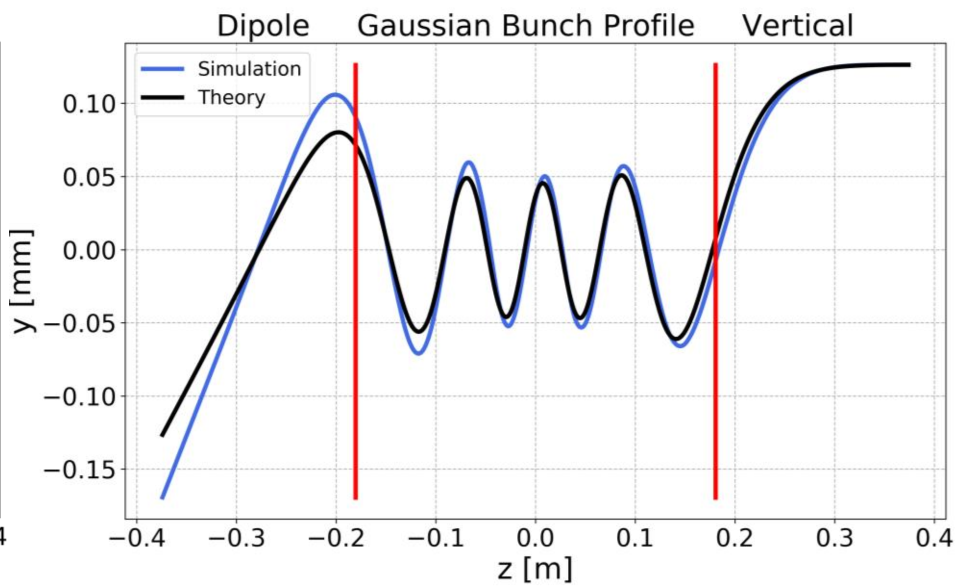
Outline

- Theoretical Oscillation
- Uniform Longitudinal Profile
- **Gaussian Longitudinal Profile**
  - Drift Space
  - **Dipole**
  - Arc Quadrupole
- Conclusions

Gaussian Longitudinal Profile: Dipole

$A_{xgoal} = 0.0$ mm $A_{ygoal} = 0.1$ mm at bucket start





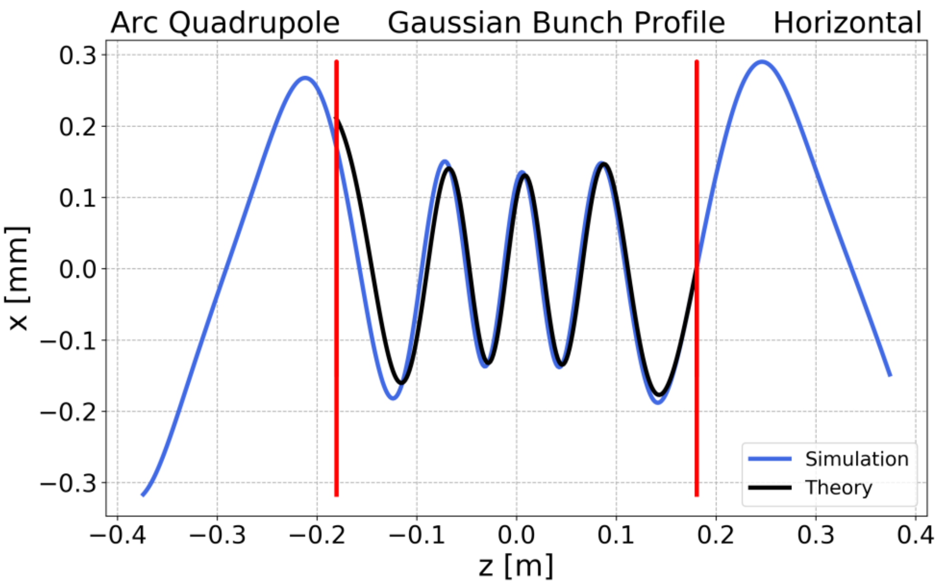
- Good agreement between simulation and theoretical prediction
- In horizontal plane, the electrons cannot move due to the presence of the dipolar magnetic field

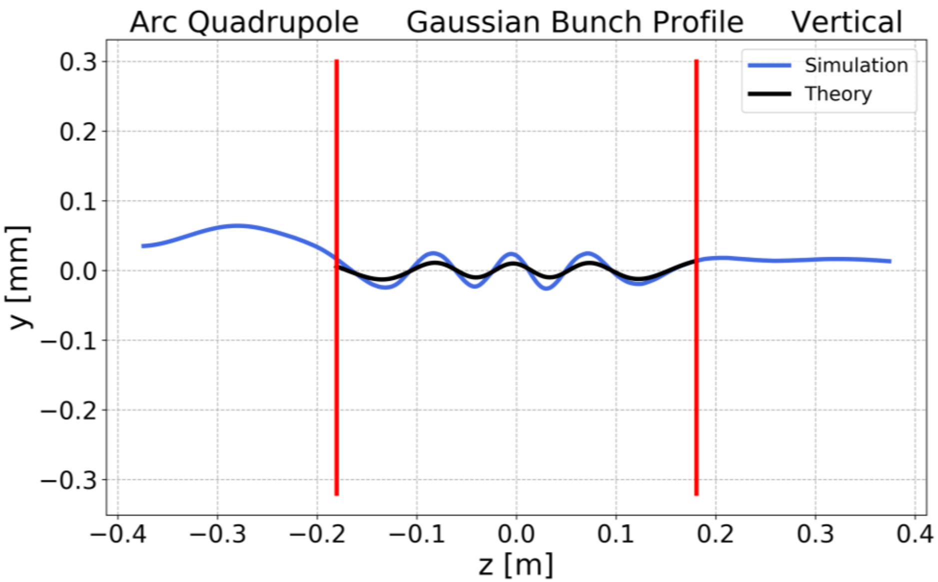
Outline

- Theoretical Oscillation
- Uniform Longitudinal Profile
- **Gaussian Longitudinal Profile**
  - Drift Space
  - Dipole
  - **Arc Quadrupole**
- Conclusions

Gaussian Longitudinal Profile: Quadrupole

$A_{x\text{goal}} = 0.1$ mm $\quad A_{y\text{goal}} = 0.0$ mm $\quad$ at bucket start

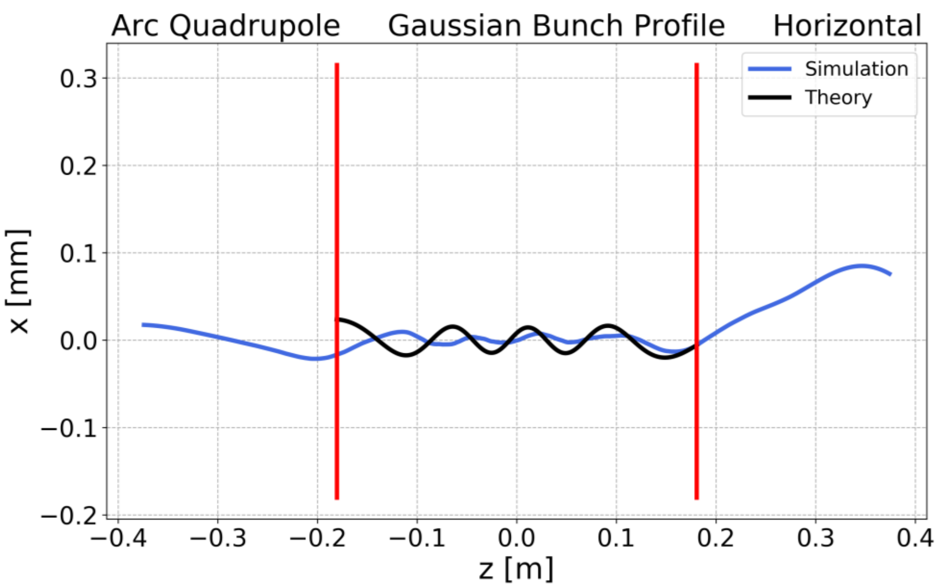


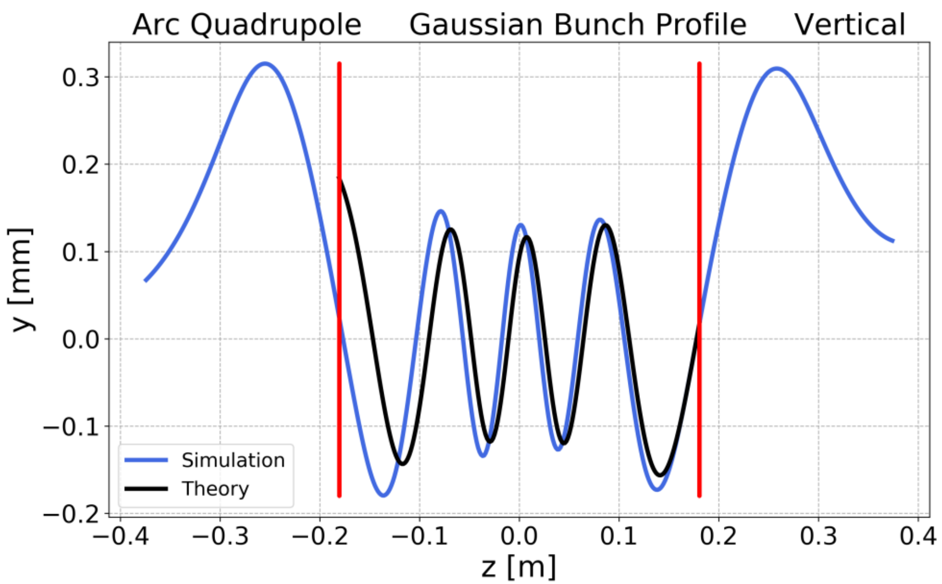


- The magnetic field force on the electrons is not negligible compared to the electric field outside the range $\pm 2\sigma_z$
- Good agreement between simulation and theoretical prediction inside the range $\pm 2\sigma_z$

Gaussian Longitudinal Profile: Quadrupole

$A_{x\text{goal}} = 0.0$ mm $\qquad A_{y\text{goal}} = 0.1$ mm $\qquad$ at bucket start

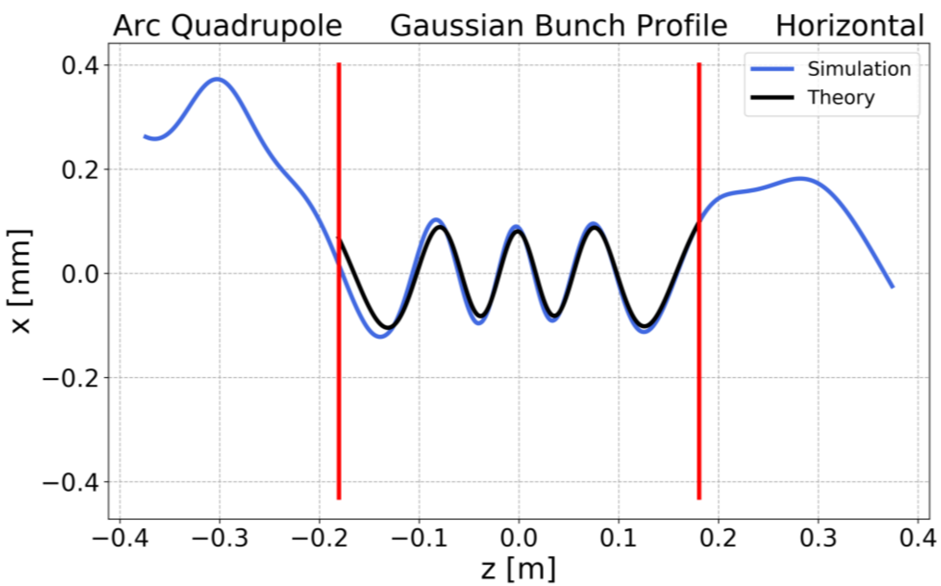


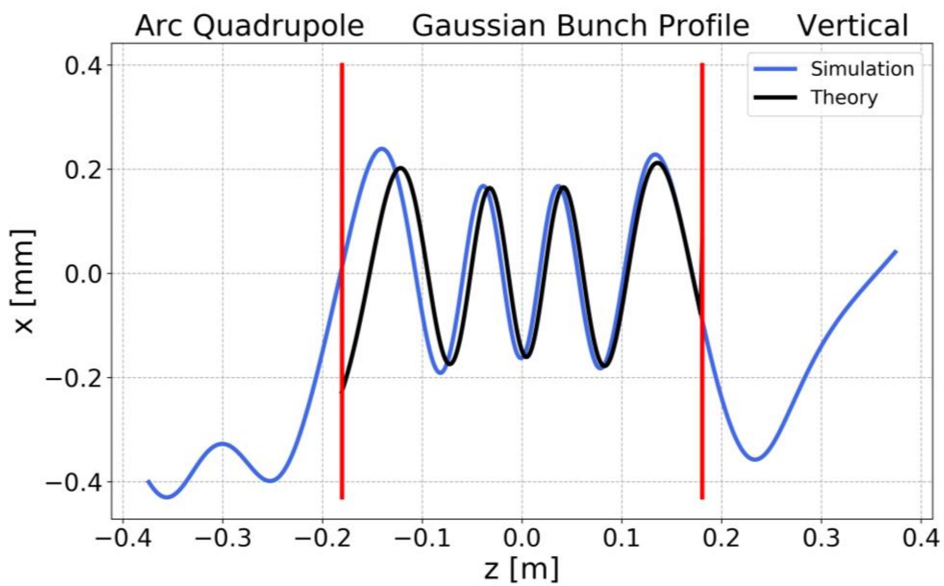


- Good agreement between simulation and theoretical prediction inside the range $\pm 2\sigma_z$

Gaussian Longitudinal Profile: Quadrupole

$A_{xgoal} = 0.1$ mm $A_{ygoal} = 0.1$ mm at bucket start





- Good agreement between simulation and theoretical prediction inside the range $\pm 2\sigma_z$
- The equations work well also when the electron oscillates in both the planes

Outline

- Theoretical Oscillation
- Uniform Longitudinal Profile
- Gaussian Longitudinal Profile
- **Conclusions**

Conclusions

- No bugs in the code: we find the frequency we expect (when λ_z is uniform);
- When λ_z is Gaussian we can compute local frequencies (for estimating Δz);
- This is also valid in the presence of a magnetic field

Future Developments

- Study the oscillation of all electrons

Thanks for your attention

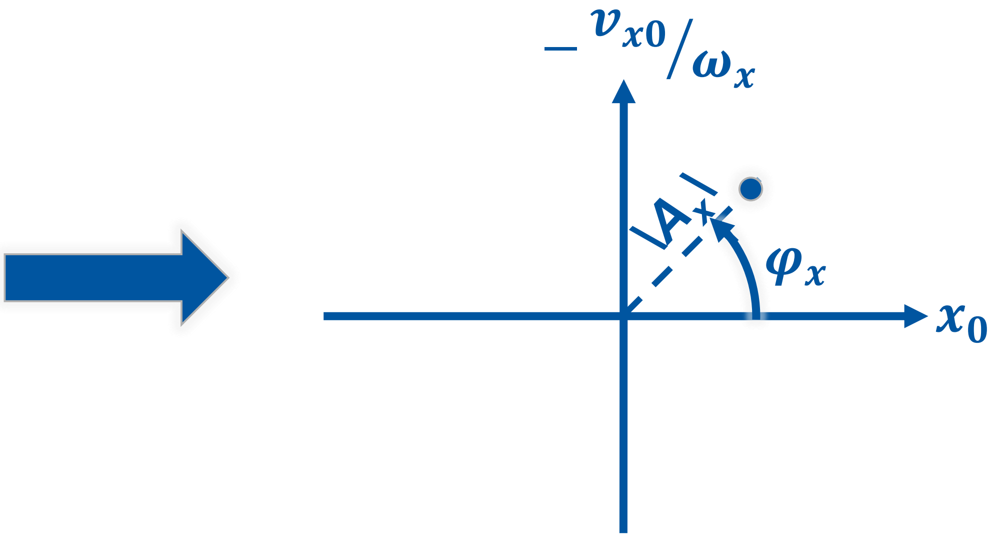
Extended Presentation

- /eos/user/l/lusabato/e_cloud_studies/milestones/2019-02-13_electron_oscillation_8.pptx

Appendix: Amplitude - Phase

$$\begin{cases} {A_x}^2 = {x_0}^2 + \left(\frac{v_{x0}}{\omega_x}\right)^2 & \textbf{(12)} \\ \varphi_x = -arctan_{IV}\left(\frac{v_{x0}/\omega_x}{x_0}\right) = \begin{cases} -arctan\left(\frac{v_{x0}/\omega_x}{x_0}\right) & x_0 > 0 \\ \pm\pi - arctan\left(\frac{v_{x0}/\omega_x}{x_0}\right) & x_0 < 0 \end{cases} & \textbf{(13)} \end{cases}$$

$$\begin{cases} A_x \cos(\varphi_x) = x_0 & \textbf{(10)} \\ A_x \sin(\varphi_x) = -\frac{v_{x0}}{\omega_x} & \textbf{(11)} \end{cases}$$



Appendix: Invariant (2.1)

An Ordinary Differential Equation (ODE) of the second order can be written as a system of two ODE of the first order:

$$\ddot{x} + {\omega_x}^2 x = 0 \qquad \textbf{(6)} \qquad \Rightarrow \qquad \begin{cases} \dfrac{dx}{dt} = v_x & \textbf{(6.1)} \\ \dfrac{dv_x}{dt} = -{\omega_x}^2 x & \textbf{(6.2)} \end{cases}$$

α*(6.1) + β*(6.2): $$\alpha \frac{dx}{dt} + \beta \frac{dv_x}{dt} = \alpha v_x - \beta {\omega_x}^2 x$$

$$\frac{d}{dt}(\alpha x + \beta v_x) = \alpha v_x - \beta {\omega_x}^2 x$$

In order to have this quantity $(\alpha x + \beta v_x)$ constant with the time:

$$\alpha v_x - \beta {\omega_x}^2 x = 0 \qquad \Rightarrow \qquad \beta = \alpha \frac{v_x}{{\omega_x}^2 x} \qquad \textbf{(21)}$$

Substituting (14) in the invariant:

$$(\alpha x + \beta v_x) = \frac{\alpha}{x}\left(x^2 + \frac{{v_x}^2}{{\omega_x}^2}\right) \qquad \textbf{(22)}$$

Appendix: Invariant (2.2)

In order to have the same invariant (13), we can choose α:

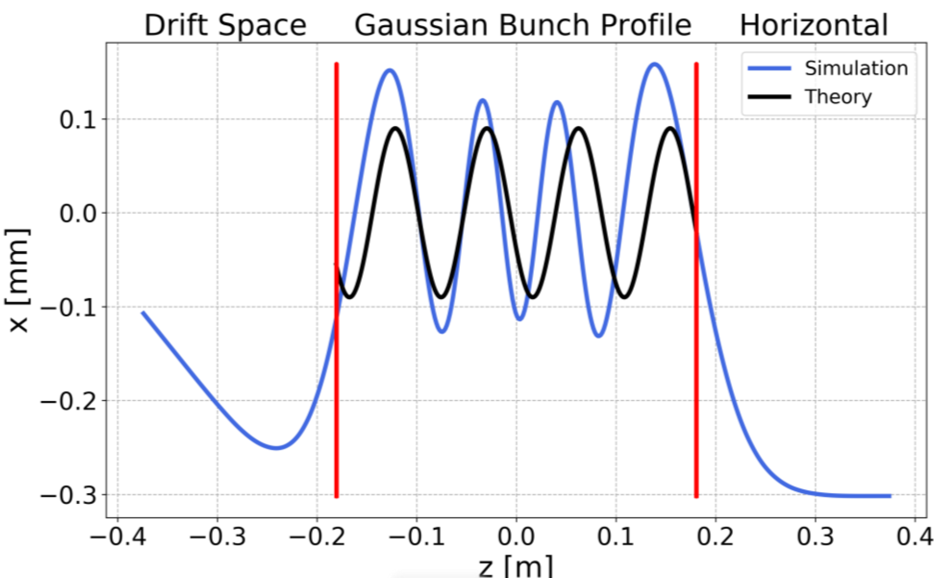
$$\alpha = \frac{1}{2} m_e {\omega_x}^2 x \quad \textbf{(23)} \quad \Rightarrow \quad \beta = \frac{1}{2} m_e v_x \quad \textbf{(24)}$$

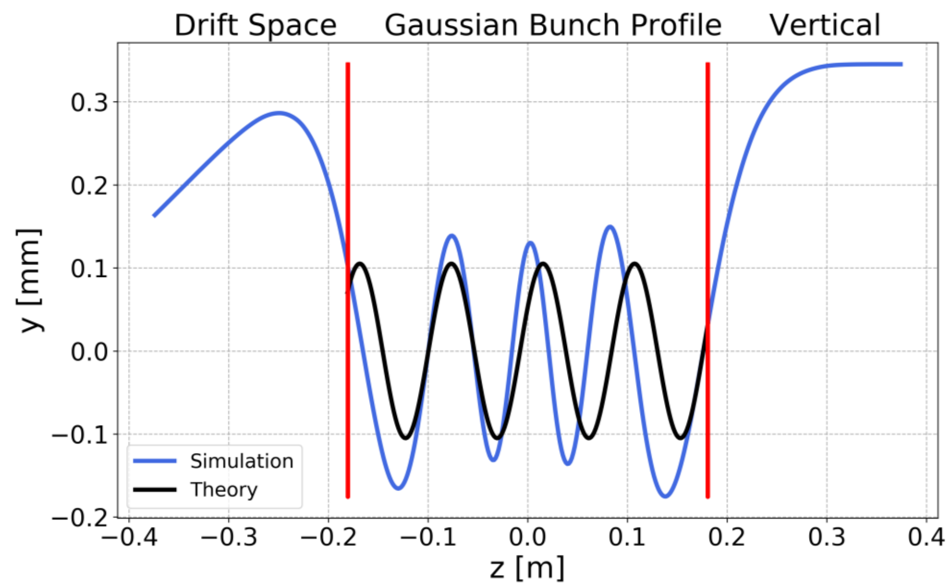
$$(\alpha x + \beta v_x) = \frac{1}{2} m_e {\omega_x}^2 \left(x^2 + \frac{{v_x}^2}{{\omega_x}^2} \right) \quad \textbf{(25)}$$

✔

Gaussian Longitudinal Bunch Profile: Drift Space

$A_{x\mathrm{goal}} = 0.1 \text{ mm}$ $\quad A_{y\mathrm{goal}} = 0.1 \text{ mm}$ $\quad$ at bunch start





$$x(z) = A_x \cos\left[\frac{\omega_x}{c}(z - z_0) + \varphi_x\right] \qquad \textbf{(9)}$$

Oscillation Amplitude:

$$A_x^{\,2} = x_0^{\,2} + \left(\frac{v_{x0}}{\omega_x}\right)^2 \qquad \textbf{(12)}$$

Traslation in *z* axis:

$z_0 = 2\sigma_z$

Oscillation phase:

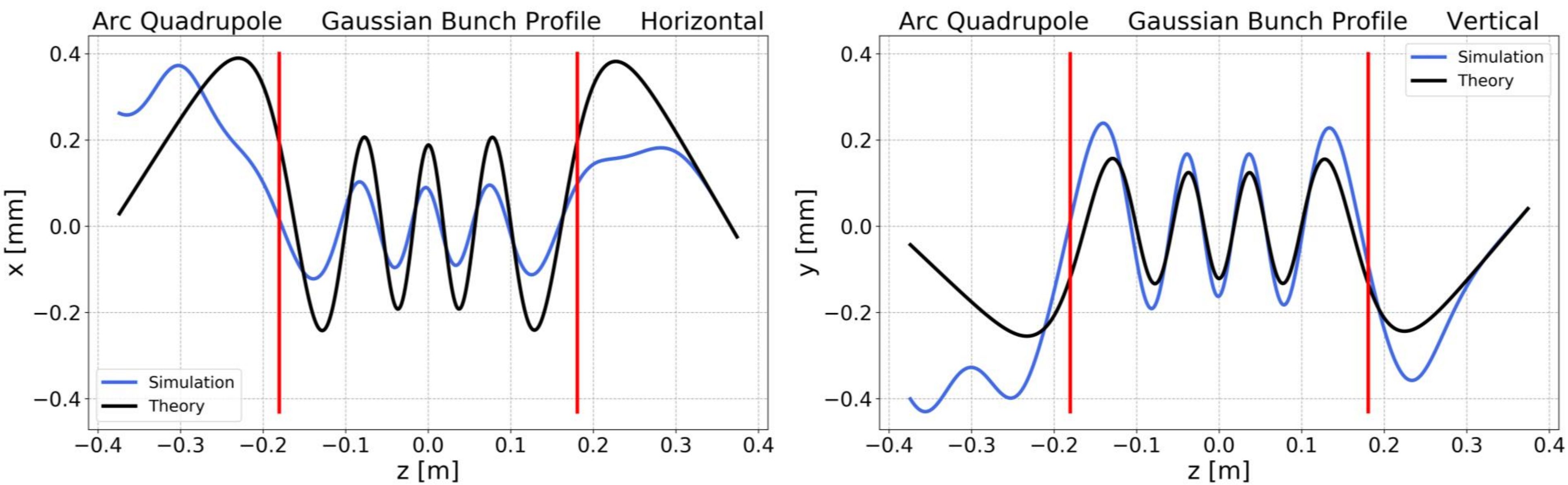
$$\varphi_x = -arctan_{IV}\left(\frac{v_{x0}/\omega_x}{x_0}\right) \qquad \textbf{(13)}$$

Oscillation Angular Frequency:

$$\omega_x = \sqrt{\frac{q_e \lambda_z}{2\pi\varepsilon_0 m_e \sigma_x(\sigma_x + \sigma_y)}} \qquad \textbf{(3)} \qquad \lambda_z = \frac{q_e N_b}{L} \qquad \textbf{(4)}$$

Gaussian Longitudinal Bunch Profile: Quadrupole

$A_{x\text{goal}} = 0.1$ mm $\quad A_{y\text{goal}} = 0.1$ mm $\quad$ at bucket start



- Discrepancy between the theoretical predictions and the simulations
- The magnetic field force on the electrons might be non-negligible compared to the electric field:
  1. considering only the part of the bunch where there are more protons (in the longitudinal centre, for example in the range $\pm 2\sigma_z$)
  2. coupling between the planes