A collection of fun numerical methods that [might] lead us to Lattice QCD

yes i think they're fun

Based on: *Lattice QCD: a practical guide*; SUPA lecture series by Christine Davies, and G. P. Lepage, *Lattice QCD for Novices,* hep-lat/0506036

Vithyaban Anjelo Narendran @ Graduate Symposium 2024

Quick Intro

Immediately starts assuming things

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- Assume you know of QCD
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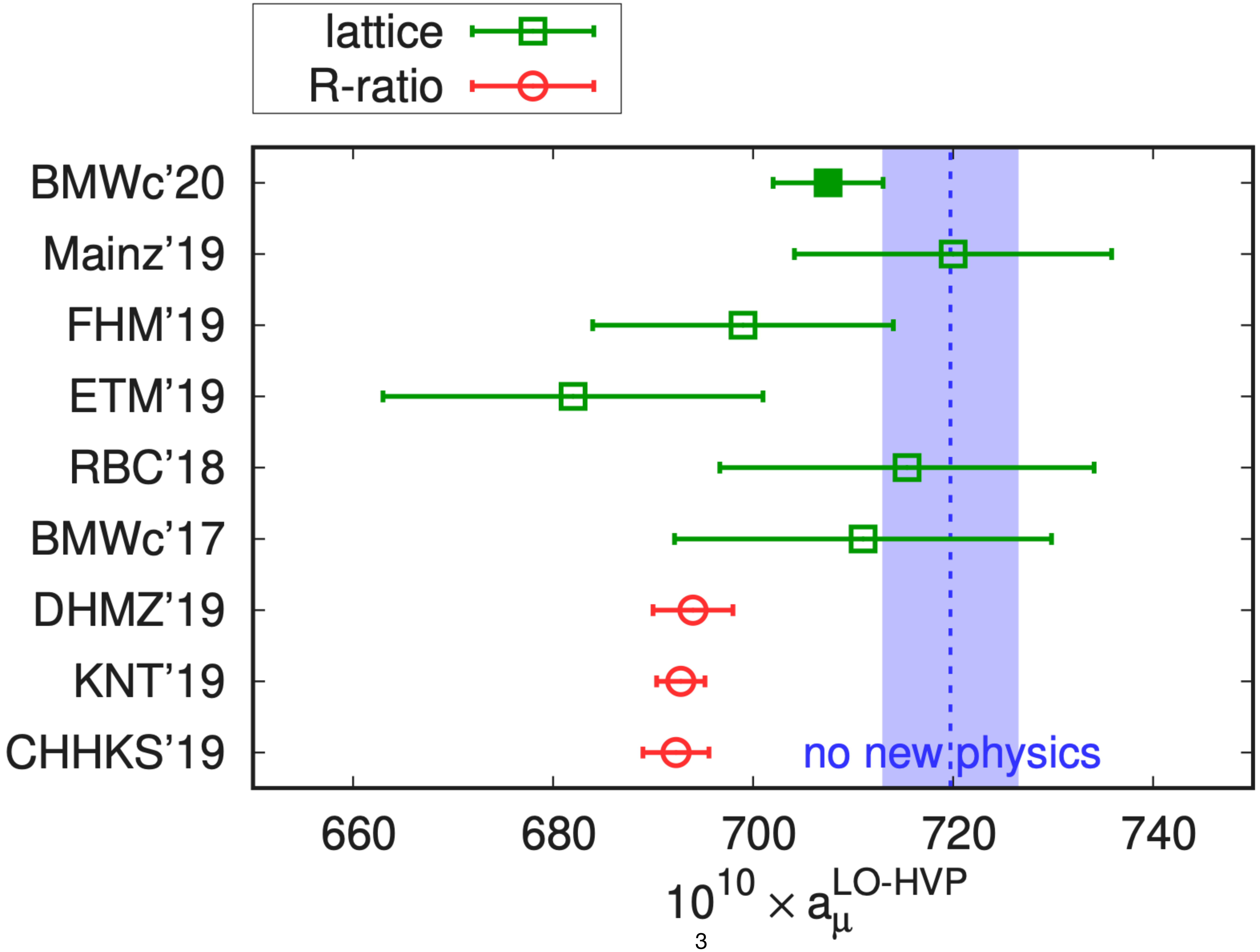
    - Power series expansion in α_s, the QCD coupling constant.
        - Blows up if α_s is large.
- Calculate numerically the properties we want to know about in QCD - masses and properties of the bound states.

Muon g-2

Houston, we have a result!



Borsanyi, S., et al. Leading hadronic contribution to the muon magnetic moment from lattice QCD. *Nature* **593**, 51–55 (2021)

Quantum Mechanics

1 particle, 1 dimension

$\hbar = c = 1$

Kinetic Energy

Hamiltonian

$$\hat{H} = \frac{p^2}{2m} + V(x)$$

Potential

$$[\hat{x}, \hat{p}] = i$$

Position and Momentum Operators don't commute

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- Solve Schrödinger's equation.
- Find Eigenfunctions, Eigenvalues
- Party!

Path Integral Formulation

Transition Amplitude and Action

$$K\left(x, t; x_i, t_i\right) = \left\langle x(t) \mid x_i\left(t_i\right)\right\rangle = \int \mathcal{D}x(t) e^{iS[x]}$$

Path Integral Formulation

Transition Amplitude and Action

Possible Path

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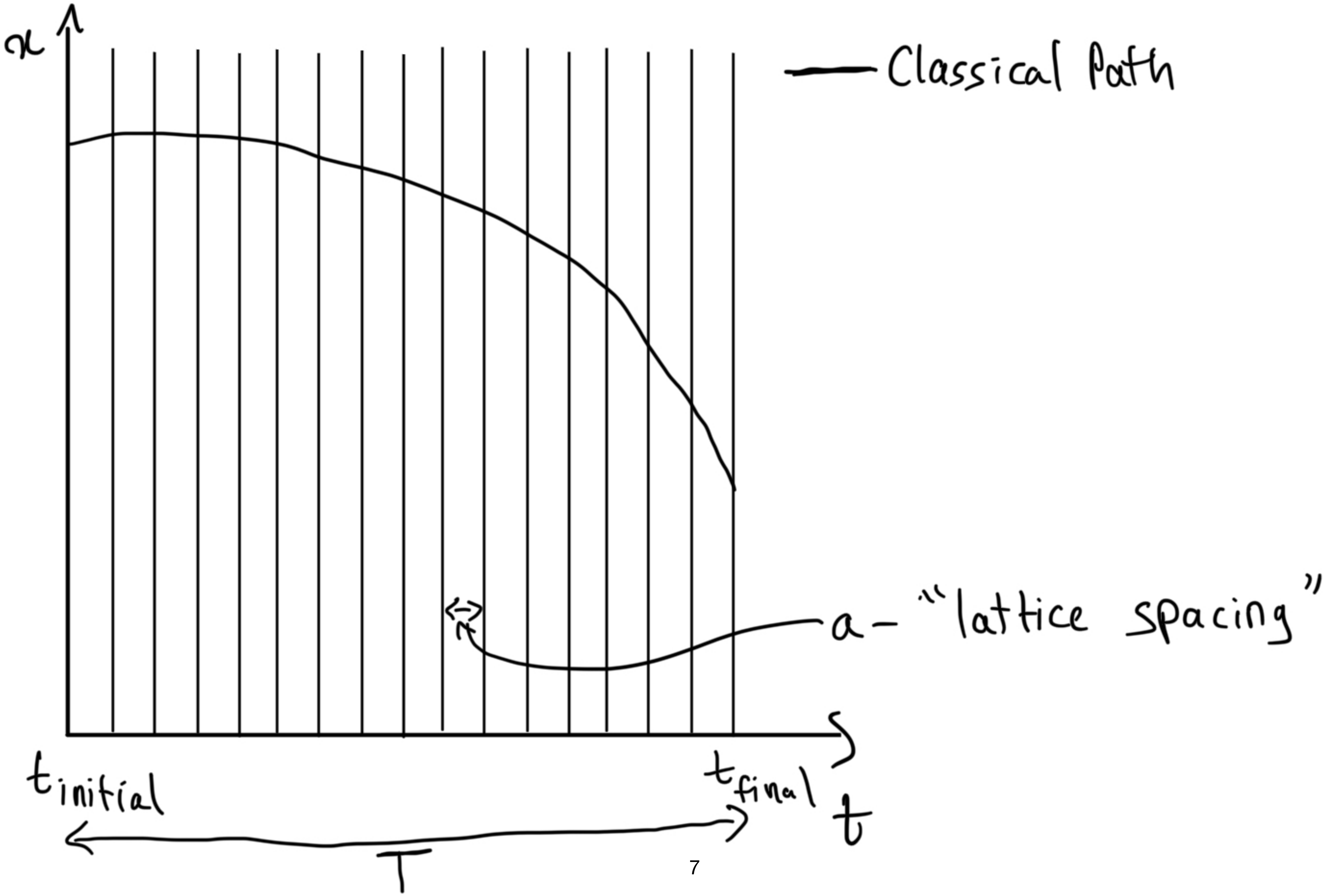
$$S[x] \equiv \int_{t_i}^{t} dtL(x, \dot{x}) \equiv \int dt\left[\frac{m\dot{x}(t)^2}{2} - V(x(t))\right]$$

Lagrangian

Classical Paths and Quantum Fluctuations



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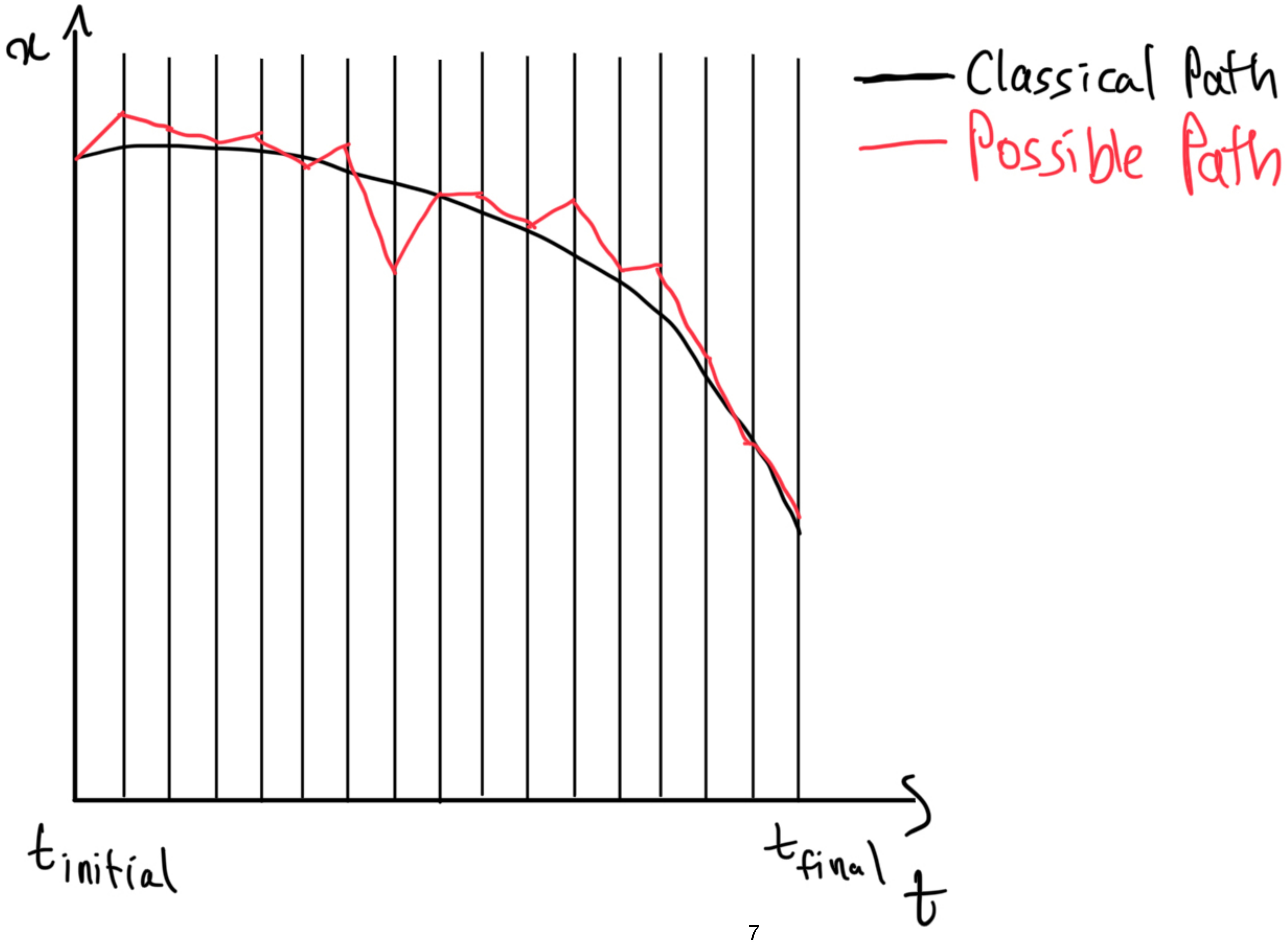

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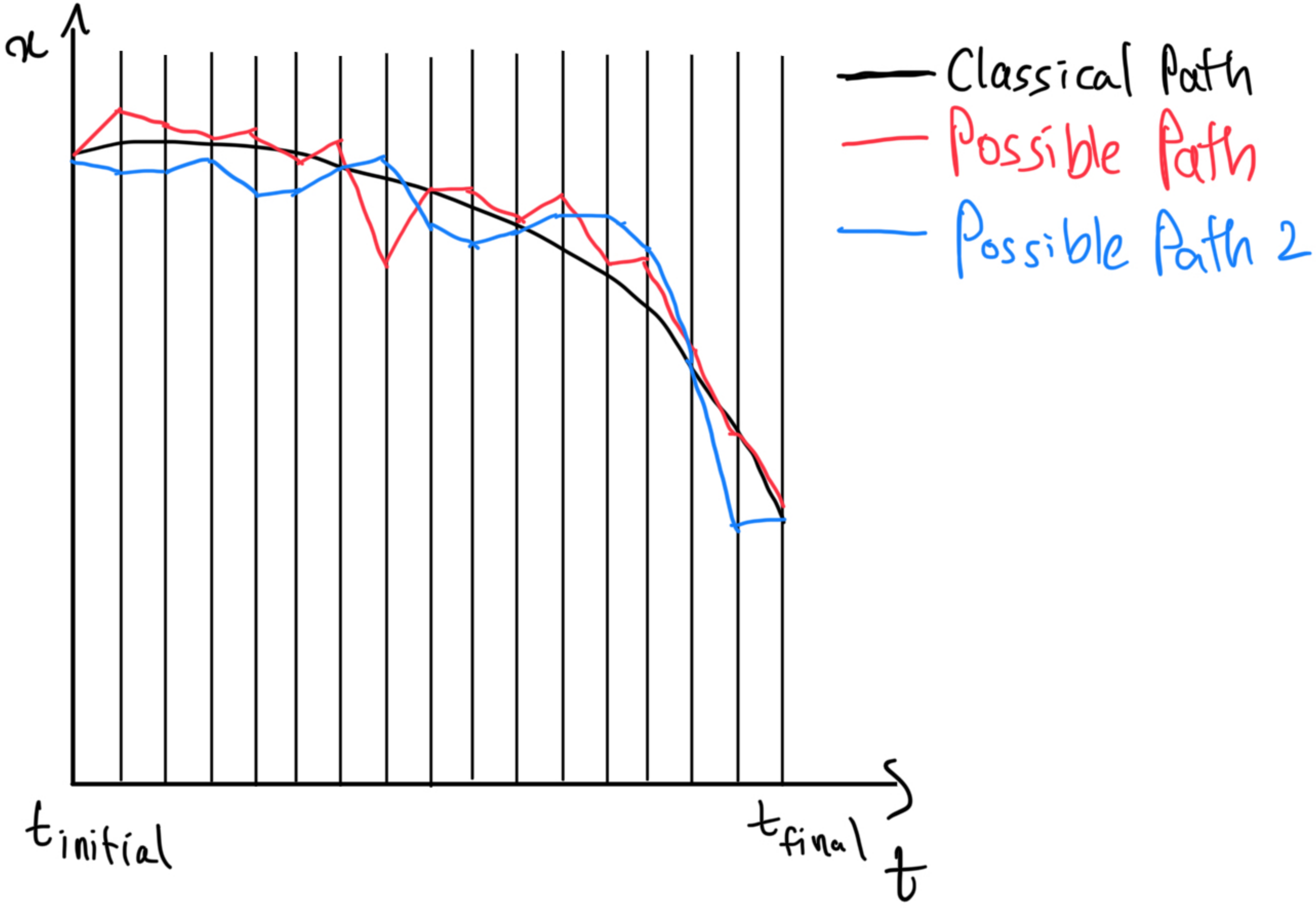

Connecting to the Eigenstates of the Hamiltonian

Inserting Hamiltonian via time evolution

$$\left\langle x \left| \exp\left(-iH(T)\right) \right| x_i \right\rangle = \sum_n \langle x \mid n \rangle \exp\left(-iE_n T\right) \left\langle n \mid \mid x_i \right\rangle$$

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Inserting Hamiltonian via time evolution

$$\sum_n \langle n \mid \mid n \rangle = 1$$

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$$= \sum_n \psi_n^*\left(x_i\right) \psi_n(x) \exp\left(-iE_n T\right)$$

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Energy Eigenstates in the Sum

$$= \sum_n \psi_n^* \left(x_i\right) \psi_n(x) \exp\left(-iE_n T\right)$$

We Rotate Time

$x_i = x_f = x$

Minkowski -> Euclidean space-time ($t \to -it$)

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If we have large T, smallest E gets picked out -> Extract E_0

Transition Amplitudes -> Eigenstates of the Hamiltonian -> Physics

Excited States

$$\frac{\left\langle x(T) \middle| x(t_2) x(t_1) \middle| x(0) \right\rangle}{\langle x(T) \mid x(0) \rangle} = \frac{\int \mathscr{D}x\, x(t_2) x(t_1) e^{-S[x]}}{\int \mathscr{D}x\, e^{-S[x]}}$$

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Average $x(t_1)x(t_2)$
on each path

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Average over all paths generated

Plug a complete set of Eigenstates in as before:

$$= \left|\left\langle E_0 | x | E_1 \right\rangle\right|^2 e^{-\left(E_1 - E_0\right)\left(t_2 - t_1\right)}, t_2 - t_1 \to \infty$$

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Matrix Element

Excitation Energy between ground and first excited state

Calculating the Path Integral Numerically

Discretise Time

Divide up the line from $t_i \rightarrow t_f$ to a set of points

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$$\int \mathscr{D}x(t) = \int dx_1 dx_2 \ldots dx_n$$

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Value of x at t_n

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Integrate over all the possible values of x for each x_j

n-dimensional integral

Calculating the Path Integral Numerically

Discretising the Action

Call $t_{j+1} - t_j = a$, the lattice spacing

$$\int_{t_j}^{t_{j+1}} dtL \approx a\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{a}\right)^2 - \frac{1}{2}\left(V\left(x_{j+1}\right) + V\left(x_j\right)\right)\right]$$

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$\dot{x}(t)^2$ between t_{j+1} and t_j

Euler Method

Calculating the Path Integral Numerically

Discretising the Action

Call $t_{j+1} - t_j = a$, the lattice spacing

A discretisation of the potential

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$\dot{x}(t)^2$ between t_{j+1} and t_j

Euler Method

Moving to QCD

The Path Integral in QCD

Quark, anti-quark fields

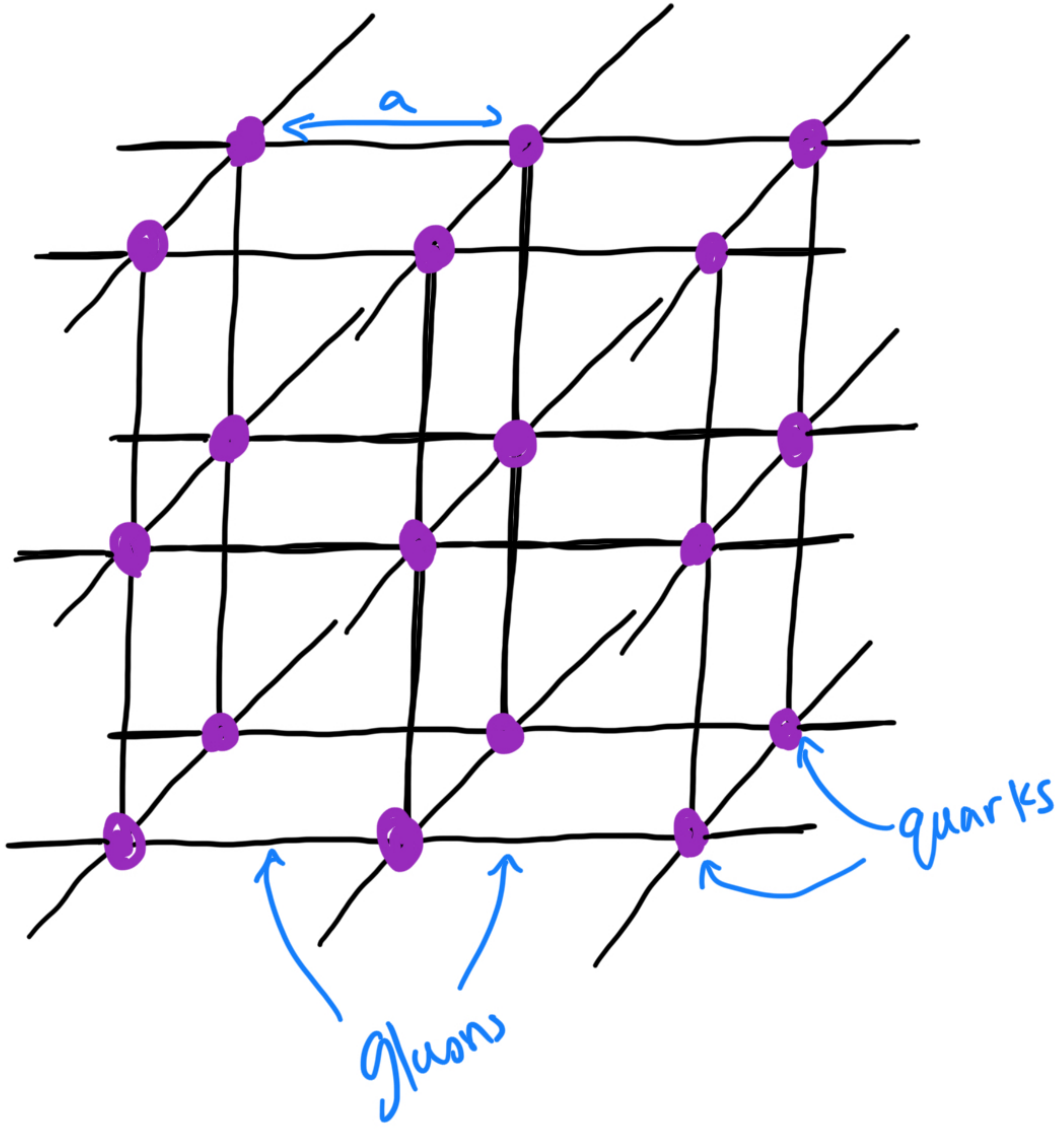
$$\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_{QCD}}$$

Gluon Fields

Weighted by the exponential of the QCD action

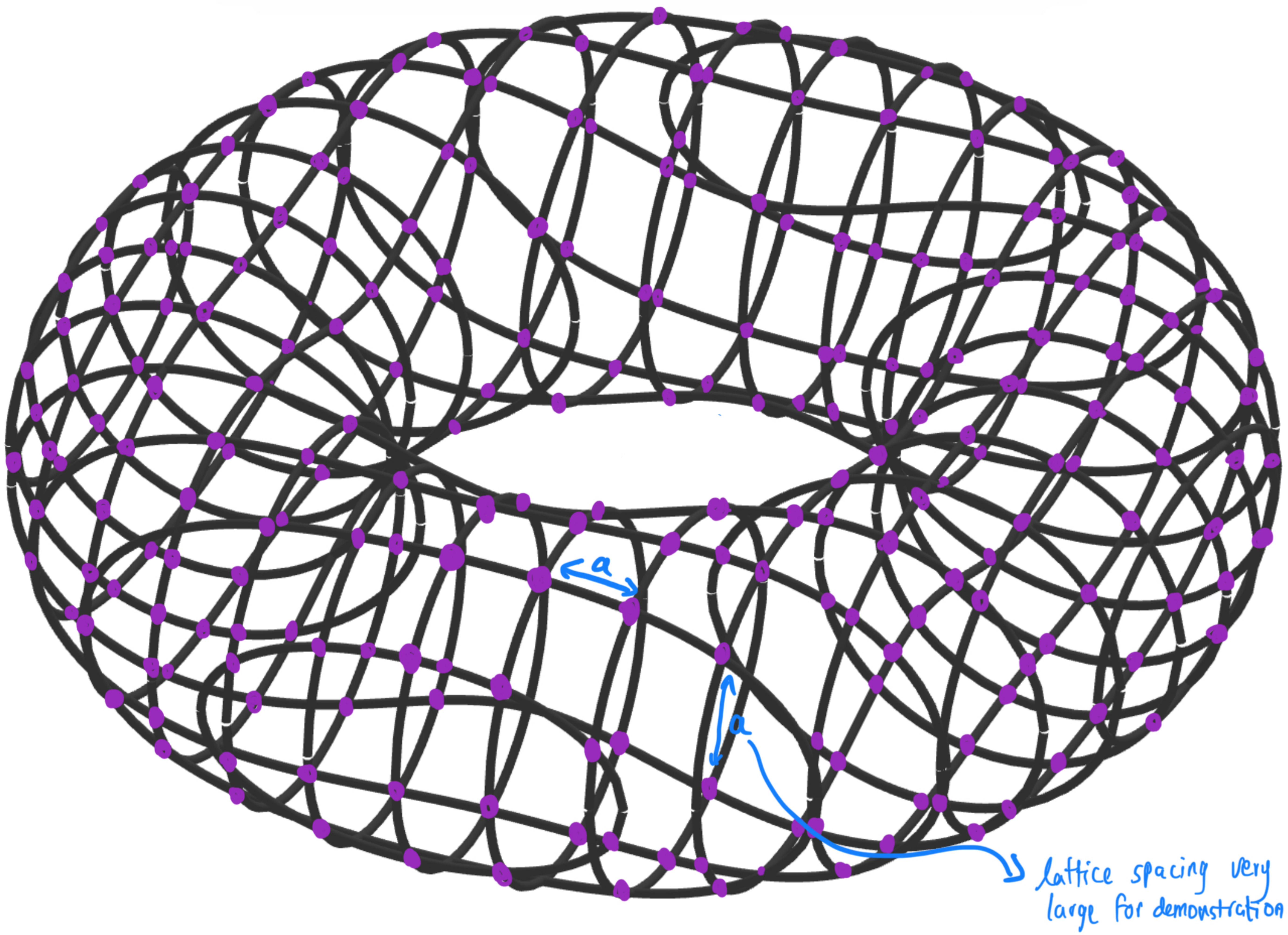
Moving to QCD

Build a Lattice



Moving to QCD

Impose Periodic Boundary Conditions



4D Doughnut?

Rendering error: Try installing additional dimensions via: sudo apt-get xtra-dims

Finishing Up

- Path Integrals for Eigenstates, and useful physics
- Calculating excited states (particles)
- Discretisation
- Make-up of the lattice
- There are so very many cool things in this field
- Big thanks to Christine Davies - Physicist in Lattice QCD

More on the Lattice

Local Gauge Invariance of QFTs

$$\mathscr{L}_{QED}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \bar{\psi}(x)(\gamma \cdot D + m)\psi(x)$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

$$D_\mu = \partial_\mu + ieA_\mu(x)$$

U(1) rotations: $\Omega(x) = e^{ie\omega(x)}$

$$\psi^{(g)} = e^{ie\omega(x)}\psi$$

$$\bar{\psi}^{(g)}(x) = e^{-ie\omega(x)}\bar{\psi}(x)$$

Rotate about a point

$$A_\mu^{(g)} = A_\mu - \partial_\mu\omega(x)$$

Photon field ‘picks up’ the difference of the gauge transformation - at the ends of a ‘Link’

More on the Lattice

QCD Lagrangian

$$\mathscr{L}_{QCD}(x) = -\frac{1}{4}F^{a}_{\mu\nu}(x)F^{\mu\nu,a}(x) + \bar{\psi}(x)(\gamma \cdot D + m)\psi(x)$$

More on the Lattice

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